On the isodiametric and isominwidth inequalities for planar bisections

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Abstract. For a given planar convex body $K$, a bisection of $K$ is a decomposition of $K$ into two closed sets $A, B$ so that $A \cap B$ is an injective continuous curve connecting exactly two boundary points of $K$. Consider a bisection of $K$ minimizing, over all bisections, the maximum diameter (resp., maximum width) of the sets in the decomposition.

In this note, we study some properties of these minimizing bisections and prove inequalities extending the classical isodiametric and isominwidth inequalities. Furthermore, we address the corresponding reverse optimization problems and establish inequalities similar to the reverse isodiametric and reverse isominwidth inequalities.

1. Introduction

The siblings Alice and Bob are deeply sad due to the loss of their uncle Charlie, who recently passed away. Soon, they will be awarded with his heritage consisting of a countryside piece of ground. They have to divide this terrain into two connected pieces of ground, which must be equal according to some even rule or fairness. In this paper, we will try to solve their issues, when the rule is either that the diameter or the minimum width of each of the pieces of ground is as small as possible (and so, the largest distance in the two pieces is minimized, or the eventual use of an agrarian harvester is optimized).

Let $\mathcal{K}^2$ be the family of planar convex bodies (recall that, as usual, a convex body is a convex compact set) with non-empty interior. Throughout this paper, for a given compact set $A \subset \mathbb{R}^2$, we will denote its area (or 2-dimensional Lebesgue measure) by $A(A)$, its diameter (largest Euclidean distance between two points in $A$) by $D(A)$, and its (minimum) width (shortest distance between two parallel lines containing $A$ between them) by $w(A)$.

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For a given $K \in \mathcal{K}^2$, a bisection of $K$ will be any pair of closed sets $(K_1, K_2)$ satisfying that

(i) $K = K_1 \cup K_2$,

(ii) $K_1 \cap K_2 = l([-1, 1])$, where $l: [-1, 1] \to K$ is an injective and continuous curve and whose endpoints $l(-1), l(1)$ are the only points of the curve in the boundary $\text{bd}(K)$ of $K$.

Some examples of (straight or curved) bisections are depicted in Figure 1.

![Figure 1. Some bisections for the ellipse.](image1)

For $K \in \mathcal{K}^2$, let $\mathcal{B}(K)$ be the set of all the bisections of $K$. We will denote the infimum of the maximum bisecting diameter of $K$ by

\[(1.1) \quad D_B(K) := \inf_{(K_1, K_2) \in \mathcal{B}(K)} \max\{D(K_1), D(K_2)\}.
\]

In some sense, $D_B(K)$ can be understood, for each $K \in \mathcal{K}^2$, as the optimal value for the diameter functional when considering bisections of $K$. We will see in Lemma 2.2 that such an infimum is in fact a minimum. We will study in this work the bisections of $K$ which provide $D_B(K)$, which will be called minimizing bisections of $K$, obtaining also an isodiametric-type inequality relating $D_B(K)$ and $A(K)$.

Our motivation mainly emanates from a paper by Miori et al. [24]. That paper focuses on bisections into two regions of equal area minimizing the maximum bisecting diameter in the setting of centrally symmetric planar convex bodies. Among other results, they prove that for every set in this family, there always exists a minimizing bisection determined by a line segment (see Proposition 4 in [24]), and describe in Theorem 5 of [24] the optimal set for this problem (that is, the set of fixed area with the minimum possible value for the maximum bisecting diameter). Such an optimal set is (see Example 2.3 in [24])

\[Q = \left\{(x_1, x_2) \in \mathbb{R}^2 : -\frac{1}{\sqrt{5}} \leq x_1 \leq \frac{1}{\sqrt{5}} \text{ and } (x_1 \pm \frac{1}{\sqrt{5}})^2 + x_2^2 \leq 1\right\},\]

see Figure 2.

![Figure 2. The optimal set $Q$.](image2)
Moreover, the classical isodiametric inequality of Bieberbach [6] states that for a given $K \in \mathcal{K}^2$, we have that

$$A(K) \leq \frac{\pi}{4} D(K)^2,$$

with equality if and only if $K$ is an Euclidean disk. Keeping this in mind, we obtain the following isodiametric-type inequality.

**Theorem 1.1.** Let $K \in \mathcal{K}^2$. Then,

$$\frac{A(K)}{D_B(K)^2} \leq 2 \arctan \left( \frac{3}{4} \right),$$

with equality if and only if $K = Q$.

Our Theorem 1.1 extends some results from [24] in the following way. In that work, the authors also demonstrate that for general planar convex bodies, the minimum value for the maximum bisecting diameter when considering bisections into two equal-area regions by line segments is attained by a centrally symmetric set (see Theorem 6 in [24]). If for $K \in \mathcal{K}^2$ we denote by

$$\tilde{\mathcal{B}}(K) = \{(K_1, K_2) \in \mathcal{B}(K) : K_1 \cap K_2 \text{ is a line segment}, A(K_1) = A(K_2)\}$$

and

$$\tilde{D}_B(K) = \inf_{(K_1, K_2) \in \tilde{\mathcal{B}}(K)} \max\{D(K_1), D(K_2)\},$$

then inequality (1.4) below follows from the results in [24] (although it is not explicitly stated in that paper):

$$\frac{A(K)}{\tilde{D}_B(K)^2} \leq 2 \arctan \left( \frac{3}{4} \right),$$

with equality if $K = Q$ (moreover, the only bisection of $Q$ in $\tilde{B}(Q)$ providing the value $\tilde{D}_B(Q)$ is the bisection $(Q^+, Q^-)$, where $Q^+ = Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ and $Q^- = Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$). Note that our Theorem 1.1 yields inequality (1.4): on the one hand, we consider arbitrary bisections, determined by curves which are not necessarily line segments. And on the other hand, we allow the regions of the bisections to have different areas. In other words, we focus on $\mathcal{B}(K)$ instead of $\tilde{B}(K)$. This makes our approach completely general in this setting.

**Remark 1.2.** Previous Theorem 1.1 implies that, if we prescribe the enclosed area, the convex body with the minimum possible value for $D_B$ is precisely $Q$, up to rigid motions (see Remark 1.5). Furthermore, in Remark 3.2 we will characterize the minimizing bisections of $Q$: a given bisection $(Q_1, Q_2) \in \mathcal{B}(Q)$ is minimizing if and only if $Q_1 \cap Q_2 = l([-1, 1])$, where

$$\{l(-1), l(1)\} = \left\{ \left( \pm \frac{1}{\sqrt{5}}, 0 \right) \right\}$$

and

$$l([-1, 1]) \subset \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + \left( x_2 \pm \frac{2}{\sqrt{5}} \right)^2 \leq 1 \right\}.$$
Surprisingly enough, the optimal set in the general situation, described in Theorem 1.1, is still the same set as in the optimal set in (1.4). This fact strengthens the idea that central symmetry is an inherent property for this optimization problem. On the other hand, we want to point out that the proof of our Theorem 1.1 cannot be carried out with the same arguments from Theorem 6 in [24], where the authors focus on bisections given by line segments and providing equal-area subsets. While the former restriction is not so significant (see our Lemma 2.1), the latter one entails a substantial reduction in the proof of Theorem 6 in [24], since it directly implies that the optimal set can be assumed to be centrally symmetric. In contrast, in the general case, the proof of our Theorem 1.1 moves around the choice of two non-parallel supporting lines at the endpoints of the line segment providing the minimizing bisection, and one cannot reduce to the simpler centrally symmetric case until the very last step.

Finally, it is worth mentioning that questions regarding the maximum bisecting diameter (firstly treated in [24]) have given rise to several works in the last years. In [11] we can find some improvements for the centrally symmetric case, and some related problems for divisions into three or more regions have been studied in [10], [9]. Moreover, we also point out that these questions have been partially treated for surfaces in $\mathbb{R}^3$ [12], [13]. Essentially, whenever there exists an isodiametric inequality, one can establish the corresponding isodiametric inequality for bisections. Although we focus in this work in the planar case, in Section 7 there are some considerations on the isodiametric-type problem for bisections in $\mathbb{R}^n$, and also in the spherical and the hyperbolic spaces.

Apart from studying the diameter, we also consider in this work the analogous problem for the width functional (which is, in some sense, the geometric functional reverse to the diameter). Recall that by replacing the diameter with the width in the classical isodiametric inequality, Pál showed that

$$A(K) \geq \frac{1}{\sqrt{3}} w(K)^2,$$

with equality if and only if $K$ is an equilateral triangle [25]. Our aim is obtaining a similar isominwidth inequality for bisections of a planar convex body. For this purpose, given $K \in \mathcal{K}^2$, we can define, analogously to $D_B(K)$, the infimum of the maximum bisecting width by

$$w_B(K) := \inf_{(K_1, K_2) \in B(K)} \max \{w(K_1), w(K_2)\}.$$

We will prove in Section 4 the following inequality.

**Theorem 1.3.** Let $K \in \mathcal{K}^2$. Then,

$$\frac{A(K)}{w_B(K)^2} \geq \frac{4}{\sqrt{3}},$$

with equality if and only if $K$ is an equilateral triangle $T$. 
The techniques employed to prove Theorem 1.3 are based on a nice combination of Pál’s inequality (1.5) and Bang’s inequality on Tarski’s plank problem [2]. We note that we will establish an isominwidth-type inequality for bisections in $\mathbb{R}^n$ in Section 7.

**Remark 1.4.** In analogy with Remark 1.2, Theorem 1.3 implies that if we prescribe the enclosed area, the corresponding equilateral triangle $T$ is the convex body with the largest possible value for $w_B$. That value is attained by the bisection determined by a line segment passing through the midpoints of two edges of $T$ (see the proof of Theorem 1.3).

**Remark 1.5.** Basic properties of the area, diameter and width imply the invariance under dilations and rigid motions of the quotients $A/D_B^2$, $A/D_B^2$ and $A/w_B^2$. Therefore, the uniqueness regarding the different optimal sets has to be understood up to dilations and rigid motions.

Notice that the study of the reverse counterparts to some geometric inequalities has increasingly gained interest in the last years (see [3], [1], [14] and references therein). In the case of the classical isodiametric inequality (1.2), a reverse inequality cannot be stated directly since the isodiametric quotient $A(K)/D(K)^2$, for $K \in \mathcal{K}^2$, cannot be bounded from below by any constant different from 0 (it suffices to consider very thin rectangles with area approaching zero). However, Behrend treated this problem finding such lower bound for the family of sets in $\mathcal{K}^2$ that maximizes that quotient in their affine class. More precisely, we will say that $K \in \mathcal{K}^2$ is in Behrend position if

$$\frac{A(K)}{D(K)^2} = \sup_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{D(\phi(K))^2},$$

where $\text{End}(\mathbb{R}^2)$ denotes the set of affine endomorphisms of $\mathbb{R}^2$ [3]. Therefore, if $K$ is in Behrend position, the above quotient achieves the maximum value among all the affine transformations of $K$. This approach allows to obtain an interesting reverse isodiametric inequality: for every $K \in \mathcal{K}^2$ in Behrend position, we have that

$$A(K) \geq \frac{\sqrt{3}}{4} D(K)^2,$$

with equality if and only if $K$ is an equilateral triangle [3]. Moreover, if we restrict $K$ to be centrally symmetric (that is, $K = x - K$ for some $x \in \mathbb{R}^2$), then

$$A(K) \geq \frac{1}{2} D(K)^2,$$

with equality if and only if $K$ is a square ([3], see also [19]). Following these ideas (also used by Ball for obtaining the first reverse isoperimetric inequality [1]), we will establish an analogous inequality to (1.8) for the infimum of the maximum bisecting diameter. In order to do this, we will say that $K \in \mathcal{K}^2$ is in Behrend-
bisecting position if

\[
\frac{A(K)}{D_B(K)^2} = \sup_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{D_B(\phi(K))^2}.
\]

In Section 5 we give some necessary conditions for a set $K$ to be in Behrend-bisecting position. In particular, and contrary to intuition, we will see that an equilateral triangle is not in Behrend-bisecting position. In fact, Proposition 5.6 gives a characterization of the unique triangle in Behrend-bisecting position, being an isosceles triangle whose angle different from the other two angles equals $\arccos(\sqrt{2/3})$. Apart from this, our Theorem 1.6 establishes the following reverse isodiametric inequality, which is not sharp in general.

**Theorem 1.6.** Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Then,

\[
\frac{A(K)}{D_B(K)^2} \geq \frac{\sqrt{3}}{4}.
\]

Moreover, the restriction to centrally symmetric convex bodies in Behrend-bisecting position allows to improve inequality (1.11), as shown in our Theorem 1.7.

**Theorem 1.7.** Let $K \in \mathcal{K}^2$ be centrally symmetric and in Behrend-bisecting position. Then,

\[
\frac{A(K)}{D_B(K)^2} \geq \frac{\sqrt{3}}{2}.
\]

In this setting, we also remark that Proposition 5.11 characterizes the parallelograms in Behrend-bisecting position: those are precisely the rectangles composed of two squares sharing an edge. Note that, in particular, a parallelogram formed by joining two equilateral triangles is not in Behrend-bisecting position.

We would also like to note that the proof of Theorem 1.6 (resp., Theorem 1.7) is inspired by the proof of Theorem 1.4 in [19] (resp., Proposition 1.3 in [19]) to reprove Behrend’s inequality (1.8). In essence, we provide the corresponding necessary condition of Behrend-bisecting position, see Lemma 5.4 (resp., the necessary condition of Behrend-bisecting position for centrally symmetric convex bodies, see Lemma 5.10), which differs from the conditions for being in Behrend position (see Proposition 5.2).

The same spirit of the previous results leads us to study a reverse isominwidth inequality for minimizing bisections of type $A(K)/w_B(K)^2 \leq \alpha$, for some $\alpha \in \mathbb{R}$. We will follow an approach similar to [19], considering again affine classes of sets in $\mathcal{K}^2$. In this sense, recall that $K \in \mathcal{K}^2$ is in isominwidth optimal position if

\[
\frac{A(K)}{w(K)^2} = \inf_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{w(\phi(K))^2}.
\]

The restriction to these suitable affine representatives of planar convex bodies yields, as in the case of the diameter functional, to the following result: for any set $K \in \mathcal{K}^2$ in isominwidth optimal position, it holds that

\[
A(K) \leq w(K)^2,
\]
Isodiametric and isominwidth inequalities for planar bisections

with equality if and only if $K$ is a square (see Theorem 1.6 in [19]). Our aim is obtaining an analogous inequality to (1.14) for the infimum of the maximum bisecting width for sets in a certain special position. Thus, given $K \in \mathcal{K}^2$, we will say that $K$ is in isominwidth-bisecting position if

\[(1.15) \quad \frac{A(K)}{w_B(K)^2} = \inf_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{w_B(\phi(K))^2}.\]

We will derive in Section 6 some necessary and sufficient conditions for being in isominwidth-bisecting position, concluding with our Theorem 1.8, which follows again from Bang’s inequality [2] and inequality (1.14).

**Theorem 1.8.** Let $K \in \mathcal{K}^2$ be in isominwidth-bisecting position. Then,

\[(1.16) \quad \frac{A(K)}{w_B(K)^2} \leq 4,\]

with equality if and only if $K$ is a square $C$.

**Remark 1.9.** We point out that $w_B(C)$ is attained by the bisection determined by a segment, which is parallel to an edge of $C$, that divides $C$ into two equal-area subsets.

The paper is organized as follows. In Section 2 we obtain some general properties of the minimizing bisections for the maximum bisecting diameter and the maximum bisecting width. In particular, Lemma 2.1 shows that there always exists a minimizing bisection given by a line segment, which allows to focus only on this type of bisections along this work. In Section 3 we prove Theorem 1.1, determining the corresponding optimal set (of fixed area) for the maximum bisecting diameter by a constructive argument. Section 4 is devoted to proving Theorem 1.3, which follows directly from Lemma 4.1. Sections 5 and 6 treat the reverse inequalities under the approach of affine representatives of planar convex bodies. In Section 5 we establish Theorem 1.6, which requires a detailed study concerning the Behrend-bisecting position, and Section 6 contains the proof of Theorem 1.8. Finally, in Section 7 we explore how to extend the isodiametric and isominwidth inequalities for bisections in the Euclidean space of higher dimension (Subsections 7.1 and 7.2), as well as in the spherical and hyperbolic spaces (Subsection 7.3).

**Notation**

We now establish some notation used throughout this paper. The Euclidean distance in $\mathbb{R}^2$ will be denoted by $d$, and the Hausdorff distance for planar compact sets will be denoted by $d_H$. Given two points $x, y \in \mathbb{R}^2$, $[x, y]$ will represent the line segment with endpoints $x$ and $y$. For every $K \in \mathcal{K}^2$, $\text{Ext}(K)$ will stand for the set of extreme points of $K$, i.e., if $x \in \text{Ext}(K)$, then $x \in [y, z] \subset K$ implies $x = y$ or $x = z$. For any planar compact set $A$, we denote by $\text{conv}(A)$ and $\text{span}(A)$ the convex hull and the linear hull of $A$, respectively. Moreover, we denote by $A^\perp$ the orthogonal complement of $A$, i.e., $A^\perp = \{x \in \mathbb{R}^2 : x^Ty = 0, \forall y \in A\}$. 
Furthermore, for $K \in \mathcal{K}^2$ and $u \in \mathbb{R}^2 \setminus \{0\}$, the Steiner symmetrization $s_u(K)$ of $K$ with respect to $\text{span}(u)$ is defined as the only symmetric set with respect to $\text{span}(u)$ such that each segment $(tu + u^\perp) \cap s_u(K)$ has the same length than $(tu + u^\perp) \cap K$ for every $t \in \mathbb{R}$, \cite{1}, \cite{26}. It is well known that $s_u(K) \in \mathcal{K}^2$ and

\begin{equation}
(1.17) \quad \Lambda(s_u(K)) = \Lambda(K), \quad D(s_u(K)) \leq D(K).
\end{equation}

Finally, a sequence $\{K_n\}_{n \in \mathbb{N}}$ in $\mathcal{K}^2$ is said to be bounded if there exists $K \in \mathcal{K}^2$ satisfying that $K_n \subseteq K$, for $n \in \mathbb{N}$.

2. Properties of minimizing bisections

In this section we will obtain some interesting properties for the minimizing bisections of the two functionals $D_B(K), \, w_B(K)$ we are considering. Lemma 2.1 shows that there is always one of these bisections given by a line segment, extending Proposition 4 in \cite{24}, and Lemma 2.3 further proves that the subsets of that bisection are in equilibrium, in some sense. Besides, we also show in Lemma 2.2 that the infima in (1.1) and (1.6) are attained (and so they are actually minima).

**Lemma 2.1.** Let $K \in \mathcal{K}^2$ and $\rho > 0$. For any bisection of $K$ with maximum bisecting diameter (resp., width) equal to $\rho$, there exists another bisection of $K$ given by a line segment with maximum bisecting diameter (resp., width) smaller than or equal to $\rho$.

**Proof.** Consider $(K_1, K_2) \in \mathcal{B}(K)$ determined by an injective continuous curve $l: [-1, 1] \to K$ with $l(-1), \, l(1) \in \text{bd}(K)$. Assume that $\max\{D(K_1), D(K_2)\} = \rho$ (resp., $\max\{w(K_1), w(K_2)\} = \rho$). Call $M_1 := \text{bd}(K) \cap K_1$ and $M_2 := \text{bd}(K) \cap K_2$. Since $M_i \subset K_i$, then $D(M_i) \leq D(K_i)$ and $w(M_i) \leq w(K_i)$, for $i = 1, 2$.

Notice that the line segment $[l(-1), l(1)]$ determines the bisection of $K$ with subsets $\text{conv}(M_1)$ and $\text{conv}(M_2)$. We claim that $D(M_i) = D(\text{conv}(M_i)), \, i = 1, 2$. On the one hand, $M_i \subset \text{conv}(M_i)$ implies that $D(M_i) \leq D(\text{conv}(M_i))$. And on the other hand, $\text{Ext}(\text{conv}(M_i)) \subset M_i$, since any point in $\text{Ext}(\text{conv}(M_i))$ cannot be expressed as a strict convex combination of points in $\text{conv}(M_i)$. Furthermore, since the diameter in $\mathbb{R}^2$ is always attained by a pair of extreme points, it follows that

\[ D(\text{conv}(M_i)) = D(\text{Ext}(\text{conv}(M_i))) \leq D(M_i). \]

We also have that $w(M_i) = w(\text{conv}(M_i)), \, i = 1, 2$, as a direct consequence of the fact that $M_i$ is contained between two parallel lines if and only if $\text{conv}(M_i)$ is contained between those lines. Then, $\text{conv}(M_1), \, \text{conv}(M_2)$ are two subsets of $K$ providing a bisection of $K$, satisfying

\[ \max\{D(\text{conv}(M_1)), D(\text{conv}(M_2))\} \leq \max\{D(K_1), D(K_2)\} = \rho, \]

as well as

\[ \max\{w(\text{conv}(M_1)), w(\text{conv}(M_2))\} \leq \max\{w(K_1), w(K_2)\} = \rho, \]

Finally, a sequence $\{K_n\}_{n \in \mathbb{N}}$ in $\mathcal{K}^2$ is said to be bounded if there exists $K \in \mathcal{K}^2$ satisfying that $K_n \subseteq K$, for $n \in \mathbb{N}$.
and so, \((\text{conv}(M_1), \text{conv}(M_2))\) is a bisection of \(K\) given by a line segment with maximum bisecting diameter (resp., width) smaller than or equal to \(\rho\), as stated. \(\square\)

**Lemma 2.2.** Let \(K \in \mathbb{K}^2\). Then,

\[
D_B(K) = \min_{(K_1, K_2) \in B(K)} \max\{D(K_1), D(K_2)\},
\]

and

\[
w_B(K) = \min_{(K_1, K_2) \in B(K)} \max\{w(K_1), w(K_2)\}.
\]

**Proof.** We will focus on \(D_B(K)\), since the case of \(w_B(K)\) is analogous. Note that Lemma 2.1 allows to consider only bisections by line segments in order to compute \(D_B(K)\). Then, in view of (1.1), let \(\{[a_i, b_i]\}_{i \in \mathbb{N}} \subset K\) be a sequence of line segments providing bisections \((K_{1,i}, K_{2,i})\) of \(K\), such that

\[
D_B(K) = \lim_{i \to \infty} \max\{D(K_{1,i}), D(K_{2,i})\}.
\]

Since \(\{K_{1,i}\}_{i \in \mathbb{N}}\) is a bounded sequence of convex bodies (notice that \(K_{1,i} \subset K\), for \(i \in \mathbb{N}\)), the Blaschke selection theorem (see Theorem 1.8.7 in [27]) implies the existence of a subsequence \(\{K_{1,s}\}\) and a subset \(K_1 \in \mathbb{K}^2\) such that \(K_1 \subset K\) and \(\lim_{i \to \infty} K_{1,i} = K_1\) in Hausdorff metric. Considering now the corresponding subsequence \(\{K_{2,i}\}\) of \(\{K_{2,i}\}\), it is clear that \(\lim_{i \to \infty} K_{2,i} = K_2\), where \(K_2 = K \setminus K_1\). Note that, without loss of generality, we can assume that the subsequences are the sequences themselves. In particular, we also obtain that \(\lim_{i \to \infty} [a_i, b_i] = [a, b]\), for certain \(a, b \in K\), with \([a, b] = K_1 \cap K_2\), and so \((K_1, K_2)\) is a bisection of \(K\). Since the diameter is a continuous functional with respect to the Hausdorff metric, we have that \(\lim_{i \to \infty} D(K_{1,i}) = D(K_1)\) and \(\lim_{i \to \infty} D(K_{2,i}) = D(K_2)\), which implies that \(D_B(K) = \max\{D(K_1), D(K_2)\}\), as stated. \(\square\)

**Lemma 2.3.** Let \(K \in \mathbb{K}^2\). There exists a bisection \((K_1, K_2)\) of \(K\) minimizing the maximum bisecting diameter (resp., width) of \(K\) such that \(D_B(K) = D(K_1) = D(K_2)\) (resp., \(w_B(K) = w(K_1) = w(K_2)\)).

**Proof.** This is a consequence of the continuity of the diameter and the width functionals. Taking into account Lemmas 2.1 and 2.2, let \((K_1, K_2)\) be a bisection of \(K\) minimizing the maximum bisecting diameter (resp., width), determined by the line segment \(L = K_1 \cap K_2\). Fix \(u\) an orthogonal vector to \(\text{span}(L)\), and let \(t_1 < 0 < t_2\) be such that \(K \cap (tu + L) \neq \emptyset\) when and only when \(t \in [t_1, t_2]\). Moreover let \(K_i^t = K \cap \{su + L : s \in [t_1, t]\}\) and \(K_i^t = K \cap \{su + L : s \in [t, t_2]\}\), for every \(t \in [t_1, t_2]\), so that \(K_0^t = K_1, i = 1, 2\). In particular, we have that \(K \cap (t, tu + L) \subset \text{bd}(K), i = 1, 2\), and thus \(K_1^{t_2} = K_2^{t_1} = K\).

For \(i = 1, 2\), let \(f_i, g_i : [t_1, t_2] \to [0, D(K)]\) be such that \(f_1(t) = D(K_1^t), f_2(t) = D(K_2^t), g_1(t) = w(K_1^t), g_2(t) = w(K_2^t)\). By direct inclusion of sets, we have that \(f_i\) and \(g_i\) are non-decreasing, whereas \(f_2\) and \(g_2\) are non-increasing. Moreover, these four functions are continuous, with \(f_1(t_2) = D(K) = f_2(t_1)\) and \(g_1(t_2) = w(K) = g_2(t_1)\).
If \( f_1(0) = f_2(0) \) (resp., \( g_1(0) = g_2(0) \)), then \((K_1, K_2)\) is a minimizing bisection with \( D_B(K) = D(K_1) = D(K_2) \) (resp., \( w_B(K) = w(K_1) = w(K_2) \)), as desired. Otherwise, let us suppose without loss of generality that \( f_1(0) < f_2(0) = D(K_2) = D_B(K) \) (resp., \( g_1(0) < g_2(0) = w(K_2) = w_B(K) \)). Since
\[
 f_1(t_2) = D(K) \geq D(K_2^t) = f_2(t_2),
\]
(resp., \( g_1(t_2) \geq g_2(t_2) \)), the Bolzano theorem implies that there exists \( t_0 \in [0, t_2] \) such that \( f_1(t_0) = f_2(t_0) \) (resp., \( g_1(t_0) = g_2(t_0) \)). By using the monotonicity of the functions, we have that
\[
 D(K_1) = f_1(0) \leq f_1(t_0) = f_2(t_0) \leq f_2(0) = D(K_2) = D_B(K)
\]
and
\[
 w(K_1) = g_1(0) \leq g_1(t_0) = g_2(t_0) \leq g_2(0) = w(K_2) = w_B(K),
\]
thus \( D(K_1^t) = D(K_2^t) \leq D_B(K) \) (resp., \( w(K_1^t) = w(K_2^t) \leq w_B(K) \)), and hence \((K_1^t, K_2^t)\) is a minimizing bisection of \( K \) providing subsets of equal diameters (resp., widths), as desired. \( \square \)

**Remark 2.4.** In fact, Lemma 2.3 proves that for every minimizing bisection determined by a line segment \( l \), there exists another minimizing bisection \((K_1^l, K_2^l)\) with \( D(K_1^l) = D(K_2^l) \) (resp., \( w(K_1^l) = w(K_2^l) \)) and determined by a line segment parallel to \( l \).

**Remark 2.5.** A minimizing bisection \((K_1, K_2)\) for \( D_B \) with subsets of equal diameters as in Lemma 2.3 might be degenerate, that is, \( K_1 \) or \( K_2 \) might be reduced to a line segment. For instance, let \( T \in K^2 \) be an equilateral triangle of vertices \( p_i, i = 1, 2, 3 \). Then \((T, [p_1, p_2])\) is a minimizing bisection with \( D_B(T) = D(T) = D([p_1, p_2]) \). This is not the case for the minimizing bisections for \( w_B \) with subsets of equal widths, which have to split any convex body into two non-degenerate subsets, since the width of a line segment trivially vanishes.

### 3. The isodiametric inequality

In this section we will prove our Theorem 1.1, providing an isodiametric-type inequality involving \( D_B \). As we will see, the proof of this result is constructive, yielding the corresponding optimal set. We first prove Lemma 3.1.

**Lemma 3.1.** There exists a maximizer \( K_0 \in K^2 \) of the quotient \( \Lambda(K)/D_B(K)^2 \), with \( D_B(K_0) \) attained by a bisection of \( K_0 \) which is determined by a line segment \([(-a, 0), (a, 0)]\), \( a > 0 \), such that \( K_0 \) is symmetric with respect to the line \( L = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \} \).

**Proof.** Consider the supremum
\[
 \gamma := \sup_{K \in K^2} \frac{\Lambda(K)}{D_B(K)^2} > 0.
\]
Taking into account that the area and the diameter functionals are homogeneous with respect to dilations (see Remark 1.5), we can normalize to unit area and so

\[ \gamma = \sup_{K \in \mathcal{K}^2} \frac{1}{D_B(K)^2}, \]

Let \( \{K_n\} \) be a sequence in \( \mathcal{K}^2 \) with \( \lim_{n \to \infty} D_B(K_n)^2 = \gamma^{-1} \), and \( A(K_n) = 1 \) for every \( n \in \mathbb{N} \). We claim that \( D_B(K_n) \leq C \), for a certain \( C > 0 \). Otherwise, \( D_B(K_n) \) would tend to infinity, which contradicts the positivity of \( \gamma \). Hence \( D(K_n) \leq 2D_B(K_n) \leq 2C \) and so, after a suitable translation of each \( K_n \), we can assume that \( \{K_n\} \) is a bounded sequence. Hence, by the Blaschke selection theorem, there exists a subsequence of \( \{K_n\} \) convergent to some \( \tilde{K} \in \mathcal{K}^2 \) in Hausdorff metric. By continuity, it follows that

\[ \frac{1}{D_B(\tilde{K})^2} = \gamma, \]

and so \( \tilde{K} \) is a maximizer of the quotient.

By Lemma 2.1, we can now suppose without loss of generality that \( D_B(\tilde{K}) \) is given by a bisection \( (K_1, K_2) \) of \( \tilde{K} \), with \( K_1 = \tilde{K} \cap H^+, \ K_2 = \tilde{K} \cap H^-, \ K_1 \cap K_2 = \{(-a, 0), (a, 0)\} \), for some \( a \in (0, D(\tilde{K})/2] \), where \( H^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \) and \( H^- = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\} \). Call \( e_2 = (0, 1) \in \mathbb{R}^2 \). By applying Steiner symmetrization \( s_{e_2} \) with respect to the vertical line \( \text{span}(e_2) \), we easily get that \( s_{e_2}(\tilde{K}) = s_{e_2}(K_1) \cup s_{e_2}(K_2) \) and \( s_{e_2}(K_1) \cap s_{e_2}(K_2) = \{(-a, 0)\} \). Denoting \( K_0 := s_{e_2}(\tilde{K}) \) and \( K_{0,i} := s_{e_2}(K_i) \), we have by (1.17) that \( A(K_{0,i}) = A(K_i) \) and \( D(K_{0,i}) \leq D(K_i) \), \( i = 1, 2 \), and so \( A(K_0) = A(\tilde{K}) \) and \( D_B(K_0) \leq D_B(\tilde{K}) \).

Since \( \tilde{K} \) is a maximizer of the quotient \( A(\tilde{K})/D_B(\tilde{K})^2 \), then necessarily \( K_0 \) is also a maximizer, which possesses the desired symmetry by construction. \( \square \)

**Proof of Theorem 1.1.** Consider an arbitrary \( K \in \mathcal{K}^2 \). We will apply several transformations to \( K \), without decreasing the enclosed area, arriving at the end of the process at the set \( Q \in \mathcal{K}^2 \), which satisfies \( A(Q) \geq A(K) \) and \( D_B(Q) = D_B(K) \). This will prove the maximality of \( Q \).

Let us suppose without loss of generality that \( (K_1, K_2) \) is a bisection of \( K \) providing \( D_B(K) \), with \( K_1 = K \cap H^+, \ K_2 = K \cap H^-, \ K_1 \cap K_2 = \{(-a, 0), (a, 0)\} \), for some \( a \in [0, D_B(K)/2] \), where \( H^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \) and \( H^- = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\} \), in view of Lemma 3.1. We can also assume that \( K \) is symmetric with respect to the vertical line \( L = \{x \in \mathbb{R}^2 : x_1 = 0\} \) and, by Lemma 2.3, that \( D(K_1) = D(K_2) = D_B(K) \).

Since \( K \) is convex and compact, and \((a, 0) \in \text{bd}(K)\), then there exists a supporting line \( M_+ \) to \( K \) at \((a, 0)\). Due to the symmetry of \( K \), the symmetric line of \( M_+ \) with respect to \( L \) is also a supporting line at \((-a, 0)\), and we call it \( M_- \). By flipping the situation if necessary, we can assume that the slope of \( M_+ \) is non-negative, and so \( M_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = m(x_1 - a)\} \), for some \( m \geq 0 \). Additionally, let \( B_k = B((\pm a, 0), D_B(K)) \) be the closed balls centered at \((\pm a, 0)\) and of radius \( D_B(K) \). Since \( D(K_1) = D_B(K) \) and \((\pm a, 0) \in K_1 \), it follows that \( K_1 \) is necessarily contained in the symmetric lens \( B_+ \cap B_- \), for \( i = 1, 2 \).
If $M_+$ is not vertical, we have that $K_2$ is contained in the triangle $T$ determined by $M_+, M_-$, and the horizontal line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ (see Figure 3).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure3.png}
\caption{If $M_+$ is not vertical, then $K_2$ is contained in the triangle $T$.}
\end{figure}

Then $D(T) = \max\{2a, \delta\} \geq D(K_2) = D_B(K)$, where
\[\delta = d((a,0), (0,-ma)) = a\sqrt{1 + m^2}.\]

We will distinguish two possibilities. If $2a > \delta$, then $2a = D(T) \geq D_B(K)$ (and so $D_B(K) = 2a$). In this case, it is straightforward checking that the area of $B_+ \cap B_-$ equals $D_B(K)^2 (4\pi - 3\sqrt{3})/6$, and so
\begin{equation}
\frac{A(K)}{D_B(K)^2} \leq \frac{A(B_+ \cap B_-)}{D_B(K)^2} = \frac{4\pi - 3\sqrt{3}}{6}.
\end{equation}

On the other hand, if $2a \leq \delta$, then $\delta = D(T) \geq D_B(K)$, which implies that $m \geq a^{-1}\sqrt{D_B(K)^2 - a^2}$. Let us estimate the isodiametric quotient of $K$ in this case.

Let $R(a,m)$ be the planar region contained between $M_+, M_-, B_+$ and $B_-$, with the dependance on $a$ and $m$ explained above. Since $K \subseteq R(a,m)$, then $A(K) \leq A(R(a,m))$. Moreover, let $R(a, +\infty)$ be the planar region contained between $B_+, B_-$ and the vertical lines passing through $(\pm a,0)$. Let us check that $A(R(a,m)) < A(R(a, +\infty))$, for every $m \geq a^{-1}\sqrt{D_B(K)^2 - a^2}$ (and $2a \leq \delta$). Due to the symmetry of these regions, we can focus on the corresponding areas contained in $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$. The only region $R_1$ (resp., $R_2$) contained in $R(a, m)$ (resp., $R(a, +\infty)$) which is not in $R(a, +\infty)$ (resp., $R(a, m)$) is the one contained between $M_+, (a,0) + L, B_-$, and $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ (resp., $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$). It can be checked that the condition $m \geq a^{-1}\sqrt{D_B(K)^2 - a^2}$ implies that the rotation centered at $(a,0)$ of angle $\pi$ maps strictly $R_1$ onto $R_2$, and so $A(R(a,m)) < A(R(a, +\infty))$. Note also that the construction of $R(a, +\infty)$ implies that $D_B(R(a, +\infty)) = D_B(K)$ (the bisection of $R(a, +\infty)$ given by the subsets $R_+ = R(a, +\infty) \cap H^+$ and $R_- = R(a, +\infty) \cap H^-$ satisfies $D(R_+) = D(R_-) = D_B(K)$, see [24]).
Let us now compute the maximum value for $A(R(a, +\infty))/D_B(K)^2$, when $a > 0$. It is straightforward checking that

$$A(a) := A(R(a, +\infty)) = 4 \int_0^a \sqrt{D_B(K)^2 - (x+a)^2} \, dx$$

$$= 2 \left( 2a \sqrt{D_B(K)^2 - 4a^2} - a \sqrt{D_B(K)^2 - a^2} + D_B(K)^2 \arctan \left( \frac{2a}{\sqrt{D_B(K)^2 - 4a^2}} \right) \right)$$

$$- D_B(K)^2 \arctan \left( \frac{a}{\sqrt{D_B(K)^2 - a^2}} \right).$$

For simplicity, call $b = a/D_B(K)$ (which corresponds to a normalization for having $D_B(K)$ equal to 1 by an appropriate dilation). Then, well-known properties of dilations give

$$A(b) = 2 \left( 2b \sqrt{1 - 4b^2} - b \sqrt{1 - b^2} + \arctan \left( \frac{2b}{\sqrt{1 - 4b^2}} \right) - \arctan \left( \frac{b}{\sqrt{1 - b^2}} \right) \right),$$

which attains its maximum value (as a function on $b$) only at $b = 1/\sqrt{5}$, and so, for any $b > 0$,

$$A(b) \leq A(1/\sqrt{5}) = 2 \arctan \left( \frac{3}{4} \right).$$

Thus

$$\frac{A(K)}{D_B(K)^2} \leq \frac{A(R(a, +\infty))}{D_B(K)^2} \leq 2 \arctan \left( \frac{3}{4} \right),$$

which gives a bound greater than the one obtained in (3.1), yielding the desired inequality (1.3). The proof finishes by noting that if $M_+$ is vertical, then $K \subset R(a, +\infty)$, which gives the same inequality (3.2). Equality above only holds when $A(b)$ is maximum, namely for $R(1/\sqrt{5}, +\infty)$, which coincides with $Q$ by definition. $\square$

The following Remark 3.2 gives a description of the minimizing bisections of the optimal set $Q$: they are determined by curves contained in a particular region of $Q$.

**Remark 3.2.** Let $(Q_1, Q_2)$ be a minimizing bisection of $Q$ determined by a curve $l: [-1, 1] \to Q$, and let $q_+ = (0, 2/\sqrt{5}) \in \partial Q$, $q_- = (0, -2/\sqrt{5}) \in \partial Q$, $p_+ = (1/\sqrt{5}, 0) \in \partial Q$, $p_- = (-1/\sqrt{5}, 0) \in \partial Q$. Recall that $D_B(Q) = 1 = d(p_-, q_+)$. Since $d(q_+, q_-) > D_B(Q)$, each of these two points must belong to a different subset of the bisection. We can assume that $q_+ \in Q_1$, $q_- \in Q_2$. Then it necessarily follows that $Q_1 \subseteq B(q_+, D_B(Q))$ and $Q_2 \subseteq B(q_-, D_B(Q))$. Those inclusions immediately imply that $\{((-1, 1), l(1)) = \{p_-, p_+\}$, and also that $l([-1, 1])$ is contained in the intersection $L$ of those balls (see Figure 4), that is,

$$l([-1, 1]) \subseteq \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + \left( x_2 \pm \frac{2}{\sqrt{5}} \right)^2 \leq 1 \right\}.$$
Figure 4. Any curve joining $p_-$ and $p_+$ and contained in $L$ provides a minimizing bisection of $Q$.

Remark 3.3. The reader will realize that the line segment $[(-a,0),(a,0)]$ does not give a minimizing bisection of $R(a,m)$ in the proof of Theorem 1.1 for some values of the parameters $a,m$. Indeed, in every step of the proof of Theorem 1.1, we replace the set by another one with greater (or equal) area. This process starts with $K$ and ends with $Q = R(1/\sqrt{5},+\infty)$, and the corresponding horizontal line segment provides a minimizing bisection for these two sets, whereas in the middle of the process, that line segment does not give necessarily a minimizing bisection of $R(a,m)$ in general. For instance, for $K = R(a,\sqrt{3})$, with $D_B(K) > 2a$, the bisection determined by the line segment $[(-a,0),(a,0)]$ is not minimizing, since it can be improved by a different line segment (placed slightly above).

4. The isominwidth inequality

In this section we will consider the problem analogous to the one studied in Section 3, but for the width functional. We will start by proving that $w_B(K) = w(K)/2$, for any $K \in \mathcal{K}^2$, by using the following celebrated result by Bang on Tarski’s plank problem [2]: for $K \in \mathcal{K}^2$, and $p,q \in \text{bd}(K)$, let $(K_1,K_2)$ be the bisection given by the line segment $[p,q]$. Then

$$w(K_1) + w(K_2) \geq w(K).$$

(4.1)

Lemma 4.1. Let $K \in \mathcal{K}^2$. Then, $w_B(K) = w(K)/2$.

Proof. Let $L_1$, $L_2$ be two parallel supporting lines of $K$ such that $d(L_1,L_2) = w(K)$, and let $u \in S^1$ be an orthogonal vector to these lines. Consider $p,q \in \text{bd}(K)$ such that $[p,q] = K \cap L$, where $L$ is a line parallel to $L_i$ which lies at distance $w(K)/2$ from each line $L_i$, $i = 1,2$. Moreover, let $(K_1,K_2)$ be the bisection determined by the line segment $[p,q]$. Note that $L$ and $L_i$ are supporting lines of $K_i$, for $i = 1,2$, and so $w(K_i) \leq w(K)/2$. Thus, $\max\{w(K_1),w(K_2)\} \leq w(K)/2$ and hence $w_B(K) \leq w(K)/2$. On the other hand, in view of Lemmas 2.1, 2.2 and 2.3, let $(\widetilde{K}_1,\widetilde{K}_2)$ be a bisection of $K$ where $w_B(K)$ is attained, given by a
line segment and satisfying $w_B(K) = w(\tilde{K}_1) = w(\tilde{K}_2)$. Then (4.1) implies that $w(K) \leq w(\tilde{K}_1) + w(\tilde{K}_2) = 2w_B(K)$, and so $w_B(K) \geq w(K)/2$, yielding the desired equality.

Now we are able to prove immediately the main result of this section, which is Theorem 1.3, providing a sharp upper bound for $w_B$.

**Proof of Theorem 1.3.** By Lemma 4.1 and Pal’s inequality (1.5), we directly have that

$$\frac{\Lambda(K)}{w_B(K)^2} = 4 \frac{\Lambda(K)}{w(K)^2} \geq \frac{4}{\sqrt{3}}.$$ 

Moreover, in order to have equality, we must have equality in (1.5), hence implying that $K$ is an equilateral triangle $\mathcal{T}$. Additionally, note that $w(\mathcal{T})$ equals any of the three heights of $\mathcal{T}$, and the width corresponding to any other different direction will be strictly greater. Since $w_B(\mathcal{T}) = w(\mathcal{T})/2$ by Lemma 4.1, this implies that any minimizing bisection $(\mathcal{T}_1, \mathcal{T}_2)$ of $\mathcal{T}$ must satisfy that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a line segment whose endpoints are the midpoints of two edges of $\mathcal{T}$. □

5. The Behrend-bisecting position and the reverse isodiametric inequality

As commented in the introduction, we will now focus on a reverse isodiametric-type inequality for $D_B$. The following definitions and results arise mainly from some ideas in [3]. For every $K \in \mathcal{K}^2$, let

$$V_K := \{ u \in S^1 : \exists x \in K \text{ such that } x + D(K)[0, u] \subset K \}$$

be the set of diametrical directions of $K$ (that is, the directions for which $D(K)$ is attained). Moreover, we will say that $u \in S^1$ is a bisector of $K$ if $u$ is the direction of a line segment providing a minimizing bisection $(K_1, K_2)$ of $K$ with $D(K_1) = D(K_2)$. We will denote by $B_K$ the set of bisectors of $K$. Note that $B_K$ contains the directions which determine suitable minimizing bisections by line segments for $D_B$.

The next result establishes that the supremum in the definition of the Behrend-bisecting position (1.10) is actually a maximum.

**Lemma 5.1.** Let $K \in \mathcal{K}^2$. Then, there exists $\phi \in \text{End}(\mathbb{R}^2)$ such that $\phi(K)$ is in Behrend-bisecting position.

**Proof.** We can assume, after a suitable translation of $K$, that $rB_2^2 \subseteq K$ for some $r > 0$, where $B_2^2$ is the planar Euclidean unit ball centered at the origin. Call

$$\rho := \sup_{\phi \in \text{End}(\mathbb{R}^2)} \frac{\Lambda(\phi(K))}{D_B(\phi(K))^2}.$$
Since $A$ and $D_B^2$ are homogeneous functionals of degree two, we can suppose without loss of generality that $|\det(\phi)| = 1$, $\Lambda(K) = 1$, and hence

\begin{equation}
(5.1) \quad \inf_{\phi \in \End(\mathbb{R}^2)} D_B(\phi(K)) = \frac{1}{\sqrt{p}}.
\end{equation}

Consider a sequence $\{\phi_i\}_{i \in \mathbb{N}} \subset \End(\mathbb{R}^2)$ such that $|\det(\phi_i)| = 1$, for $i \in \mathbb{N}$, and

$$D_B(\phi_i(K)) \to \frac{1}{\sqrt{p}} \quad \text{when } i \to \infty.$$  

We can additionally assume that all the endomorphisms $\phi_i$ are linear, since $D_B$ is invariant under translations, and that there exists $C > 0$ such that $D_B(\phi_i(K)) \leq C$ for every $i \in \mathbb{N}$. Since $(0, 0) \in \phi_i(K)$ and $D(\phi_i(K)) \leq 2 D_B(\phi_i(K)) \leq 2 C$, for all $i \in \mathbb{N}$, then $\{\phi_i(K)\}_{i \in \mathbb{N}}$ is a bounded sequence (since $(0, 0) \in \phi_i(K)$, we actually have that $\phi_i(K) \subseteq 2C B_2^2$). Hence the Blaschke selection theorem implies that there exists a subsequence (which will be denoted as the original one) such that $\phi_i(K) \to K_0$ when $i \to \infty$, for some $K_0 \in K^2$. Let us furthermore observe that if $\phi_i = (a_{jk}^i)_{1 \leq j, k \leq 2} \in \mathbb{R}^{2 \times 2}$, since $r B_2^2 \subseteq K$ and $\phi_i(K) \subseteq 2C B_2^2$, then it follows that $|a_{jk}^i| \leq 2C/r$ for $1 \leq j, k \leq 2$ and $i \in \mathbb{N}$. Thus $\{\phi_i\}_{i \in \mathbb{N}}$ is bounded with respect to the so-called induced norm (or operator norm) $\|\cdot\|_\text{op}$ for linear endomorphisms, and so there exists a subsequence (which will be denoted again as the original one) such that $\phi_i \to \phi_0$ when $i \to \infty$, for some $\phi_0 \in \End(\mathbb{R}^2)$. Moreover, $|\det(\phi_0)| = 1$, with $\phi_i(K) \to K_0 = \phi_0(K)$ when $i \to \infty$. We will now prove that $D_B(\phi_0(K)) = 1/\sqrt{p}$, which will imply that $\phi_0(K)$ is in Behrend-bisecting position, as desired.

First of all, since each $\phi_i$ is linear and non-singular (recall that $|\det(\phi_i)| = 1$), we have that $\phi_i$ is bijective. Fix $u_i \in B_{\phi_i(K)}$, and let $x_i \in K$, $\mu_i > 0$ be such that the line segment $\phi_i(x_i) + \mu_i [0, u_i] \subset \phi_i(K)$ provides a minimizing bisection of $\phi_i(K)$, for each $i \in \mathbb{N}$. Let $\phi(K_i^1)$, $\phi(K_i^2)$ be the subsets of that bisection, satisfying $D_B(\phi_i(K)) = D(\phi_i(K_i^1)) = D(\phi_i(K_i^2))$ for every $i \in \mathbb{N}$. Since $\phi_i$ is a bijection, we will have that $(K_i^1, K_i^2)$ is a bisection of $K$ and moreover, we can consider $y_i \in K$ such that $\phi_i(y_i) = \phi_i(x_i) + \mu_i u_i$, for every $i \in \mathbb{N}$. Since $[x_i, y_i] \subset K$, for $i \in \mathbb{N}$, the sequence $\{[x_i, y_i]\}_{i \in \mathbb{N}}$ is bounded, and we can assume that $[x_i, y_i] \to [x_0, y_0]$ when $i \to \infty$, for some $x_0, y_0 \in K$. Let $(K^1_j, K^2_j)$ be the bisection of $K$ given by $[x_0, y_0]$, and let us see that $\phi_j(K^1_i) \to \phi_0(K^0_0)$ when $i \to \infty$, for $j = 1, 2$. By the subadditivity of Hausdorff distance $d_H$, it is clear that

\begin{equation}
(5.2) \quad d_H(\phi_i(K^1_i), \phi_0(K^0_0)) \leq d_H(\phi_i(K^1_i), \phi_0(K^0_1)) + d_H(\phi_0(K^0_1), \phi_0(K^0_0)).
\end{equation}

Note that, since $K$ is compact, then $K \subseteq \delta B_2^2$, for some $\delta > 0$, and thus $K^1_j \subseteq \delta B_2^2$ for every $i \in \mathbb{N}$ and $j = 1, 2$. Then, for $x \in K^1_i$,

$$\|\phi_i(x) - \phi_0(x)\| = \|\phi_i - \phi_0\| \|x\| \leq \delta \|\phi_i - \phi_0\|_\text{op},$$

which implies that $d_H(\phi_i(K^1_i), \phi_0(K^0_1)) \to 0$ when $i \to \infty$, for $j = 1, 2$. On the other hand, we claim that $d_H(\phi_0(K^1_j), \phi_0(K^0_0)) \leq \|\phi_0\|_\text{op} d_H(K^1_j, K^0_0)$. Consider $\varepsilon_j :=
Lemma 5.4. Let \( d_H(K_j^1, K_j^2) \), which tends to 0 when \( i \to \infty \), for \( j = 1, 2 \). It follows that \( K_j^1 \subseteq K_j^2 + \varepsilon_j^j B_2^2 \), and by applying \( \phi_0 \) we get \( \phi_0(K_j^1) \subseteq \phi_0(K_j^2) + \varepsilon_j^j \| \phi_0 \| \|B_2^2 \|. \) Analogously, we will get \( \phi_0(K_j^1) \subseteq \phi_0(K_j^2) + \varepsilon_j^j \| \phi_0 \| \|B_2^2 \| \). These two inclusions yield the claim, by the definition of \( d_H \), which implies that \( d_H(\phi_0(K_j^1), \phi_0(K_j^2)) \to 0 \) when \( i \to \infty \), for \( j = 1, 2 \). Taking into account (5.2), we conclude that \( \phi_0(K_j^1) \to \phi_0(K_j^2) \) when \( i \to \infty \), for \( j = 1, 2 \). Therefore, \( D(\phi_0(K_j^2)) = 1/\sqrt{\rho} \), for \( j = 1, 2 \), and so \( D_B(K_0) \leq 1/\sqrt{\rho} \). But if this inequality is strict, we get a contradiction with (5.1), so equality must hold, which finishes the proof. \( \square \)

The proof of the following characterization of the Behrend position for a convex body can be found in [19] (equivalence (ii) was already proved by Behrend [3]).

Proposition 5.2. Let \( K \in \mathbb{K}^2 \). The following statements are equivalent.

(i) \( K \) is in Behrend position.

(ii) For every \( u \in S^1 \), there exists \( v \in V_K \) such that \( |u^T v| \geq 1/\sqrt{2} \).

(ii') For every \( u \in S^1 \), there exists \( v \in V_K \) such that \( |u^T v| \leq 1/\sqrt{2} \).

(iii) There exist \( u_i \in V_K \) and \( \lambda_i \geq 0, i = 1, 2, 3 \), such that \( \sum_{i=1}^{3} \lambda_i (u_i u_i^T) = I_2 \), where \( I_2 \) denotes the identity matrix of degree two.

Remark 5.3. Condition (ii) (resp., (ii')) in Proposition 5.2 means that for any fixed \( u \in S^1 \), there exists a diametrical direction \( v \in V_K \) contained in the double cone (resp., outside the double cone) with apex at 0 and vectors making an angle of at most \( \pi/4 \) radians with respect to \( \pm u \). Condition (iii) states that the identity matrix of degree two admits a decomposition as a non-negative linear combination of matrices of rank one, by means of three certain diametrical directions of \( K \) (cf. [19] and the references therein for further details and connections with other results).

The following result establishes some conditions derived from being in Behrend-bisecting position (the reader may compare it with Proposition 5.2). The proof is inspired by the ideas from Lemma 3.2 in [19].

Lemma 5.4. Let \( K \in \mathbb{K}^2 \) be in Behrend-bisecting position. For every \( u \in S^1 \) and every \( w \in B_K \), with \( (K_1^w, K_2^w) \) being the corresponding minimizing bisection of \( K \), we have that

(i) there exists \( v \in V_{K_1^w} \cup V_{K_2^w} \) such that \( |u^T v| \geq 1/\sqrt{2} \), and

(ii) there exists \( v \in V_{K_1^w} \cup V_{K_2^w} \) such that \( |u^T v| \leq 1/\sqrt{2} \).

Proof. We start by proving (i). Let us assume that for every \( v \in V_{K_1^w} \cup V_{K_2^w} \) we have that \( |u^T v| < 1/\sqrt{2} \). Hence every \( v \in V_{K_1^w} \cup V_{K_2^w} \) makes an angle \( \theta \) with the line \( u^w \) satisfying

\[
\theta = \frac{\pi}{2} - \arccos(u^T v) = \arcsin(u^T v) < \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}.
\]
and so $\cos^2 \theta > 1/2$. More precisely, since $K$ is compact (as well as $K^\prime_w$, for $i = 1, 2$), there exists $\delta > 0$ such that for every $v \in V_{K^\prime_w} \cup V_{K^\prime_w}$ making angle $\theta$ with respect to $u^\perp$, we have

\begin{equation}
\cos^2 \theta > \frac{1}{2} (1 + \delta).
\end{equation}

After a suitable rotation of $K$, we can assume that $u = (1, 0)$. For small $\varepsilon > 0$, consider the endomorphism of $\mathbb{R}^2$ given by the matrix

$$A_\varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & 1 - \varepsilon \end{pmatrix}.$$ 

Using elementary trigonometry and calculus, we can see that the length of any line segment $\ell$, making angle $\theta$ with $u^\perp$, varies under $A_\varepsilon$ according to the formula

\begin{equation}
||A_\varepsilon(\ell)|| = ||\ell|| \sqrt{1 - 2 \varepsilon \cos^2 \theta + \varepsilon^2 \cos^2 \theta} = ||\ell|| (1 - \varepsilon \cos^2 \theta + O(\varepsilon^2)).
\end{equation}

Let $K^\prime = A_\varepsilon(K)$ and $(K^\prime_w)^\prime = A_\varepsilon(K^\prime_w)$, for $i = 1, 2$ (since $A_\varepsilon$ is bijective, then $((K^\prime_w)^\prime, (K^\prime_w)^\prime)$ is a bisection of $K^\prime$). As $A_\varepsilon$ is close to the identity matrix for small $\varepsilon$, and $K, K^\prime_w, K^\prime_w$ are compact sets, for every $w \in D((K^\prime_w)^\prime) \cup D((K^\prime_w)^\prime)$ making an angle $\theta^\prime$ with $u^\perp$ it is possible to choose $\delta^\prime > 0$ small enough such that

\begin{equation}
\cos^2 \theta^\prime > \frac{1}{2} (1 + \delta^\prime).
\end{equation}

Let $A_\varepsilon(\ell)$ be the line segment in $K^\prime$ with $||A_\varepsilon(\ell)|| = \max\{D((K^\prime_w)^\prime), D((K^\prime_w)^\prime)\}$, being $\ell$ the corresponding line segment in $K$, making angle $\theta^\prime$ with $u^\perp$. Then, equation (5.5) implies that there exists $\delta''$ such that

$$\cos^2 \theta'' > \frac{1}{2} (1 + \delta''),$$

since $A_\varepsilon^{-1}$ is also close to the identity matrix. Thus, taking into account (5.4) and the fact that $w \in B_K$, we have

$$\frac{A(K^\prime)}{D_B(K^\prime)^2} \geq \frac{A_\varepsilon(K)}{D_B(K^\prime)^2} \frac{1 - \varepsilon}{(1 - \varepsilon \cos^2 \theta'' + O(\varepsilon^2))^2} \geq \frac{A(K)}{D_B(K)^2} \frac{1 - \varepsilon}{1 - 2 \varepsilon \cos^2 \theta'' + O(\varepsilon^2)}.$$ 

for $\varepsilon$ small enough, contradicting the fact that $K$ is in Behrend-bisecting position.

On the other hand, (ii) follows directly from (i), since (ii) holds for $u \in S^1$ if (i) holds for $u^\prime \in S^1 \cap u^\perp$ (and vice versa). □
**Remark 5.5.** We will now see that, in contrast to Proposition 5.2, the necessary condition in Lemma 5.4 for \( K \) to be in Behrend-bisecting position is not sufficient. Let \( K^\theta \in K^2 \) be the isosceles triangle with angle \( \theta \) different from the other two angles. Assume that \( \theta \in [0, \pi/3] \), with \( p_1 \) the vertex of angle \( \theta \), and \( p_2, p_3 \) the other two vertices. For any minimizing bisection \((K^\theta_1, K^\theta_2)\) of \( K^\theta \) determined by a line segment, we can assume that \( p_1 \in K^\theta_1 \) and \( p_2, p_3 \in K^\theta_2 \) (otherwise, the diameter of one of the subsets will be equal to \( D(K^\theta) \), and so the bisection will not be minimizing). By a suitable rescaling, we can suppose without loss of generality that \( p_2 = (1,0) \), \( p_3 = (-1,0) \), and \( p_1 = (0, \tan((\pi - \theta)/2)) \).

The distance from \( q_\lambda = (1 - \lambda) p_1 + \lambda p_2 \) (see Figure 5) to \( p_1 \) equals \( \lambda \sqrt{1 + \tan((\pi - \theta)/2)^2} \), whereas to \( p_3 \) equals \( \sqrt{(1 + \lambda)^2 + (1 - \lambda)^2 \tan((\pi - \theta)/2)^2} \).

![Figure 5. An isosceles triangle \( K^\theta \) and an arbitrary bisection of \( K^\theta \).](image)

Since the bisection is minimizing, these two distances must coincide, and so the value of \( \lambda \) must be equal to

\[
\lambda_m = \lambda_m(\theta) = \frac{1 + \tan((\pi - \theta)/2)^2}{2(\tan((\pi - \theta)/2)^2 - 1)}
\]

An analogous reasoning for the points of the edge \([p_1, p_3]\) yields that the only minimizing bisection by a line segment is given by the horizontal segment

\[
\left[ (\lambda_m, 1 - \lambda_m \tan(\pi - \theta)/2)), (\lambda_m, (1 - \lambda_m \tan(\pi - \theta)/2)) \right].
\]

In this case,

\[
\lambda_m \left( \pm 1, -\tan\left(\frac{\pi - \theta}{2}\right) \right) \in V_{K^\theta_1} \quad \text{and} \quad \left( \pm (\lambda_m + 1), (1 - \lambda_m) \tan\left(\frac{\pi - \theta}{2}\right) \right) \in V_{K^\theta_2}.
\]

It can be checked that for \( \theta \in [\pi/6, \pi/3] \), the triangles \( K^\theta \) satisfy the assumption in Lemma 5.4, by a direct analysis of the positions of the vectors of \( V_{K^\theta_1} \cup V_{K^\theta_2} \). However, not all of those triangles are in Behrend-bisecting position. Note that the isodiametric quotient

\[
\frac{\Lambda(K^\theta)}{D_B(K^\theta)^2} = \frac{\tan((\pi - \theta)/2)}{\lambda_m^2 (1 + \tan((\pi - \theta)/2)^2)} = 2 \cos^2(\theta) \sin(\theta)
\]
attains its maximum value in the interval $[0, \pi/3]$ only if $\theta = \theta_M := \arccos(\sqrt{2}/3)$ ($\approx 35.26^\circ$), with maximum value

$$\frac{A(K^{\theta_M})}{D_B(K^{\theta_M})^2} = \frac{4}{3\sqrt{3}},$$

which implies that $K^{\theta_M}$ is the only isosceles triangle among $K^\theta$, with $\theta \in [0, \pi/3]$, which is a candidate for being in Behrend-bisecting position.

Proposition 5.6 below proves that, in fact, the unique triangle in Behrend-bisecting position is $K^{\theta_M}$ from Remark 5.5.

**Proposition 5.6.** The unique triangle in Behrend-bisecting position is $K^{\theta_M}$ from Remark 5.5.

**Proof.** First of all, we will see that, in the class of isosceles triangles, the isodiametric quotient is uniquely maximized by $K^{\theta_M}$. Let $K^\theta \in K^2$ be now the isosceles triangle with different (largest) angle $\theta \in [\pi/3, \pi]$. Let $p_1$ be the vertex of angle $\theta$, and let $p_2, p_3$ be the other two vertices. For any minimizing bisection $(K_1^\theta, K_2^\theta)$ of $K^\theta$ determined by a line segment, we can now assume that $p_1, p_2 \in K_1^\theta$ and that $p_3 \in K_2^\theta$, and so $d(p_1, p_2) \leq D_B(K^\theta)$. In particular, if we consider the bisection given by the line segment $[p_1, (1/2)(p_2 + p_3)]$, then $D(K_1^\theta) = D(K_2^\theta) = d(p_1, p_2)$, and so $D_B(K^\theta) = d(p_1, p_2)$. Call $a = d(p_1, p_2)$ and $b = d(p_2, p_3)$. Then, basic computations show that $b = 2a \sin(\theta/2)$ and

$$\frac{A(K^\theta)}{D_B(K^\theta)^2} = \frac{\frac{1}{2}(2a \sin(\frac{\theta}{2})) \sqrt{a^2 - a^2 \sin(\frac{\theta}{2})^2}}{a^2} = \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right) = \frac{\sin \theta}{2},$$

and hence,

$$\frac{A(K^\theta)}{D_B(K^\theta)^2} \leq \frac{A(K^{\pi/2})}{D_B(K^{\pi/2})^2} = \frac{1}{2} \leq \frac{A(K^{\theta_M})}{D_B(K^{\theta_M})^2},$$

taking into account Remark 5.5.

Now consider an arbitrary triangle $K \in K^2$. We can assume that $K = \text{conv}(\{p_1, p_2, p_3\})$ for some $p_i \in \mathbb{R}^2$, $i = 1, 2, 3$, with $D(K) = d(p_1, p_2)$. Let $\alpha_i > 0$ be the angle at vertex $p_i$, for $i = 1, 2, 3$, with $\alpha_1 \leq \alpha_2 \leq \alpha_3$. For any minimizing bisection $(K_1, K_2)$ of $K$, we can assume that $p_1 \in K_1$ and that $p_2, p_3 \in K_2$ (otherwise, the bisection will not be minimizing). Call $q_\lambda = (1 - \lambda)p_1 + \lambda p_3$, and let $\lambda_m \in [0, 1]$ be such that the distance $d_1$ from $q_{\lambda_m}$ to $p_1$ is the same than $p_2$. Analogously, consider $r_\mu := (1 - \mu)p_1 + \mu p_2$, and let $\mu_m \in [0, 1]$ be such that the distance $d_2$ from $r_{\mu_m}$ to $p_1$ is the same than $p_3$. In this case, and since the distance from $p_1$ to $p_3$ is not larger than to $p_2$, we clearly have that $d_1 \geq d_2$, and hence the line segment with endpoints $q_{\lambda_m}$ and $r_{\mu_m}$ provides a minimizing bisection of $K$, with subsets $K_1 = \text{conv}(\{p_1, q_{\lambda_m}, r_{\mu_m}\})$ and $K_2 = \text{conv}(\{p_2, p_3, q_{\lambda_m}, r_{\mu_m}\})$ satisfying $D(K_1) = D(K_2) = d_1$. Let $p'_3$ be the point in the ray from $p_1$ to $p_3$ which is at the same distance from $p_1$ than from $p_2$, and consider the isosceles triangle $K' = \text{conv}(\{p_1, p_2, p'_3\})$ (see Figure 6). Then we clearly have that $K \subseteq K'$.
Isodiametric and isominwidth inequalities for planar bisections

Moreover, a bisection attaining $D_B(K')$ is given again by the line segment with endpoints $q_{\lambda_m}$ and $(1 - \lambda_m) p_1 + \lambda_m p_2$, with $D_B(K') = D_B(K) = d_1$. Hence

$$\frac{A(K)}{D_B(K)^2} \leq \frac{A(K')}{D_B(K')^2},$$

which implies that the isodiametric quotient of $K$ is always maximized by the isodiametric quotient of an isosceles triangle whose angle different from the other two angles is not larger than $\pi/3$ (because $\alpha_1 \leq \pi/3$). Taking into account Remark 5.5 (and the fact that any planar triangle can be obtained by applying an appropriate affine endomorphism to $K$), we conclude that the unique triangle in Behrend-bisecting position is the isosceles triangle $K^{\theta_M}$ from Remark 5.5.

In view of Proposition 5.6, and taking into account the results from [3], it is natural to conjecture the following optimal reverse isodiametric-type inequality for bisections.

**Conjecture 5.7.** Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Then,

$$\frac{A(K)}{D_B(K)^2} \geq \frac{4}{3\sqrt{3}},$$

with equality if and only if $K$ is the isosceles triangle whose angle different from the other two angles equals $\arccos(\sqrt{2/3})$.

The following proof is strongly inspired by the original proof of Behrend [3] for showing (1.8).

**Corollary 5.8.** Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Given $w \in B_K$, let $(K^{w_1}, K^{w_2})$ be the corresponding minimizing bisection of $K$. Then there exist $u_1, u_2 \in V_{K^{w_1}} \cup V_{K^{w_2}}$ such that $|u_1^T u_2| \leq 1/2$.

**Proof.** Call $e_1 = (1,0)$ and $e_2 = (0,1)$. By applying a proper rotation, we can assume that $e_1 \in V_{K^{w_1}} \cup V_{K^{w_2}}$. Then, for $e_2 \in S^1$, by Lemma 5.4 (i), there exists $u = (\cos \alpha, \sin \alpha) \in V_{K^{w_1}} \cup V_{K^{w_2}}$ such that $|e_2^T u| \geq 1/\sqrt{2}$, which implies that $\alpha \in [\pi/4, 3\pi/4]$. We can assume that $\alpha \in [\pi/4, \pi/2]$, by reflecting $K$ with respect to span($e_2$) if necessary. If $\alpha \geq \pi/3$, then $|e_2^T u| \leq 1/2$, which proves the statement for $u_1 = e_1$ and $u_2 = u$. So assume that $\alpha < \pi/3$, and note that, taking into account the previous argument, we can assume that $(\cos \mu, \sin \mu) \notin V_{K^{w_1}} \cup V_{K^{w_2}}$ for
Lemma 5.10. Let \( \mu \in [\pi/3, 2\pi/3] \). Consider the vector \( \tilde{u} = (\cos(\pi/3 + \pi/4), \sin(\pi/3 + \pi/4)) \) \( \in \mathbb{S}^1 \).

Again by Lemma 5.4 (i), there exists \( v = (\cos \beta, \sin \beta) \in V_{K^{w}_1} \cup V_{K^{w}_2} \) such that \( |\tilde{u}^T v| \geq 1/\sqrt{2} \). This necessarily implies that \( 2\pi/3 < \beta \leq \pi/3 + \pi/2 = 5\pi/6 < \pi \).

In particular, the angle between \( u \) and \( v \) is at least \( 2\pi/3 - \pi/3 = \pi/3 \) and at most \( 5\pi/6 - \pi/4 = 7\pi/12 < 2\pi/3 \), and thus we have that \( |u^T v| \leq 1/2 \), as desired (just take \( u_1 = u \) and \( u_2 = v \)).

We are now able to prove Theorem 1.6.

Proof of Theorem 1.6. Since \( K \) is in Behrend-bisecting position, for any \( w \in B_K \) with \((K^w_1, K^w_2)\) the corresponding minimizing bisection, then Corollary 5.8 states that there exist \( u_1, u_2 \in V_{K^{w}_1} \cup V_{K^{w}_2} \) such that \( |u_1^T u_2| \leq 1/2 \). Since \( D(K^w_1) = D(K^w_2) = D_B(K) \), there exist \( x_1, x_2 \in K \) such that \( x_1 + D_B(K)[0, u_1], x_2 + D_B(K)[0, u_2] \subset K \) (note that each of these segments is contained in \( K^w_1 \) or \( K^w_2 \)).

Now we will use an argument from the proof of Theorem 1.4 in [19]. Since \( K \) is convex, then \( C := \text{conv}(\{x_1 + D_B(K)[0, u_1], x_2 + D_B(K)[0, u_2]\}) \) is contained in \( K \), and so \( A(C) \leq A(K) \). In this situation, a result by Groemer [20] (see Theorem 2 in [4]) states that \( A(C) \) is minimal if both segments have a common endpoint, and thus, straightforward computations give

\[
A(K) \geq A(C) \geq A(\text{conv}(\{D_B(K)[0, u_1], D_B(K)[0, u_2]\})) = \frac{D_B(K)^2}{2} \sqrt{1 - (u_1^T u_2)^2} \geq \frac{\sqrt{3}}{4} D_B(K)^2,
\]

which completes the proof.

\( \square \)

5.1. The centrally symmetric case

As in [3], we will also focus on the centrally symmetric case (considering always the origin as center of symmetry), pursuing an isodiametric-type inequality for bisecions in this setting. The following result was proven in Proposition 3.1 of [11] (cf. Proposition 4 in [24]).

Lemma 5.9. Let \( K \in \mathbb{K}^2 \) be centrally symmetric. Then, there exists a minimizing bisection \((K_1, K_2)\) of \( K \) such that \( K_1 \cap K_2 = [-p, p] \), for some \( p \in \text{bd}(K) \). Consequently, \( K_1 = -K_2 \).

The above Lemma 5.9 allows to obtain a necessary condition for a given centrally symmetric convex body to be in Behrend-bisecting position.

Lemma 5.10. Let \( K \in \mathbb{K}^2 \) be centrally symmetric and in Behrend-bisecting position. For every \( w \in B_K \) with \((K^w_1, -K^w_2)\) as the corresponding minimizing bisection of \( K \), we have that \( K^w_1 \) and \(-K^w_2\) are in Behrend position.

Proof. Since \( K \) is in Behrend-bisecting position and \( w \in B_K \), Lemma 5.4 (ii) implies that for every \( u \in \mathbb{S}^1 \), there exists \( v \in V_{K^{w}_1} \cup V_{-K^{w}_2} = V_{K^{w}_1} = V_{-K^{w}_2} \), such that \( |u^T v| \leq 1/\sqrt{2} \). By Proposition 5.2, we obtain that \( K^w_1 \) is in Behrend position, as well as \(-K^w_2\).

\( \square \)
We can now prove Theorem 1.7, which establishes an isodiamic inequality for bisections in the centrally symmetric case.

Proof of Theorem 1.7. Let $(K_1, K_2)$ be a minimizing bisection of $K$. We can assume by Lemma 5.9 that $K_2 = -K_1$. As $K$ is centrally symmetric and in Behrend-bisecting position, Lemma 5.10 yields that $K_1$ (and also $K_2 = -K_1$) is in Behrend position. Thus (1.8) implies that

\[
\frac{A(K)}{D_R(K)^2} = \frac{A(K_1) + A(K_2)}{D_R(K)^2} = \frac{A(K_1)}{D(K_1)^2} + \frac{A(K_2)}{D(K_2)^2} \geq \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}. \quad \square
\]

We will now proceed as in Proposition 5.6, but focusing on the affine class of the square, i.e., the parallelograms, which are centrally symmetric. Proposition 5.11 shows that the only parallelogram in Behrend-bisecting position is the rectangle $[-1, 1] \times [-2, 2]$ (up to dilations and rigid motions, see Remark 1.5).

Proposition 5.11. The unique parallelogram in Behrend-bisecting position is the rectangle $[-1, 1] \times [-2, 2]$.

Proof. Let $K \subset \mathbb{R}^2$ be a parallelogram, and let $[-p, p]$ be a line segment determining a minimizing bisection $(K_1, K_2)$ of $K$, for some $p \in \text{bd}(K)$. Since $K$ is in Behrend-bisecting position, then $K_1$ (and $K_2 = -K_1$) is in Behrend position, by Lemma 5.10. We will distinguish two possibilities:

If $p$ is a vertex of $K$, then $K_1$ and $K_2$ are triangles. Since the only triangle in Behrend position is the equilateral one [3], then the only candidate in this case is the parallelogram $P$ formed by two congruent equilateral triangles joined by a common edge, with isodiamic quotient $A(P)/D_R(P)^2 = \sqrt{3}/2$, in view of (5.6).

If $p$ is not a vertex of $K$, then $K_1$ is a quadrangle in Behrend position with two parallel edges. We can assume that $K_1 = \text{conv}([p_1, p_2, p_3, p_4])$, where $p_i \in \mathbb{R}^2$, $i = 1, \ldots, 4$. Proposition 5.2 implies that there exist at least two different vectors $v_1, v_2 \in V_{K_1}$, and so $K_1$ contains at least two different diametrical segments. Since $K_1$ is a quadrangle with two parallel edges, then necessarily one of the diagonals of $K_1$, namely $[p_1, p_3]$, is a diametrical segment. Denote by $h_1$ (resp., $h_2$) the distance from $p_2$ (resp., $p_4$) to $[p_1, p_3]$. Then $h_1 + h_2 \leq d(p_2, p_4) \leq D(K_1)$, and so

\[
A(K_1) = \frac{1}{2} D(K_1) (h_1 + h_2) \leq \frac{D(K_1)^2}{2}.
\]

Since $K_2 = -K_1$, we will also have that $A(K_2) \leq D(K_2)^2/2$. Then,

\[
\frac{A(K)}{D_R(K)^2} = \frac{A(K_1) + A(K_2)}{D_R(K)^2} = \frac{A(K_1)}{D(K_1)^2} + \frac{A(K_2)}{D(K_2)^2} \leq \frac{1}{2} + \frac{1}{2} = 1.
\]

Moreover, we have equality above if and only if $h_1 + h_2 = D(K_1)$. This is equivalent to the fact that $[p_2, p_4]$ is orthogonal to $[p_1, p_3]$, i.e., when $K_1$ (and thus $K_2$) is a square. This implies that $K = K_1 \cup K_2$ is a rectangle of the form $[-1, 1] \times [-2, 2]$. Since this rectangle has isodiamic quotient greater than or equal to the isodiamic quotient of $P$, the statement holds. \quad \square
Remark 5.12. A remarkable consequence from Proposition 5.11 is that the necessary condition in Lemma 5.10 is not sufficient (analogously to Remark 5.5): the parallelogram consisting of two equilateral triangles touching in a common edge, both of them in Behrend position [3], is not in Behrend-bisecting position.

The previous Proposition 5.11 suggests that the inequality from our Theorem 1.7 is not sharp, leading us to the following conjecture.

Conjecture 5.13. Let $K \in \mathcal{K}^2$ be centrally symmetric and in Behrend-bisecting position. Then,

$$\frac{A(K)}{D_B(K)^2} \geq 1,$$

with equality if and only if $K = [-1,1] \times [-2,2]$.

6. The isominwidth-bisecting position and the reverse isominwidth inequality

In this section we will establish a reverse isominwidth inequality, following the same scheme as in Section 5. In order to obtain such an inequality, we will focus on the planar convex bodies in isominwidth-bisecting position, defined by equality (1.15). Our first observation is that the infimum in (1.15) is actually a minimum, and so, for any given $K \in \mathcal{K}^2$ there exists an affine representative in isominwidth-bisecting position (we will omit the proof of this fact since it is completely analogous to Lemma 5.1). Notice also that $w_B(K) = w(K)/2$ by Lemma 4.1, and so

$$(6.1) \quad \min_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{w_B(\phi(K))^2} = 4 \min_{\phi \in \text{End}(\mathbb{R}^2)} \frac{A(\phi(K))}{w(\phi(K))^2}.$$  

This equality immediately gives the following Corollary 6.1, which states a new equivalence for the planar convex bodies in isominwidth optimal position, defined by (1.13) and introduced in [19] (see Theorem 5.3 in [19] for some other related equivalences).

Corollary 6.1. Let $K \in \mathcal{K}^2$. The following statements are equivalent.

(i) $K$ is in isominwidth-bisecting position.

(ii) $K$ is in isominwidth optimal position.

Finally, we can prove Theorem 1.8.

Proof of Theorem 1.8. By Corollary 6.1, $K$ is in isominwidth optimal position, and taking into account Lemma 4.1 and (1.14) we conclude that

$$\frac{A(K)}{w_B(K)^2} = 4 \frac{A(K)}{w(K)^2} \leq 4.$$  

The equality case follows directly from the corresponding equality case in (1.14). □
7. Other spaces

In this section we briefly discuss how most of the above definitions and posed problems can be extended to other spaces. We also point out some of the technical difficulties that we find in order to go on solving these problems in those settings.

7.1. Isodiametric and isominwidth bisecting inequalities in \( \mathbb{R}^n \)

Let \( K \subset \mathbb{R}^n \) be a convex body with non-empty interior, and denote by \( V(K) \) the \( n \)-dimensional volume of \( K \) in \( \mathbb{R}^n \). Recall that the diameter \( D(K) \) of \( K \) is the maximum distance between any two points of \( K \), whereas the minimum width \( w(K) \) is the minimum distance between two parallel hyperplanes containing \( K \) between them. We first extend the notion of bisection previously introduced for the planar setting. Let \( B_2^n \) be the Euclidean unit ball of \( \mathbb{R}^n \).

For a convex body \( K \) in \( \mathbb{R}^n \), a bisection of \( K \) will be any pair of closed sets \( (K_1, K_2) \) satisfying that

(i) \( K = K_1 \cup K_2 \),

(ii) \( K_1 \cap K_2 = l(B_2^{n-1}) \), where \( l : B_2^{n-1} \to K \) is an injective and continuous map such that \( l(B_2^{n-1}) \cap \text{bd}(K) = l(\text{bd}(B_2^{n-1})) \).

We will denote by \( B(K) \) the set of all bisections of \( K \). We can now define the infimum of the maximum bisecting diameter of \( K \) by

\[
D_B(K) := \inf_{(K_1, K_2) \in B(K)} \max\{D(K_1), D(K_2)\}.
\]

A remarkable difference with respect to the planar case is that it is not clear now whether this infimum is a minimum. The reason is that for an arbitrary bisection \( (K_1, K_2) \) of \( K \) in \( \mathbb{R}^n \), for \( n \geq 3 \), the set \( K_1 \cap K_2 \cap \text{bd}(K) \) is, in general, \( n \)-dimensional, and so it will not induce a bisection by a hyperplane (cf. Lemma 2.1). This suggests that an appropriate approach could be focusing on bisections by hyperplanes, which will imply that the infimum is actually a minimum, by using the Blaschke selection theorem.

We now sketch that the corresponding isodiametric quotient is bounded above. For a given convex body \( K \) in \( \mathbb{R}^n \), let \( x, y \in K \) be points such that \( d(x, y) = D(K) \), and let \( (K_1, K_2) \in B(K) \). Then at least two points from \( \{x, (x + y)/2, y\} \) belong to one of the sets \( K_1 \) or \( K_2 \). Since the distance between any pair of those three points is at least \( D(K)/2 \), then we can conclude that \( \max\{D(K_1), D(K_2)\} \geq D(K)/2 \), and so \( D_B(K) \geq D(K)/2 \), which together with the classical isodiametric inequality in \( \mathbb{R}^n \) (see (1.2) for the planar case) gives

\[
\frac{V(K)}{D_B(K)^n} \leq 2^n \frac{V(K)}{D(K)^n} \leq 2^n \frac{V(B_2^n)}{D(B_2^n)^n},
\]

thus showing that this quotient is bounded above by an absolute positive constant. Hence the supremum of \( V(K)/D_B(K)^n \) over the convex bodies in \( \mathbb{R}^n \) is finite and it would be interesting to characterize the sets attaining such a value, as done in Theorem 1.1.
Analogously, we can define the infimum of the maximum bisecting width of $K \subset \mathbb{R}^n$ by

$$w_B(K) := \inf_{(K_1, K_2) \in B(K)} \max\{w(K_1), w(K_2)\}.$$  

Using analogous ideas to the ones exhibited in Lemma 4.1, we can see that $w(K) = 2w_B(K)$. Notice that this implies that the previous infimum is in fact a minimum: if $w(K)$ is attained between two parallel supporting hyperplanes $H_1$, $H_2$, then $w_B(K)$ will be attained by the bisection of $K$ given by the hyperplane $(H_1 + H_2)/2$. Moreover, taking into account that

$$V(K) / w(K)^n \geq 2 / \sqrt{3n!}$$

(see Theorem 6.2 in [5]), we can conclude that

$$\frac{V(K)}{w_B(K)^n} = 2^n \frac{V(K)}{w(K)^n} \geq \frac{2^{n+1}}{\sqrt{3n!}},$$

thus showing that this quotient is bounded below by an absolute constant.

**Remark 7.1.** Some interesting results regarding the isodiametric quotient in compact convex surfaces of $\mathbb{R}^3$ can be found in [12], [13].

### 7.2. Reverse isodiametric and isominwidth bisecting inequalities in $\mathbb{R}^n$

Using the same ideas commented in the introduction for the planar case, and the same definitions from Subsection 7.1, we can see that the quotient $V(K)/D_B(K)^n$ cannot be bounded below by any positive constant, when considering arbitrary convex bodies $K \subset \mathbb{R}^n$. However, we can develop the same approach from Section 5: we can say that a convex body $K \subset \mathbb{R}^n$ is in Behrend-bisecting position if

$$\frac{V(K)}{D_B(K)^n} = \sup_{\phi \in \text{End}(\mathbb{R}^n)} \frac{V(\phi(K))}{D_B(\phi(K))^n}.$$  

The ideas from Lemma 3.2 in [19] allow to obtain a result analogous to Lemma 5.4, which will lead to the following consequence, see Theorem 1.4 in [19]: if $K$ is a convex body in $\mathbb{R}^n$ in Behrend-bisecting position, then

$$\frac{V(K)}{D_B(K)^n} \geq \frac{1}{\sqrt{n!} n^{n/2}}.$$  

As we noted in (1.11), this inequality is not sharp.

Analogously, the quotient $V(K)/w_B(K)^n$ cannot be bounded above by any positive constant (just consider a very flat convex body $K$ in $\mathbb{R}^n$). However, if we assume that $K$ is in isominwidth-bisecting position, i.e., if

$$\frac{V(K)}{w_B(K)^n} = \inf_{\phi \in \text{End}(\mathbb{R}^n)} \frac{V(\phi(K))}{w_B(\phi(K))^n},$$
then one could prove that this quotient is bounded above. More precisely,
\[
\frac{V(K)}{w_B(K)^n} \leq 2^n,
\]
with equality if and only if \(K\) is a cube (this follows from Theorem 1.6 in \([19]\), which is an extension of (1.14) to higher dimensions, together with \(w_B(K) = w(K)/2\), as in Lemma 4.1, and the fact that \(K\) is in isominwidth position if and only if \(K\) is in isominwidth-bisecting position, as in Corollary 6.1).

7.3. Isodiametric inequality in the spherical and hyperbolic space

The study of geometric inequalities can be also done in the spherical space \(S^n\) and in the hyperbolic space \(H^n\) of dimension \(n\). In this setting, some interesting results have been obtained in the last years \([16]\), \([22]\), \([21]\), \([8]\). In this general context, a set \(K\) is called convex if for any \(x, y \in K\), the shortest geodesic segment joining \(x\) and \(y\) is contained in \(K\) (in the spherical case, it is additionally required that \(K\) is contained in a hemisphere). Moreover, one can naturally define the notions of spherical diameter and the spherical area, as well as the corresponding hyperbolic analogues. The notions of spherical and hyperbolic width are not so natural, see \([23]\), \([15]\), or \([17]\). In particular, the isodiametric and isominwidth inequalities have been proven in the 2-dimensional spherical and hyperbolic cases, when \(K\) is centrally-symmetric \([21]\), and very recently, the isodiametric inequality has been obtained in the general case \([8]\). Some other related considerations in the hyperbolic case can be found in \([18]\).

We would like to note that, in \(S^n\) and \(H^n\), we cannot assure that the diameter of a given set \(A\) is attained by a pair of its extreme points (they can be defined as the points of \(A\) which do not belong to the relative interior of any geodesic segment contained in \(A\)), as it occurs in \(\mathbb{R}^n\). For instance, consider the region \(A_\delta\) in \(S^2\) given by
\[
\text{conv}_{S^2}\left(\{(-1,0,0)\} \cup \{\sin \theta(0,1,0) + \cos \theta(\delta,0,\sqrt{1-\delta^2}) : \theta \in [-\pi/4,\pi/4]\}\right),
\]
for some \(\delta \in [1/4,3/4]\). It is easy to see that \(A_\delta\) is a geodesic triangle in \(S^2\) and the spherical diameter of \(A_\delta\) is equal to the distance between the points \((-1,0,0), (\delta,0,\sqrt{1-\delta^2}) \in A_\delta\), but it is clear that \((\delta,0,\sqrt{1-\delta^2})\) is not an extreme point of \(A_\delta\).

Of course, for a given set \(A\) contained in \(S^n\) or in \(H^n\) we can consider the problems studied in Sections 3 and 4. From the previous example \(A_\delta \subset S^2\), it is not clear that an analogous result to Lemma 2.1 can be obtained in this setting. Notice that if we substitute a general bisection of a given set \(A \subset S^2\) by the bisection determined by the maximum arc with the same endpoints, the corresponding diameters of the new subsets can be greater than the former ones, since they are not necessarily attained by extreme points of the subsets.

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References


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