Journal of Topology and Analysis
(2021)

DOI: 10.1142/S1793525321500709

Morse–Bott theory on posets and a homological Lusternik–Schnirelmann theorem

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Received 16 March 2021
Revised 8 October 2021
Accepted 20 November 2021
Published 30 December 2021

We develop Morse–Bott theory on posets, generalizing both discrete Morse–Bott theory for regular complexes and Morse theory on posets. Moreover, we prove a Lusternik–Schnirelmann theorem for general matchings on posets, in particular, for Morse–Bott functions.

Keywords: Morse–Bott functions; posets; Lusternik–Schnirelmann theorem.

AMS Subject Classification: 57R70, 37B35, 55M30, 58E05, 06A07

1. Introduction

Since its introduction, Morse Theory has been an active field of research with connections with many different areas of Mathematics. That interaction has led to several adaptations of Morse Theory to different contexts, for example: PL versions by Banchoff [2, 3] and by Bestvina and Brady [7] and a purely combinatorial approach by Forman [19, 20]. Nowadays, not only pure mathematics benefit from that interaction, but also applied mathematics [25] due to the importance of discrete settings.

Roughly speaking, Morse Theory addresses the study of the topology (homology, originally) of a space by breaking it into "elementary" pieces. That is achieved by the so-called Fundamental or Structural Theorems of Morse Theory, which assert

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that the object of study (for example a smooth manifold or a simplicial complex) has the homotopy type of a CW-complex with a given cell structure determined by the criticality of a Morse function defined on it [19, 33].

Originally, Morse Theory began with the definition of Morse function itself, that is, a smooth function with nondegenerate critical points. So, the only critical objects allowed were points. That was overcome by Morse–Bott theory [9], which broadened the class of critical objects by including nondegenerate critical submanifolds. The attempt to avoid the condition that the critical objects must be nondegenerate led to the introduction of Lusternik–Schnirelmann theory [30].

In the context of Morse Theory, Morse inequalities guarantee that the number of critical points of a Morse function $f\colon X \to \mathbb{R}$ is an upper bound for the homological complexity of the space X. The role of the Morse inequalities in the setting of Lusternik–Schnirelmann theory is played by the so-called Lusternik–Schnirelmann theorem, which asserts that the weighted sum of the number of critical objects is an upper bound for the category of the space [30].

Recent works have shown that it is possible to approach important problems regarding posets by using topological methods. See for example Barmak and Minian's work on the realizability of groups as the automorphism groups of certain posets [5] or Stong's work on groups explaining the way in that the homotopy type of the poset of nontrivial p-subgroups ordered by inclusion determines algebraic properties of the group [37]. Moreover, it is expected that recent discrete analogues of some classical concepts from differential topology will shed light on their original counterparts [22]. Therefore, it makes sense to study the topology of finite spaces by means of some version of Morse theory adapted to this context. This approach was first introduced by Minian [34] and later continued by Kukiela [29].

An invariant of a space X is usually defined as the smallest number of open sets that cover X and that satisfy certain properties, such as being elementary in a certain sense, for instance acyclic or contractible (see [23, 26] for more examples). More generally, and analogously, a categorical invariant of an object X (such as a simplicial complex) can be defined as the smallest number of subobjects needed to cover X and that verify certain properties (see for example [14, 15, 31, 39]). In vague terms, such an invariant provides a certain measure of the complexity of an object. For example, the Lusternik–Schnirelmann category measures, in a particular manner, how far is a space from being contractible.

This work addresses two aims. First, to develop Morse–Bott theory in the context of finite spaces, generalizing both Morse theory for posets introduced by Minian [34] and discrete Morse–Bott theory for complexes [18]. In particular, we prove: an integration result for matchings, the Fundamental Theorems of Morse–Bott theory in this setting and several generalizations of Morse inequalities for arbitrary matchings. Second, we introduce a homological LS-category and we prove the Lusternik–Schnirelmann theorem.

We now describe the main motivations for the introduction of a homological LS-category. The first one is the absence of a discrete Lusternik–Schnirelmann theorem for arbitrary matchings, not even in the simplicial setting, and second, the lack in the literature of a Lusternik–Schnirelmann theorem even for Morse (acyclic) matchings. Nevertheless, several attempts were made. On the one side, a subset of the authors in joint work with Scoville proved a Lusternik–Schnirelmann theorem for a notion of simplicial category and acyclic matchings in the context of simplicial complexes [17]. However, in order to do so, they developed another notion of criticality which leads to a different and non-equivalent definition of discrete Morse function. On the other side, first Scoville and Aaronson [1], then Tanaka [38], and afterwards Knudson and Johnson [28], approached the task by defining another categorical invariant while keeping the usual definition of discrete Morse function.

The organization of the paper is as follows. In Sec. 2 we recall some definitions and standard results about posets, finite topological spaces and regular complexes. Section 3 is devoted to the study of Morse–Bott theory in the context of posets. In Sec. 4 we prove the Fundamental Theorems of Morse–Bott Theory in this setting and exploit some of their consequences. In Sec. 5 we introduce a new notion of homological LS-category and prove the corresponding Lusternik–Schnirelmann theorem for general matchings. Finally, Sec. 6 is devoted to working out a complete example to illustrate the main ideas, definitions and results in the paper.

Following the suggestion of an anonymous referee, we sketch several proofs which follow closely the ones in the simplicial setting due to Forman.

2. Finite Spaces, Posets and Simplicial Complexes

This section is devoted to introducing the objects we will work with. Most of the material is well established in the literature, for further details or proofs the reader is referred to [4, 6, 8, 13, 16, 34, 40].

2.1. *Finite spaces and posets*

It is well known that finite posets (partially ordered sets) and finite T_0-spaces are in bijective correspondence. If $(X, \leq)$ is a poset, a topology $\mathcal{T}$ on X is given by taking the sets

$$U_x := \{y \in X : y \leq x\}$$

as a basis. On the other hand, if X is a finite T_0-space, define for each $x \in X$ the *minimal open set* U_x as the intersection of all open sets containing x. Then X may be given an order by defining $y \leq x$ if and only if $U_y \subset U_x$. It is easy to see that these correspondences are mutual inverses of each other. Moreover a map between posets $f\colon X \to Y$ is order preserving if and only if it is continuous when considered as a map between the associated finite spaces. All posets will be assumed to be finite and by finite space we will mean T_0-space. We will use the notion of finite T_0-space and poset interchangeably.

We need to introduce some terminology.

Definition 2.1. A *chain* in a poset X is a subset $C \subseteq X$ such that if $x, y \in C$, then either $x \leq y$ or $y \leq x$.

Definition 2.2. The *height* of a poset X is the maximum length of the chains in X, where the chain $x_0 < x_1 < \cdots < x_n$ has length n. The height $h(x)$ of an element $x \in X$ is the height of U_x with the induced order.

Definition 2.3. A poset X is said to be *homogeneous* of degree n if all maximal chains in X have length n. A poset is *graded* if U_x is homogeneous for every $x \in X$. In that case, the *degree* of x, denoted by $\deg(x)$, is its height.

We will denote both the height and degree of an element by superscripts, for example $x^{(p)}$.

Let X be a finite poset, $x, y \in X$. If $x < y$ and there is no $z \in X$ such that $x < z < y$, we write $x \prec y$.

For $x \in X$ we also define $\widehat{U}_x := \{w \in X : w < x\}$ as well as $F_x := \{y \in X : y \geq x\}$ and $\widehat{F}_x := \{y \in X : y > x\}$.

2.2. *The McCord functors*

We now recall McCord functors between posets and simplicial complexes (see [32]). Given a poset X, we define its *order complex* $\mathcal{K}(X)$ as the simplicial complex whose k-simplices are the non-empty k-chains of X. Furthermore, given an order preserving map $f\colon X \to Y$ between posets, we define the simplicial map $\mathcal{K}(f)\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ given by $\mathcal{K}(f)(x) = f(x)$.

Conversely, if K is a simplicial complex, we define the face poset of K, as the poset $\Delta(K)$ of simplices of K ordered by inclusion. Given a simplicial map $\phi\colon K \to L$ we define the order preserving map $\Delta(\phi)\colon \Delta(K) \to \Delta(L)$ given by $\Delta(\phi)(\sigma) = \phi(\sigma)$ for each simplex σ of K.

The face poset functor can be defined analogously for regular CW-complexes. That is, given a regular CW-complex K, $\Delta(K)$ is the poset of cells of K ordered by inclusion. Given a cellular map $\phi\colon K \to L$ we define the order preserving map $\Delta(\phi)\colon \Delta(K) \to \Delta(L)$ given by $\Delta(\phi)(\sigma) = \phi(\sigma)$ for each cell σ of K.

Note that for the simplicial complex K, $\mathcal{K}\Delta(K)$ is $\mathrm{sd}(K)$, the first barycentric subdivision of K. By analogy, the first subdivision of a finite poset X is defined as $\Delta\mathcal{K}(X)$.

Theorem 2.1. *The following statements hold*:

(1) *Let X be a finite T_0-space. Then there is a map $\mu_X\colon |\mathcal{K}(X)| \to X$ which is a weak homotopy equivalence.*
(2) *Let K be a simplicial complex. Then there is a map $\mu_K\colon |K| \to \Delta(K)$ which is a weak homotopy equivalence.*

The maps $\mu_X\colon |\mathcal{K}(X)| \to X$ and $\mu_K\colon |K| \to \Delta(K)$ will be referred as McCord maps. For details and a proof of the result above see [4].

2.3. *Cellular poset homology*

We shall consider a special kind of posets called cellular. They were first introduced by Farmer [13] and then recovered by Minian [34]. Farmer's definition is more general while Minian's one is more adequate for our purposes. That is why we present the latter one.

Definition 2.4 ([34]). The poset X is *cellular* if it is graded and for every $x \in X$, $\widehat{U}_x$ has the homology of a $(p-1)$-sphere, where p is the degree of x.

Let X be a cellular poset. We denote by $H_*(X)$ the singular homology of X. Unless stated otherwise, homology will be considered with integer coefficients. However, the constructions work as well for homology modules with coefficients in any principal ideal domain. We recall the construction due to Farmer [13] and Minian [34] of a "cellular homology theory" for cellular posets.

Definition 2.5. Given a finite graded poset X, we define $X^{(p)}$ as the subposet of elements of degree less or equal than p, i.e.

$$X^{(p)} = \{x \in X\colon\ \deg(x) \leq p\}.$$

Given the cellular poset X, there is a natural filtration by the degree

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} = X,$$

which allows to define a *cellular chain complex* (C_*, d) as follows:

$$C_p(X) = H_p(X^{(p)}, X^{(p-1)}) = \bigoplus_{\deg(x)=p} H_{p-1}(\widehat{U}_x),$$

which is a free abelian group with one generator for each element of X of degree p. The differential $d\colon C_p(X) \to C_{p-1}(X)$ is defined as the composition

$$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\ \partial\ } H_{p-1}(X^{(p-1)}) \xrightarrow{\ j\ } H_{p-1}(X^{(p-1)}, X^{(p-2)}),$$

where j is the canonical map induced by the inclusion and ∂ is the connecting homomorphism coming from the long exact sequence associated to the pair $(X^{(p)}, X^{(p-1)})$. It can be shown (see [34]) that the differential can be written as $d(x) = \sum_{w \prec x} \epsilon(x, w)w$ where the incidence number $\epsilon(x, w)$ is the degree of the map

$$\widetilde{\partial}\colon \mathbb{Z} = H_{p-1}(\widehat{U}_x) \to H_{p-2}(\widehat{U}_w) = \mathbb{Z},$$

which coincides with the connecting morphism of the Mayer–Vietoris sequence associated to the covering $\widehat{U}_x = (\widehat{U}_x - \{w\}) \cup U_w$ (see [34]).

Theorem 2.2 ([34, Theorem 3.7]). *Let X be a cellular poset, then*

$$H_*(C_*(X)) \cong H_*(X).$$

2.4. *Homologically admissible posets*

We recall the notion of homologically admissible posets introduced by Minian [34]. We denote by $\mathcal{H}(X)$ the Hasse diagram associated to the poset X.

Definition 2.6 ([34]). Let X be a poset. An edge $(w, x) \in \mathcal{H}(X)$ is homologically admissible if $\widehat{U}_x - \{w\}$ is acyclic. A poset is *homologically admissible* if all its edges are homologically admissible.

The importance of homologically admissible posets lies, partially, in the following result.

Lemma 2.1 ([34, Remark 3.9]). *If (w, x) is a homologically admissible edge of a cellular poset X, then the incidence number $\epsilon(x, w)$ is 1 or -1.*

Remark 2.1. The face posets of regular CW-complexes are homologically admissible [34, Remark 2.6]. However, not every homologically admissible poset is the face poset of a regular CW-complex [34, Example 2.7].

Lemma 2.2 ([34]). *Let X be a poset. If X is homologically admissible, then it is cellular.*

Remark 2.2. In Lemma 2.2 it is assumed that the empty set is not acyclic.

2.5. *Euler characteristic*

Definition 2.7. Let X be a finite graded poset of degree n. Denote by $X^{(=p)}$ the elements of degree p in X. The *graded Euler-Poincaré characteristic* of X is defined as the number:

$$\chi_g(X) = \sum_{p=0}^{n} (-1)^p \# X^{(=p)}.$$

It is clear that given a poset X of the form $X = \Delta(K)$ for a finite simplicial complex K, then $\chi_g(X) = \chi(\mathcal{K}(X))$. Moreover, as a consequence of Minian's result (Theorem 2.2), the standard homological argument (see for example [27, pp. 146–147]) proves that for a finite cellular poset X, $\chi_g(X) = \chi(\mathcal{K}(X))$. However, this does not hold in general for finite posets as the following example illustrates.

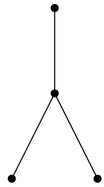

Fig. 1. A poset where $\chi_g(X) \neq \chi(\mathcal{K}(X))$.

Example 2.1. Consider the poset X represented in Fig. 1. Due to the homotopic invariance of χ, $\chi(\mathcal{K}(X)) = 1$ because X is contractible by removing beat points (see [4]). However, $\chi_g(X) = 2$.

3. Dynamics and Morse–Bott Functions for Posets

In this section, we generalize the notion of discrete Morse–Bott function, introduced by Forman in the setting of complexes [18], to the context of posets. It also generalizes that of Morse functions on posets defined by Minian [34].

3.1. *Morse functions*

We recall the definition of Morse function for posets introduced by Minian [34].

Definition 3.1. Let X be a finite poset. A *Morse function* is a function $f\colon X \to \mathbb{R}$ such that, for every $x \in X$, we have

$$\#\{y \in X\colon x \prec y \text{ and } f(x) \geq f(y)\} \leq 1$$

and

$$\#\{z \in X\colon z \prec x \text{ and } f(z) \geq f(x)\} \leq 1.$$

If f is a Morse function, an element $x \in X$ is said to be *critical* if

$$\#\{y \in X\colon x \prec y \text{ and } f(x) \geq f(y)\} = 0$$

and

$$\#\{z \in X\colon z \prec x \text{ and } f(z) \geq f(x)\} = 0.$$

The set of critical points of f is denoted by $\mathrm{crit} f$. The images of the critical points are called *critical values* and the real numbers which are not critical are called *regular values*. The points which are not critical are said to be *regular points*.

3.2. *Matchings*

Forman [21] introduced combinatorial vector fields. It is easy to see that this notion can be substituted by the concept of matching introduced to the context of discrete Morse Theory by Chari [10].

Definition 3.2. A *matching* in a poset X is a subset $\mathcal{M} \subset X \times X$ such that

- $(x, y) \in \mathcal{M}$ implies $x \prec y$;
- each $x \in X$ belongs to at most one element in $\mathcal{M}$.

Given a poset X, let us denote by $\mathcal{H}(X)$ its associated Hasse diagram. If $\mathcal{M}$ is a matching in X, write $\mathcal{H}_{\mathcal{M}}(X)$ for the directed graph obtained from $\mathcal{H}(X)$ by reversing the orientations of the edges which are not in $\mathcal{M}$. Any node of $\mathcal{H}(X)$ not incident with any edge of $\mathcal{M}$ is called *critical*. The set of all critical nodes of $\mathcal{M}$ is denoted by $C_{\mathcal{M}}$.

Definition 3.3. Let $\mathcal{M}$ be a matching on a poset X and let $x^{(p)}$ and $\tilde{x}^{(p)}$ be two elements of X. An $\mathcal{M}$-path γ of index p from $x^{(p)}$ to $\tilde{x}^{(p)}$ is a sequence:

$$\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}^{(p)}$$

such that for each $i = 0, 1, \ldots, r-1$ with $r \geq 1$:

(1) $(x_i, y_i) \in \mathcal{M}$,
(2) $x_i \neq x_{i+1}$.

A $\mathcal{M}$-*cycle* γ in $\mathcal{H}_{\mathcal{M}}(X)$ is a closed $\mathcal{M}$-path in $\mathcal{H}_{\mathcal{M}}(X)$ seen as a directed graph. And the matching $\mathcal{M}$ is said to be a *Morse matching* if $\mathcal{H}_{\mathcal{M}}(X)$ is acyclic.

3.3. *Critical subposets*

In this subsection, we develop the notion of critical subposet (*chain recurrent set*) by means of matchings generalizing the analogous notion introduced by Forman [18] in the context of discrete Morse Theory.

Definition 3.4. Let $\mathcal{M}$ be a matching on X. We say that $x^{(p)} \in X$ is an element of the *chain recurrent set* $\mathcal{R}$ if one of the following conditions holds:

- x is a critical point of $\mathcal{M}$.
- There is a $\mathcal{M}$-cycle γ in $\mathcal{H}_{\mathcal{M}}(X)$ such that $x \in \gamma$.

The chain recurrent set decomposes into disjoint subsets Λ_i by means of the equivalence relation defined as follows:

(1) If x is a critical point, then it is only related to itself.
(2) Given $x, y \in \mathcal{R}$, $x \neq y$, $x \sim y$ if there is a $\mathcal{M}$-cycle γ such that $x, y \in \gamma$.

Let $\Lambda_1, \ldots, \Lambda_k$ be the equivalence classes of $\mathcal{R}$. The $\Lambda_i's$ are called *basic sets*. Each Λ_i consists of either a single critical point of $\mathcal{M}$ or a union of $\mathcal{M}$-cycles.

Example 3.1. Consider the finite model of $\mathbb{R}\mathrm{P}^2$ depicted in Fig. 2 (see [4, Example 7.1.1]). There is a critical point which is also a basic set, depicted with a cross. Moreover, the dashed and dotted arrows represent another two basic sets, each consisting of one cycle.

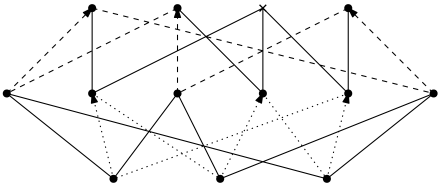

Fig. 2. A finite model of $\mathbb{R}\mathrm{P}^2$.

3.4. *Integration of matchings*

When working in the differentiable category, Morse theory generalizes naturally to Morse–Bott Theory. The purpose of this subsection is to generalize Minian's integration result for matchings [34, Lemma 3.12] to the context of Morse–Bott functions and arbitrary matchings.

We introduce some auxiliary notation. For each edge $(x, y) \in \mathcal{M}$, we say that x is the *source* of the edge and y is the *target*. For convenience, we define the *source* and *target maps* (only defined for elements in the matching $\mathcal{M}$) as follows: given $(x, y) \in \mathcal{M}$, $s(y) = x$ and $t(x) = y$.

Definition 3.5. Given a matching on a finite poset X a function $f\colon X \to \mathbb{R}$ is said to be a *Morse–Bott* or *Lyapunov function* if it is constant on each basic set and it is a Morse function away from the chain recurrent set.

We say that the *critical values* of a Morse–Bott function are the images of the basic sets. The ideas of Forman's proof of [18, Theorem 2.4] generalize to the context of graded posets.

Theorem 3.1 (Integration of matchings). *Let X be a finite graded poset and let $\mathcal{M}$ be a matching in X. Then there exists a Morse–Bott function $f\colon X \to \mathbb{R}$ such that*

(1) *If $x^{(p)} \notin \mathcal{R}$ and $x \prec y^{(p+1)}$, then*

$$\begin{cases} f(x) < f(y) & \text{if } (x, y) \notin \mathcal{M}, \\ f(x) \geq f(y) & \text{if } (x, y) \in \mathcal{M}. \end{cases}$$

(2) *If $x^{(p)} \in \mathcal{R}$ and $x \prec y^{(p+1)}$, then*

$$\begin{cases} f(x) = f(y) & \text{if } x \sim y, \\ f(x) < f(y) & \text{if } x \nsim y. \end{cases}$$

Proof. First of all, we extend the equivalence relation $\sim$ to all of X as follows: if $x \notin \mathcal{R}$, then $\{x\}$ is an equivalence class. Second, we define an auxiliary map $d\colon X \to \mathbb{N}$ given by

$$\begin{aligned} d(x) = \max\{s\colon \exists\, &\mathcal{M}\text{-path} \\ &\gamma\colon x = x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)} = \tilde{x}^{(p)} \\ &\text{such that the } x_i's \text{ in } \gamma \text{ include elements from exactly} \\ &s \text{ distinct equivalence classes}\}. \end{aligned} \tag{1}$$

Third, we define $D = \max_{x \in X} d(x)$. Now, we define the function $f\colon X \to \mathbb{R}$ inductively on the degree of the poset. Given $x^{(p)} \in X$, we define $f(x)$ as follows:

(F1) If x is a critical point of $\mathcal{M}$, then $f(x) = p$.

(F2) If $x \in s(\mathcal{M})$, then

$$f(x) = p + \frac{d(x)}{2 \cdot D}.$$

Note that this guarantees that

$$p < f(x) \leq p + \frac{1}{2}$$

due to $d(x) \geq 1$ in this case.

(F3) If $x \in t(\mathcal{M})$, then there exists $w^{(p-1)}$ such that $t(w) = x$ and $f(w)$ was defined in (2). We set $f(x) = f(w)$ and it follows that

$$p - 1 \leq f(x) \leq p - \frac{1}{2}.$$

It remains to check that f satisfies the desired properties. We split the verification in cases:

(1) Assume that $x^{(p)} \notin \mathcal{R}$ and $x < y^{(p+1)}$.

(a) If $t(x) = y$, then $f(y) = f(x)$, so

$$f(x) \geq f(y).$$

(b) If $t(x) \neq y$, we consider several cases again:

(i) If y is a critical point, then

$$f(y) = p + 1 > p + 1/2 \geq f(x).$$

(ii) If $y \in s(\mathcal{M})$, then

$$f(y) > p + 1 > p + 1/2 \geq f(x).$$

(iii) If $y \in t(\mathcal{M})$, then there exists a unique $\tilde{x}^{(p)} \neq x$ such that $t(\tilde{x}) = y$. Since $x \notin \mathcal{R}$, there are two cases:

(A) If $x \in t(\mathcal{M})$, then

$$f(y) = f(\tilde{x}) \geq p > p - 1/2 \geq f(x).$$

(B) If $x \in s(\mathcal{M})$ and $\gamma\colon x \prec \cdots$ is any $\mathcal{M}$-path beginning at x, then

$$\tilde{\gamma}\colon \tilde{x} \prec y \succ x \prec \cdots$$

is a $\mathcal{M}$-path beginning at $\tilde{x}$. Moreover, since $x \notin \mathcal{R}$, x is not an element of any closed $\mathcal{M}$-path. Therefore,

$$d(\tilde{x}) \geq d(x) + 1,$$

hence

$$f(y) = f(\tilde{x}) > f(x).$$

(2) Assume that $x^{(p)} \in \mathcal{R}$ and

$$\gamma\colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots x_r^{(p)} = x_0^{(p)}$$

is a non-stationary closed $\mathcal{M}$-path. Then for each i, j, $0 \leq i, j \leq r-1$, $d(x_i) = d(x_j)$, hence $f(x_i) = f(x_j)$. Moreover, by the definition of f, $f(y_i) = f(x_i)$, so f is constant on each non-stationary closed $\mathcal{M}$-path.

(3) Suppose $x^{(p)} \in \mathcal{R}$ and $y^{(p+1)} > x$, $y \nsim x$. We want to prove that $f(y) > f(x)$.

(a) If y is a critical point, then

$$f(y) \geq p + 1 > p + 1/2 \geq f(x).$$

(b) If $y \in s(\mathcal{M})$, then

$$f(y) \geq p + 1 \geq p + 1/2 \geq f(x).$$

(c) Suppose $y \in t(\mathcal{M})$. Since $y \nsim x$ and $y \neq t(x)$, then there exists a unique $\tilde{x}^{(p)} \neq x^{(p)}$ such that $t(\tilde{x}) = y$.

(i) if x is a critical point or $x \in t(\mathcal{M})$,

$$f(y) = f(\tilde{x}) > p \geq f(x).$$

(ii) If $x \in s(\mathcal{M})$ and $\gamma\colon x \prec \cdots$ is any $\mathcal{M}$-path starting at x, then

$$\tilde{\gamma}\colon \tilde{x} \prec y \succ x \prec \cdots$$

is a $\mathcal{M}$-path beginning at $\tilde{x}$. Moreover, $\tilde{x}$ is not equivalent to any element of γ, since, otherwise, x and y would be contained in a non-stationary closed path, which contradicts $y \nsim x$. Thus, $d(\tilde{x}) \geq d(x) + 1$, which implies

$$f(y) = f(\tilde{x}) > f(x). \qquad \square$$

3.5. *Morse–Smale matchings*

In this subsection, we generalize the notion of Morse–Smale vector field from the context of simplicial complexes [18] to the setting of finite spaces.

Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. A $\mathcal{M}$-cycle γ is *prime* if there do not exist a natural number $n > 1$ and a $\mathcal{M}$-cycle $\widetilde{\gamma}$ such that γ is the concatenation of $\widetilde{\gamma}$ n times (see [18, Definition 5.3] for details).

An equivalence relation on the set of $\mathcal{M}$-cycles is defined as follows. Two $\mathcal{M}$-cycles γ and $\widetilde{\gamma}$ are equivalent if $\widetilde{\gamma}$ is the result of varying the starting point of γ (see [18, p. 631] for an example). An equivalence class of $\mathcal{M}$-cycles is called a *closed $\mathcal{M}$-orbit*. The equivalence class of γ is denoted by $[\gamma]$. The concepts of *prime closed $\mathcal{M}$-orbit* and *index of a closed $\mathcal{M}$-orbit* are defined as expected (see [18] for details).

A special kind of matching which will play an important role is the following. In a certain sense, it controls the complexity of the chain recurrent set.

Definition 3.6. Let X be a homologically admissible poset. A matching $\mathcal{M}$ on X is a *Morse–Smale matching* if the chain recurrent set $\mathcal{R}$ consists only of critical points and pairwise disjoint prime closed $\mathcal{M}$-orbits.

4. Fundamental Theorems and Consequences

The purpose of this section is to prove the Fundamental Theorems of Morse Theory for Morse–Bott functions on posets and obtain some consequences.

4.1. *Fundamental theorems*

In what follows, we extend the equivalence relation defined in Subsection 3.3 from $\mathcal{R}$ to all of X by saying that a point which is not critical is an equivalence class on its own.

Definition 4.1. Given a finite poset X, a point $x \in X$ and a matching $\mathcal{M}$ on X, we define

$$\partial[x] = \{w \in X \colon w \prec \tilde{x} \text{ for some } \tilde{x} \sim x \text{ but } w \nsim \tilde{x}\}.$$

Example 4.1. Consider the poset depicted in Fig. 2. In Fig. 3 we show $\partial[x]$ for any x in the dashed cycle of Fig. 2.

The lemma below follows from the definition of matching.

Lemma 4.1. *Let γ be a cycle of index p and let $u^{(p-1)} \in X$, $\tilde{v}^{(p)} \in X$, $w^{(p+1)} \in X$ and $r^{(p+2)} \in X$ such that $u, \tilde{v}, w, r \notin \gamma$. Then it holds the following*:

$$t(u) \notin \gamma, \quad t(\tilde{v}) \notin \gamma, \quad s(w) \notin \gamma \quad \textit{and} \quad s(r) \notin \gamma.$$

We introduce the following definition: given a finite poset X and a Morse–Bott function $f \colon X \to \mathbb{R}$, for each $a \in \mathbb{R}$ we write

$$X_a = \bigcup_{f(x) \leq a} U_x.$$

Our next result is a homological collapsing theorem for Morse–Bott functions. As a consequence of Lemma 4.1, the elements of a cycle cannot be connected by arrows with elements which are not in the cycle. Therefore, the result below follows from [16, Theorem 4.2.2].

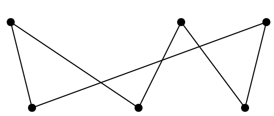

Fig. 3. Example of $\partial[x]$.

Theorem 4.1. *Let X be a finite homologically admissible poset and let $f\colon X \to \mathbb{R}$ be a Morse–Bott function. If $[a, b]$ contains no critical values, then $i\colon X_a \hookrightarrow X_b$ induces an isomorphism in homology.*

In this generalized context, we also have a result which explains what happens when a critical value is reached.

Theorem 4.2. *Let X be a finite homologically admissible poset and let $f\colon X \to \mathbb{R}$ be a Morse–Bott function. If $f(x) \in [a, b]$ is a critical value and there are no other values of f in $[a, b]$, then $X_b = X_a \cup [x]$.*

Proof. There are two cases to consider. First, assume that $[x]$ is a critical point, then the results reduces to [16, Theorem 4.2.8]. So, assume $[x]$ is a cycle of index p. Let $\tilde{f}\colon X/\!\sim \to \mathbb{R}$ denote the function induced by f on the set of equivalence classes. We may assume that $\tilde{f}$ is injective, that $\tilde{f}([x]) > a$ and that the only critical subposet in $f^{-1}([a, b])$ is $[x]$.

Since $[x]$ is a cycle and $f(x)$ is a critical value, then given $y^{(p+1)} \succ \tilde{x}$ and $y \notin [x]$ with $\tilde{x} \in [x]$, $f(y) > f(\tilde{x})$. Hence, $f(y) > b$ and Lemma 4.1 guarantees that $f(z) > b$ for every $z > \tilde{x}$, $z \notin [x]$. Therefore, $[x] \cap X_a = \emptyset$. Given any $w^{(p-1)} \prec \tilde{x}^{(p)}$, $\tilde{x} \in [x]$ and $w \notin [x]$ or $w^{(p)} \prec \tilde{x}^{(p+1)}$, $\tilde{x} \in [x]$ and $w \notin [x]$, due to the criticality of $[x]$, it holds that $f(w) < f(\tilde{x})$. Therefore, $f(w) < a$ and $w \in X_a$. Hence, $\partial[x] \subset X_a$. That is, $X_b = X_a \cup_{\partial[x]} [x]$. □

4.2. *Morse–Bott inequalities*

In this subsection, we generalize Morse–Bott inequalities from the context of CW-complexes [18, Theorem 3.1] to the setting of posets. This result can be seen as a combinatorial analogue of a theorem due to Conley [24, Theorem 1.2; 11]. Again, we assume that our coefficients are any principal ideal domain R. From now on the poset X is assumed to be homologically admissible.

Given a subposet $Y \subset X$ we denote by $\bar{Y}$ the subposet $\cup_{x\in Y} U_x$ and by $\dot{Y} = \bar{Y} - Y$.

Definition 4.2. For each $k \geq 0$, we define

$$m_k = \sum_{\text{basic sets } \Lambda_i} \operatorname{rank} H_k(\bar{\Lambda}_i, \dot{\Lambda}_i).$$

Observe that in the particular case we have a Morse matching, then the basic sets are just critical points and m_k is the number of critical points of index k.

Lemma 4.2. *If the index of the basic set Λ_i is p, then $H_k(\bar{\Lambda}_i, \dot{\Lambda}_i) = 0$ unless $k = p, p+1$. Moreover, if Λ_i is just a critical point $x^{(p)}$, then $H_k(\bar{\Lambda}_i, \dot{\Lambda}_i) = 0$ for $k \neq p$ and the ring of coefficients, R, for $k = p$.*

Proof. For convenience, during the proof we will denote $\Lambda_i = \Lambda$. Since all the posets involved are cellular we can use cellular homology. Consider the Homology Long Exact Sequence for the pair $(\bar{\Lambda}, \dot{\Lambda})$:

$$\cdots \longrightarrow H_p(\bar{\Lambda}) \longrightarrow H_p(\bar{\Lambda}, \dot{\Lambda}) \longrightarrow H_{p-1}(\dot{\Lambda}) \longrightarrow H_{p-1}(\bar{\Lambda}) \xrightarrow{j} H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) \xrightarrow{\partial} H_{p-2}(\dot{\Lambda}) \xrightarrow{\cong} H_{p-2}(\bar{\Lambda}) \longrightarrow H_{p-2}(\bar{\Lambda}, \dot{\Lambda}) \longrightarrow H_{p-3}(\dot{\Lambda}) \longrightarrow \cdots$$

First of all, the homomorphism $H_k(\dot{\Lambda}) \to H_k(\bar{\Lambda})$ is an isomorphism for $k \leq p-2$, so $H_k(\bar{\Lambda}, \dot{\Lambda}) = 0$ for $k \leq p-2$. Second, we have that

$$H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) = \mathrm{Ker}\partial = \mathrm{Im} j \cong \frac{H_{p-1}(\bar{\Lambda})}{\ker j} \cong \frac{H_{p-1}(\bar{\Lambda})}{\mathrm{Im}\, i}.$$

Third, the homomorphism $i : H_{p-1}(\dot{\Lambda}) \to H_{p-1}(\bar{\Lambda})$ induced by the inclusion is surjective by the construction of cellular homology. Therefore, $H_{p-1}(\bar{\Lambda}, \dot{\Lambda}) = 0$. Fourth, if Λ is just a critical point $x^{(p)}$, then $H_k(\bar{\Lambda}, \dot{\Lambda}) = H_k(U_x, \widehat{U}_x)$ and by cellularity of X and the Homology Long Exact Sequence for the pair $(U_x, \widehat{U}_x)$ the result follows. □

We denote by b_k the Betti number in degree k with coefficients in R. Taking into account the ideas involved in the proof of [18, Theorem 3.1] and our Theorems 4.1 and 4.2 yields the Strong Morse–Bott inequalities:

Theorem 4.3 (Strong Morse–Bott inequalities). *Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. Then, for every $k \geq 0$:*

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0.$$

From the standard argument (see [33, p. 30]), we obtain the Weak Morse inequalities.

Corollary 4.1 (Weak Morse–Bott inequalities). *Let X be a homologically admissible poset and let $\mathcal{M}$ be a matching on X. Then*

(1) *For every $k \geq 0$, $m_k \geq b_k$;*
(2) $\chi(X) = \sum_{k=0}^{\deg(X)} (-1)^k b_k = \sum_{k=0}^{\deg(X)} (-1)^k m_k$.

4.3. *Morse–Smale matchings*

In this section, we generalize [19, Sec. 7] to the context of homologically admissible posets while improving some of the results even in the case of simplicial or regular CW-complexes.

Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. We denote by c_k the number os critical points of index k and by A_k the number of prime closed $\mathcal{M}$-orbits of index k.

Recall that the coefficient ring R is a principal ideal domain. Hence, the Structure Theorem for finitely generated modules over a principal ideal domain guarantees that

$$H_k(X) \cong R^{b_k} \oplus \frac{R}{(r_1)} \oplus \ldots \oplus \frac{R}{(r_{\eta_k})}. \tag{2}$$

Combining the proof of [19, Theorem 7.1] with our Pitcher strengthening of Morse inequalities [16, Corollary 5.2.3] we obtain the following improvement of [19, Theorem 7.1], taking torsion into account, measured by η_k.

Theorem 4.4. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. Let the coefficients R be a principal ideal domain. Then, for every $k \geq 0$:*

$$A_k + \sum_{i=0}^{k} (-1)^i c_{k-i} \geq \eta_k + \sum_{i=0}^{k} (-1)^i b_{k-i}.$$

Proof. We begin with a matching $\mathcal{M}_0 = \mathcal{M}$ and we iterate the following procedure. Given a closed $\mathcal{M}_i$-orbit $\{[\gamma]_i\}_i$:

$$\gamma_i \colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)},$$

we define a new matching $\mathcal{M}_{i+1} = \mathcal{M}_i - (x_0, y_0)$. We iterate this process until there are no closed orbits left. We call the obtained matching $\mathcal{M}^*$. Observe that $\mathcal{M}^*$ is acyclic or Morse and $m_p^* = c_p + A_p + A_{p-1}$, where m_p^* denotes the number of critical points of index p of the matching $\mathcal{M}^*$. By [16, Corollary 5.2.3], it holds that

$$\sum_{i=0}^{k} (-1)^i m_{k-i} \geq \eta_k + \sum_{i=0}^{k} (-1)^i b_{k-i},$$

which implies the result we want to prove. □

Definition 4.3. Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. Endow each element of X with an orientation. Let γ be an $\mathcal{M}$-path

$$\gamma \colon x_0^{(p)} \prec y_0^{(p+1)} \succ x_1^{(p)} \prec y_1^{(p+1)} \succ \cdots \prec y_{r-1}^{(p+1)} \succ x_r^{(p)}.$$

We define the multiplicity of γ by

$$\prod_{i=0}^{r-1} -\langle d_{(p+1)} y_i, x_i \rangle_p \langle d_{(p+1)} y_i, x_{i+1} \rangle_p,$$

where d is the cellular boundary operator and $\langle \bullet\, , \bullet \rangle_p$ is the inner product on $C_p(X)$ such that the degree p elements of X are mutually orthogonal.

Remark 4.1. Observe that the multiplicity of a path is always 1 or -1 due to Lemma 2.1.

The generalization of [18, Lemma 4.6] to our context is straightforward. As a consequence, [18, Theorem 7.3] generalizes to our setting with the same proof.

Theorem 4.5. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. Let the coefficients ring R be a principal ideal domain. Let λ_i denote a basic set consisting of a single closed orbit $[\gamma]$ of index p. Then*

$$H_k(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong 0 \quad \textit{for } k \neq p, p+1,$$

$$H_p(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong \begin{cases} R & \textit{if } m(\gamma) = 1, \\ \dfrac{R}{2R} & \textit{if } m(\gamma) = -1, \end{cases}$$

$$H_{p+1}(\bar{\Lambda}_i, \dot{\Lambda}_i) \cong \begin{cases} R & \textit{if } m(\gamma) = 1, \\ 0 & \textit{if } m(\gamma) = -1. \end{cases}$$

Therefore, combining the Strong Morse–Bott inequalities (Theorem 4.3) with Theorem 4.5 and taking into account that fields have no torsion, we obtain a generalization of [18, Corollary 7.4] to our setting.

Theorem 4.6. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. Let the coefficients ring be the field $\mathbb{R}$. Denote by A'_p the number of closed $\mathcal{M}$-orbits of index p and multiplicity 1. Then, for every $k \geq 0$:*

$$A'_k + \sum_{i=0}^{k} (-1)^i c_{k-i} \geq \sum_{i=0}^{k} (-1)^i b_{k-i}(\mathbb{R}).$$

Remark 4.2. While [18, Corollary 7.4] refined [18, Theorem 7.2], Theorem 4.6 does not refine our improved Theorem 4.4. They are complementary results.

5. Homological Lusternik–Schnirelmann Theorem

The purpose of this section is to prove a Lusternik–Schnirelmann theorem for general matchings and a suitable definition of homological category.

5.1. *Homological chain category*

Let (C_*, ∂) denote a free chain complex of abelian groups. It is *bounded* if only finitely many of the C_p are nonzero. Moreover, if each term C_p is finitely generated, then we define the *rank* of C_* as $\operatorname{rank}(C_*) = \sum_p \operatorname{rank}(C_p)$.

Definition 5.1. Let (C_*, ∂) be a free chain complex of abelian groups. We define its *homological chain category*

$$\operatorname{hccat}(C_*) = \inf \left\{ \begin{matrix} \operatorname{rank}(B_*)\colon B_* \text{ bounded subcomplex of } C_* \text{ and the} \\ \text{inclusion } i\colon B_* \hookrightarrow C_* \text{ is a quasi-isomorphism} \end{matrix} \right\}.$$

Let X be a topological space. We denote by $S_*(X)$ its singular chain complex. For all the definitions that follow we consider coefficients in $\mathbb{Z}$.

Definition 5.2. Let X be a topological space. We define its *homological chain category* $\mathrm{hccat}(X) = \mathrm{hccat}(S_*(X))$.

We introduce a homological lower bound for $\mathrm{hccat}(X)$ analogous to the Pitcher strengthening of Morse inequalities [16, Corollary 5.2.3]. Recall from Eq. (2) the definition of η_k.

Proposition 5.1. *Let X be a topological space with finitely generated homology. Then*

$$\sum_k b_k + 2\sum_k \eta_k \leq \mathrm{hccat}(X).$$

Proof. Let us denote by (B_*, ∂) a bounded chain complex whose homology is isomorphic to $H_*(X)$. By standard algebra (see, for example [36, Theorem 4.11]), we have $b_k + \eta_k + \eta_{k-1} \leq \mathrm{rank}\,(B_k)$. Now the result follows by a sum indexed by the dimension. □

Corollary 5.1. *Let X be a homologically admissible poset or a CW-complex with finitely generated homology. Then*

$$\chi(X) \leq \mathrm{hccat}(X).$$

In fact, the bound given by Proposition 5.1 is the best possible as a consequence of the following result due to Pitcher [35, Lemma 13.2].

Proposition 5.2. *Let (C_*, ∂) be a free chain complex with singular homology groups $H_k(X)$, $k = 0, 1, \ldots$. Then there exists a free chain complex (L, ∂^L) such that*

(1) *For every $k \geq 0$, the group L_k has rank $b_k + \eta_k + \eta_{k-1}$.*
(2) *There exists a monomorphism $i\colon L \hookrightarrow C$ which is a chain map.*
(3) *The monomorphism $i\colon L \hookrightarrow C$ is a quasi-isomorphism.*

Corollary 5.2. *Let X be a topological space with finitely generated homology. Then*

$$\mathrm{hccat}(X) = \sum_k b_k + 2\sum_k \eta_k.$$

Moreover, observe that a topological X is acyclic if and only if $\mathrm{hccat}(X) = 1$.

As a consequence of [12, Example 1.33] we have the following result relating the homological chain category to the Lusternik–Schnirelmann category.

Proposition 5.3. *Let K be a simply connected CW-complex with finitely generated homology groups such that there exists $n \in \mathbb{N}$ satisfying $H_n(K) \neq 0$ and $H_p(K) = 0$ for $p > n$. Then*

$$\mathrm{cat}(K) \leq \mathrm{hccat}(K).$$

The result does not necessarily hold if we remove the simply connectedness hypothesis, as the following example shows.

Example 5.1. Consider the Poincaré homology 3-sphere M. Observe that $\mathrm{hccat}(M) = \mathrm{hccat}(\mathbb{S}^3) = 2$. However, $\mathrm{cat}(M) \geq 3$ [23].

5.2. *Homological Lusternik–Schnirelmann theorem*

In this subsection, we state and prove a Lusternik–Schnirelmann theorem for the homological chain category and general matchings on posets.

Theorem 5.1. *Let X be a homologically admissible poset and let $\mathcal{M}$ be a Morse–Smale matching on X. Then*

$$\mathrm{hccat}(X) \leq \sum_{\text{basic sets } \Lambda_i} \mathrm{hccat}(\Lambda_i).$$

In particular, if $\mathcal{M}$ is a Morse matching on X, then $\mathrm{hccat}(X)$ *is a lower bound for the number of critical elements of $\mathcal{M}$.*

Proof. We will define another Morse matching $\mathcal{M}^*$ by perturbing $\mathcal{M}$. The idea is to replace each prime closed orbit by two critical points. This will be achieved by removing exactly one of the edges of the matching in each closed orbit. By repeating the technique used in the proof of Theorem 4.4, we obtain a Morse matching $\mathcal{M}^*$ satisfying $m_p^* = c_p + A_p + A_{p-1}$, where m_p^* denotes the number of critical points of index p of the matching $\mathcal{M}^*$ (see Subsection 4.3 for the definition of A_p).

Recall that $C_*(X)$ denotes the cellular chain complex of X. We define a map $V\colon C_p(X) \to C_{p+1}(X)$ as follows:

$$V(x) = \begin{cases} -\epsilon(y,x)y & \text{if there exists } y \in X \text{ with } (x,y) \in \mathcal{M}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Following the ideas of Minian [34], define the discrete flow operator $\phi\colon C_p(X) \to C_p(X)$ as $\phi = \mathrm{id} + dV + Vd$. The ϕ-invariant chains

$$C_p^{\phi}(X) = \{c \in C_p(X)\colon \phi(c) = c\}$$

form a well-defined subcomplex of $(C_*(X), d)$ [34]. Moreover, the inclusion of $(C_*^{\phi}(X), d)$ into $(C_*(X), d)$ induces isomorphisms in homology and $C_p^{\phi}(X)$ is isomorphic to the free abelian group spanned by the critical p-elements of X [34]. As a consequence

$$\mathrm{hccat}(C_*(X)) \leq \sum_p m_p^* = \sum_p c_p + A_p + A_{p-1}. \tag{3}$$

There are two kinds of basic sets for $\mathcal{M}$: critical points and disjoint closed $\mathcal{M}$-orbits. Observe that if Λ_i is a critical point, then $\mathrm{hccat}(\Lambda_i) = 1$ while if Λ_i is a closed orbit, then $\mathrm{hccat}(\Lambda_i) = 2$. So, from Eq. (3), it follows that

$$\mathrm{hccat}(C_*(X)) \leq \sum_{\text{basic sets } \Lambda_i} \mathrm{hccat}(\Lambda_i).$$

Finally, observe that $\mathrm{hccat}(X) = \mathrm{hccat}(C_*(X))$ due to the isomorphism between cellular homology and singular homology for cellular posets (Theorem 2.2). □

Remark 5.1. In the proof of Theorem 5.1, Eq. (3) could also be derived as a consequence of combining our Pitcher strengthening of Morse-inequalities [16, Corollary 5.2.3] applied to the matching $\mathcal{M}^*$ with Corollary 5.2.

As a consequence of [16, Theorem 3.3.6], we obtain the following corollary.

Corollary 5.3. *Let X be a homologically admissible poset and let $f\colon X \to \mathbb{R}$ be a Morse function. Then* $\mathrm{hccat}(X)$ *is a lower bound for the number of critical points of f.*

Remark 5.2. Let K be a simplicial complex or, more generally, a regular CW-complex K. Recall that its face poset $\Delta(K)$ is a homologically admissible poset. Moreover, the chain complex $C_\bullet(\Delta(K), d)$ where d is the cellular boundary operator coincides with the chain complex $C_\bullet(K, \partial)$ where ∂ is the cellular -or simplicial in case K is a simplicial complex- boundary operator. Therefore, $\mathrm{hccat}(\Delta(K)) = \mathrm{hccat}(K)$. Hence, we have in particular a simplicial homological Lusternik–Schnirelmann theorem.

6. A Worked Out Example

In this section, we work out a full example to illustrate the main ideas in the paper.

6.1. *Morse–Smale matching on a homologically admissible poset*

Consider the homologically admissible poset depicted in Fig. 4. We will denote it by X.

Moreover, in Fig. 4 we also exhibit a matching $\mathcal{M}$ on X. The crosses represent critical points, the dashed edges with the circles represent a periodic orbit, and the arrows represent the matched elements. Observe that the matching $\mathcal{M}$ is Morse–Smale since it only has a prime orbit, therefore disjoint from the others, and three critical points (see Definition 3.6).

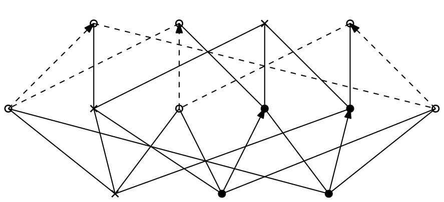

Fig. 4. The homologically admissible poset X with a matching.

6.2. *Integration of the matching*

We proceed to illustrate how to integrate the matching $\mathcal{M}$ on X following the proof of Theorem 3.1. First, in Fig. 5 we show the values of the map $d\colon X \to \mathbb{N}$ given by Eq. (1).

Observe that $d(x)$ is greater than one only for $x \in s(\mathcal{M})$. We compute $D = \max_{x\in X} d(x) = 5$. Now, we will build a Morse–Bott or Lyapunov function $f\colon X \to \mathbb{R}$. Recall that for $x^{(p)} \in X$, we define $f(x)$ following (F1), (F2) and (F3) in p. 10.

We show the values of the Morse–Bott function $f\colon X \to \mathbb{R}$ in Fig. 6.

6.3. *Fundamental theorems*

Recall that given the poset X and the Morse–Bott function $f\colon X \to \mathbb{R}$, for each $t \in \mathbb{R}$ we have the level subposet

$$X_t = \bigcup_{f(x)\le t} U_x.$$

We will illustrate the structural theorems in this context (Theorems 4.1 and 4.2), which describe the changes in the level subposets X_t as $t \in \mathbb{R}$ increases.

We begin the analysis of the level subposets. First, the minimum value of f is attached at a critical element (see Fig. 7(a)).

As we reach the value $t = 3/10$, the inclusion $i\colon X_0 \to X_{3/10}$ induces an isomorphism in homology. Observe that $X_{3/10}$ is acyclic (see Fig. 7(b)).

Next, we reach the value $t = 1/2$ (see Fig. 8(a)). The inclusion $i\colon X_{3/10} \to X_{1/2}$ induces an isomorphism in homology and $X_{1/2}$ is still acyclic.

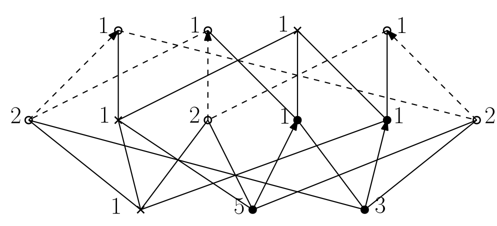

Fig. 5. Values of the map $d\colon X \to \mathbb{N}$.

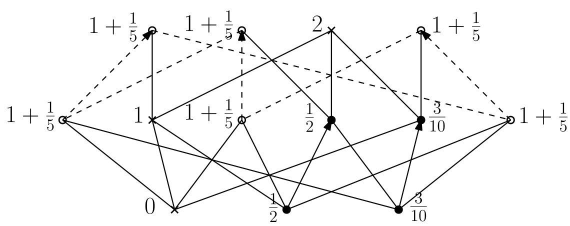

Fig. 6. Values of the map $f\colon X \to \mathbb{R}$.

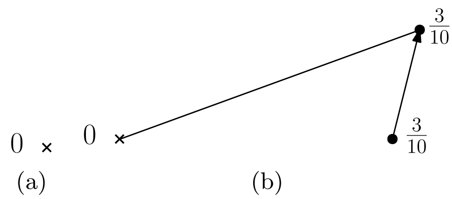

Fig. 7. Level subposets X_0 and $X_{3/10}$.

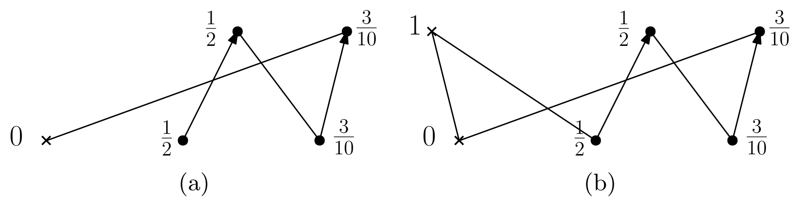

Fig. 8. Level subposets $X_{1/2}$ and X_1.

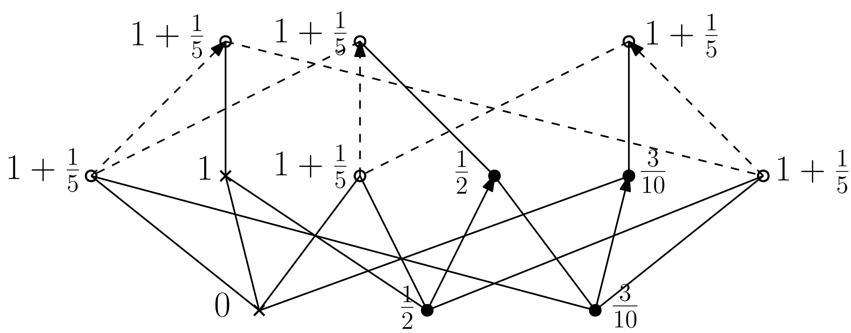

Fig. 9. Level subposet $X_{1+1/5}$.

As we reach the value $t = 1$ (see Fig. 8(b)), which is critical, the inclusion $i\colon X_{1/2} \to X_1$ no longer induces an isomorphism in homology. The level subposet X_1 has the same homology as $\mathbb{S}^1$.

The next value is $t = 1 + 1/5$ (see Fig. 9). Despite corresponding to a basic set, the homology does not change.

Finally, we reach the value $t = 2$ (see Fig. 6), which is critical and produces a change in the homology since $X_2 = X$ has homology: $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(X;\mathbb{Z}) \cong \mathbb{Z}_2$ and $H_k(X;\mathbb{Z}) \cong 0$ for $k \geq 2$.

6.4. *Homological inequalities*

In this last subsection, we provide the explicit computations which are necessary to check the homological inequalities (Theorems 4.3, 4.4 and 5.1). First of all, $\mathrm{hccat}(X) = 3$. We provide the remaining information in tables (Tables 1, 2 and 3)

Table 1. Computation of $H_k(\bar{\Lambda}_i, \dot{\Lambda}_i)$.

$H_k(\bar{\Lambda}_i, \dot{\Lambda}_i)$	0	1	2	$k \geq 3$
Λ_0	$\mathbb{Z}$	0	0	0
Λ_1	0	$\mathbb{Z}$	0	0
Λ_2	0	$\mathbb{Z}$	$\mathbb{Z}$	0
Λ_3	0	0	$\mathbb{Z}$	0

Table 2. Computation of $H_k(\Lambda_i)$.

$H_k(\Lambda_i)$	0	1	2	$k \geq 3$
Λ_0	$\mathbb{Z}$	0	0	0
Λ_1	$\mathbb{Z}$	0	0	0
Λ_2	$\mathbb{Z}$	$\mathbb{Z}$	0	0
Λ_3	$\mathbb{Z}$	0	0	0

Table 3. Computation of several homological and combinatorial invariants.

	0	1	2	3	$i \geq 4$
β_i	1	0	0	0	0
η_i	0	1	0	0	0
c_i	1	1	1	0	0
A_i	0	1	0	0	0
m_i	1	3	1	0	0
$\mathrm{hccat}(\Lambda_i)$	1	1	1	2	1

and assuming we work with integer coefficients. We introduce some notation regarding the basic sets: Λ_0 denotes the critical point of degree zero, Λ_1 denotes the critical point of degree one, Λ_2 denotes the orbit and Λ_3 denotes the critical point of degree two.

Acknowledgments

The first and the fourth authors were partially supported by MINECO Spain Research Project MTM2015-65397-P and Junta de Andalucía Research Groups FQM-326 and FQM-189. The second author was partially supported by MINECO-FEDER research project MTM2016-78647-P. The third author was partly supported by Ministerio de Ciencia, Innovación y Universidades, grant FPU17/03443. The second and third authors were partially supported by Xunta de Galicia ED431C 2019/10 with FEDER funds. All the authors were partially supported by MINECO Spain Research Project PID2020 114474GB-100.

References

1. S. Aaronson and N. A. Scoville, Lusternik-Schnirelmann category for simplicial complexes, *Illinois J. Math.* **57**(3) (2013) 743–753.
2. T. F. Banchoff, Critical points and curvature for embedded polyhedra, *J. Differ. Geom.* **1** (1967) 245–256.

3. T. F. Banchoff, Critical points and curvature for embedded polyhedra II, *Prog. Math.* **32** (1983) 32–55.
4. J. A. Barmak, *Algebraic Topology of Finite Topological Spaces and Applications*, Vol. 2032 (Springer, 2011).
5. J. A. Barmak and E. G. Minian, Automorphism groups of finite posets, *Discrete Math.* **309**(10) (2009) 3424–3426.
6. J. A. Barmak and E. G. Minian, Strong homotopy types, nerves and collapses, *Discrete Comput. Geom.* **47**(2) (2012) 301–328.
7. M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, *Invent. Math.* **129**(3) (1997) 445–470.
8. E. D. Bloch, Polyhedral representation of discrete Morse functions, *Discrete Math.* **313**(12) (2013) 1342–1348.
9. R. Bott, Nondegenerate critical manifolds, *Ann. Math.* (2) **60** (1954) 248–261.
10. M. K. Chari, On discrete Morse functions and combinatorial decompositions, *Discrete Math.* **217**(1–3) (2000) 101–113.
11. Ch. Conley, *Isolated Invariant Sets and the Morse Index*, Vol. 38 (Amer. Math. Soc., 1978).
12. O. Cornea, G. Lupton, J. Oprea and D. Tanré, *Lusternik-Schnirelmann Category*, Vol. 103 (Amer. Math. Soc., 2003).
13. F. D. Farmer, Cellular homology for posets, *Math. Japon.* **23** (1979) 607–613.
14. D. Fernández-Ternero, E. Macías-Virgós, E. Minuz and J. A. Vilches, Discrete topological complexity, *Proc. Amer. Math. Soc.* **146**(10) (2018) 4535–4548.
15. D. Fernández-Ternero, E. Macías-Virgós, E. Minuz and J. A. Vilches, Simplicial Lusternik-Schnirelmann category, *Publ. Mat.* **63**(1) (2019) 265–293.
16. D. Fernández-Ternero, E. Macías-Virgós, D. Mosquera-Lois, N. A. Scoville and J. A. Vilches, Fundamental theorems of Morse theory on posets, preprint (2020), arXiv:2002.06933.
17. D. Fernández-Ternero, E. Macías-Virgós, N. A. Scoville and J. A. Vilches, Strong discrete Morse theory and simplicial L-S category: A discrete version of the Lusternik-Schnirelmann theorem, *Discrete Comput. Geom.* **63**(3) (2020) 607–623.
18. R. Forman, Combinatorial vector fields and dynamical systems, *Math. Z.* **228**(4) (1998) 629–681.
19. R. Forman, Morse theory for cell complexes, *Adv. Math.* **134**(1) (1998) 90–145.
20. R. Forman, A user's guide to discrete Morse theory, *Sémin. Lothar. Comb.* **48** (2002) B48c.
21. R. Forman, Bochner's method for cell complexes and combinatorial Ricci curvature, *Discrete Comput. Geom.* **29**(3) (2003) 323–374.
22. R. Forman, Topics in combinatorial differential topology and geometry, in *Geometric Combinatorics* (Amer. Math. Soc., 2007), pp. 133–205.
23. R. H. Fox, On the Lusternik-Schnirelmann category, *Ann. Math.* (2) **42** (1941) 333–370.
24. J. M. Franks, *Homology and Dynamical Systems*, Vol. 49 (Amer. Math. Soc., 1982).
25. R. W. Ghrist, *Elementary Applied Topology* (CreateSpace Independent Publishing Platform, 2014).
26. J. C. Gómez-Larrañaga, F. González-Acuña and W. Heil, Old and new categorical invariants of manifolds, *Bol. Soc. Mat. Mex., III. Ser.* **20**(2) (2014) 177–198.
27. A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2002).
28. L. Johnson and K. Knudson, Min-max theory for cell complexes, *Algebra Colloq.* **27**(3) (2020) 447–454.

29. M. Kukiela, The main theorem of discrete Morse theory for Morse matchings with finitely many rays, *Topology Appl.* **160**(9) (2013) 1074–1082.
30. L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels. I: Espaces à un nombre fini de dimensions*, Actualités Scientifiques et Industrielles. 188. Exposés sur l'analyse mathématique et ses applications III. Publiés par J. Hadamard (Hermann & Cie, 1934).
31. E. Macías-Virgós and D. Mosquera-Lois, Homotopic distance between functors, *J. Homotopy Relat. Struct.* **15**(3–4) (2020) 537–555.
32. M. C. McCord, Singular homology groups and homotopy groups of finite topological spaces, *Duke Math. J.* **33** (1966) 465–474.
33. J. W. Milnor, *Morse Theory*. Based on Lecture Notes by M. Spivak and R. Wells, Vol. 51 (Princeton University Press, 1963).
34. E. G. Minian, Some remarks on Morse theory for posets, homological Morse theory and finite manifolds, *Topology Appl.* **159**(12) (2012) 2860–2869.
35. E. Pitcher, Inequalities of critical point theory, *Bull. Amer. Math. Soc.* **64** (1958) 1–30.
36. V. V. Prasolov, *Elements of Homology Theory*, Vol. 81 (Amer. Math. Soc., 2007). Translated from the Russian by Olga Sipacheva.
37. R. E. Stong, Group actions on finite spaces, *Discrete Math.* **49** (1984) 95–100.
38. K. Tanaka, Lusternik-Schnirelmann category for cell complexes and posets, *Illinois J. Math.* **59**(3) (2015) 623–636.
39. K. Tanaka, Lusternik-Schnirelmann category for categories and classifying spaces, *Topology Appl.* **239** (2018) 65–80.
40. M. L. Wachs, Poset topology: Tools and applications, in *Geometric Combinatorics* (Amer. Math. Soc., 2007), pp. 497–615.