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Article

Specialised Knowledge for Teaching Geometry in a Primary Education Class: Analysis from the Knowledge Mobilized by a Teacher and the Knowledge Evoked in the Researcher

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Abstract: Deepening the specialised knowledge of mathematics teachers is necessary for teaching, and it is one of our concerns as researchers; accordingly, in this study we analyze the knowledge that a teacher mobilizes when giving a first lesson on geometry to a class of fifth year of primary education at a Spanish school. We provide this analysis with a complementary perspective that leads us to broaden our focus of attention to other aspects of classroom management, thus identifying the *specialised knowledge evoked in the researcher by the opportunities* observed in the classroom. We take *opportunities* to mean classroom situations that occur as a result of the teacher's management for which the researcher considers that an alternative management of such situations would be supported by relevant specialised knowledge for the teaching of the given content. Considering the possible alternative management of specific classroom situations makes this specialised knowledge emerge, which, together with that mobilized by the teacher, generates specialised knowledge useful for guiding the content of the training of primary education teachers.

Keywords: *specialised knowledge evoked in the researcher by the opportunities*; mathematics teachers' specialised knowledge; geometry; primary education; case study

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1. Introduction

One of the lines of research in mathematics education is to deepen the knowledge of the mathematics teacher. Characterizing this knowledge from a teacher's professional practice is of fundamental interest for the authors and the working group SIDM (Research Seminar in the Didactics of Mathematics, University of Huelva) in which this research is framed. We share the view of [1] (p. 58) that, "*an essential object of investigation is what and how a mathematics teacher knows or should know about mathematics*".

Geometry in compulsory education is commonly identified with the handling of quantities, formulas and geometric shapes, focusing on measurement in primary education, and becoming analytical in secondary education. However, it is not usually related to asking questions, obtaining models and identifying relationships and structures [2]. On the other hand, various studies have revealed the difficulties experienced by prospective primary teacher (henceforth PPT) in this area [3–5], which could come from their own training in primary education and continue later in their professional development.

These antecedents show us the importance of addressing the problem of teaching and learning geometry in primary education, and the need to focus on investigating what and how a primary education teacher knows or should know about mathematics when teaching geometry. Thus, this study involves the observation of the practice of a teacher of the last year of primary education while teaching geometry, in particular, Euclid's fifth postulate, focusing on the specialised knowledge that this teacher mobilizes when teaching the observed classes. The interpretation of this knowledge is crucial to understanding *how this teacher knows mathematics*, which, in turn, provides us with information to partially answer the question: *how should one know mathematics in order to teach a class of the last cycle of Primary Education?* Based on this analysis, we provide a complementary perspective by focusing on "details" associated with the teacher's class management, such as situations or moments that emerge in the classroom as a result of explanations or interactions with students, and the use of the materials available in the classroom.

Thus, we broaden our focus of attention to include these "details", considering the specialised knowledge that all this evokes in the researcher, and allowing us to reflect on the specialised knowledge that could support an alternative management of the classroom situation, without assessing why it is not employed. This new analysis of the specialised knowledge evoked in the researcher is linked to the teacher's class management, as it arises from the teacher's own practice, and gives greater emphasis to our role as teachers trainer and researchers, in the sense that the purpose of our research is to improve teacher training.

To understand this new approach, let us look at the following example from one of the teacher's classes observed under this study. When asking students about straight lines and their relative positions, in order to elicit their previous knowledge, a girl simulated with her fingers two segments that intersected at a point. The student then spread her fingers apart vertically to obtain a representation of two crossed segments that did not, however, intersect, therefore making the third dimension ostensibly explicit. As researchers, we consider this student's gesture, which was left unaddressed by the teacher, an *opportunity* to ask ourselves what specialised knowledge would support an alternative management of this situation. Answering this question lends a greater role to our *theoretical sensitivity* [6] and to the triangulation of experts [7]. The *specialised knowledge evoked in the researcher by the opportunities* [8–10] is a construction of the researcher interwoven with the situation managed by the teacher, which faithfully reflects its nature.

In this study, we present part of a broader investigation with access to all of the geometry classes taught by a teacher in the fifth year of primary education.

2. Mathematics Teachers' Specialised Knowledge: MTSK as a Theoretical Study Lens

We adopt the definition of knowledge provided by [11], considering the knowledge of a teacher as information available to solve problems, in the broad sense of the expression. [12] highlight two fundamental elements: information available to use that discards all that does not make sense for the task being carried out; and information that is not necessarily correct that allows us to position ourselves in the search for the understanding of knowledge (it may not be showed in the observed episode, but not being showed does not necessarily imply that it is not available). In this study, this question becomes of special interest, as, in observing the class, we try to identify: (a) the specialised knowledge *mobilized by the teacher*, without judging whether or not it is correct; and (b) the *specialised knowledge evoked in the researcher by the opportunities*. On the other hand, as we are observing practice, the fact that certain knowledge is not mobilized does not mean it is not possessed, as it may constitute knowledge that the teacher decides not to use.

Based on the work of [13,14], ref. [15] refined their proposals and formulated the analytical model Mathematics Teachers' Specialised Knowledge (MTSK), which considers specialization as the axis of knowledge of the teacher of mathematics in all its domains, subdomains and categories, and does not refer to any other science or profession, considering such professional knowledge to be what the teacher needs and uses [11] by the nature of mathematics teaching. On the other hand, it is rooted in mathematics itself, leaving out aspects of Pedagogy and general Psychology, it includes beliefs and the affective domain, and it covers all educational levels, from early childhood education [16,17] to university education [18].

MTSK divides knowledge into three main domains: *Mathematical Knowledge* (MK), *Pedagogical Content Knowledge* (PCK), and, finally, *beliefs on mathematics and mathematics teaching and learning*, which are not considered in this article (Figure 1).

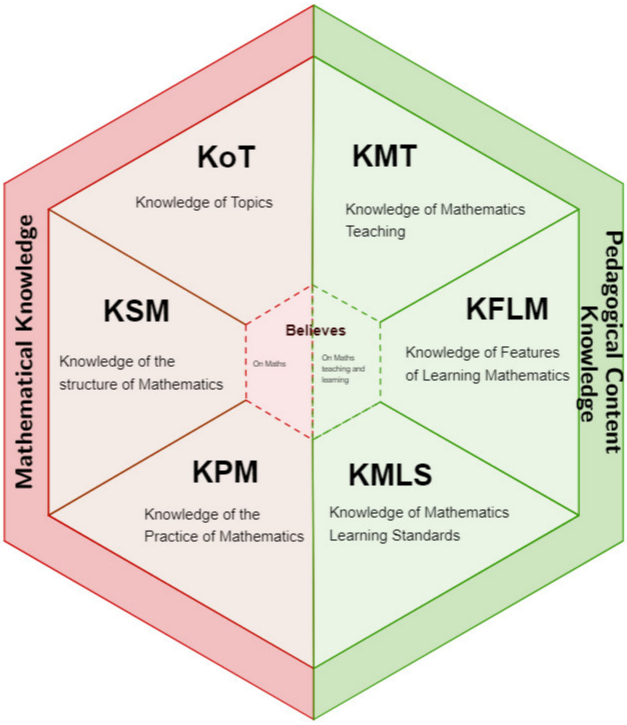


Figure 1. Mathematics teachers' specialised knowledge (elaborated from [15]).

Mathematical Knowledge (MK) integrates the knowledge of the mathematical discipline that is taught, in our case, geometry. It considers three sub-domains: (deep) knowledge of the mathematical content itself and its intra-conceptual relationships (*Knowledge of Topics*, KoT); knowledge of the inter-conceptual relationships between these knowledges (*Knowledge of the Structure of Mathematics*, KSM); and knowledge of how to produce and proceed in mathematics (*Knowledge of Practices in Mathematics*, KPM).

The *Knowledge of Topics* (KoT) considers the knowledge of mathematical contents, processes and procedures in themselves, approached from what [19] conceives as deep knowledge of fundamental mathematics. Taking as references the areas (numbers and operations, algebra, geometry, measurement, data analysis, and probability) proposed by [20], it achieves an international characterization that might vary slightly according to the curriculum of each country. A teacher must know contents, processes and school procedures, and the different situations in which they are presented, not only in relation to the mathematical discipline, but also as interrelated elements, understanding their properties and the processes that justify them, and the possible representations and procedures, from how to when and why.

We distinguish four specific categories: *Definitions, properties and foundations*; *phenomenology*; *registers of representation* and *procedures*. The category of *definitions, properties and foundations* considers the set of properties that make a topic definable and give it sense and meaning, in addition to alternative ways of doing so. For example, in this category, we can place knowledge of the properties of regular polygons, which, in turn, define them, or the definition of a convex angle, as opposed to a concave angle, derived through its properties. Within this sub-domain, the model considers intra-conceptual connections, that is, relationships between concepts or processes belonging to the same topic. For example, the definition of perpendicularity between straight lines, which implies the division of the plane into four regions, or the relationship that exists between the definition of angle and their classification according to position and measure (e.g., opposite angles, congruent angles), which allows us to demonstrate the property of the sum of the magnitudes of the interior angles of any triangle.

Phenomenology, the second category we consider in the KoT sub-domain, encompasses the teacher's knowledge of the phenomena relating to a topic that can generate mathematical knowledge and its uses and applications. We can interpret phenomenology from two different perspectives, one that identifies the knowledge that a teacher has about the applications that a topic can have, and one that the teacher has about models attributable to a topic. For example, we can consider the Pythagorean theorem as knowledge that allows us to construct algebraic irrational numbers, as we can also consider in this category the differentiation between the place value of figures in a number expressed in the Decimal Numbering System (DNS) and the grouping value; that is to say, in the number 2748, we can say that the number 7 occupies the hundreds place, which means that when regrouping from thousand to thousand, 7 groups of a hundred have remained without generating a group of a thousand.

Registers of representation bring together the teacher's knowledge of the different ways in which a topic can be represented [21], and the transformations between them, as well as associated notation and vocabulary. For example, if we treat a straight segment, we can draw it based on its properties, define it verbally with those properties, find elements of reality that could represent them, and use algebraic language to describe it analytically in relation to a coordinate system, whether Cartesian, rectangular, polar, etc. Included here is the knowledge that allows us to move from one representation to another without loss of its properties or characteristics, even within the same register, identified by [22] as mathematical flexibility. In addition, there is also the knowledge that certain mathematical constructs are not representable if not through cognitive metaphors [23].

In the *procedures* category, we have know-how, the practical knowledge of mathematics, which includes the indicators of *how, when* and *why* and the *characteristics of the result*. For example, knowledge of the decimal metric system and its structure would be behind the knowledge of all the previous indicators in the procedures for changing units of measurement, and in the consequent differences between the changing of units corresponding to the measurement of the magnitudes of length, surface area and volume (we consider the magnitudes of length, surface area and volume as measurable characteristics of geometric constructs approached from the composition and decomposition of said constructs, which is why we include them in the knowledge of geometry [3]).

Knowledge of the Structure of Mathematics (KSM) contemplates connections, referring to the teacher's knowledge of the relationships between different contents [12]. This subdomain has its origin in the work of [24], also considered for the development of Horizon Content Knowledge (HCK) subdomain in MKT model [14]. HCK considers all types of connections, including intra-conceptual ones and connections to content from other disciplines (HCK with regard to topics [25]). MTSK includes intra-conceptual connections in KoT subdomain (see KoT description above), and inter-conceptual ones in KSM, but connections to content from other disciplines are excluded. We distinguish four categories, referred, as indicates before, all of them to inter-conceptual connections, within this subdomain. *Connections of increased complexity* relate current contents with others that will be part of later moments in education. We can see clear examples in kindergarten teachers or the first grade of primary education when the teacher works on the designation of objects and sets and on classification with an eye on, for example, natural, rational or real numbers. *Connections of simplification* relate current contents with others that have already been covered, though not necessarily from the previous school year. An example of this would be the relationship that be drawn between the definition of the geometric figure of a circular sector and the interpretation of a fraction of the surface of a circle delimited by two radii (part of a whole, measurement). *Transverse connections* are those existing between different contents that have qualities in common. For example, the relationship that can be established between the distance function on a plane and in space, and the definition of the heights of a triangle and the equivalence existing between the area calculated with one or another height. *Auxiliary connections* are established between knowledges that are related to each other as support for a certain purpose. For example, using the division of a circumference into arcs of equal length in order to construct a regular polygon, or using two parallel segments to generate all the quadrilaterals that have at least two parallel sides.

Knowledge of Practices in Mathematics (KPM) is directly related to Shulman's syntactic knowledge [13]. These practices refer to the actions put into play when a mathematical action is carried out. Thus, we will consider argumentation [26] in the *knowledge of the processes associated with problem-solving as a means of producing mathematics* and *knowledge of ways of validating and demonstrating*, that is, mainly how results are demonstrated, the fundamental difference in illustrating one or more cases in which a statement is fulfilled, and the role of examples and counterexamples in the validation of results and in the generation of definitions (*forms of validation*). The *role of symbols and use of formal language*, *hierarchy and planning as a way of proceeding with the resolution of mathematical problems*, *the particular procedures for mathematical work*, such as modelling, and the *necessary and sufficient conditions for generating definitions* are the remaining indicators of this sub-domain. Table 1 shows the categories and indicators of the domain MK.

Table 1. Categories and indicators of the MK domain [15].

Subdomains	Categories/Indicators [1]
	Categories
Knowledge of Topics KoT	Procedures: How, why and when to do, and characteristics of the result
	Definitions, properties and foundations
	Phenomenology Registers of representation
Knowledge of the Structure of Mathematics KSM	Connections of increased complexity
	Connections of simplification Transverse connections Auxiliary connections
	Indicators
Knowledge of Practices in Mathematics KPM	*Knowledge of the processes associated with problem-solving as a means of producing mathematics* *Knowledge of ways of validating and demonstrating* *Role of symbols and use of formal language* *Hierarchy and planning as a way of proceeding with the resolution of mathematical problems* *Particular procedures for mathematical work* *Necessary and sufficient conditions for generating definitions*

[1]: Categories and indicators are tools proposed by the analytical model MTSK to operationalize the identification and characterization of each subdomain. Categories are identified in all subdomains, except for KPM, which remains at the indicator (are expressed in italics in order to distinguish from categories) level, as it is still in the characterization process.

In *Pedagogical Content Knowledge* (PCK), we distinguish three sub-domains, referring to the knowledge of mathematics teaching, the characteristics of the learning process of students and learning standards.

Knowledge of Mathematics Teaching (KMT) considers knowledge of aspects relating to the teacher's main activity: Teaching, professional practice directly linked to the classroom, although this model also considers other contexts and situations specific to the work of the teacher and that have the same importance as part of the teacher's development as an education professional [27] (p. 46). This subdomain arose as a result of the reflection on Knowledge of Content and Teaching (KCT) in the MKT model [14]. For example, this subdomain considers the knowledge of metaphors [23,28] as a powerful resource for the teaching learning process [19] (to see more details of its evolution and foundation see [15]).

We distinguish three categories within this sub-domain. The first is *theories of mathematics teaching* associated with mathematical content, which can be personal or formal, and involve knowledge based on observation and experience, or published research relating to mathematics teaching. The category of *teaching resources (physical and digital)*, on the other hand, represents knowledge of the mathematical characteristics (mathematical potential and limitations) of resources and materials. For example, the limitation of the non-isometric rectangular geo-plane in representing equilateral triangles. Finally, *strategies, techniques, tasks and examples* lead us to knowledge that the teacher has of the adequacy of these elements given the teaching intention at a given moment, including both the potential for the teaching of mathematics that certain sequences of tasks and examples may have and their limitations, and the obstacles generated in a particular group of students.

Knowledge of Features of Learning Mathematics (KFLM) considers the knowledge of aspects relating to the learning of the students, and the chosen categorization is supported by the initial proposal of [13]. Here, we distinguish four categories. The first, *theories of mathematical learning*, whether they are formal or personal theories of the observed teacher, considers the teacher's knowledge about the students' ways of learning mathematical content, including theories on cognitive development for mathematics. The second, s*trengths and weaknesses in learning mathematics*, represents the teacher's knowledge about the strengths of students concerning mathematical content, but also about errors, obstacles and difficulties in the students' mathematical thinking. Also included in this category would be knowledge about strengths deriving from potentialities that may be exploited according to the class group, for example, considering whether they have a facility for a certain type of representation, such as symbolization for the calculation of the area of a surface, and its use to obtain this measure.

The *ways students interact with mathematical content* relate to the teacher's knowledge about the procedures and strategies of the students, including knowledge about the language and representations that they usually use. The fourth and last category refers to *emotional aspects of learning mathematics*, including the interests and expectations of students with respect to mathematics, as well as whether they consider a specific content, process or procedure easy or not.

Knowledge of Mathematics Learning Standards (KMLS) considers knowledge of both the official curriculum in force in each country at a given time and the standards defined by research groups or professional mathematics teaching and learning associations (such as the National Council of Teachers of Mathematics—NCTM). The temporal aspect of this sub-domain is motivated by the curriculum itself, that is, what content is situated in a given course. We distinguish three categories within this sub-domain. *Expected learning outcomes* include the teacher's knowledge of what is expected, in the context of the official curriculum, for a student to learn in a given course. *Expected level of conceptual or procedural development* refers to knowledge about the level to be reached for that content at a specific academic time. An example of this are the different contexts in which the subtraction of numbers is treated, and the different properties that are gradually added, from natural numbers to real numbers. *Sequencing of topics* a certain school moment differs from the KSM by referring to a temporal rather than conceptual question. Table 2 shows the categories of the domain PCK.

Table 2. Categories of the PCK domain [15].

Subdomains	Categories [1]
Knowledge of Mathematics Teaching KMT	Theories of mathematics teaching Teaching resources (physical and digital) Strategies, techniques, tasks and examples
Knowledge of Features of Learning Mathematics KFLM	Theories of mathematical learning Strengths and weaknesses in learning mathematics Ways students interact with mathematical content Emotional aspects of learning mathematics
Knowledge of Mathematics Learning Standards KMLS	Expected learning outcomes Expected level of conceptual or procedural development Sequencing of topics

[1]: Categories are tools proposed by the analytical model MTSK to operationalize the identification and characterization of each subdomain.

3. Methodology

The trigger for this study is our interest as researchers and teacher educators in understanding the specialised knowledge desirable for teaching geometry in primary education. To understand this knowledge, we consider that the classroom of a teacher with many years of experience is a privileged environment, and, therefore, we access Jimena's classroom with her 25 students in the 5th year of primary education. Jimena is a teacher specialised in social sciences, with 35 years of experience divided between early childhood education and primary education. She has a special interest in the teaching of geometry at various stages, believing that it is largely forgotten in various courses, as it is usually taught towards the end of the year, and that it is one of the topics that can be most contextualized in the students' environment.

Our aim was to understand the specialised knowledge needed to teach geometry. However, during the study, we observed that focusing only on the interpretation of the knowledge mobilized by the teacher was not enough. During the observation, various situations occurred, sometimes attended to by Jimena, sometimes not, but always a consequence of the teacher's management of the class. We consider that, if she had decided to attend to them, geometric concepts typical of the educational stage could have been addressed, which, together with the knowledge actually mobilized, would allow access to further specialised knowledge needed to teach geometry at that stage. For this reason, we introduced the methodological element of *opportunities*, understood as those moments or situations that arise in the class, as a consequence of the actions of the teacher, and that allow the researcher to reflect on the specialised knowledge that would have supported an alternative management of those same situations.

Thus, emerges the *knowledge evoked in the researcher by the opportunities*, as specialised knowledge that the researcher interprets as being able to support an alternative management of the *opportunities* arising in the class. This interpretation is the consequence of a dialogic relationship [10] between the theoretical proposals, which make up the *theoretical sensitivity* of the researcher [6], and the reality of the fifth-year primary education class that we observed. The triangulation of experts [7] is a key validation tool in this interpretation.

Consistent with this methodological positioning, we formulate the following research objectives: (a) identify the specialised knowledge for the teaching of geometry that Jimena mobilizes in her class, in particular, when she deals with Euclid's fifth postulate; and (b) interpret the specialised knowledge that could have supported an alternative management of Jimena's class when dealing with such content.

We position ourselves in a paradigm close to the interpretive one [29], aware of the special role we play as researchers. We adopt an instrumental type case study research design [30], as we intended to understand a singular reality (Jimena's classroom with her 5th-year primary school students as a particular case for studying the specialised knowledge for teaching geometry), in depth, and in its natural environment (the classroom within its school, without modifying its natural course). This deep understanding is achieved via an initial observation of the specialised knowledge mobilized, subsequent interpretation of *the knowledge evoked in the researcher by the opportunities*, and validation by the triangulation of experts [7].

The information gathering technique used for this study is non-participant classroom observation, which is carried out in the class described, using a video recording of the class and taking field notes [31]. For this study, we have selected the beginning of the first lesson on geometry in 5th year of primary education. This lesson stops at a contextualization that the textbook proposes at the beginning of the topic, centered on a reading that tells the story of a hypothetical disciple of Euclid dealing with the following question, “Did you know that, through a point outside a path, only one parallel path can be drawn?”.

The analysis of the information begins with the transcription of the session. In it, the vocabulary and expressions used by the actors are faithfully reproduced, including the gestures of the students or the teacher when they are considered significant for the analysis.

We have carried out a detailed interpretive analysis [32], taking as a reference the corresponding field notes that, by themselves, are already the researcher’s interpretation [31].

We consider *evidence of knowledge* when the researcher interprets that there is sufficient certainty to indicate that the informant possesses such knowledge [33]. We also see an *indication of knowledge* when there is a suspicion of it, which could become evidence upon further investigation. In this study, in addition, we rely on *opportunities*, as already defined.

The analysis of specialised knowledge identified in Jimena’s classroom has been carried out following the natural course of the class, in two phases. In the first phase, we have identified the knowledge mobilized by Jimena, from *indications* and *evidences*. Such degrees of certainty allow us to respond to the first objective of the investigation: To understand the specialised knowledge in geometry that the teacher mobilizes during our observation.

In the second phase, we have identified the *opportunities*, and we have interpreted the specialised knowledge that would allow alternative management (*knowledge evoked in the researcher by the opportunities*), responding to the second objective of the investigation. The research team participates collaboratively, accomplishing the triangulation of experts [7] in the whole process.

The analysis instrument has been the MTSK model itself, considering its domains, sub-domains, categories and indicators [15].

Finally, we integrated both analyses to obtain the mapping of the MTSK in geometry in relation to the content selected in the observed 5th-year class. In this way, we increasingly moved away from the specific data to finally obtain a global vision of the whole.

3.1. Description of the Session: Only One Parallel Path Can Be Traced through a Point Outside a Path

We present a part of the transcript of the first lesson on geometry in the 5th year of primary education that Jimena is teaching (Table 3). They begin by reading the introductory text of the corresponding chapter of the textbook, where a fictitious conversation between Euclid and a supposed disciple is dramatized, in which they argue about the affirmation that, *through a point outside a path, only one parallel path can be traced*.

Table 3. Part of the transcript of the first lesson on geometry.

(They start by reading the introductory text).

Teacher (T): Where does it say it? There, at the end, at the end, "By the way, did you know that, through a point outside a path, only one parallel path can be drawn?" Let's see, if someone thinks for a moment to draw, because now we are dealing with geometry and need to think about what we are seeing. A path (indicates a path with two fingers) and then a point, and he says that I can only draw one parallel straight line. Is that true?

Student (S): Yes.

T: Doesn't anyone see a path in their head?

S: Yes, I do.

T: It's so simple . . . A path, a point (draws two parallel segments on the board, simulating a path, and an exterior point).

S: Ah, I thought it was a path (pointing to the drawing on the board).

T: Sure, outside... How? Have I made a mistake? Have I not read it well . . . Yes. Outside, out of the path. It says that we can only draw one parallel straight line.

S: Yes, yes. There it is a parallel path (points to the drawing of the two parallel segments without considering the point).

T: Let's see, S., a path, if at an outside point, outside the path, we can only draw one parallel line. Yes or no? I'm going to start drawing a lot of lines (draws lots of segments that go through that point).

S: Straight lines?

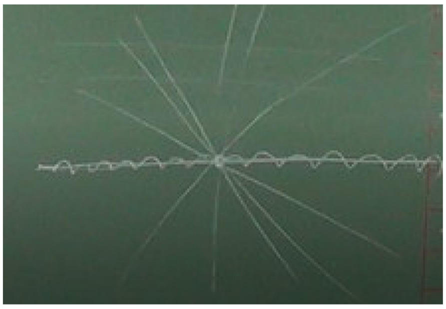
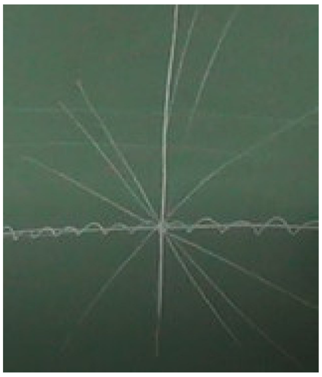
T: They all go through the point, don't they? (points out the one that seems parallel) So on to infinity, I can draw an infinite number of straight lines that pass through the point, but he says that there is only one that is parallel (points to it), and I am going to draw it like that with a wavy line so you can see it.

T: Well, that's one of the things that Euclid thought.

S: But, Miss, there's another way, from the path, draw a line down.

T: From the path, I draw a line down (does it), but it is not parallel.

S: Ah, no, that's perpendicular.

T: So, Euclid was right. Look, it seems like a tongue twister, but we only have to draw what we are told, either in our brains or on paper. In geometry, there's a lot to draw, isn't there? So, is it true what Euclid says, that, on a path, if we draw an exterior point, there is only one line parallel to that path? Yes? Are you sure? Can there be two parallel lines at the same point?" (. . .)

3.2. *The Specialised Knowledge in Jimena's Class for Teaching Geometry*

We organize the analysis of the session described in the previous section in two parts. In the first, we highlight the specialised knowledge that the teacher, Jimena, mobilizes when she teaches Euclid's fifth postulate. In the second part, we describe what the *opportunities* are, and what specialised knowledge would support an alternative approach to them, which we call *specialised knowledge evoked in the researcher by the opportunities*. The set of knowledge elements that we express in an interpretive way in the following sections are shown in a synthesized way in Table 4. This table is organized to indicate the sub-domains and MTSK category in the rows and the specialised knowledge mobilized by the teacher and the knowledge evoked in the researcher in the columns, in relation to the corresponding sub-domain and category.

Table 4. Specialised knowledge to teach Euclid's fifth postulate which emerges from Jimena's session, in relation to the corresponding MTSK sub-domain and category.

	Categories	Specialised Knowledge Jimena Mobilizes	*Opportunities and Specialised Knowledge Evoked in the Researcher by the Opportunities*
KoT	*Definition, properties and foundations*	– Jimena indicates that the relevant mathematical content that the reading has is the question that arises.	– The question, "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" (*opportunity*), would allow an alternative management by relating it to the statement of Euclid's fifth postulate. – Jimena considers that, even though the book does not indicate it, the path to which it refers must be straight for the statement to make sense (*opportunity*). Asking the students about this possibility might bring up the idea of the generalization of parallelism. – The exemplification of drawing a part of the set of straight lines that pass through a point in the plane (*opportunity*) would allow an alternative management by including this concept and its relationship in the generation of a plane in two and three dimensions. – *Did you know that, through a point outside a path, only one parallel path can be drawn?* (*opportunity*). An alternative management would start from the related question, "*Would the conjecture hold true in space?*" with the relative positions of the straight lines in the plane and in space.
	Registers of representation	– Jimena uses different registers of representation, as, in addition to drawing on the board, she uses two fingers to point out the path to which the textbook refers.	– Did you know that, through a point outside a path, only one parallel path can be drawn? (opportunity). An alternative management would start from the related question, "Would the conjecture hold true in space?" with the relative positions of the straight lines in the plane and in space, which cannot be exemplified with a graphical representation on the blackboard.
	Phenomenology		– The exemplification of drawing a part of the set of straight lines that pass through a point in the plane (*opportunity*) would allow an alternative management by including this concept and its relationship in the generation of a plane in two and three dimensions. – "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" (*opportunity*). An alternative management would start from the question, "*Would the conjecture hold true in space?*" To connect the relative positions of the straight lines in the plane and in space.

Table 4. *Cont.*

	Categories	Specialised Knowledge Jimena Mobilizes	*Opportunities and Specialised Knowledge Evoked in the Researcher by the Opportunities*
KPM	*Forms of validation*	– Jimena shows her students that they are wrong by using an example that contradicts the proposition.	
	Role of symbols and use of formal language		– The question, "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" (*opportunity*) formulates, in common language, Euclid's fifth postulate. The comparison of this sentence with the correct statement of the axiom would generate a discussion in the classroom about the importance of formal language in Mathematics. – The students interpret each segment to be a path. However, Jimena represents other paths that pass through the indicated point with one segment instead of two (*opportunity*). The question, "*What is a path in this drawing?*" would support the importance of the good use of mathematical language.
	Necessary and sufficient conditions for generating definitions		– The students interpret each segment to be a path. However, Jimena represents other paths that pass through the indicated point with one segment instead of two (*opportunity*). The question, "*What is a path in this drawing?*", would support an alternative management starting from the mathematical practice of defining.
KSM	*Connections of complexity*		– Jimena believes that the path referred to in the book must be straight for the statement to make sense (*opportunity*). An alternative management would include asking the students about this possibility in order to provoke reflection on the generalization of the concept, making the connection with non-Euclidean geometries.
KMT	*Theories of mathematics teaching*	– Jimena provokes reflection by asking. She knows that this can generate significant knowledge of the content in question.	
	Strategies, techniques, tasks and examples	– Jimena uses exemplification of the possible options to help students understand. – She knows the power of examples to reinforce a claim or to show that it is false. – She knows that graphical representation in geometry aids the understanding of concepts.	

Table 4. *Cont*.

	Categories	Specialised Knowledge Jimena Mobilizes	*Opportunities and Specialised Knowledge Evoked in the Researcher by the Opportunities*
	Teaching resources (physical and digital)	– She knows the importance of using different registers of representation for a better understanding of geometry.	– Drawing the path with two parallel segments has the students interpret each segment as a path (*opportunity*). An alternative management would relate to an exhaustive representation of the construct, using ad hoc registers.
KFLM	*Strengths and weaknesses in learning mathematics*	– Jimena knows the difficulty her students have in drawing and mental representation in order to consider all the conditions given to obtain the result.	– Drawing the path with two parallel segments has the students interpret each segment as a path (*opportunity*). An alternative management would relate to an exhaustive representation of the construct by knowing the strengths and weaknesses of the students in understanding it.
KMLS	Sequencing of topics	– Jimena knows that the fifth Euclid's postulate is not contained in the curriculum for this course, but she uses it as a trigger for reasoning and future understanding.	– *"Did you know that, through a point outside a path, only one parallel path can be drawn?"* (*opportunity*). An alternative management would start from the related question, "*Would the conjecture given by the book hold true in space?*" with the relative positions of the straight lines in the plane and in space.

4. Analysis

4.1. The Specialised Knowledge Mobilized by Jimena

Jimena explains that the relevant mathematical content that the reading has is the question it poses, "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" (KoT, *properties and foundations*), although she does not make explicit that it is Euclid's fifth postulate reformulated in common language. She seems to be aware of the power of the claim even though she knows it is not contained in the curriculum for this year (KMLS, *Sequencing of topics*), and asks her students about it in order to provoke reflection. Therefore, her knowledge of mathematics teaching (KMT) is reflected here, in the categories of *theories of mathematics teaching* and *strategies, techniques, tasks and examples*, considering that provoking such reflection can generate significant knowledge of the content at hand. She creates the situation for her students to discuss the possible options, asking them to think and try to draw in their minds what they are proposing, stating that, "*now we are dealing with geometry and need to think about what we are seeing*", indicating the path with her two fingers, in addition to using the blackboard (KoT, *registers of representation*; KMT, *teaching resources (physical and digital)*.

In addition, despite the fact that the reading does not mention it, she considers that the only way for the cited statement to even make sense (KoT, *definitions, properties and foundations*) is for the path to which it refers to be straight, which she makes clear with her following comment stating, "*I can only draw one parallel line*", at which point she stops used the term "path" to indicate that which can be traced through the point parallel to the first path (KoT, *definitions, properties and foundations*).

The fact that Jimena considers that the only option is for the path to be straight is reinforced by the drawing made on the blackboard (KoT, *registers of representation*), which, in turn, implies an intra-conceptual relationship (KoT, *definitions, properties and foundations*) in the representation of the path evoking a flat figure, although she does not specify it and maintains only two parallel segments of equal length.

After completing the drawing, she asks, "So, is it true what Euclid says, that, on a path, if we draw an exterior point, there is only one line parallel to that path? Yes? Are you sure? Can there be two parallel lines at the same point?" We observe her knowledge about the ways of learning of her students (theories of mathematical learning) and their strengths and weaknesses in learning mathematics (KFLM) in her insisting on the need to represent the concept in their brains or on paper, as she seems aware that, both when drawing and in mental representation, her students have difficulties in considering all the conditions given to obtain the result. Likewise, she elicits a task (drawing in order to understand geometry) that implies she has knowledge of strategies, techniques, tasks and examples for deepening the understanding of geometry of her students in general (KMT).

When a student states that, "*there is another way, that is, drawing a line down from the path*", she draws the proposed straight line (interpreting that they mean perpendicular to those previously drawn to indicate the path) in order to show them that they were wrong by using an example that contradicts what was proposed by the student. She thus shows her knowledge of being able to prove the falsity of a claim by using a counterexample (KPM, *forms of validation*). She also reveals her knowledge about the power of examples (KMT, *theories of mathematics teaching*) to show whether a statement is correct or not.

4.2. The Specialised Knowledge Evoked in the Researcher by the Opportunities in Jimena's Class

The fact that the book indicates as a question, "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" generates an *opportunity*, as it evokes knowledge relating to Euclid's fifth postulate (KoT, *definitions, properties and foundations*), although it does not make it explicit. The fact that it is formulated in common language makes an alternative management emerge, that of stating the axiom itself in order to generate a discussion in the classroom about the *role of symbols and the use of formal language in mathematics* (KPM).

Jimena considers that the only way for the book's statement to make sense is for the path to which it refers to be straight (*opportunity*). An alternative management might be to ask the students to consider non-straight segments (Figure 2), and thus bring into play reasoning on the idea of the generalization of parallelism from straight lines to curves.

This would mean considering *definitions, properties and foundations* (KoT) of non-Euclidean geometries. In addition, knowledge emerges connecting this generalization from parallelism in Euclidean geometry to the same in non-Euclidean geometries (KSM, *connections of complexity*).

Figure 2. Generalization of the concept of parallelism.

Drawing the path with two parallel segments has the students interpret each segment as a path, "*Yes, yes. There it is a parallel path (points to the drawing of the two parallel segments without considering the point)"*. When Jimena draws other paths that pass through the indicated point, she represents them with one segment instead of two, (*opportunity*), as the students might rightly think that there are infinite paths of different widths, depending on where the second segment that represents the second straight line is placed. An alternative management might start with the question, "*What is a path in this drawing?*", supported by evoked knowledge relating to the mathematical practice of defining (KPM, *necessary and sufficient conditions to generate definitions*) and the importance of the proper use of mathematical language (KPM, *role of symbols and use of formal language*). In addition, this alternative management would relate to an exhaustive representation of the construct by knowing the *strengths and weaknesses in learning mathematics* (KFLM) of the students in understanding it. This representation would also require knowledge about ad hoc material and virtual resources (KMT, *teaching resources (physical and digital)*) for the better understanding of the students.

To verify that the axiom is fulfilled in a certain case, Jimena uses the example of the pencil of straight lines that pass through a point in the plane: "*I can draw an infinite number of straight lines that pass through the point [...] but he says that there is only one that is parallel (points to it), and I am going to draw it like that with a wavy line so you can see it*". She does not make explicit in her verbal register the pencil of straight lines (*opportunity*). An alternative management might include this concept, from which a discussion could be triggered in the class about the generation of the plane and about would happen in space. This management would be supported by specialised knowledge relating to the definition of the pencil of straight lines and the plane (KoT, *definitions, properties and foundations*). As any two non-coincident lines define a plane, knowledge of *phenomenology* (KoT) would also be involved, when relating the set of lines with the definition of a plane in two and three dimensions.

Would the conjecture given by the book hold true in space? (*opportunity*). An alternative management might start with the question, "*Did you know that, through a point outside a path, only one parallel path can be drawn?*" related to the relative positions of the straight lines in space, which cannot be shown graphically on the blackboard, and which could generate a new discussion. This management would be supported by the knowledge of the connection of topics (KMLS, *sequencing of topics*), of the *registers of representation* (KoT) and of *definitions, properties and foundations* (KoT). We would also have a connection between the relative positions of the lines in the plane and in space (KoT, *phenomenology*).

5. Discussion of the Results and Conclusions

The fundamental weight of knowledge of geometry mobilized by Jimena falls within the domain of *pedagogical content knowledge*. Regarding *mathematical knowledge* domain, *knowledge of topics* (KoT) seems the most present, with most of the rest of the sub-domains and categories being less relevant. However, if we consider *the specialised knowledge evoked in the researcher by the opportunities*, the variety of knowledge covers practically all the MTSK sub-domains.

At times, exemplification serves Jimena both to verify a particular case of an affirmation and to demonstrate the falsehood of other result with a counterexample (KPM, *forms of validation*). As an example of this, we consider two situations in her teaching. The first is when she shows, through graphic representation, the result of "*through a point outside a path, only one parallel path can be drawn*", allowing her students to empirically verify the veracity of the statement in a particular case. The second is when a student says that there is another (straight line), referring to the vertical that passes through the point (with the path drawn horizontally) and the teacher substantiates the falsehood of the statement by drawing the student's proposal and allowing the class to see that it is perpendicular, contradicting the characteristic that it must fulfil: To be parallel.

As in the previous case, and it seems following her own statement that, "*we only have to draw what we are told, either in our brains or on paper... In geometry, there's a lot to draw, isn't there?*", Jimena bases her knowledge on certain definitions and properties through this medium. We can see a strong connection, in this case, between her knowledge of *definitions, properties and foundations* and *registers of representation*, both KoT categories.

Using an example that contradicts the generality of a statement to refute possible erroneous proposals of the students will result in the enrichment of their knowledge about how to produce mathematics. [34] (p.2) said of geometry that, "*it began as a set of rules and empirical knowledge, obtained experimentally*", while [35] (p.41) interprets procedural contents to be essential for "*the construction of the geometric knowledge of children up to 12 years of age*". As we are at a stage where analytical formalization is still far off, we interpret these two ideas to coincide with the observed knowledge of the registers of representation, when Jimena emphasizes that, "*there is a lot of drawing in geometry*".

Regarding the *knowledge of mathematics teaching* (KMT), we observe that she promotes, in her students, reflection on contents as a teaching strategy (*strategies, techniques, tasks and examples*), probably because she knows that it can generate significant knowledge of the contents. She insists, on various occasions, that geometry consists of "*thinking about what we are seeing*". She constantly uses the blackboard as a resource (KMT, *teaching resources (physical and digital)*), knowing that drawing is essential for a better understanding of geometry. A connection is observed between her knowledge of *definitions, properties and foundations* (KoT) and her knowledge about teaching (KMT) and learning standards (KMLS), as her knowledge of the content regarding the statement that *only one parallel straight line passes through a point outside of a straight line* emphasizes the importance of reflecting on it for the contents of the straight line they are about to discuss, even though she knows that it is not part of the content of the curriculum for that year.

We observe a connection between her KoT (*registers of representation*) and her KFLM (*strengths and weaknesses in learning mathematics*), because, on various occasions (e.g., when trying to represent straight lines parallel to one given by a fixed point), she knows that they have difficulties in considering all the conditions given to obtain the result in order to correctly represent the mathematical object. On many occasions, she uses examples from everyday life to achieve a better understanding (KMT), so we know that she knows the power of using these in her teaching in order to stimulate better learning. Thus, the first conclusion we can draw is how important the KoT in the knowledge of geometry is in primary education.

Knowledge of the most appropriate registers of representation to represent certain constructs is then fundamental, on various occasions, as different *opportunities* appear related to possible difficulties in the understanding of the students. The exhaustive representation of a construct, while, in principle, being freehand and only pretending to represent a sketch of the construct, generates *opportunities* to support an alternative management based on the knowledge of the *strengths and weaknesses in learning mathematics* of the students (KMT) with respect to the construct itself, in relation to *definitions, properties and foundations*, and to the *registers of representation* (KoT).

In general, we can talk about the importance that should be given to the context in which we place the treated knowledge, in this case, in space or on a plane, and differentiation between the mathematical object and the physical object used to represent it (*registers of representation*, *procedures*), as the lack of resources in the reality that surrounds us to represent, for example, a line or a plane, may generate identifications of concepts with what they are not [35].

The choice of the MTSK theoretical framework, based on our consideration that all the teacher's knowledge is specialised, has allowed access to information subdivided into clear sub-domains, categories and indicators that have been highly refined by previous research. In addition, by focusing on the mathematical knowledge base necessary to support specialised didactic knowledge, it has allowed us to establish relationships between the aforementioned sub-domains, categories and indicators to obtain conclusions that respond to objectives. We have identified the specialised knowledge for the teaching of geometry that Jimena mobilizes in her class, in particular, when she deals with Euclid's fifth postulate, we have interpreted the specialised knowledge that might have supported an alternative management of Jimena's class when dealing with such content, and we have characterized the specialised knowledge necessary for the teaching of Euclid's fifth postulate.

Thus, returning to the beginning of this study, given that one of the lines of research in mathematics education is to deepen knowledge of mathematics teachers, characterizing them by their professional practice, and considering the need to know what and how they know and should know about mathematics [1], with this research we provide a first approach to characterizing the specialised knowledge desirable for teaching Euclid's fifth postulate.

We consider that this work raises new research opportunities for the future, as, in fact, observing knowledge from the potential perspective of *opportunities* would consequently result in a new approach to research supported by the framework of MTSK. Based on the case study of this article, we have understood that, by including in the observation the *specialised knowledge evoked in the researcher by the opportunities*, this has become part of the specialised knowledge that supports and creates a foundation, in the sense of [19], for the knowledge treated in the classroom.

There are various studies that have used the *specialised knowledge evoked in the researcher by the opportunities*, including at other educational levels, such as early childhood education [10] or in the training of primary education teachers [9,36]. These studies use it as a necessary construct for the design of training tasks for PPT, based on real classroom practice. Starting from the consideration of the teacher's professional tasks as the backbone of their initial training, [37] define a training task as one that takes professional tasks as a reference, and implies the systematic analysis of real classroom situations that make practice the focus of training for PPT (professional tasks constitute the teacher's system of activities characterizing their practice [38]). To design training tasks for initial teacher training, [36] propose the selection of powerful fragments of specialised knowledge that can be reflected on in a student teacher training classroom. Considering the *specialised knowledge evoked in the researcher by the opportunities*, they include, in the literal transcription, elements that can enrich reflection on additional aspects of specialised knowledge for teaching geometry in an initial teacher training classroom. The training task is thus made up by adding activities oriented to stimulate the PPT to mobilize elements of specialised knowledge from the different MTSK sub-domains. The use of the *specialised knowledge evoked in the researcher by the opportunities* in the design of training tasks allows the possible alternative management of specific situations, to promote the mobilization of specialised knowledge, to be worked on with PPT.

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References

1. Flores-Medrano, E.; Escudero-Ávila, D.I.; Montes, M.; Aguilar, A.; Carrillo, J. Nuestra modelación del conocimiento especializado del maestro de matemáticas, el MTSK. In *Un Marco Teórico para el Conocimiento Especializado del Maestro de Matemáticas*; Carrillo, J., Contreras, L.C., Climent, N., Eds.; Publicaciones de la Universidad de Huelva: Huelva, Spain, 2014; pp. 57–72. Available online: https://www.researchgate.net/publication/271205348_Nuestra_modelacion_del_conocimiento_especializado_del_profesor_de_matematicas_el_MTSK (accessed on 1 July 2021).
2. Barrantes, M.; Blanco, L. A study of prospective primary teachers' conceptions of teaching and learning school Geometry. *J. Math. Teach. Educ.* **2006**, *9*, 411–436. [CrossRef]
3. Liñán-García, M.M.; Contreras, L.C.; Climent, N.; Montes, M.A.; Carrillo, J.; Muñoz-Catalán, M.C. Features of the Specialised Knowledge in Geometry of Prospective Primary Teachers at the Beginning of Their Training. In *Student Teaching: Perspectives, Opportunities and Challenges*; Jung, L., Ed.; Nova Publishers: New York, NY, USA, 2019; pp. 57–107.
4. López, E.M.; Guerrero, A.C.; Carrillo, J.; Contreras, L.C. La resolución de problemas en los libros de texto: Un instrumento para su análisis. *Av. Investig. Educ. Mat.* **2015**, *8*, 73–94. [CrossRef]
5. Zazkis, R.; Leikin, R. Exemplifying definitions: A case of a square. *Educ. Stud. Math.* **2008**, *69*, 131–148. [CrossRef]
6. Strauss, A.; Corbin, J. Grounded Theory Methodology. In *Handbook of Qualitative Research*; Denzin, N.K., Lincoln, Y.S., Eds.; Sage Publications: Thousand Oaks, CA, USA, 1994; pp. 217–285.
7. Flick, U. *An Introduction to Qualitative Research*, 3rd ed.; Sage Publications: Thousand Oaks, CA, USA, 2007.
8. Liñán-García, M.M. Conocimiento Especializado en Geometría en un aula de 5º de Primaria. Ph.D. Thesis, Universidad de Huelva, Huelva, Spain, 2017. Available online: http://rabida.uhu.es/dspace/handle/10272/14230 (accessed on 1 July 2021).
9. Montes, M.A.; Carrillo, J.; Contreras, L.C.; Liñán-García, M.M.; Barrera-Castarnado, V.J. Estructurando la formación inicial de Maestros de Matemáticas: Una propuesta desde el modelo MTSK. In *Investigación Sobre el Maestro de Matemáticas: Formación, Práctica de Aula, Conocimiento y Competencia Profesional*; Badillo, E., Climent, N., Fernández, C., González, M.T., Eds.; Ediciones Universidad Salamanca: Salamanca, Spain, 2019. Available online: https://eusal.es/index.php/eusal/catalog/view/978-84-1311-073-8/5054/4201-1 (accessed on 1 July 2021).
10. Muñoz-Catalán, M.C.; Ramírez-García, M.; Joglar-Prieto, N.; Carrillo, J. Mathematics Teachers' Specialised Knowledge to promote Algebraic Reasoning in Early Childhood Education as from a task of additive decomposition. *J. Study Educ. Dev.* **2021**, 1–21. [CrossRef]
11. Schoenfeld, A.H. *How We Think*; Routledge: New York, NY, USA, 2010.
12. Montes, M.; Aguilar, A.; Carrillo, J.; Muñoz-Catalán, M.C. MTSK: From common and horizon knowledge to knowledge of topics and structures. In Proceedings of the CERME 8, Antaya, Turkey, 6–10 February 2013; Ubuz, B., Haser, C., Mariotti, M.A., Eds.; Middle East Technical University: Ankara, Turkey, 2013; pp. 3185–3194. [CrossRef]
13. Shulman, L.S. Those who understand: Knowledge growth in teaching. *Educ. Res.* **1986**, *15*, 4–14. [CrossRef]
14. Ball, D.L.; Thames, M.H.; Phelps, G. Content knowledge for teaching: What makes it special? *J. Teach. Educ.* **2008**, *59*, 389–407. [CrossRef]
15. Carrillo, J.; Climent, N.; Montes, M.; Contreras, L.C.; Flores-Medrano, E.; Escudero-Ávila, D.; Vasco, D.; Rojas, N.; Flores, P.; Aguilar-González, A.; et al. The Mathematics Teachers' Specialised Knowledge (MTSK) model. *Res. Math. Educ.* **2018**, *20*, 236–253. [CrossRef]
16. Ribeiro, C.M.; Muñoz-Catalán, M.C.; Liñán-García, M.M. Discutiendo el conocimiento matemático especializado del maestro de infantil como génesis de aprendizajes futuros. In Proceedings of the Fourth International Congress ETM, San Lorenzo de El Escorial, Spain, 30 June–4 July 2014; Gómez-Chacón, I., Escribano, J., Kuzniak, A., Richard, P.R., Eds.; Available online: http://www.mat.ucm.es/imi/ETM4/ETM4libro-final.pdf (accessed on 1 July 2021).
17. Muñoz-Catalán, M.C.; Liñán-García, M.M.; Ribeiro, M. El conocimiento especializado para enseñar la operación de resta en Educación Infantil. *Cad. Pesqui.* **2017**, *24*, 4–19. [CrossRef]
18. Vasco, D.L. Conocimiento especializado del maestro de álgebra lineal: Un estudio de casos en el nivel universitario. Ph.D. Thesis, Universidad de Huelva, Huelva, Spain, 1 July 2021. Available online: http://rabida.uhu.es/dspace/handle/10272/11901 (accessed on 1 July 2021).
19. Ma, L. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*; Lawence Erlbaum: Mahwah, NJ, USA, 1999.
20. NCTM. *Principles and Standards for School Mathematics*; NCTM: Reston, VA, USA, 2000.
21. Duval, R. Un análisis cognitivo de problemas de comprensión en el aprendizaje de las matemáticas. In *Comprensión y Aprendizaje en Matemáticas: Perspectivas Semióticas Seleccionadas Universidad Distrital Francisco José de Caldas*; Duval, R., Sáenz-Ludlow, A., Eds.; Editorial Universidad Distrital Francisco José de Caldas: Bogotá, Colombia, 2016; pp. 61–94. Available online: http://funes.uniandes.edu.co/12213/1/Duval2016Un.pdf (accessed on 1 July 2021).
22. Star, J.; Rittle-Johnson, B. Making algebra work: Instructional strategies that deepen student understanding, within and between representations. *ERS Spectr.* **2009**, *27*, 11–18. Available online: http://nrsharvardedu/urn (accessed on 1 July 2021).
23. Lakoff, G.; Núñez, R. *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being*; Basic Books: New York, NY, USA, 2000.
24. Schwab, J.J. *Science, Curriculum and Liberal Education*; University of Chicago Press: Chicago, IL, USA, 1975.

25. Ball, D.L.; Bass, H. With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. In Proceedings of the Paper Prepared Based on Keynote Address at the 43rd Jahrestagung für Didaktik der Mathematik, Oldenburg, Germany, 2–6 March 2009.
26. Planas, N.; Morera, L. La argumentación escolar: Dos ejemplos para la formación del profesorado. In *El Desarrollo de Competencias en la Clase de Ciencias y Matemáticas*; Badillo, E., García, L., Marbà, A., Briceño, M., Eds.; Universidad de Los Andes, Vicerrectorado Administrativo: Mérida, Venezuela, 2012; pp. 275–299.
27. Escudero-Ávila, D.I. Una Caracterización del Conocimiento Didáctico del Contenido Como Parte del Conocimiento Especializado del Profesor de Matemáticas de Secundaria. Ph.D. Thesis, Universidad de Huelva, Huelva, Spain, 2015. Available online: http://rabida.uhu.es/dspace/handle/10272/11456 (accessed on 1 July 2021).
28. Font, V.; Acevedo, J.I. Fenómenos relacionados con el uso de metáforas en el discurso del profesor. *El caso de las gráficas de funciones. Ensen. Cienc.* **2003**, *21*, 405–418.
29. Bassey, M. *Creating Education through Research*; Open University Press: Buckingham, UK, 1999.
30. Stake, R.E. Case Studies. In *Handbook of Qualitative Research*; Denzin, N.K., Lincoln, Y., Eds.; Sage Publications: Thousand Oaks, CA, USA, 2000; pp. 435–454.
31. Brymann, A. *Social Research Methods*, 2nd ed.; Oxford University Press: Oxford, UK, 2004.
32. Kvale, S. *Interviews: An Introduction to Qualitative Research Interviewing*; Sage: Thousand Oaks, CA, USA, 1996.
33. Escudero-Ávila, D.; Moriel-Junior, J.; Muñoz-Catalán, M.C.; Flores-Medrano, E.; Flores, P.; Rojas, N.; Aguilar, A. Aportaciones metodológicas de investigaciones con MTSK. In Proceedings of the II Jornadas del Seminario de Investigación de Didáctica de la Matemática de la Universidad de Huelva, Huelva, Spain, 15–16 September 2015; Carrillo, J., Contreras, L.C., Montes, M., Eds.; Available online: http://rabida.uhu.es/dspace/bitstream/handle/10272/12509/Reflexionando_sobre_el_conocimiento.pdf?sequence=2 (accessed on 1 July 2021).
34. Puig Adam, P. *Curso de Geometría Métrica I: Fundamentos*; Euler: Madrid, Spain, 1986.
35. Canals, M.A. La Geometría en las primeras edades escolares. *Suma Rev Sobre Enseñanza Aprendiz. Mat.* **1997**, *25*, 31–44.
36. Barrera-Castarnado, V.J.; Liñán-García, M.M. El conocimiento especializado en matemáticas en las tareas formativas para la formación inicial de profesores de educación primaria. In *Escenarios Educativos Investigadores: Hacia una Educación Sostenible*; Romero Rodríguez, J.M., Ramos Navas-Parejo, M., Rodríguez Jimenez, C., Sola Reche, J.M., Eds.; Dyckinson: Madrid, Spain, 2021; pp. 934–946.
37. Joglar-Prieto, N.; Liñán-García, M.M.; Contreras, L.C. MTSK en la formación del profesorado de Educación Primaria. In *Actualización del MTSK*; Climent, N., Contreras, L.C., Montes, M., Eds.; Dyckinson: Madrid, Spain, 2021.
38. Muñoz-Catalán, M.C.; Gavilán, J.M.; Liñán-García, M.M. La escritura como recurso para promover el conocimiento especializado del estudiante para maestro en matemáticas. In *Escribir en las Disciplinas: Intervenciones para Desarrollar los Géneros Académicos y Profesionales en la Educación Superior*; Gallardo-Saborido, A.J., Núñez-Román, F., Eds.; Síntesis: Madrid, Spain, 2017; pp. 39–41.