Universidad de Sevilla

Positivity conditions and hook+column sequences of plethystic coefficients

Condiciones de positividad y sucesiones gancho+columna de coeficientes pletísticos

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Abstract

In this work we aim to study and better understand the coefficients of the plethystic operation on the symmetric functions. We will try to be as self-contained as possible, beginning with the basic definitions of symmetric functions and some proven formulas on them. We give a condition on the positivity of the coefficient $[s_\mu](p_n \circ s_\lambda)$ of the plethysm between a power sum function p_n and a Schur function s_λ, namely, $\lambda \subseteq \mu$. We then study the coefficients of the Schur expansion of a plethystic family of functions $s_{n_1} \circ s_{n_2} \circ ... \circ s_{n_k} = \sum a_\lambda s_\lambda$ when λ is a partition of the form $(\alpha, 2^\beta, 1^\gamma)$ (hook+column). We completely characterize the case $s_2 \circ s_n \circ s_m$. Fixing a γ, and letting β vary, we associate a sequence of coefficients $(a_0, a_1, ..., a_\beta, ...)$ to each function f, and prove that $f = s_2 \circ s_2 \circ \cdots s_2 \circ s_n \circ s_m$ always yields a symmetric sequence. Finally, we make some remarks and conjectures regarding unimodality and asymptotic normality of these sequences.

Keywords: symmetric functions, Littlewood-Richardson coefficients, plethysm
MSC: 05E05, 05E18, 05A17

Resumen

En este trabajo pretendemos estudiar y entender la operación pletística sobre las funciones simétricas. Intentaremos ser autocontenidos, empezando por las definiciones más básicas de qué es una función simétrica y enunciando resultados conocidos sobre ellas. Damos una condición de positividad del coeficiente $[s_\mu](p_n \circ s_\lambda)$ del pletismo entre una función suma de potencias p_n y una función de Schur s_λ, concretamente $\lambda \subseteq \mu$. Estudiaremos seguidamente la expansión sobre la base de Schur the una familia pletística de funciones $s_{n_1} \circ s_{n_2} \circ ... \circ s_{n_k} = \sum a_\lambda s_\lambda$ para particiones λ de la forma $(\alpha, 2^\beta, 1^\gamma)$ (gancho+columna). Caracterizamos completamente el caso $s_2 \circ s_n \circ s_m$. Fijando un γ y dejando β variar, asociamos una sucesión finita $(a_0, a_1, ..., a_\beta, ...)$ de coeficientes a cada función f, y probamos que $f = s_2 \circ s_2 \circ \cdots s_2 \circ s_n \circ s_m$ siempre está asociada a una sucesión simétrica. Finalmente, damos algunos comentarios y conjeturas sobre la unimodalidad y la normalidad asintótica de dichas sucesiones.

Résumé

Dans ce travail nous avons l'intention d'étudier et de mieux comprendre le pléthysme des fonctions symétriques. Nous commençons par donner les définitions les plus basiques des fonctions symétriques, et par énoncer les principaux résultats les concernant, pour en donner une présentation complète. Nous donnons une condition sur la positivité du coefficient $[s_\mu](p_n \circ s_\lambda)$ du pléthysme d'une fonction symétrique somme de puissances p_n avec une fonction de Schur s_λ, plus concrètement, $\lambda \subseteq \mu$. Puis, nous étudions le développement dans la base de Schur d'une famille de pléthysmes $s_{n_1} \circ s_{n_2} \circ ... \circ s_{n_k} = \sum a_\lambda s_\lambda$ quand λ est une partition de la forme $(\alpha, 2^\beta, 1^\gamma)$ (équerre+colonne). Nous donnons une formule explicite pour le cas $s_2 \circ s_n \circ s_m$. En fixant γ et en faisant varier β, on peut associer à chaque fonction f une suite finie $(a_0, a_1, ..., a_\beta, ...)$ de ses coefficients. Nous montrons que les suites associées à $f = s_2 \circ s_2 \circ \cdots s_2 \circ s_n \circ s_m$ sont toujours symétriques. Finalement, nous faisons quelques remarques sur l'unimodalité et normalité asymptotique de ces suites.

Contents

1 Introduction

Algebraic combinatorics and combinatorial algebra are the fields of mathematics which apply algebraic methods to problems in combinatorics and vice versa. Perhaps the main link between the two fields is representation theory. Let us explore a problem in algebraic combinatorics as to motivate this work.

To a given combinatorial object, one can associate certain algebraic structures. For instance, consider the set $\mathtt{L}$ of leaves of the tree $\mathtt{T}$ of Figure 1, together with the inherited structure of the tree. With the given interpretation of $\mathtt{T}$ as a family tree, $\mathtt{L}$ is the set of me, my siblings and my cousins.

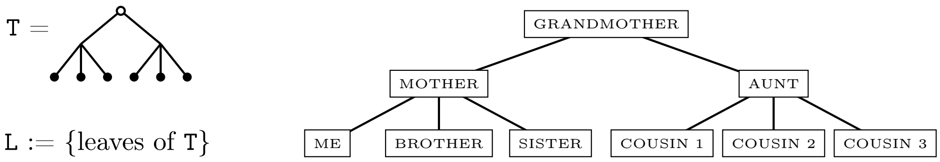


Figure 1: A rooted unlabeled tree $\mathtt{T}$ and a possible interpretation as a family tree.

One algebraic structure associated to this tree is the structure-preserving permutation group on its leaves, which we denote by $\mathrm{Aut}(\mathtt{L})$. The word *structure-preserving* is key. It means that we can't shuffle around the leaves as we please. In plain words, there are many legal ways to draw the family tree – we can swap the node ME with the node BROTHER – but not all ways are legal – we can't swap the node ME with the node GRANDMOTHER.

We can swap the node MOTHER with the node AUNT, if we also move me, my siblings and my cousins accordingly. The ways of permuting the set {MOTHER, AUNT} give us a copy of $\mathbb{S}_2$ inside $\mathrm{Aut}(\mathtt{L})$. Inside each of those two clusters of nodes, we now find a copy of $\mathbb{S}_3$ (corresponding to the permutations on {ME, BROTHER, SISTER} and {COUSIN 1, COUSIN 2, COUSIN 3} respectively). The resulting group is called the wreath product[1] of $\mathbb{S}_3$ and $\mathbb{S}_2$, and denoted by $\mathrm{Aut}(\mathtt{L}) = \mathbb{S}_3 \wr \mathbb{S}_2$.

Representation theory of finite groups now associates each element of $\mathbb{S}_3 \wr \mathbb{S}_2$ to a certain linear transformation of the vector space $\mathbb{C}[\mathtt{L}] = \mathrm{span}_{\mathbb{C}}\{l \in \mathtt{L}\}$. The vector subspace which is invariant under all these transformations is a representation of $\mathrm{Aut}(\mathtt{L})$, which in this case is $S^2(S^3(\mathbb{C}[\mathtt{L}]))$. The wreath product of symmetric groups is translated to composition of Schur functors S^n when talking in the language of representations [4, 19, 29, 32].

There exist a class of special representations which are the irreducible representations, $\{S^\lambda\}_\lambda$. Any given representation can be decomposed into irreducibles. Our goal in this work is to study the decomposition into irreducibles of the composition of Schur functors. For example, for our tree, we want to be able to understand the numbers a_λ appearing in the following equation:

$$S^2(S^3(\mathbb{C}[\mathtt{L}])) = \bigoplus_\lambda a_\lambda S^\lambda(\mathbb{C}[\mathtt{L}]).$$

However, we won't be working with representations. We will be translating the problem into the language of symmetric functions. We can do this two ways: via the Frobenius characteristic map (ch) [1, 27] or via Pólya's cycle index ($Z_{(\cdot)}$) [7, 32].

[1] Pólya called it the Kranz product [7] in [23].

$$\begin{tikzcd}
\mathbb{S}_3 \wr \mathbb{S}_2 \arrow[r, mapsto, "\mathbf{1}\uparrow_{(\cdot)}^{\mathbb{S}_6}"] \arrow[dr, mapsto, bend right, "Z_{(\cdot)}"'] & S^2(S^3(\mathbb{C}[\mathtt{L}])) \arrow[d, mapsto, "\mathrm{ch}"] \\
& s_2 \circ s_3
\end{tikzcd}$$

Either way, what we obtain is called the *plethysm* of Schur symmetric functions s_2 and s_3. The previous equation is translated to

$$s_2 \circ s_3 = \sum_{\lambda} a_\lambda s_\lambda.$$

The coefficients a_λ are left unchanged. They are called *plethystic coefficients.* Although their definition is rather natural, the understanding and computing of plethystic coefficients is a notoriously hard problem, featured in Stanley's list of major open problems in algebraic combinatorics [33]. Plethysm was first introduced in the context of invariant theory [16], and has recently been the focus of intense investigations [8, 13, 20].

We will be focusing on a particular case of irreducible representation of $\mathbf{GL}(V)$ which is of central importance in mathematics: $S^n(V)$, the space of homogeneous polynomials of degree k in n variables. In this work we investigate the plethystic coefficients appearing in the iterated plethysm $S^{n_1}(S^{n_2}(\dots(S^{n_k}(V))))$. An instance of this problem is the famous *Foulkes's conjecture* [9]. It claims that for any $a \geq b$, the coefficient of $S^\lambda(V)$ in $S^a(S^b(W))$ is always greater than or equal to its coefficient in $S^b(S^a(W))$.

In [15], Langley and Remmel studied the decomposition into irreducibles of the representation $S^a(S^b(V))$. Since this problem quickly becomes intractable, they restricted the question to a particular family of partitions that they called n-hook+column partitions. For $n = 1$, it turns out that one can naturally index the hook+column partitions by two integers, making it possible to associate several integer sequences to any representation. The resulting sequences, that we call hook+column sequences, are the center of this work.

In section 4, we compute explicitly the hook+column sequences of $S^2(S^a(S^b(V)))$, generalizing the result of Langley and Remmel. We show that the hook+column sequences of $S^2(S^2(\dots(S^2(V)))$ are symmetric (Theorem 4.28). In order to prove this symmetry, we construct a bijection $\lambda \mapsto \lambda^R$ such that the multiset $\mathtt{D}_\lambda$ of partitions appearing in the expansion of s_λ over the Schur basis is invariant under our bijection. This is reminiscent of a recent work by Grinberg [10].

As a preparation for our work, in section 3 we present some general lemmas that bound the sizes of the parts of the partitions λ such that the $S^\lambda(V)$ appears with positive multiplicity in $S^\mu(V) \otimes S^\nu(V)$. This follows the spirit of the recently achieved and hightly celebrated result by Paget and Wildom [22], and improves already known bounds [37]. Thanks to the SXP rule [17, 36], we get a beautiful lower bound on the partitions λ such that s_λ appears in $p_n \circ s_\mu$.

We also provide a Sage notebook in which every important result and example from sections 3 and 4 are coded, and which hopefully showcases the importance of fast formulas for plethystic coefficients, in contrast to the current available computation methods.

2 Preliminary Concepts

We will introduce the subject of symmetric functions following Stanley's and Sam's expositions in [28, 32]. Let $\mathbb{S}_n$ be the symmetric group on n letters. It acts on $\mathbb{Z}[x_1, ..., x_n]$ by

$$\sigma f(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)}).$$

We define the ring of symmetric functions on n letters as the fixed subset $\Lambda(n) = \{f \ : \ \sigma f = f \ \forall \sigma\}$ of $\mathbb{Z}[x_1, ..., x_n]$. Similarly, let $\mathbb{S}_\infty$ be the group of permutations on $\mathbb{N}$ and R be the ring of formal series of bounded degree. Hence, elements of R can be infinite sums, but only in a finite number of degrees. Then, $\mathbb{S}_\infty$ acts on R in the same way as $\mathbb{S}_n$ on $\mathbb{Z}[x_1, ..., x_n]$. The **ring of symmetric functions** is the fixed subset

$$\Lambda := \{f \ : \ \sigma f = f \quad \forall \sigma\} \subseteq R.$$

One can check that this is a subring of R. In fact, Λ is a (graded) $\mathbb{Z}$-algebra. We will note by $\Lambda_d(n)$ (resp. Λ_d) the subset of functions of degree d in $\Lambda(n)$ (resp. Λ). That is, $\Lambda(n) = \bigoplus_d \Lambda_d(n)$ and $\Lambda = \bigoplus_d \Lambda_d$.

Definition 2.1. Here are some notorious families in Λ. Set $p_0 = e_0 = h_0 = 1$ and let, for $k \geq 1$,

- **Power sum** symmetric functions: $p_k = \sum_{i \geq 1} x_i^k$.
- **Elementary** symmetric functions: $e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$.
- **Completely homogeneous** symmetric functions: $h_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$.

As an exercise, check that $h_2 = e_2 + p_2$.

2.1 Partitions

A **partition** λ of $n \in \mathbb{N}$ is a (weakly) decreasing sequence $(\lambda_1, \lambda_2, ...)$ of natural numbers such that $\sum \lambda_i = n$. Note that it must necessarily stabilize to 0.

Most often, we will omit the trail of zeros, writing $(4, 2, 2, 2)$ instead of $(4, 2, 2, 2, 0, 0, ...)$. Also, we may write $(4, 2^3)$ instead of $(4, 2, 2, 2)$. We write $\lambda \vdash n$ and say that the **size** of λ is $|\lambda| = n$. We define the **length** $l(\lambda)$ of λ as the number of nonzero entries. Let $m_i(\lambda)$ denote the number of entries of λ that are equal to i; the multiplicity of i in λ.

Usually, we try to visualize partitions, using what is called a **Young diagram** (or sometimes a tabloid or a Ferrers diagram). Using the French convention[1], this is a bottom-left justified set of boxes (cells) in which the lower-most row has λ_1 cells, the next one λ_2 and so on.

$\lambda = (6, 3, 1) =$

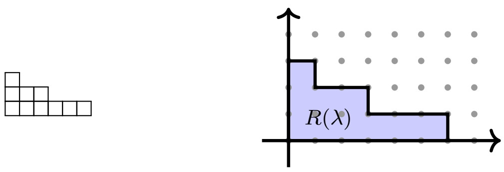


Figure 2: An example of a Young diagram of a partition λ and its region.

Starting with the bottom-left corner as (0,0), we will use coordinates to refer to the cells, following the usual cartesian order of (column, row). We can then consider the subset of $\mathbb{N}^2$ made

[1] The French convention for Young diagrams is to draw bottom-left justified boxes. It is certainly more natural when using coordinates to refer to the cells. With the English convention, the cells are top-left justified, and the coordinate system is matrix-like [19].

of the points which are the coordinates of some cell of λ. We call this the **region** of λ, and denote it by $R(\lambda)$. Refer to Figure 2 for an example. Note that throughout this work we represent the region of λ as a subset of the plane in which each integer point in $R(\lambda)$ is the bottom-left corner of a 1-by-1 square.

We can also do some basic operations on these diagrams. The first one we will introduce is the **transpose** λ' of λ, which is the result of flipping λ over its main diagonal.

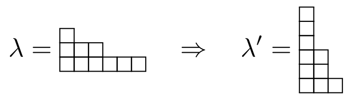


Figure 3: The transpose of a partition.

Next, we will define two orderings on the set of partitions: We will say that λ is a **subset** of μ, and denote it by $\lambda \subseteq \mu$, if $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. The **dominance order** will let $\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots \mu_i$ for all i. It will be important to notice that for two partitions μ and λ of the same size, we have $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$ (see [19, 32]).

Define the **sum** of two partitions λ and μ as the partition $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, ...)$. For a given $n \in \mathbb{N}$, let $n\lambda = \lambda + \lambda + \cdots + \lambda$ (n times). Define the **union** of two partitions λ and μ as the partition with rows $\lambda_1, \lambda_2, ...$ and $\mu_1, \mu_2, ...$ in descending order.

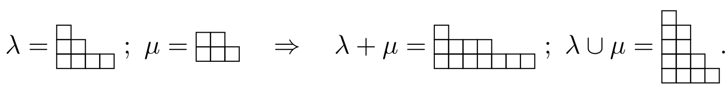


Figure 4: Sum and union of partitions.

One property that we will use throughout this work is that $\lambda' + \mu' = (\lambda \cup \mu)'$. As an exercise, check that this equality does indeed hold.

Note 2.2. We have that $\lambda \subseteq \mu$ if and only if $R(\lambda) \subseteq R(\mu)$. However, please note that $\lambda \cup \mu$ is drastically different to the partition whose region is $R(\lambda) \cup R(\mu)$, where the union is the usual union of sets.

We can also generalize the concept of partition to **skew partitions** [19, 32], which are given by the "inverse" of a sum: if $\mu \subseteq \lambda$, then the Young diagram of λ/μ is defined as the diagram of λ but with the cells of μ removed. Refer to Figure 5, where we let λ and μ be defined as in Figure 4.

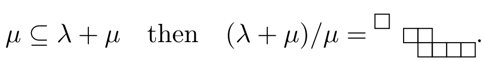


Figure 5: Reading the row sizes of $(\lambda + \mu)/\mu$, one can recover λ.

Some special types of partitions for this work need to be defined. Classically, a lot of work has been centered around row partitions (partitions of shape (α)) and column partitions (partitions of shape (1^γ)). The next most complicated type of partition is a **hook**, which is a partition of shape $(\alpha, 1^\gamma)$. Only recently [15], the protagonist partitions of this thesis were studied. They are called **hook+columns**, and they are partitions of shape $(\alpha, 2^\beta, 1^\gamma)$. Notice that letting $\beta = \gamma = 0$ we recover a row partition, and the same can be said of column partitions and hooks by letting $\beta = 0, \alpha = 1$ and $\beta = 0$ respectively.

2.2 Bases of Λ

Partitions help us encode different bases of the algebra of symmetric functions. Recall from 2.1 the definition of p_k, e_k and h_k. For any partition λ, let p_λ be the product $p_{\lambda_1} \cdots p_{\lambda_{l(\lambda)}}$. Define e_λ and h_λ similarly. We get that $\{e_\lambda\}_\lambda$ is a basis of Λ as a $\mathbb{Z}$-module, and so is $\{h_\lambda\}_\lambda$, whereas $\{p_\lambda\}_\lambda$ defines only a basis as a $\mathbb{Q}$-module (see [28] or example 2.4).

But there is another basis of Λ that is far more important to study, Schur functions. Indeed, even though these functions will be more difficult to define, they are "irreducible" in some sense. They open a window between the study of symmetric functions and representation theory, thanks to the Frobenius characteristic map. This bijective map sends the irreducible representations of the symmetric group to Schur functions. And so, the results on this field shed light on representation theory. An introduction to said relationship can be found at [1].

To define them, we need to first talk about (Young) tableaux.

A **tableau** of shape λ is a map $T : R(\lambda) \to \mathbb{N}$, which we represent by writing the number $T(c, r)$ inside of cell (c, r). A **semi standard Young tableau (SSYT)** is a tableau of shape λ such that the entries are weakly increasing going left to right in each row, and strictly increasing going from bottom to top in each column. The **weight** of T is the sequence $\big(m_1(T), m_2(T), ...\big)$ where $m_i(T)$ is the number of cells with entry i.

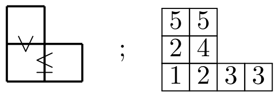


Figure 6: On the left, the rules of semi standard tableaux. On the right, an example of a SSYT.

The **Schur function** of shape λ is defined as $s_\lambda = \sum x^T$ where the sum is over all SSYT T of shape λ, and where $x^T = x_1^{m_1(T)} x_2^{m_2(T)} \cdots$. Sometimes, x^T is also called the weight of T. It is not clear from the definition that Schur functions are symmetric functions. A proof of this fact can be found in [27].

Example 2.3. Let us compute an explicit expression for $s_{2,1}$. There are four distinct types of tableaux of shape $\ydiagram{1,2}$, namely:

$$\begin{ytableau} 3 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 1 \end{ytableau}$$

$$s_{\ydiagram{1,2}} = s_{2,1} = 2x_1x_2x_3 + x_1x_2^2 + x_1^2x_2 + \cdots$$

We sum over the weights of those tableaux and all their possible symmetries.

Example 2.4. Let us compute the expression of s_2 over the power sum basis. There are two types of tableaux of shape $\ydiagram{2}$, with corresponding weights x_1^2 and x_1x_2. And thus,

$$s_{\ydiagram{2}} = s_2 = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + \cdots$$

It is an homogeneous function of degree 2, so only $p_{1,1}$ and p_2 can appear in its expansion. Those two functions are

$$p_{\ydiagram{1,1}} = p_{1,1} = p_1 \cdot p_1 = (x_1 + x_2 + \cdots) \cdot (x_1 + x_2 + \cdots) = x_1^2 + 2x_1x_2 + x_2^2 + \cdots \ ;$$

$$p_{\ydiagram{2}} = p_2 = x_1^2 + x_2^2 + x_3^2 + \cdots \ .$$

So, we have the equality $s_2 = \frac{1}{2}(p_{1,1} + p_2)$.

As announced, $\{s_\lambda\}_\lambda$ is a $\mathbb{Z}$-algebra basis[2] of Λ [27]. Given the inherited importance of decomposing symmetric functions into "irreducibles", we will need to define the **support**

$$\mathrm{supp}(f) := \{\lambda \ : \ [s_\lambda]f \neq 0\}$$

for any symmetric function f, where $[s_\lambda]f$ denotes the coefficient of s_λ in the decomposition of f over the Schur basis. We will usually write $[\lambda]f$ instead of $[s_\lambda]f$, in order to simplify our notation.

One can also define the **skew Schur functions**, $\{s_{\lambda/\mu}\}_{\lambda,\mu}$, indexed by skew partitions , as $s_{\lambda/\mu} = \sum x^T$, again summing over all SSYT T of shape λ/μ (where the SSYT rules apply to consecutive cells).

Understanding how all these basis coexist, and how to change a function from one basis to another, is a core subject in the theory of symmetric functions. We need to state some well-known results that we will use in this work. See [28] for an in-depth study of basis changes.

Theorem 2.5. *Let z_ν be the number $\prod_i m_i(\nu)! \cdot i^{m_i(\nu)}$. We have the following identities:*

$$h_n = \sum_{\nu \vdash n} \frac{p_\nu}{z_\nu} \quad ; \quad e_n = \sum_{\nu \vdash n} (-1)^{l(\nu)} \frac{p_\nu}{z_\nu}.$$

Theorem 2.6 (Jacobi-Trudi identity). *Set $h_i = 0 = e_i$ if $i \leq 0$. Then,*

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j}^{|\lambda|} = \det(e_{\lambda'_i - i + j})_{i,j}^{|\lambda|}.$$

2.3 n-core and n-quotient

Another two constructions involving partitions are the n-core and n-quotient. For any $n \in \mathbb{N}$, given the quotient and the core, one can always construct the original partition [19]. That is, they encode all the information of the partition.

We will construct them two ways. In this section, we will construct them the classical way (although avoiding the abstractness of the definition in [19]) and in the next section, introducing the abacus, following [12].

One more definition is needed. Removing a **rim hook** (or border strip) of length m from a partition λ is choosing a subset μ of λ such that λ/μ is a connected skew partition of size m and such that it contains no 2×2 subdiagram (a 2 by 2 square of cells in the Young diagram).

Definition 2.7 (n-quotient). Given a number $n \in \mathbb{N}$, and a partition λ (of any size), the n-quotient is defined as the n-tuple

$$\lambda^* = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(n-1)}),$$

where $\lambda^{(i)}$ is made of the cells (k, j) in λ such that $c_k := \lambda'_k + k + 1 \equiv i \ (n)$ and $r_j := \lambda_j + j \equiv i \ (n)$.

Note that c_k only depends on the column and r_j on the row.

Definition 2.8 (n-core). Given a number $n \in \mathbb{N}$, and a partition λ (of any number), the n-core is defined as the partition $\tilde{\lambda}$ which remains after removing (step by step) every rim hook of length n from λ (in no particular order[3]).

These definitions of n-quotient and n-core are not exactly the classical ones. However, they have an advantage: they give us an algorithm to compute them. The algorithm to compute the n-quotient is the following:

[2] In particular, the fact that $[s_\lambda](s_\mu s_\nu)$ is an integer comes from Theorem 2.13.

[3] The order does not matter [19].

1. Draw the Young diagram of λ.

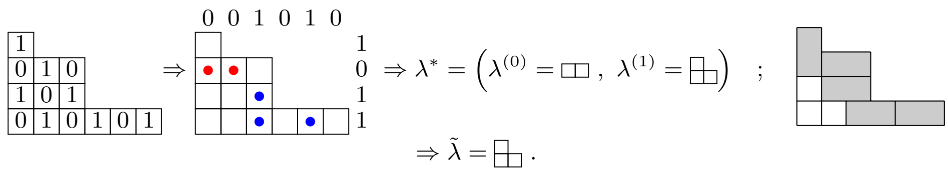
2. Construct the tableau $(k, j) \mapsto (k + j) \pmod n$. In other words, $T(k, j)$ is the orthogonal distance from (k, j) to the $(0, 0)$ cell, modulo n.

3. Now, r_j is the right-most entry in row j, whereas c_k is the top-most entry of column k plus 1 and modulo n.

4. Forget the entries of the tableau, and look for all cells with a $i = 0$ both as its column number and as its row number. These cells will form a partition, namely $\lambda^{(0)}$.

5. Repeat the previous step for $i = 1, ..., n - 1$ to compute $\lambda^{(1)}, ..., \lambda^{(n-1)}$.

Example 2.9. Let $n = 2$ and $\lambda = (6, 3, 3, 1)$. Then,



We will take a little detour here, and compute the 2-quotient of an arbitrary hook+column $\nu = (\alpha, 2^\beta, 1^\gamma)$, as we will need it later. In fact, we will only need to consider the case when $N = |\nu|$ is even. This will become apparent in section 4. The first three steps of the discussed algorithm will result in something resembling Figure 7.

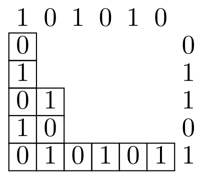


Figure 7: The first steps of the 2-quotient algorithm performed to $\nu = (6, 2^2, 1^2)$.

As we supposed $N = |\nu|$ to be even, we will have $\alpha \equiv \gamma$ modulo 2. We are somewhat restricted:

$$\begin{cases} r_0 \equiv \alpha + 1 & \text{mod } 2, \\ r_i \equiv i + 1 & \text{mod } 2 \quad \text{for } 1 \leq i \leq \beta, \\ r_i \equiv i & \text{mod } 2 \quad \text{for } \beta + 1 \leq i. \end{cases} \quad ; \quad \begin{cases} c_0 \equiv \alpha + \beta + 1 & \text{mod } 2, \\ c_1 \equiv \beta & \text{mod } 2, \\ c_j \equiv j + 1 & \text{mod } 2 \quad \text{for } 2 \leq j. \end{cases}$$

We may now ask ourselves which cells (k, j) verify $(c_k, r_j) = (0, 0)$ or $(1, 1)$. These cells will form $\nu^{(0)}$ and $\nu^{(1)}$ respectively. As everything depends on the parity of two variables (α, β), we are left with four cases. As to not clutter the page with symbols and operations, we went ahead and record the results in Table 1.

In particular, when we let $\gamma = 0$, then $\alpha \equiv 0$ modulo 2 and we get

$$\nu^* = \left(\left(1^{\lceil \frac{\beta}{2} \rceil}\right), \; \left(\frac{N}{2} - \beta, 1^{\lfloor \frac{\beta}{2} \rfloor}\right) \right).$$

α mod 2	β mod 2	$\nu^{(0)}$	$\nu^{(1)}$
0	0	$\left(1^{\frac{\beta}{2}}\right)$	$\left(\frac{\alpha}{2},\ 1^{\frac{\beta+\gamma}{2}}\right)$
0	1	$\left(1^{\frac{\beta+\gamma+1}{2}}\right)$	$\left(\frac{\alpha}{2}, 1^{\frac{\beta-1}{2}}\right)$
1	0	$\left(\frac{\alpha+1}{2}, 2^{\frac{\beta}{2}}, 1^{\frac{\gamma-1}{2}}\right)$	$\emptyset$
1	1	$\left(\frac{\alpha-3}{2}\right)$	$\left(2^{\frac{\beta-1}{2}},\ 1^{\frac{\gamma-1}{2}}\right)$

Table 1: The 2-quotient of a hook+column $\nu = (\alpha, 2^\beta, \gamma)$ of even size.

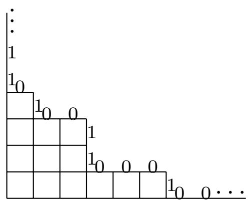
2.4 Abaci

Another way of computing the n-core and n-quotient requires the notion of a balanced abacus, which was first introduced in [12]. That the two constructions coincide will be apparent. A **balanced abacus** is a function $w : \mathbb{Z} \to \{0, 1\}$ such that

$$\begin{cases} w(n) = 1 \text{ for } n \ll 0 \\ w(n) = 0 \text{ for } n \gg 0 \\ \#\{n \geq 0 : w(n) = 1\} = \#\{n < 0 : w(n) = 0\} \qquad \text{(balanced criterion)} \end{cases}$$

We will use the word $..., w(n-1), w(n), w(n+1), ...$ to represent the abacus and will mark $w(0)$ with a bar.

One can easily go from Young diagrams to balanced abaci with the following correspondence: a 1 in the abacus encodes a vertical line, and a 0 encodes a horizontal one. Reading from left to right, the abacus spells the outline of the Young diagram. See Figure 8.

$\mu = (6, 3, 3, 1) =$ 

$\longleftrightarrow \quad ..., 1, 1, 0, 1, 0, 0, \bar{1}, 1, 0, 0, 0, 1, 0, 0, ...$

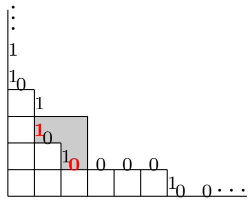
Figure 8: The partition $\mu = (6, 3, 3, 1)$ and its corresponding abacus.

Note that removing a rim hook of length n in the Young diagram corresponds to swapping a 0 and a 1 that are *exactly* n letters apart in the word. Refer to Figure 9.

Given w and $n \in \mathbb{N}$, we can define the n**-runner** of the abacus as the n-tuple of abaci $(w^0, ..., w^{n-1})$ such that w^i is made of every ith entry of the original abacus (mod n). For instance, the 2-runner of the running example is

$$\begin{cases} w^0 = ..., 1, 0, 0, \bar{1}, 0, 0, 0, ... \\ w^1 = ..., 1, 1, 0, 1, 0, 1, 0, ... \end{cases}$$

The abacus of the n-core of w can be now computed by pushing all the 1s to the left in the n-runner



$$\longleftrightarrow \quad ..., 1, 1, 0, 1, {\color{red}\mathbf{1}}, 0, \bar{1}, {\color{red}\mathbf{0}}, 0, 0, 0, 1, 0, 0, ...$$

Figure 9: Removing a rim hook and its effect on the abacus.

as far as they can go, and then reassembling the word back in the same order as before.

$$\begin{cases} ..., 1, 1, 0, \bar{0}, 0, 0, 0, ... \\ ..., 1, 1, 1, 1, 0, 0, 0, ... \end{cases} \Rightarrow ..., 1, 1, 0, 1, \bar{0}, 1, 0, 0, ... \longleftrightarrow \ \tilde{\mu} = \yng(1,2)$$

And to compute the n-quotient, simply place the "bar" in each of the n-runner abaci as to make them balanced.

$$\begin{cases} w^{(0)} = ..., 1, 0, \bar{0}, 1, 0, 0, 0, ... \\ w^{(1)} = ..., 1, 1, 0, 1, \bar{0}, 1, 0, ... \end{cases} \longleftrightarrow \mu^* = \left(\mu^{(0)} = \yng(2)\ ,\ \mu^{(1)} = \yng(1,2)\right)$$

This way of computing the n-core an the n-quotient lets itself to the following formula [19, 35]:

$$|\mu| = |\tilde{\mu}| + n \cdot |\mu^*|,$$

where $|\mu^*| = |\mu^{(0)}| + \cdots + |\mu^{(n-1)}|$. Indeed, the n-quotient encodes how many positions does each 1 have to move when we "push everything to the left as far as it goes". And shifting a 1 one position to the left in the n-runner translates to removing a rim hook of length n. So, in order to compute the n-core, we are starting with μ and are eliminating $n \cdot |\mu^*|$ cells in the process. As $|\tilde{\mu}|$ cells remain, we arrive to the announced formula.

2.5 The LR rule

The Littlewood-Richardson rule (LR rule) is used to compute the product of Schur functions. It is an extremely important theorem in representation theory (and, consequently, in the theory of symmetric functions). The first published proof of the LR rule, in 1934, was incomplete, and it lasted unsolved for four decades.

> The Littlewood-Richardson rule helped to get men on the moon, but it was not proved until after they had got there. The first part of this story might be an exaggeration (Gordon James, 1986, [12] p. 117).

Nowadays, many different proofs of this rule can be found, and in many contexts[4]. We will introduce some definitions before the result.

Definition 2.10. A **word** w is a finite sequence of natural numbers, $w = w_1 w_2 ... w_n$. Let $m_i(w)$ be the number of w_j equal to i, and let $\big(m_1(w), m_2(w), ...\big)$ be the **weight** of w. A **prefix** is

[4] Here we list some of them: a well-known proof using Jeux de Taquin [27], a classical combinatorial proof by Remmel and Shimozono [25], a specially short and easy-to-follow proof by Stembridge for both the skew formula and the product rule [34], and a more implicit proof with another point of view, but equally easy to read, by Knutson, Tao and Woodward [14].

a subsequence $w_1 w_2 ... w_m (m \leq n)$. We say that w is a **lattice permutation** (on the left)[5] if $m_i(v) \geq m_{i+1}(v)$ for every i and for every prefix v of w.

Definition 2.11. Given a (skew) tableau T, the **reverse reading (row) word** is the sequence of entries of T in the following order: start with row 0 and list the entries from right to left, move on to row 1 and list the entries from right to left, etc.

Example 2.12. The reverse reading word of the following skew tableau is a lattice permutation:

$$\begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 2 & 2 \\ \cline{1-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 \\ \cline{3-4} \end{array} \quad ; \quad w = 1121223.$$

Theorem 2.13 (LR rule). *We have:*

- $\mu \not\subseteq \lambda$ *implies* $[s_\lambda](s_\mu \cdot s_\nu) = 0$
- *If* $\mu \subseteq \lambda$*, then* $[s_\lambda](s_\mu \cdot s_\nu)$ *is the number of skew SSYT of shape* λ/μ *such that the reverse reading word is a lattice permutation* w *of weight* ν*.*

Example 2.14. For instance, $[s_{(4,3,2,1)}] s_{(3,2,1)} \cdot s_{(3,1)}$ is equal to 3, corresponding to the following tableaux:

$$\begin{array}{|c|ccc}\cline{1-1} 2 \\ \cline{1-2} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{2-3} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{3-4} & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{4-4} \end{array} \quad ; \quad \begin{array}{|c|ccc}\cline{1-1} 1 \\ \cline{1-2} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{2} \\ \cline{2-3} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{3-4} & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{4-4} \end{array} \quad ; \quad \begin{array}{|c|ccc}\cline{1-1} 1 \\ \cline{1-2} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{2-3} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{2} \\ \cline{3-4} & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} \\ \cline{4-4} \end{array} \quad .$$

We define the **Littlewood-Richardson coefficients** as $c^\lambda_{\mu,\nu} := [s_\lambda](s_\mu \cdot s_\nu)$. It is widely known [27] that these coefficients also verify $c^\lambda_{\mu,\nu} := [s_\nu] s_{\lambda/\mu}$, giving us a tool to work with skew Schur functions.

Define also the generalized LR coefficient $c^\lambda_{\mu^0,\mu^1,\ldots,\mu^{n-1}}$ as the number $[\lambda](s_{\mu^0} \cdot s_{\mu^1} \cdots s_{\mu^{n-1}})$. In particular, for $n = 2$, we recover the usual LR coefficient $c^\lambda_{\mu^0,\mu^1}$. Note that, by definition, $c^\lambda_{\mu^0,\mu^1,\ldots,\mu^{n-1}} \neq 0$ if and only if $\lambda \in \text{supp}(s_{\mu^0} \cdot s_{\mu^1} \cdots s_{\mu^{n-1}})$.

Immediately from the theorem, we get the following lemma.

Lemma 2.15. *If* $c^\lambda_{\mu,\nu} \neq 0$ *then* $\mu \cup \nu \trianglelefteq \lambda \trianglelefteq \mu + \nu$*. Moreover, both extremes are attained.*

Proof. From the LR rule, we can construct $\lambda \in \text{supp}(s_\mu \cdot s_\nu)$ by starting with μ and adding some cells that can be filled in a particular way. By simple inspection, if any cells are to be added in the first row, then *all* of them must be fillable with 1s, of which there are a total of ν_1 available. Then, what is the biggest possible value for λ_1? It is $\mu_1 + \nu_1$. A similar argument is used for every other row. Also, we have constructed $\mu + \nu$ while verifying the LR conditions, so $\mu + \nu$ is in $\text{supp}(s_\mu \cdot s_\nu)$.

For the lower bound, the same analysis column by column reveals that $\lambda' \trianglelefteq \mu' + \nu'$, which is equivalent to $\mu \cup \nu = (\mu' + \nu')' \trianglelefteq \lambda$. ■

2.6 Plethysm

Plethysm is an operation in the algebra Λ. Many different definitions can be given in various different contexts. Here, our definition will be an axiomatic one.

[5] Similarly, a lattice permutation on the right verifies $m_i(v) \geq m_{i+1}(v)$ for every *sufix* v of w

Denote the plethysm of f and g by $f \circ g$. The notion of plethysm comes from the composition of Schur functors (representations of $\mathbf{GL}(V)$). Let's say that our functions $h \in \Lambda$ are functions in the variables $x_1, x_2, ...$. That is, h denotes $h(x_1, x_2, ...)$.

If g is a sum of monic terms, $g = g_1 + g_2 + \cdots$ then $f \circ g(x_1, x_2, ...) = f(g_1, g_2, ...)$.

Example 2.16. p_m is always a sum of monic terms, $p_m = x_1^m + x_2^m + \cdots$. And thus,

$$p_n \circ p_m = p_n(x_1^m, x_2^m, ...) = (x_1^m)^n + (x_2^m)^n + \cdots = p_{nm}.$$

Example 2.17. Any $f \in \Lambda$ with positive integers as coefficients can be expressed as a sum of monic terms. For instance,

$$2p_2 = 2x_1^2 + 2x_2^2 + \cdots = x_1^2 + x_1^2 + x_2^2 + x_2^2 + \cdots .$$

Consequently,

$$p_n \circ 2p_2 = p_n(x_1^2, x_1^2, x_2^2, x_2^2, ...) = 2p_{2n}.$$

More precisely, we can define the plethysm by means of four axioms.

Definition 2.18. Plethysm, denoted by $\circ$, is the operation $\Lambda \times \Lambda \to \Lambda$ verifying

1. $p_n \circ p_m = p_{nm}$ for all $n, m \in \mathbb{N}$.
2. For any $f \in \Lambda$, the map $g \mapsto g \circ f$ is a $\mathbb{Z}$-algebra homomorphism on Λ.
3. For any $f \in \Lambda$, the equality $p_n \circ f = f \circ p_n$ holds.

This definition give us a set of rules to operate with plethysm. For any $f, g, h \in \Lambda$, $n, m \in \mathbb{N}$ and constants a, b,

1. $p_n \circ p_m = p_{nm}$.
2. $(af \pm bg) \circ h = a(f \circ h) \pm b(g \circ h)$.
3. $(af \cdot g) \circ h = a(f \circ h) \cdot (g \circ h)$.
4. $p_n \circ (af \pm bg) = a(p_n \circ f) \pm b(p_n \circ g)$.
5. $p_n \circ (af \cdot g) = a(p_n \circ f) \cdot (p_n \circ g)$.
6. $f \circ (ag) = a(f \circ g)$.
7. $p_n \circ f = f \circ p_n$.

It is a hard open problem in algebraic combinatorics to understand the resulting coefficients of the plethystic operation expressed over the Schur basis. Many formulas have been proven for the simpler cases (for instance, see [15]). But even this next problem remains open:

Problem (Prob. 9 from Stanley's List [33]). *Find a combinatorial interpretation of the plethystic coefficients $[s_\lambda](s_a \circ s_b)$, thereby combinatorially reproving that they are nonnegative.*

If this is the first time hearing about plethysm, the reader may wonder why is this a relevant problem. As briefly announced before, the notion of plethysm comes from invariant theory and representation theory. Recall that we motivated the introduction of Schur functions by saying that they are the image of the irreducible representations of a group under a bijection we called the *Frobenius characteristic map*. Without entering in too much detail[6], if $S^\lambda(V)$ and $S^\mu(V)$ are

[6] An interested reader may read [1, 27].

two irreducible representations of $\mathbf{GL}(V)$ then they map to s_λ and s_μ respectively. And their composition $S^\lambda(S^\mu(V))$ maps to $s_\lambda \circ s_\mu$. Another natural operation on representations, namely the tensor product $S^\lambda(V) \otimes S^\mu(V)$, has a nice combinatorial interpretation: it maps to the product $s_\lambda \cdot s_\mu$, and thus the LR rule returns the coefficients in its decomposition into irreducibles. Due to the success of the LR rule, our long-time goal is to find an analog for our problem.

Another unsolved problem related in nature is to combinatorially understand the Kronecker product which comes from the restriction to $\mathbf{GL}(V) \times \mathbf{GL}(W)$ of the representation $S^\lambda(V \otimes W)$ of $\mathbf{GL}(V \otimes W)$.

2.7 The SXP rule

This next rule will be extremely useful for us. Proofs of this proposition can be found in [17, 36].

Theorem 2.19 (SXP rule). *For any partitions λ, μ and any $n \in \mathbb{N}$,*

$$[\mu](p_n \circ s_\lambda) = \mathrm{sgn}_n(\mu) \;\cdot\; [\lambda]\left(s_{\mu^{(0)}} \;\cdot\; s_{\mu^{(1)}} \;\cdots\; s_{\mu^{(n-1)}}\right),$$

where $\circ$ denotes the plethysm, $\mu^* = \left(\mu^{(0)}, \mu^{(1)}, ..., \mu^{(n-1)}\right)$ is the n-quotient of μ, and the sign function is defined below.

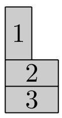
Example 2.20. Our work will primarily use this rule for hook+columns and for $n = 2$. We computed in section 1 the 2-quotient of every hook+column of even size. The SXP rule will be simplified to equalities of the type

$$\left[\,\ydiagram{6,2,2,2,2,1,1}\,\right](p_2 \circ s_\lambda) = \pm[\lambda]\left(s_{\ydiagram{1,1}} \cdot s_{\ydiagram{3,1,1,1}}\right),$$

where the right-most product is done by the LR rule.

We have proven, in section 2.4, that $|\mu| = |\tilde{\mu}| + n \cdot |\mu^*|$. Because of this formula, in the SXP rule, $\tilde{\mu}$ needs to be empty. Otherwise, the resulting coefficient would be zero. To see this, let's ask which partitions μ can ever appear in $\mathrm{supp}(p_n \circ s_\lambda)$, for some λ. By the basic properties of plethysm, we have $|\mu| = n \cdot |\lambda|$. And if λ has to be in $\mathrm{supp}(s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}})$, then the LR rule imposes $|\lambda| = |\mu^*|$. So then, $|\mu| = n \cdot |\mu^*|$. Therefore, only partitions μ of a multiple of n and of empty n-core will not vanish.

It remains to define the sign function. For this matter, we will use the Young diagram approach. For a partition $\lambda \vdash nk$ of empty n-core, we can "decompose" the Young diagram in its successive rim hooks of length n (the ones we remove to compute the n-core). Define the resulting skew partitions $\sigma_1, ..., \sigma_k$ corresponding to the removed rim hooks of length n, in no particular order (see Figure 10).



$$\Rightarrow \begin{cases} \sigma^1 = \ydiagram{2,2,1,1}/\ydiagram{2,2} & ; \quad l(\sigma^1) = 2 \\ \sigma^2 = \ydiagram{2,2}/\ydiagram{2} & ; \quad l(\sigma^2) = 1 \\ \sigma^3 = \ydiagram{2} & ; \quad l(\sigma^3) = 1 \end{cases}$$

Figure 10: Let $\lambda = (2^2, 1^2) \vdash 6$ and $n = 2$. We write i in the cells of the ith removed rim hook.

We define the sign function for such partitions as

$$\mathrm{sgn}_n(\lambda) = \prod_{i=1}^{k} (-1)^{l(\sigma^i)-1}.$$

Example 2.21. In the example of Figure 10, $\mathrm{sgn}_2\left(\ydiagram{1,1,2,2}\right) = (-1)^1 \cdot (-1)^0 \cdot (-1)^0 = -1$. Hence, as the 2-quotient of $\ydiagram{1,1,2,2}$ is $(\ydiagram{1,1}, \ydiagram{1})$ we get $\left[\ydiagram{1,1,2,2}\right](p_2 \circ s_{\ydiagram{1,2}}) = -[\ydiagram{1,2}](s_{\ydiagram{1,1}} \cdot s_{\ydiagram{1}}) = -1$.

The SXP rule lets us immediately identify some partitions of $\mathrm{supp}(p_n \circ s_\lambda)$, making it extremely useful. For instance, we get this next lemma.

Lemma 2.22. *For any $n \in \mathbb{N}$, we have $s_{n\lambda} \in \mathrm{supp}(p_n \circ s_\lambda)$.*

Proof. Let $\mu = n\lambda$. We are going to prove that the n-core of μ is empty and that λ is an element of $\mathrm{supp}(s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}})$, which will yield the result thanks to the SXP rule.

To begin with, we need to visualize μ. Picture its diagram as an stretched version of λ. Any inner corner[7] of λ, when streched, will result in a rim hook of length n. In particular, this shows that the n-core of μ is empty (by an inductive argument).

For the second part, we will need to compute the n-quotient of μ. Let us follow an example: let $\lambda = (3, 2, 2)$ and $n = 3$. The first steps of the n-quotient algorithm will result in the following diagram:

$$\begin{array}{cccccccccc|c}
0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & & \\
\hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & & & & & 1 \\
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{0} & & & & & 0 \\
\multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & & 2 \\
\hline
\end{array}$$

As each row is of length a multiple of n, the number r_j associated with the jth row is congruent with j modulo n. And the difference in length of two given rows is also a multiple of n, so we will find that the numbers c_k associated with the columns come in groups of n in which all numbers 0, 1, ..., $n-1$ appear exactly once.

And thus,

$$\mu^{(0)} = (\lambda_2, \lambda_{n+2}, \lambda_{2n+2} \ldots),$$

$$\mu^{(1)} = (\lambda_3, \lambda_{n+3}, \lambda_{2n+3} \ldots),$$

$$\vdots$$

$$\mu^{(n-1)} = (\lambda_1, \lambda_{n+1}, \lambda_{2n+1} \ldots).$$

From Lemma 2.15, $\lambda = \mu^{(0)} \cup \mu^{(1)} \cup \cdots \cup \mu^{(n-1)} \in \mathrm{supp}(s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}})$, as desired. ∎

Using similar techniques, one can show the following lemma, which we leave as an exercise.

Lemma 2.23. *Given $n \in \mathbb{N}$ and a partition λ of length $l(\lambda) =: l$, define ν as the partition $n \cdot (\lambda/1^{(l)}) + 1^{(nl)}$. Then $\nu \in \mathrm{supp}(p_n \circ s_\lambda)$.*

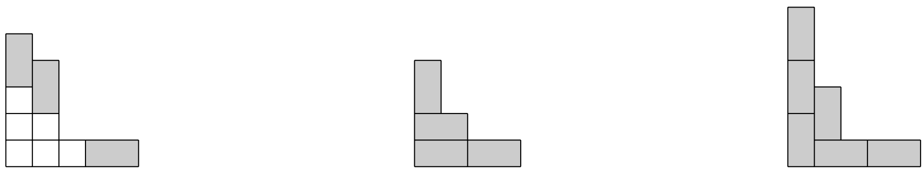
Example 2.24. Let us give an example of the construction involved in the lemma: let $\lambda = (3, 2)$ and $n = 3$. Then $\nu = (7, 4, 1, 1, 1, 1) \in \mathrm{supp}(p_3 \circ s_{(3,2)})$. We obtain ν by eliminating the first column, then multiplying each row by n, and adding $(1^{(nl)})$ as the first column:

$$\lambda = \ydiagram{2,3} \longrightarrow \ydiagram{1,2} \longrightarrow \ydiagram{3,6} \longrightarrow \ydiagram{1,1,1,1,4,7} = \nu.$$

[7] An inner corner of λ is a cell of λ whose removal leaves a valid partition. Intuitively, the inner corners are the top-right corners of the diagram.

Again, in this work, we will primarily deal with hook+column partitions. Hence, we would like to know how to compute the sign function $\mathrm{sgn}_2(\lambda)$ for $n = 2$ and $\lambda = (\alpha, 2^\beta, 1^\gamma) \vdash 2k$.

If $\gamma \equiv \beta \equiv 1 \pmod 2$, then I claim that the 2-core is not empty and thus the sign function is not defined. Indeed, start by removing the right part of the 2s, using rim hooks of shape . One cell will remain. Notice that both γ and α are odd (as λ is a partition of an even number). After the removal of enough rim hooks, the 2-core is guaranteed to be . Refer to Figure 11a. In any other case,



(a) $\gamma \equiv \beta \equiv 1 \pmod 2$ (b) $\gamma \equiv 0 \pmod 2$ (c) $\gamma \equiv 1$, $\beta \equiv 0 \pmod 2$

Figure 11: The three distinct cases of sign computations of a hook-column.

- If $\gamma \equiv 0 \pmod 2$, we can start by taking away the 1s part, and make it so every other rim hook is of shape , and thus the sign depends on the parity of $\frac{\gamma}{2}$. In the example of Figure 11b, $\gamma = 2$ and thus $\mathrm{sgn}_2(\lambda) = (-1)^{\frac{\gamma}{2}} = (-1)^1 = -1$.
- If $\gamma \equiv 1 \pmod 2$ and $\beta \equiv 0 \pmod 2$, then we can start by taking away the right part of the 2s, using $\frac{\beta}{2}$ rim hooks of shape . We are left with a hook shape, that will have $\frac{\gamma+\beta+1}{2}$ rim hooks of shape , and the rest of shape . So the sign will depend on $\frac{\beta}{2} + \frac{\gamma+\beta+1}{2} \pmod 2 \equiv \frac{\gamma+1}{2} \pmod 2$. In the example of Figure 11c, $\gamma = 3$ hence $\mathrm{sgn}_2(\lambda) = (-1)^{\frac{\gamma+1}{2}} = (-1)^2 = 1$.

Summing up: the 2-sign only depends on $\gamma = m_1(\lambda)$ modulo 4 (see Table 2).

$m_1(\lambda)$ mod 4	0	1	2	3
$\mathrm{sgn}_2(\lambda)$	+	−	−	+

Table 2: The sign function for $n = 2$ and λ a hook+column of even size, empty 2-core.

2.8 Schur functions evaluations

One of the main tools for discussing plethystic and similar coefficients is the evaluation of symmetric functions in different alphabets [15, 26]. For us, an **alphabet** X will be a collection of variables $x_1, x_2, ...$ indexed by a set of **letters**, $\{1, 2, ...\} \subseteq \mathbb{N}$ (finite or otherwise). Identify Λ with $\mathbb{Q}[p_1, p_2, ...]$, that is, the $\mathbb{Q}$-algebra spanned by the power sum symmetric functions. Given a symmetric function f, we will denote by $f[X]$ the evaluation of f in the alphabet X, which is the image of f under the morphism sending the basis $\{p_k\}_k$ to $\{x_1^k + x_2^k + ...\}_k$. Up until now, we were thinking of symmetric functions as evaluated on an alphabet X on infinite letters. We could also identify X with $p_1[X]$ and then $f[X]$ is just $f \circ X$.

Example 2.25. From the axioms of plethysm, for $c \in \mathbb{N}$ and two alphabets X, Y, we have

$$p_2[X+Y] = p_2 \circ (p_1[X] + p_1[Y]) = p_2(x_1, y_1, x_2, y_2, \dots) = x_1^2 + y_1^2 + x_2^2 + y_2^2 + \dots = p_2[X] + p_2[Y];$$

$$p_2[cX] = p_2 \circ (cp_1[X]) = p_2(x_1, \overset{c \text{ times}}{\cdots}, x_1, x_2, \overset{c \text{ times}}{\cdots}, x_2, \ldots) = cp_2[X].$$

As we can see in the previous example, the evaluation morphism have some linearity properties. We extend those properties by letting $0 = p_k[0] = p_k[X - X] = p_k[X] + p_k[-X]$, hence $p_k[-X] = -p_k[X]$.

> **Note 2.26.** One needs to be careful with the notation. We will sometimes want to evaluate a function f on the alphabet $cx_1, cx_2, ...$ for some integer c. For that, we will write $f[tX]|_{t=c}$. In general, this will not be equal to $p_k[cX]$. In particular, $-p_k[X] = p_k[-X] \neq p_k[tX]|_{t=-1} = (-1)^k p_k[X]$.

It is key to remark that plethystic calculus is not trivial, and it often gives raise to some unexpected consequences. We get the following lemma.

Lemma 2.27. *Let X be an alphabet. Then, $s_\lambda[-X] = (-1)^{|\lambda|} s_{\lambda'}[X]$.*

Proof. Using the Jacobi-Trudi identity from 2.6, we get

$$s_\lambda[-X] = \det(h_{\lambda_i - i + j})_{i,j=1}^{|\lambda|}[-X].$$

Let's look at one of these terms. As $-X = -p_1(x_1, x_2, ...)$,

$$h_\mu \circ (-p_1) = \prod_i h_{\mu_i} \circ (-p_1) \overset{2.5}{=} \prod_i \left(\sum_{\nu \vdash \mu_i} \frac{p_\nu}{z_\nu} \circ (-p_1) \right) =$$

$$= \prod_i \left(\sum_{\nu \vdash \mu_i} (-1)^{l(\nu)} \frac{p_\nu}{z_\nu} \right) \overset{2.5}{=} \prod_i \left((-1)^{l(\mu_i)} e_{\mu_i} \right) = (-1)^{|\mu|} e_\mu.$$

But then, $s_\lambda[-X] = (-1)^{|\lambda|} \det(e_{\lambda_i - i + j})_{i,j=1}^{|\lambda|}[X] = (-1)^{|\lambda|} s_{\lambda'}[X]$ again by Jacobi-Trudi. ■

So what does it mean to evaluate a Schur function on a negative alphabet?[8] Recall that $s_\lambda = \sum x^T$ where the sum is over the SSYT T of shape λ. Well, the correct interpretation of the above lemma is that when we use negative alphabets we switch the rules of the SSYT, to be *weakly* increasing in the columns and *stricly* increasing in the rows. In Figure 12, we list some valid SSYT. Rather naturally, the following lemma holds.

$$\begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 2 & 3 & 3 & \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array} \; ; \; \begin{array}{|c|c|c|} \hline -1 & & \\ \hline -4 & -3 & \\ \hline -5 & -4 & \\ \hline -5 & -4 & -2 \\ \hline \end{array} \; ; \; \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline -1 & & & \\ \hline -2 & 2 & 3 & \\ \hline -2 & 1 & 1 & 3 \\ \hline \end{array} \; .$$

Figure 12: Three valid SSYT with positive and/or negative letters.

Lemma 2.28. *Let X and Y be two alphabets, λ a partition. Then:*

[8]Questions like this one arise frequently in combinatorics, when something is defined naturally for the positive integers; and then has a different interpretation in the negative integers. Lemma 2.27 is just another example of Combinatorial Reciprocity, a phenomenon in which the answer to the above question is always similar. Namely, strict inequalities are converted to weak inequalities, matrices are transposed, formulas are inverted... See [2, 30] for a survey on this subject.

1. $s_\lambda[X+Y] = \sum_{\mu\subset\lambda} s_\mu[X] \cdot s_{\lambda/\mu}[Y] = \sum_{\mu,\nu} c^\lambda_{\mu,\nu}\ s_\mu[X] \cdot s_\nu[Y]$.

2. $s_\lambda[X-Y] = \sum_{\mu\subset\lambda}(-1)^{|\lambda/\mu|}\ s_\mu[X] \cdot s_{(\lambda/\mu)'}[Y] = \sum_{\mu,\nu}(-1)^{|\nu|}\ c^\lambda_{\mu,\nu}\ s_\mu[X] \cdot s_\nu[Y]$.

Note 2.29. A more general theorem, for $s_\lambda \circ (f \pm g)$ on two arbitrary symmetric functions, is stated and proven in [19].

Accordingly, from now on, if $X = \{x_i\}_{i\in I}$ and $Y = \{y_j\}_{j\in J}$ are two alphabets, then

$$s_\lambda[X-Y] = \sum_{T:R(\lambda)\to I\cup -J} x^{T^+}(-y)^{T^-}, \quad \text{where} \quad \begin{cases} x^{T^+} &= \prod_{i\in I} x_i^{m_i(T)}, \\ (-y)^{T^-} &= \prod_{j\in J}(-y_j)^{m_{-j}(T)}, \end{cases}$$

and where the rules of the tableaux T are those illustrated in Figure 12. For instance, the third tableau of said figure will yield the monomial $-x_1^2 x_2 x_3^3 y_1 y_2^2$.

2.9 The Hopf Algebra Structure of Λ

We already work with the $\mathbb{Z}$-algebra structure of Λ. Recall that a $\mathbb{Z}$-algebra is an associative ring with unit $(\Lambda, +, 0, \cdot, 1)$, containing a distinguished copy of $\mathbb{Z}$ which commutes with every element, and with $1 \in \mathbb{Z}$ being the algebra unit. In particular, a $\mathbb{Z}$-algebra is also a $\mathbb{Z}$-module.

Consider now the tensor product $\Lambda\otimes\Lambda$. In order to distinguish between the first and the second copy of Λ, we will write every function $f\otimes g$ in $\Lambda\otimes\Lambda$ as $f[X]\otimes g[Y]$, with X and Y being some alphabets $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_j\}_{j\in\mathbb{N}}$. Note that these alphabets are just placeholders, letting us identify $f[X]$ with $f[Y]$. With this in mind, we write $\Lambda\otimes\Lambda$ as $\Lambda[X]\otimes\Lambda[Y]$. This is the module of functions which are symmetric in X and Y separately. Note that we can see the usual product as a function $(\cdot) : \Lambda[X]\otimes\Lambda[Y] \ni f[X]\otimes g[Y] \mapsto fg \in \Lambda$. Similarly, the unit can be seen as a function $\mathbf{1} : \mathbb{Z} \to \Lambda$; namely the inclusion.

As it turns out, Λ also admits a $\mathbb{Z}$-coalgebra structure. That is, we are able to define a coproduct $\Delta : \Lambda \ni f \mapsto f[X+Y] \in \Lambda[X]\otimes\Lambda[Y]$ and a counit $\epsilon : \Lambda \ni f \mapsto f[0] \in \mathbb{Z}$ such that the following diagrams are commutative:

$$\begin{tikzcd} \Lambda \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Lambda\otimes\Lambda \arrow[d, "1\otimes\Delta"] \\ \Lambda\otimes\Lambda \arrow[r, "\Delta\otimes 1"'] & \Lambda\otimes\Lambda\otimes\Lambda \end{tikzcd} \qquad \begin{tikzcd} \Lambda \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Lambda\otimes\Lambda \arrow[d, "\mathbf{1}\otimes\epsilon"] \\ \Lambda\otimes\Lambda \arrow[r, "\epsilon\otimes\mathbf{1}"'] & \Lambda \end{tikzcd}$$

Indeed, with our defined coproduct and counit, for all f in Λ, it is

$$(\mathbf{1}\otimes\Delta)\Delta(f) = f[X+(Y+Z)] = f[(X+Z)+Y] = (\Delta\otimes\mathbf{1})\Delta(f),$$

$$(\epsilon\otimes\mathbf{1})\Delta(f) = f[Y] = f = f[X] = (\mathbf{1}\otimes\epsilon)\Delta(f),$$

where we identified $\mathbb{Z}\otimes\Lambda[Y]$ with Λ and $\Lambda[X]\otimes\mathbb{Z}$.

Moreover, the $\mathbb{Z}$-algebra structure is compatible with the $\mathbb{Z}$-coalgebra structure, in the sense that Δ and ϵ are $\mathbb{Z}$-algebra morphisms. This is immediate from the second axiom of plethysm. We say that $(\Lambda, \cdot, \mathbf{1}, \Delta, \epsilon)$ is a bialgebra.

Finally, Λ admits an structure of Hopf algebra $(\Lambda, \cdot, \mathbf{1}, \Delta, \epsilon, \bar\omega)$, which is a bialgebra with an antipode. An antipode is a $\mathbb{Z}$-linear function $\bar\omega : \Lambda \to \Lambda$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & \Lambda\otimes\Lambda \arrow[r, "\bar{\omega}\otimes\mathbf{1}"] & \Lambda\otimes\Lambda \arrow[dr, "(\cdot)"] & \\
\Lambda \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{Z} \arrow[r, "\mathbf{1}"] & \Lambda \\
 & \Lambda\otimes\Lambda \arrow[r, "\mathbf{1}\otimes\bar{\omega}"'] & \Lambda\otimes\Lambda \arrow[ur, "(\cdot)"'] &
\end{tikzcd}$$

In our case, define $\bar{\omega} : h_\mu \mapsto (-1)^{|\mu|} e_\mu$. As we saw in the proof of 2.27, this is the function $f[X] \mapsto f[-X]$. And thus, for any power sum function p_n, we have

$$(\cdot)(\bar{\omega}\otimes\mathbf{1})\Delta(p_n) = (\cdot)(p_n[-X+Y]) = (\cdot)(-p_n[X]+p_n[Y]) = (\cdot)(-p_n\otimes\mathbf{1}+\mathbf{1}\otimes p_n) = p_n - p_n = 0.$$

Similarly, $(\cdot)(\mathbf{1}\otimes\bar{\omega})\Delta(p_n) = 0$. In the other hand, $\mathbf{1}\epsilon(p_n) = \mathbf{1}(p_n[0]) = \mathbf{1}(0) = 0$. As every operation is an algebra morphism, proving the properties for p_n is enough. For a more detailed description of the Hopf algebra structure of Λ, see [1].

3 Some positivity conditions for the plethystic coefficients

The SXP rule allows us to combinatorially compute $p_n \circ s_\lambda$ by means of the well-understood Littewood-Richardson coefficients. In this section we study conditions that allow us to deduce that a plethystic coefficient is zero (or that it is positive) in terms of similar results for the Littlewood-Richardson coefficients. We show that if $\mu \in \mathrm{supp}(p_n \circ s_\lambda)$ then $\lambda \subseteq \mu$ (3.8). On the other hand, we show that μ is upperly bounded by a pair of partitions (3.10). Our main tool is a determination of the region where non-vanishing Littewood-Richardson coefficients are located (3.5) that we have not seen explicitly written, but that appears implicitly in the literature [26, 15], combined with the SXP rule.

The motivating question behind our tool is, given $c^\lambda_{\mu,\nu} \neq 0$, what can we say about λ? A preliminary analysis was already given before (Lemma 2.15). It also follows from the LR rule that both $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$. However, this next lemma will let us say much more, by linking the LR coefficients with Schur functions evaluations.

We will mainly work with finite alphabets. Let X_a denote an alphabet $x_1, ..., x_a$ in a letters. We will identify the alphabet $X_a + Z_b$ with X_{a+b} by renaming $z_i \mapsto x_{a+i}$.

Lemma 3.1. *Let λ be a partition. Then, $s_\lambda[X_{a+b} - Y_{c+d}] \neq 0$ if and only if there exist partitions μ_0 and ν_0 such that*

$$c^\lambda_{\mu_0,\nu_0} \neq 0 \qquad ; \qquad s_{\mu_0}[X_a - Y_c] \neq 0 \qquad ; \qquad s_{\nu_0}[Z_b - W_d] \neq 0.$$

Proof. We know from Lemma 2.28 that

$$s_\lambda[X_{a+b} - Y_{c+d}] = s_\lambda[(X_a - Y_c) + (Z_b - W_d)] = \sum c^\lambda_{\mu,\nu}\ s_\mu[X_a - Y_c] \ \cdot\ s_\nu[Z_b - W_d].$$

So if $s_\lambda[X_{a+b} - Y_{c+d}] \neq 0$ then at least one of the terms in the sum doesn't vanish.

Conversely, if $c^\lambda_{\mu_0,\nu_0}\ s_{\mu_0}[X_a - Y_c] \ \cdot\ s_{\nu_0}[Z_b - W_d] \neq 0$, then let us consider the equality

$$s_\lambda[X_{a+b} - tY_{c+d}] = \sum c^\lambda_{\mu,\nu}\ s_\mu[X_a - tY_c] \ \cdot\ s_\nu[Z_b - tW_d],$$

where t is a variable as in Note 2.26. Let $t = -1$. Every monomial in both sides of the equality is now positive (recall that LR coefficients are positive by the LR rule), so there can't be any cancellations. This means in particular that $s_\lambda[X_{a+b} - tY_{c+d}]\big|_{t=-1} \neq 0$, and so necessarily $s_\lambda[X_{a+b} - Y_{c+d}] \neq 0$. ■

Note 3.2. If both X and Y are infinite, our argument can be thought of as taking the equality $s_\lambda[X + Z - Y - W] = \sum c^\lambda_{\mu,\nu}\ s_\mu[X - Y] \ \cdot\ s_\nu[Z - W]$ and applying the antipode $\mathbf{1} \otimes \bar{\omega} \otimes \mathbf{1} \otimes \bar{\omega}$ of $\Lambda[X] \otimes \Lambda[Y] \otimes \Lambda[Z] \otimes \Lambda[W]$ on both sides. Now every monomial is positive, yielding the same result.

From Lemma 3.1, if $c^\lambda_{\mu,\nu} \neq 0$ then $s_\lambda[X_{l(\mu)+l(\nu)}] \neq 0$. In particular, $l(\lambda) \leq l(\mu) + l(\nu)$, which we already knew (Lemma 2.15). Similarly, $\lambda_1 \leq \mu_1 + \nu_1$ when evaluating on $-Y_{\mu_1+\nu_1}$. The fun part comes when we consider an alphabet mixing positive and negative letters.

If $s_\lambda[X_r - Y_c] \neq 0$ then λ does not have a (c, r) cell. (In order to see this, think of what would be the value of (c, r) in a tableau with r positive letters and c negative letters.) But if λ does not have a (c, r) cell, then its region $R(\lambda)$ must fit in an infinite hook-looking region which can be thought of as the union of r rows and c columns, which we depicted in Figure 13. In the literature, a partition not containing a (c, r) cell is sometimes called a (c, r)-hook, but the reader is advised not to use this notation too frequently, as it is already used for other concepts in the theory of symmetric functions.

But if $R(\lambda)$ fits in multiple of these regions, we can then take the intersection of them to find a smaller region for which $R(\lambda)$ is a subset.

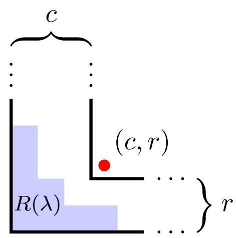
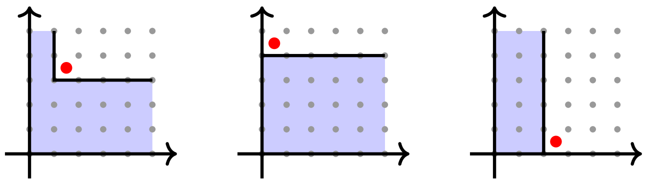
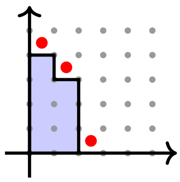


Figure 13: If $s_\lambda[X_r - Y_c] \neq 0$, then λ does not have a (c, r) cell, drawn as a point, and so $R(\lambda)$ fits in the depicted "fat-hook" region.

Example 3.3. Suppose that $s_\lambda[X_3 - Y_1] \neq 0$. This means that there exists a SSYT of shape λ and filled with the letters $\{1, 2, 3, -1\}$. One can easily see why this implies that there is no $(1, 3)$ cell in λ. Suppose we also know $s_\lambda[X_4] \neq 0$ and $s_\lambda[-Y_2] \neq 0$. Then there are no $(0, 4)$ or $(2, 0)$ cells in λ. Therefore, $R(\lambda)$ must fit inside each of the following regions:



Consequently, $R(\lambda)$ must also fit inside of $R(2^3, 1)$, *ie*, $\lambda \subseteq (2^3, 1)$:



In what comes, we will be working with subsets of $\mathbb{N}^2$. Some notation is needed. Consider the partial order in $\mathbb{N}^2$ given by $(a, b) \leq (c, d) \Leftrightarrow a \leq b$ and $c \leq d$. Let $A + B = \{a + b \ : \ a \in A \ ; \ b \in B\}$, and $(a, b) + (c, d) = (a + c, b + d)$. Let $\bar{A}$ be the complement of A in $\mathbb{N}^2$.

Definition 3.4. Let λ be a partition. We say that (c, r) is an **outer corner** if $(c, r) \notin \lambda$ but its addition to the diagram produces a valid partition. Let $\mathtt{OC}_\lambda$ be the set of outer corners of λ. Equivalently, $\mathtt{OC}_\lambda$ is the set of minima in $\overline{R(\lambda)}$ with respect to $(\leq)$.

For example, $\mathtt{OC}_{\yng(3,1,1)} = \{(0, 3), (1, 2), (3, 0)\}$. Conversely, a set of points $P \subset \mathbb{N}^2$ defines a region

$$\Theta(P) := \{q \in \mathbb{N}^2 \ : \ \forall \ p \in P \text{ either } q \leq p \text{ or } q \text{ and } p \text{ are not comparable}\}.$$

Moreover, if there is at least a point in P lying in the 0th column and at least one lying in the 0th row, then $\Theta(P)$ is the region of a partition. Think of it as the biggest partition not containing any cell of P. Refer to example 3.3 for an illustration of this fact. In particular, $\Theta(\mathtt{OC}_\mu) = R(\mu)$ for every partition μ.

Notice that for any given μ, if (c, r) is an exterior corner, then $s_\mu[X_r - Y_c] \neq 0$. In fact, there is a "canonical" SSYT of shape μ in those letters, filling the 0th column with $-c$s, the first column with $-(c-1)$s, ..., and the $(c-1)$th column with -1s; then fill the remaining cells of the 0th row with 1s, the first with 2s, ..., and the last cells with rs. Refer to Figure 14 for an example.

-3	-2					
-3	-2	-1				
-3	-2	-1				
-3	-2	-1	2	2	2	
-3	-2	-1	1	1	1	1

Figure 14: The canonical SSYT of shape $(7, 6, 3^2, 2)$ for the exterior corner $(3, 2)$.

We are now ready to state and prove the desired theorem.

Theorem 3.5. *If $c^{\lambda}_{\mu,\nu} \neq 0$ then $\overline{R(\mu)} + \overline{R(\nu)} \subseteq \overline{R(\lambda)}$. More generally, if $c^{\lambda}_{\mu^0,\mu^1,\dots,\mu^{n-1}} \neq 0$ then*

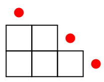
$$\sum_k \overline{R(\mu^k)} \subseteq \overline{R(\lambda)}.$$

In other words, if $\lambda \in \operatorname{supp}(s_{\mu^0} \cdot s_{\mu^1} \cdots s_{\mu^{n-1}})$ then $R(\lambda) \subseteq \Theta\left(\sum_k \mathtt{OC}_{\mu^k}\right)$.

Note 3.6. We are using a set-wise sum (Minkowski sum).

Let illustrate this theorem before giving the proof.

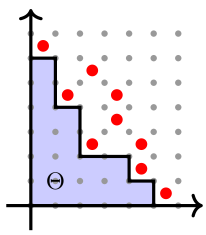
Example 3.7. Let $\mu^0 =$ , $\mu^1 =$  $= \mu^2$. We compute the exterior corners of each partition:

$\mu^0 =$  $\Rightarrow \quad \mathtt{OC}_{\mu^0} = \{(0, 2), (2, 1), (3, 0)\}.$

Similarly, $\mathtt{OC}_{\mu^1} = \{(0, 2), (1, 0)\} = \mathtt{OC}_{\mu^2}$. We now add together all possible combinations of exterior corners of our three partitions, to get 9 unique points of $\mathbb{N}^2$,

$$\sum_{k=0}^{2} \mathtt{OC}_{\mu^k} = \{(0, 6), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (3, 4), (4, 2), (5, 0)\}.$$

Plot those points in $\mathbb{N}^2$ with our system of coordinates. The shape $\Theta = \Theta(\sum_{k=0}^{2} \mathtt{OC}_{\mu^k})$ arises. Hence the region of every partition in $\operatorname{supp}(s_{\mu^0} s_{\mu^1} s_{\mu^2})$ must fit inside Θ.



Proof of 3.5. Let $\lambda \in \operatorname{supp}(s_{\mu^0} \cdots s_{\mu^{n-1}})$. Then, there exists $\nu^{n-2} \in \operatorname{supp}(s_{\mu^0} \cdots s_{\mu^{n-2}})$ such that $\lambda \in \operatorname{supp}(s_{\nu^{n-2}} s_{\mu^{n-1}})$. Take now ν^{n-2}. By the same analysis, there exists $\nu^{n-3} \in \operatorname{supp}(s_{\mu^0} \cdots s_{\mu^{n-3}})$ such that $\nu^{n-2} \in \operatorname{supp}(s_{\nu^{n-3}} s_{\mu^{n-2}})$. Iterate this process to obtain a chain of partitions

$$\lambda = \nu^{n-1}, \nu^{n-2}, \dots, \nu^1, \nu^0 = \mu^0.$$

Choose two outer corners $(c_0, r_0) \in \mathtt{OC}_{\mu^0}$ and $(c_1, r_1) \in \mathtt{OC}_{\mu^1}$. As the canonical tableau for a given corner exists, $s_{\mu^0}[X_{r_0} - Y_{c_0}] \neq 0$ and $s_{\mu^1}[X_{r_1} - Y_{c_1}] \neq 0$. In addition, we know that $\nu^1 \in \operatorname{supp}(s_{\nu^0} s_{\mu^1}) = \operatorname{supp}(s_{\mu^0} s_{\mu^1})$. And thus, by Lemma 3.1, we get $s_{\nu_1}[X_{r_0+r_1} - Y_{c_0+c_1}] \neq 0$.

Choose now an outer corner $(c_2, r_2) \in \mathtt{OC}_{\mu^2}$. As $\nu^2 \in \operatorname{supp}(s_{\nu^1} s_{\mu^2})$, we get that $s_{\nu^2}[X_{r_0+r_1+r_2} - Y_{c_0+c_1+c_2}] \neq 0$, again by Lemma 3.1.

By induction, $\nu^{n-1} = \lambda$ and so $s_\lambda[X_{\Sigma r_i} - Y_{\Sigma c_j}] \neq 0$. This means that $\left(\sum c_i, \sum r_j\right)$ is not a cell in $R(\lambda)$. Choosing any combination of corners from $\mu^0, ..., \mu^{n-1}$ will give a similar result, proving the theorem. ∎

Thanks to the SXP rule, we have a close connection between plethysm and LR coefficients. By means of said rule, we now have a powerful tool to work with plethysm, which follows as a corollary of our previous theorem. This next result is a refined version of the positivity condition found in [37], and follows the spirit of similar kind of results which have been recently achieved [22].

Corollary 3.8. *If $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$ then $\lambda \subseteq \mu$.*

Proof. Let $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$. By the SXP rule, we have $\lambda \in \operatorname{supp}\left(s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}}\right)$. Choose an outer corner $(c, r) \in \mathtt{OC}_\mu$. Hence $s_\mu[X_r - Y_c] \neq 0$. Let

$$T : R(\mu) \longrightarrow \{1, 2, ..., r, -1, -2, ..., -c\}$$

be the canonical SSYT of shape μ for said corner.

Compute the n-quotient the usual way, thus having each $\mu^{(k)}$ "embedded" inside μ's diagram. Considering the corresponding values $T(i, j)$ of the canonical tableaux at those embedded cells, we obtain a SSYT T^k of shape $\mu^{(k)}$, which we presume to correspond to the alphabet $X_{r_k} - Y_{c_k}$. Then (c_k, r_k) is an outer corner of $\mu^{(k)}$.

Furthermore, we know that no two partitions of the n-quotient share any common letters, by construction of T and the n-quotient. Consequently, $X_r - Y_c = X_{\Sigma r_k} - Y_{\Sigma c_k}$.

We began by choosing $(c, r) \in \mathtt{OC}_\mu$ and we showed $(c, r) \in \sum \mathtt{OC}_{\mu^{(k)}}$, that is $\mathtt{OC}_\mu \subseteq \sum \mathtt{OC}_{\mu^{(k)}}$. By the previous theorem and the SXP rule, (c, r) is not a cell of λ. The lemma is thus proved:

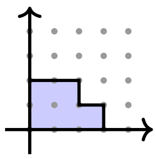
$$R(\lambda) \subseteq \Theta\left(\sum_k \mathtt{OC}_{\mu^{(k)}}\right) \subseteq \Theta\big(\mathtt{OC}_\mu\big) = R(\mu). \qquad ∎$$

Much like Lemma 3.8 offers a lower bound on the partitions $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$, we are able to find some upper bounds on that set. These kind of results prove to be useful when collecting computer data. Notice that a partition $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$ must be of size $n|\lambda|$. Hence, the maximum size of the rth row is $\left\lfloor \frac{n|\lambda|}{r} \right\rfloor$ (this falls from the very definition of a partition). Therefore,

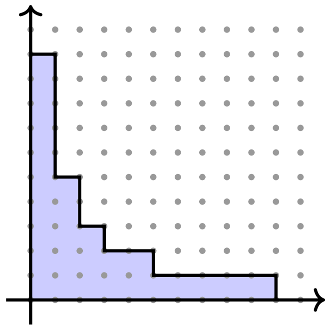
$$\mu \in \operatorname{supp}(p_n \circ s_\lambda) \quad \Rightarrow \quad \mu \subseteq \left(n|\lambda|, \left\lfloor \frac{n|\lambda|}{2} \right\rfloor, \left\lfloor \frac{n|\lambda|}{3} \right\rfloor, ...\right),$$

which is a partition, since only a finite number of terms can be non zero.

Example 3.9. Let $\lambda = \ydiagram{3,2}$, and let $n = 2$. Let $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$. Then, from 3.8, $\lambda \subseteq \mu$, and from our previous analysis, $\mu \subseteq (10, 5, 3, 2, 2, 1, 1, 1, 1, 1)$. In other words, the region of μ must satisfy the following inclusions:



$\subseteq R(\mu) \subseteq$



However, we can do better. In our example, we bounded μ_1 by 10, when the only partition of size 10 with the first row equal to 10 is the row partition (10), which clearly does not contain $\lambda = (3, 2)$. That is to say, we can optimize the upper bound by having the lower bound in consideration. More precisely, the bound on μ_r when $\lambda \subseteq \mu \vdash n|\lambda|$ becomes

$$\mu_r \leq \left\lfloor \frac{n|\lambda| - |(\lambda_{r+1}, \lambda_{r+2}, ...)|}{r} \right\rfloor =: a_r.$$

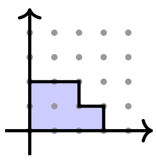
Similarly, we can bound the columns of μ,

$$\mu'_c \leq \left\lfloor \frac{n|\lambda| - |(\lambda'_{c+1}, \lambda'_{c+2}, ...)|}{c} \right\rfloor =: b_c.$$

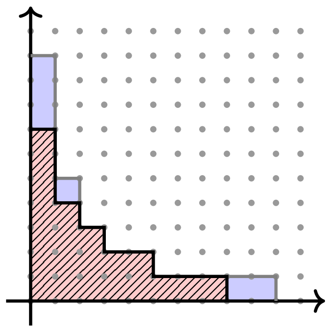
In general, the analysis on the columns will not yield the same bounding partition as the analysis on the rows, meaning that we can combine both bounds to get a more optimized one.

Lemma 3.10. *Let μ and λ be two partitions, and let $\Xi^1 = (a_1, a_2, ...)$ and $\Xi^2 = (b_1, b_2, ...)'$ with a_r and b_c defined as before. If $\mu \in \operatorname{supp}(p_n \circ s_\lambda)$ then $\mu \subseteq \Xi^1$ and $\mu \subseteq \Xi^2$. That is, $R(\mu) \subseteq R(\Xi^1) \cap R(\Xi^2)$.*

Example 3.11. Continuing our previous example, our optimization ensures that the region of μ actually lives inside the darker shaded region:



$\subseteq R(\mu) \subseteq$



4 Hook+column sequences

Following [15], we define a **hook+column** as a partition $(\alpha, 2^\beta, 1^\gamma)$ for some $\alpha, \beta, \gamma \in \mathbb{N}$. Equivalently, a partition that does not have a $(2,1)$ cell, and thus does not vanish when evaluated at $X_1 - Y_2$. Fixing the size N and γ, we can parametrize the family of hook+column partitions with one only index, β (the multiplicity of 2 in our partition). Let $\nu_\beta = (N - 2\beta - \gamma, 2^\beta, 1^\gamma)$.

Our goal is to study the coefficient of s_{ν_β} when expressing a function $s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}$ over the Schur basis. For a given γ, define

$$a_\beta = [\nu_\beta]\left(s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}\right),$$

where the size of ν_β is $N = \prod_i n_i$. We say that $\left(a_0, a_1, ..., a_{\frac{N-\gamma}{2}}\right)$ is the **hook+column sequence** of $(s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k})$ for our fixed γ. Refer to Table 3 for some examples. When $n = n_1 = n_2 = \cdots = n_k$, we introduce the notation $s_n^{\circ k} = s_n \circ s_n \circ \cdots \circ s_n$ (k times).

Function	γ	Hook+column sequence
$s_2 \circ s_4$	0	(1, 1, 0, 0)
$s_2 \circ s_3 \circ s_2$	0	(1, 2, 3, 3, 2, 1)
$s_2 \circ s_3 \circ s_2$	1	(0, 1, 1, 1, 0)
$s_2^{\circ 4} = s_2 \circ s_2 \circ s_2 \circ s_2$	2	(0, 1, 2, 2, 1, 0, 0)

Table 3: Hook+column sequences associated with some symmetric functions arising from plethysm and some γs.

Note 4.1. Define a **hook+row** as a partition $\nu' = (1^\alpha) + (\beta^2) + (\gamma)$, which is the transpose of the hook+column $\nu = (\alpha, 2^\beta, 1^\gamma)$. Equivalently, a partition that does not have a $(1,2)$ cell. Let a'_β be $[\nu'_\beta]\left(s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}\right)$ and define a hook+row sequences similarly to the previous case. There exists a relationship between hook+column sequences and hook+row sequences, via the ω involution [19]: for two homogeneous symmetric functions,

$$\begin{cases} \omega(f \circ g) = f \circ \omega(g) & \text{if degree}(g) \text{ is even,} \\ \omega(f \circ g) = \omega(f) \circ \omega(g) & \text{if degree}(g) \text{ is odd.} \end{cases}$$

Consequently, if $(s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k})|^\gamma_{\text{hook+col}} = \sum_\beta a_\beta s_{\nu_\beta}$, then applying ω to both sides gives

$$\begin{cases} \left(s_{n_1} \circ \cdots \circ s_{n_{k-1}} \circ s_{1^{n_k}}\right)|^\gamma_{\text{hook+row}} = \sum_\beta a_\beta s_{\nu'_\beta} & \text{if } n_k \text{ is even,} \\ (s_{1^{n_1}} \circ \cdots \circ s_{1^{n_{k-1}}} \circ s_{1^{n_k}})|^\gamma_{\text{hook+row}} = \sum_\beta a_\beta s_{\nu'_\beta} & \text{if } n_k \text{ is odd.} \end{cases}$$

That is, studying the hook+row sequences of the left–hand side functions is equivalent to studying the hook+column sequences of $s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}$. And so all our work could be translated to hook+row sequences.

In this section we will explore hook+column sequences. We start recalling a result of Langley and Remmel that explicitly describe for the restriction of $s_a \circ s_b$ to hook+columns. For a symmetric function $f = \sum_\lambda ([\lambda]f) \cdot s_\lambda$, let

$$(f)|_{\text{hook+col}} = \sum_{\lambda \text{ is hook+column}} ([\lambda]f) \cdot s_\lambda \quad \text{and} \quad (f)|^\gamma_{\text{hook+col}} = \sum_{\substack{\lambda \text{ is hook+column} \\ \text{and } m_1(\lambda) = \gamma}} ([\lambda]f) \cdot s_\lambda.$$

Theorem 4.2 (Langley and Remmel [15], Thm. 4.8). *For any $a, b \geq 2$, it is*

$$(s_a \circ s_b)|_{\text{hook+col}} = (s_a \circ s_b)|^{\gamma=0}_{\text{hook+col}} = \sum_{k<a} s_{(ab-2k,2^k)}.$$

Note 4.3. With our language in mind, we can rephrase this theorem. It establishes that the hook+column sequences of $s_a \circ s_b$ vanish for $\gamma \neq 0$ and is $(1, \overset{a \text{ times}}{\dots}, 1, 0, \dots)$ for $\gamma = 0$. The formula barely depends on b. In fact, all that b introduces is a trail of 0s in our hook+column sequence (see Table 4).

Function	γ	Hook+column sequence
$s_3 \circ s_2$	0	(1,1,1)
$s_3 \circ s_3$	0	(1,1,1,0)
$s_3 \circ s_4$	0	(1,1,1,0,0,0)
$s_3 \circ s_5$	0	(1,1,1,0,0,0,0)

Table 4: As b increases, the hook+column sequence of $s_3 \circ s_b$ for $\gamma = 0$ expresses a growing trail of 0s.

We will prove 4.2 after a handful of lemmas that will help avoid some confusing notation. Our first proof will use the tools developed originally in [26] and then used in [15], where this theorem is first proved. However, our exposition will not be identical, hopefully being more accessible for readers not as familiarized with the theory of symmetric functions.

Let $\Lambda(c)$ be the algebra of symmetric functions over $\mathbb{Z}[x_1, ..., x_c]$, and let $\Lambda_n(c)$ be the $\mathbb{Z}$-submodule of the (homogeneous) functions of degree n. Our first result roughly states that restricting our functions to hook+columns and evaluating on the alphabet $(1 - x - y)$ amount to the same result in practice.

This has already been proven in [15] for n-hook+columns and n-hook+rows. More generally, one can show this for (c, r)-hooks, which for us will be defined as those partitions not containing a (c, r) cell. With heavy use of Hopf algebra notation,

Lemma 4.4. *For each n, the family $\mathcal{F} = \{s_\lambda[X_r - Y_c] \ : \ \lambda \vdash n \text{ is a } (c,r)\text{-hook}\}$ is a basis of the $\mathbb{Z}$-module $\Lambda_n[X_r - Y_c] := \operatorname{Im}\Big((\mathbf{1} \otimes \bar{\omega})\tilde{\Delta}\Big)$.*

$$(\mathbf{1} \otimes \bar{\omega})\tilde{\Delta} \ : \ \Lambda_n \to \bigoplus_{i+j=n} \Lambda_i(c) \otimes \bar{\omega}\Lambda_j(r)$$
$$f \mapsto f[X_r - Y_c]$$

Proof. To begin with, $\mathcal{F}$ is generating because it is the image of a basis; a function $\sum_\lambda c_\lambda s_\lambda$ maps to $\sum_\lambda c_\lambda s_\lambda[X_r - Y_c]$ and, as seen in section 3, $s_\lambda[X_r - Y_c]$ vanishes if and only if λ is not a (c, r)-hook.

Furthermore, suppose by *reductio ad absurdum* there were some non-trivial dependence relations,

$$\sum_\lambda d_\lambda s_\lambda[X_r - Y_c] = 0.$$

Ignoring for now the coefficients d_λ, each polynomial $s_\lambda[X_r - Y_c]$ will have a smallest monomial with respect to the lexicographic order, letting $\cdots < -y_2 < -y_1 < x_1 < x_2 < \cdots$. One can check that smallest monomial will be x^T where T is the canonical tableau for the (c, r) corner as defined in section 3. This order on monomials gives, in turn, an order on the partitions λ.

Take the smallest λ such that d_λ is non-vanishing. Then the smallest monomial of $s_\lambda[X_r - Y_c]$ appears only once in the whole sum and with coefficient d_λ. In particular, it cannot be cancelled with any other monomial ($\rightarrow\leftarrow$). This shows that there is no non-trivial dependece relation. ∎

Corollary 4.5. *Let* $f \in \Lambda$. *Then,* $(f)|_{\text{hook+col}}[1 - x - y] = f[1 - x - y]$.

Furthermore, $\{s_\lambda[1 - x - y] \ : \ \lambda \vdash n$ *is hook+column*$\}$ *is a basis of* $\Lambda_n[1 - x - y]$.

Proof. The first assertion falls from section 3. Take $r = 1$, $c = 2$ in the previous theorem and evaluate on $x_1 = 1$, $y_1 = x$, $y_2 = y$ for the second assertion. ∎

Now we know that every function of Λ evaluated at $(1 - x - y)$ can be expressed as a sum of hook+columns. And there is in fact one only way of expressing that decomposition in terms of hook+columns, provided that we are working with an homogeneous function $f \in \Lambda_n$.

Lemma 4.6. *Let* $\lambda = (\alpha, 2^\beta, 1^\gamma)$ *be a hook+column with* $\alpha \geq 2$. *Then,*

$$s_\lambda[1 - x - y] = (-1)^\gamma (xy)^\beta (1 - x)(1 - y) \frac{x^{\gamma+1} - y^{\gamma+1}}{x - y}.$$

Proof. Recall what it meant to evaluate in an alphabet with positive and negative letters: in order to compute the left-hand side of the equation, we need to construct a SSYT of shape λ with three letters (fixing an order, let 1 be the letter for the variable 1, -1 for $-x$, and -2 for $-y$). We will then record the weight of every resulting tableau; their sum will yield the desired function.

Notice that we have very little freedom when filling a hook+column with only these three letters. Our only choices are in the last entries in the first two columns (see Figure 15).

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1} ? \\
\cline{1-1} \vdots \\
\cline{1-2} ? & ? \\
\cline{1-2} -2 & -1 \\
\cline{1-2} \vdots & \vdots \\
\cline{1-2} -2 & -1 \\
\hline -2 & -1 & 1 & \cdots & 1 \\
\hline
\end{array}$$

Figure 15: Our only choices when filling a hook+column with 1, -1 and -2 are represented by a question mark.

We can have 0, 1 or 2 entries equal to 1 in these cells. The rest can be filled with various quantities of -1s and -2s, resulting into weights

$$(xy - x - y + 1) \sum_{i+j=\gamma} (-1)^\gamma x^i y^j = (-1)^\gamma (1 - x)(1 - y) \frac{x^{\gamma+1} - y^{\gamma+1}}{x - y}.$$

As there are β instances of $\boxed{-2|-1}$, the desired expression arises. ∎

Lemma 4.7. *Let* $a \in \mathbb{N}$. *This equality holds:*

$$s_a[1 - x - y + xy] = \frac{1 - (xy)^a}{1 - xy}(1 - x)(1 - y).$$

Proof. We need to fill a row partition of size a with the four following entries: 1 and 2 of weight 1 and xy respectively, and $-1, -2$ with weights $-x, -y$.

-2	-1	1	$\cdots$	1	2	$\cdots$	2

The weights -2 and -1 can appear *at most* once. After those entries are fixed, we are left with a choice: we can fill the rest of the row in many ways.

- If neither -2 nor -1 appear, we can have up to a 2s. This results in tableaux of weights

$$1, xy, (xy)^2, ..., (xy)^a.$$

  Thus, if neither -2 nor -1 appear, we get the total weight

$$\sum_{i=0}^{a}(xy)^i = \frac{1-(xy)^{a+1}}{1-xy}.$$

- If -1 appears but -2 doesn't, we get a total weight of $-x\frac{1-(xy)^a}{1-xy}$.
- If -2 appears but -1 doesn't, we get a total weight of $-y\frac{1-(xy)^a}{1-xy}$.
- If both -1 and -2 appear, we get a total weight of $xy\frac{1-(xy)^{a-1}}{1-xy}$.

Adding up these four weights gives us the desired expression. ∎

We are now ready to prove the theorem.

Proof of 4.2.

$$\begin{aligned}
(s_a \circ s_b)|_{\text{hook+col}}[1-x-y] &\overset{4.5}{=} (s_a \circ s_b)[1-x-y] \\
&= s_a \circ \big(s_b[1-x-y]\big) \\
&\overset{4.6}{=} s_a\big[(1-x)(1-y)\big] \\
&\overset{4.7}{=} \frac{1-(xy)^a}{1-xy}(1-x)(1-y) \\
&= \sum_{k<a}(xy)^k(1-x)(1-y) \\
&\overset{4.6}{=} \sum_{k<a} s_{(ab-2k,2^k)}[1-x-y].
\end{aligned}$$

As hook+columns indexed Schur functions form a basis of $\Lambda_{ab}[1-x-y]$ from Lemma 4.5, this ends the proof. ∎

As an example, we are going to repeat the proof of the theorem but for $\gamma = 0$, $a = 2$. This will demonstrate our novel approach to the problem, which will lead to a similar formula for the plethysm $s_2 \circ s_a \circ s_b$ and will let us study the properties of $s_2^{\circ k} \circ s_a \circ s_b$ in general.

Recall from example 2.4 that $s_2 = \frac{1}{2}(p_2 + p_{1,1})$. From the properties of the plethysm in the power sum basis, we can write

$$2\,(s_2 \circ s_b)|^{\gamma=0}_{\text{hook+col}} = (p_2 \circ s_b)|^{\gamma=0}_{\text{hook+col}} + \big((s_b)^2\big)\big|^{\gamma=0}_{\text{hook+col}}$$

for any $b \geq 2$.

We will proceed in two steps. First, we will compute $p_2 \circ s_b$ using the SXP formula. We will then be left with a sum of products of Schur functions, which we will compute using the LR rule.

Notice that every resulting partition of $(p_2 \circ s_b)|_{\text{hook+col}}$ will be of size $N = 2b$, even. Let ν_β be the hook+column $(N - 2\beta, 2^\beta)$, as usual. By the SXP rule, Table 2 and the last remark of section 2.3,

$$[\nu_\beta]\Big(p_2 \circ s_b\Big) = \operatorname{sgn}_2(\nu_\beta) \cdot [s_b]\Big(s_{\nu_\beta^{(0)}} \cdot s_{\nu_\beta^{(1)}}\Big) = [s_b]\Big(s_{(1^{\lceil \frac{\beta}{2} \rceil})} \cdot s_{(b-\beta, 1^{\lfloor \frac{\beta}{2} \rfloor})}\Big).$$

This formula, in particular, imposes that both $\nu_\beta^{(0)}$ and $\nu_\beta^{(1)}$ must be subsets of (b). But this is only true if $\lfloor \frac{\beta}{2} \rfloor = 0$, so we can discard every term with β strictly greater than 1. For $\beta = 0$ (resp. 1), the skew partition $b/\nu_\beta^{(1)}$ will be $\emptyset$ (resp. $\square$). And so there is exactly one tableau with shape $\nu_\beta^{(0)} = \emptyset$ (resp. $\square$) and verifying the LR conditions. So far, we have

$$2\ (s_2 \circ s_b)|_{\text{hook+col}}^{\gamma=0} = s_{\nu_0} + s_{\nu_1} + \big((s_b)^2\big)\big|_{\text{hook+col}}^{\gamma=0}. \tag{1}$$

Take the square, $\big((s_b)^2\big)\big|_{\text{hook+col}}^{\gamma=0} = \sum_\beta [s_{\nu_\beta}]\big((s_b)^2\big)$. Again, every term will vanish, unless (b) is a subset of ν_β. That is, unless $b \leq 2b - 2\beta$. The coefficient $[s_{\nu_\beta}]\big((s_b)^2\big)$ will be the number of skew tableaux of shape ν_β/b and whose words are lattice permutation of type (b). Such a tableau will not exist if any cell of ν_n/b lives orthogonally below another cell, letting us discard the case $\beta \geq 2$. In any other case, there is exactly one possibility. We conclude $\big((s_b)^2\big)\big|_{\text{hook+col}}^{\gamma=0} = s_{\nu_0} + s_{\nu_1}$.

Combining this with equation (1), we have once again shown

$$(s_2 \circ s_b)|_{\text{hook+col}}^{\gamma=0} = s_{\nu_0} + s_{\nu_1},$$

which is what we get letting $a = 2$ in Theorem 4.2.

In section 4.2, we aim to generalize the theorem of Langley and Remmel to the family of iterated plethysms $s_2 \circ s_a \circ s_b$ (see 4.21). An example that beautifully illustrates our result is the following.

Example 4.8. The hook+column sequence of $s_2 \circ s_4 \circ s_2$ for $\gamma = 0$ is $(1, 2, 3, 4, 4, 3, 2, 1)$.

$$(s_2 \circ s_4 \circ s_2)|_{\text{hook+col}}^{\gamma=0} = 1s_{(16)} + 2s_{(14,2)} + 3s_{(12,2^2)} + 4s_{(10,2^3)} + 4s_{(8,2^4)} + 3s_{(6,2^5)} + 2s_{(4,2^6)} + 1s_{(2^8)}.$$

To simplify our computations, one would hope for this next equation to hold for any γ:

$$(s_2 \circ s_a \circ s_b)|_{\text{hook+col}}^{\gamma} = \big(s_2 \circ (s_a \circ s_b)|_{\text{hook+col}}^{\gamma}\big)\big|_{\text{hook+col}}^{\gamma}.$$

Unfortunately, that is not the case, as shown by this next example.

Example 4.9. $(s_2^{\circ 4})|_{\text{hook+col}}^{\gamma=0}$ is not equal to $\Big(s_2 \circ (s_2^{\circ 3})|_{\text{hook+col}}^{\gamma=0}\Big)\Big|_{\text{hook+col}}^{\gamma=0}$:

- $(s_2^{\circ 4})|_{\text{hook+col}}^{\gamma=0} = s_{\nu_0} + 3s_{\nu_1} + 8s_{\nu_2} + 13s_{\nu_3} + 13s_{\nu_4} + 8s_{\nu_5} + 3s_{\nu_6} + s_{\nu_7}$.
- $\Big(s_2 \circ (s_2^{\circ 3})|_{\text{hook+col}}^{\gamma=0}\Big)\Big|_{\text{hook+col}}^{\gamma=0} = s_{\nu_0} + 3s_{\nu_1} + 7s_{\nu_2} + 10s_{\nu_3} + 10s_{\nu_4} + 7s_{\nu_5} + 3s_{\nu_6} + s_{\nu_7}$.

However, we can relax the conditions to

$$(s_2 \circ s_a \circ s_b)|_{\text{hook+col}} = \big(s_2 \circ (s_a \circ s_b)|_{\text{hook+col}}\big)\big|_{\text{hook+col}}.$$

This equality does hold, as does this lemma:

Lemma 4.10 (Langley and Remmel [15], Thm. 4.2). *For any symmetric function f, we have* $(s_2 \circ f)|_{\text{hook+col}} = \big(s_2 \circ (f)|_{\text{hook+col}}\big)\big|_{\text{hook+col}}$.

Proof. Decomposing s_2 as before, we will have to look at $(p_2 \circ f)|_{\text{hook+col}}$ and $\big(f^2\big)\big|_{\text{hook+col}}$ separately.

For the first part, write $f = \sum a_\lambda s_\lambda$, and thus $(p_2 \circ f)|_{\text{hook+col}} = \sum a_\lambda \cdot (p_2 \circ s_\lambda)|_{\text{hook+col}}$. But then, by Corollary 3.8, any λ that doesn't disappear in that sum must be contained in a hook+column — hence it must be a hook+column itself —, so

$$(p_2 \circ f)|_{\text{hook+col}} = \sum_{\lambda \text{ is hook+col}} a_\lambda \cdot (p_2 \circ s_\lambda)|_{\text{hook+col}} = \big(p_2 \circ (f)|_{\text{hook+col}}\big)\big|_{\text{hook+col}}.$$

Let us now take a look at $\big(f^2\big)\big|_{\text{hook+col}}$. We ask whether $\Big(\big((f)|_{\text{hook+col}}\big)^2\Big)\Big|_{\text{hook+col}} = \big(f^2\big)\big|_{\text{hook+col}}$. This boils down to asking what kind of partitions μ, λ verify $[\nu_\beta](s_\mu \cdot s_\lambda) \neq 0$ for some β. The first statement of the LR rule gives us that λ and μ must be subsets of ν_β. In particular, they are also hook+columns. ■

Note 4.11. Here the lemma is stated for our particular problem, but one could change "hook+column" for "n-hook+column" or "n-hook+row" partitions and it would remain true. In [15], it is also proved in more generality, by considering $s_\mu \circ f$.

With this in mind, as $2s_2 = (p_2 + p_{1,1})$, all we need in order to understand $(s_2 \circ f)|^\gamma_{\text{hook+col}}$ is to describe $(p_2 \circ f)|^\gamma_{\text{hook+col}}$ and $(p_{1,1} \circ f)|^\gamma_{\text{hook+col}}$. Indeed, from the well–behaving properties of plethysm over the power sum basis, we get $2 \cdot s_2 \circ f = p_2 \circ f + p_{1,1} \circ f$ for every symmetric function f.

4.1 The expressions $p_2 \circ f$ and $p_{1,1} \circ f = f^2$

Recall that $p_{1,1} \circ f = (p_1 \circ f)(p_1 \circ f) = f^2$. We explore the expressions $(p_2 \circ f)|_{\text{hook+col}}$ and $(p_{1,1} \circ f)|_{\text{hook+col}} = \big(f^2\big)\big|_{\text{hook+col}}$. More precisely, if $f = \sum_\lambda ([\lambda]f) \cdot s_\lambda$, then

$$(p_2 \circ f)|_{\text{hook+col}} = \sum_\lambda ([\lambda]f) \cdot (p_2 \circ s_\lambda)|_{\text{hook+col}}.$$

From Lemma 4.10, this simplifies to

$$(p_2 \circ f)|_{\text{hook+col}} = \sum_{\lambda \text{ is hook+col}} ([\lambda]f) \cdot (p_2 \circ s_\lambda)|_{\text{hook+col}}.$$

Thereby, we only need to compute $p_2 \circ f$ when λ is a hook+column. Similarly,

$$(p_{1,1} \circ f)|_{\text{hook+col}} = \left(\sum_{\lambda \text{ is hook+col}} ([\lambda]f) \cdot s_\lambda\right)^2,$$

and so we only need to compute $s_\lambda \cdot s_\mu$ when both partitions are hook+column.

Let $\nu = (\alpha, 2^\beta, 1^\gamma)$. We want to begin decomposing $(p_2 \circ s_\lambda)|_{\text{hook+col}}$, that is, we are asking who is $[\nu](p_2 \circ s_\lambda)$. Thanks to the SXP rule, we may equivalently ask, up to sign, who is $[\lambda](s_{\nu^{(0)}} \cdot s_{\nu^{(1)}})$. Much of the work was done in section 2.3.

Looking at Table 1, we can now merge the four cases into two:

1. If $\alpha \equiv 0$ modulo 2, we have to multiply a column by a hook. This is particularly simple[9]. We leave it to the reader to check that if $\alpha \equiv 0$ (2), then

$$\mathrm{supp}(s_{\nu^{(0)}} \cdot s_{\nu^{(1)}}) = \left\{\lambda \ : \ \substack{2\lambda_1=\alpha \\ m_2(\lambda)\leq\beta/2}\right\} \sqcup \left\{\lambda \ : \ \substack{2\lambda_1=\alpha+2 \\ m_2(\lambda)\leq\frac{\beta-1}{2}}\right\}.$$

(We omitted the condition $2|\lambda| = |\nu|$ for clarity's sake.)

2. If $\alpha \equiv 1$ modulo 2, we have to discuss the parity of β. From section 2.7, if β is odd then the 2-core of ν is not empty and the SXP rule states that $[\lambda](s_{\nu^{(0)}} \cdot s_{\nu^{(1)}}) = 0$. If β is even, then $s_{\nu^{(1)}} = 1$ and so $\mathrm{supp}(s_{\nu^{(0)}} \cdot s_{\nu^{(1)}}) = \left\{\left(\frac{\alpha+1}{2}, 2^{\frac{\beta}{2}}, 1^{\frac{\gamma-1}{2}}\right)\right\}$.

This analysis returns λ for a given ν. But we are also interested in knowing ν for a given λ. Working our way backwards, (and checking Table 2 for the 2-sign function), we get

Lemma 4.12. *If $\lambda = (\lambda_1, 2^{m_2(\lambda)}, 1^{m_1(\lambda)})$ is a hook+column, then*

$$\begin{aligned}(p_2 \circ s_\lambda)|_{\text{hook+col}} \quad &= \quad (-1)^{m_1(\lambda)} \cdot \left(\sum_{\beta\geq 2m_2(\lambda)} (-1)^\beta s_{(2\lambda_1, 2^\beta, 1^\bullet)} \right. \\ - \quad s_{(2\lambda_1-1, 2^{2m_2(\lambda)}, 1^\bullet)} \quad &+ \quad \left. \sum_{\lceil\frac{\beta}{2}\rceil\geq m_2(\lambda)+1} (-1)^{\beta+1} s_{(2\lambda_1-2, 2^\beta, 1^\bullet)} \right).\end{aligned}$$

Note 4.13. For readability reasons, we omitted the multiplicity of the 1s in the previous formula. It can be recovered from the equation $2|\lambda| = |\nu|$. Also, it may seem as if the sign of each term depends on λ and β. Nonetheless, recall from Table 2 it only depends on the size of the 1's part in each term of the sum.

Example 4.14. Let's look at an example:

$$\left(p_2 \circ s_{(4,2^3,1)}\right)|_{\text{hook+col}} = s_{(8,2^7)} - s_{(8,2^6,1^2)} + s_{(7,2^6,1^3)} + s_{(6,2^8)} - s_{(6,2^7,1^2)}.$$

There is a better combinatorial interpretation of the aforementioned coefficients. In their work, Carré and Leclerc [5] express LR coefficients as the cardinalities of certain sets of "Yamanouchi domino tableaux". More precisely, they show that

$$p_2 \circ s_\lambda = \sum_\nu \mathrm{sgn}_2(\nu) c^\lambda_{\nu^{(0)}\nu^{(1)}} s_\nu = \sum_\nu \mathrm{sgn}_2(\nu) \#\mathtt{Yam}_2(\nu, \lambda) s_\nu.$$

Here, we list some of the definitions needed in order to understand this expression.

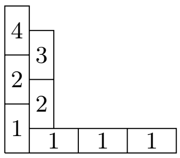
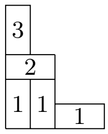
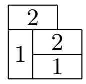
Definition 4.15. A **domino** is a 2×1 or 1×2 subdiagram of a Young diagram. A **domino diagram** of shape ν is a Young diagram of shape ν which is tiled with dominos. Consequently, a **domino tableau** is a map $T : \{\text{dominos of } \nu\} \to \mathbb{N}$ which we represent the usual way. Such a tableau is **semi standard** (SSDT) if the represented numbers are strictly increasing along each column and weakly increasing along each row of the original Young diagram. Below, examples of what is and what isn't a semi standard domino tableau. Finally, a SSDT is **Yamanouchi** if the reverse reading *column* word is lattice permutation *on the right.* The **weight** of the tableau T is defined as usual, $(m_1(T), m_2(T), \ldots)$.

The set $\mathtt{Yam}_2(\nu, \lambda)$ is the set of Yamanouchi SSDT of shape ν and weight λ.

[9] It is a special case of the LR rule known as Pieri's formula. See [19].

Note 4.16. If ν admits a domino tiling, then the 2-core of ν is empty. Also, the sign function $\mathrm{sgn}_2(\nu)$ is now redefined as $(-1)^{\#\text{ vertical dominos}}$. The reader may verify these assertions.

Example 4.17. As an illustration of the above definitions, let

$T^0 =$  ; $T^1 =$  ; $T^2 =$  .

- T^0 is a Yamanouchi SSDT of shape $\nu^0 = (7, 2^4, 1)$, reverse reading column word 42132111 and weight $\lambda^0 = (4, 2, 1, 1)$. It is an element of $\mathtt{Yam}_2(\nu^0, \lambda^0)$.
- T^1 is a Yamanouchi SSDT of shape $\nu^1 = (4, 2^2, 1^2)$, reverse reading column word 32111 and weight $\lambda^1 = (3, 1, 1)$. It is an element of $\mathtt{Yam}_2(\nu^1, \lambda^1)$.
- T^2 is not a SSDT, as the second column is not strictly increasing.

Given hook+columns ν and λ, we are very restricted when constructing an element of $\mathtt{Yam}_2(\nu, \lambda)$. If $m_1(\nu)$ is odd, there will only be one domino tiling of ν. Moreover, there is only one way of filling that tableau in a Yamanouchi way. (See T^0 in our previous example.) If $m_1(\nu)$ is even, there will be, in general, multiple ways of tiling ν; but a given domino diagram can only be filled with weight λ in one way. (See T^1 in our previous example.)

Now, one can derive the formula for $(p_2 \circ s_\lambda)|_{\text{hook+col}}$, and check that we retrieve our previous computation.

Regarding the expression $(p_{1,1} \circ f)|_{\text{hook+col}}$, it remains to compute $(s_\lambda \cdot s_\mu)|_{\text{hook+col}}$ when both λ and μ are hook+column partitions.

For this matter, we will use the Remmel and Whitney product rule [24], instead of the LR rule.

Theorem 4.18 (Remmel and Whitney, [24]). *Given the partitions μ and ν, it's*

$$s_\mu \cdot s_\nu = \sum_{T \in \mathcal{O}(\mu * \nu)} s_{\rho(T)},$$

where $\mathcal{O}(\mu * \nu)$ is the set of skew tableaux of shape

$$\mu * \nu = (\nu_1 + \mu_1, \nu_1 + \mu_2, \dots, \nu_1 + \mu_{l(\mu)}, \nu_1, \nu_2, \dots, \nu_{l(\nu)}) \Big/ (\nu_1, \overset{l(\mu)\ \text{times}}{\dots}, \nu_1)$$

satisfying the two following conditions:

1. *If in the reverse lexicographic filling L of $\mu * \nu$ we have $\boxed{y}\boxed{x}$ then in T we have y below and strictly to the right of x*
2. *If in the reverse lexicographic filling L of $\mu * \nu$ we have $\begin{array}{|c|}\hline y \\ \hline x \\ \hline \end{array}$ then in T we have y to the left and strictly to above of x*

Then we can directly compute:

Lemma 4.19. *Let $\mu = (\mu_1, 2^{m_2(\mu)}, 1^{m_1(\mu)})$ and $\nu = (\nu_1, 2^{m_2(\nu)}, 1^{m_1(\nu)})$ be hook+columns. Let*

$$\begin{cases} \alpha := \nu_1 + \mu_1, \\ m_2 := m_2(\mu) + m_2(\nu), \\ m_1 := \min(m_1(\mu), m_1(\nu)). \end{cases}$$

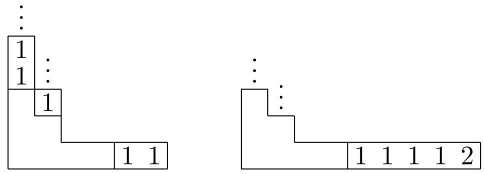
Then,

$$(s_\mu \cdot s_\nu)|_{\text{hook+col}} = \sum_{\beta=m_2}^{m_2+m_1} s_{(\alpha,2^\beta,1^\bullet)} +$$

$$+ \sum_{\beta=m_2}^{m_2+m_1+1} \chi^\beta_{(\mu,\nu)} \cdot s_{(\alpha-1,2^\beta,1^\bullet)} + \sum_{\beta=m_2+1}^{m_2+m_1+1} s_{(\alpha-2,2^\beta,1^\bullet)},$$

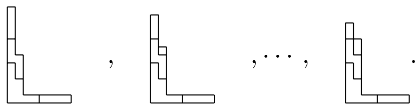
where $\chi^\beta_{(\mu,\nu)} = 1$ *if* $\beta = m_2$ *or* $\beta = m_2 + m_1 + 1$ *and* $\chi^\beta_{(\mu,\nu)} = 2$ *otherwise.*

Proof. The reader may convince him/herself just by following an example. For the sake of rigurosity, let $\lambda \in \mathrm{supp}(s_\nu \cdot s_\mu)$. Then, if $\lambda_1 < \alpha - 2$, consider λ/μ and try to fill it with weight ν and such that the reverse reading word is lattice permutation (following the LR rule). This is clearly not possible. Similarly, $\lambda_1 \not> \alpha$.



Now, case by case, the Remmel and Whitney rule is very restrictive. It is not difficult to see that:

- if $\lambda_1 = \alpha$, all of the following λ shapes appear once in the support:



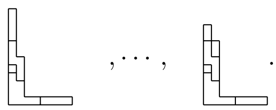
And so have the subset

$$\left\{ \left(\alpha, 2^\beta, 1^\bullet\right) \ : \ m_2 \le \beta \le m_2 + m_1 \right\} \subseteq \mathrm{supp}(s_\mu s_\nu).$$

Here, we omitted the 1's part, which can be computed by letting $|\lambda| = |\mu| + |\nu|$.

- if $\nu_1 = \alpha - 1$, then the possible shapes are:

and also



This yields the subset

$$\left\{ \left(\alpha - 1, 2^\beta, 1^\bullet\right) \ : \ m_2 \le \beta \le m_2 + m_1 + 1 \right\}.$$

Note that every partition appears twice in $\mathrm{supp}(s_\mu s_\nu)$ except when one of the two equalities hold.

- finally, if $\nu_1 = \alpha - 2$, we get, analogous to the first case:

The associated subset is

$$\left\{ \left(\alpha - 2, 2^\beta, 1^\bullet\right) \ : \ m_2 + 1 \leq \beta \leq m_2 + m_1 + 1 \right\}. \qquad \blacksquare$$

We conclude with a beautiful result. Take $\mu = \nu$ and compare Lemmas 4.19 and 4.12. In particular, note that

$$\text{supp}\left((p_2 \circ s_\lambda)|_{\text{hook+col}} \right) \subseteq \text{supp}\left(\left(s_\mu^2\right)|_{\text{hook+col}} \right).$$

Furthermore, if $m_1(\mu) = 0$, then the two sets are identical, and so are the multiplicities associated which each partition.

In fact, there is one more thing to notice; every hook+column $\nu = (\alpha, 2^\beta, 1^\gamma)$ such that $m_1(\nu) = \gamma = 0$ has a positive 2-sign in the SXP rule. Consequently, we always get the following result:

Lemma 4.20. *Let $\mu = (\alpha, 2^\beta)$. Then, $(p_2 \circ s_\mu)|_{\text{hook+col}}^{\gamma=0} = (p_{1,1} \circ s_\mu)|_{\text{hook+col}}^{\gamma=0}$.*

4.2 An explicit formula for $s_2 \circ s_a \circ s_b$ on hook+columns

Now that we have explicit formulas for computing the plethystic action of p_2 and $p_{1,1}$ on functions $f = (f)|_{\text{hook+col}}$, we can bring back 4.2 and compute the next iteration, namely, from

$$(s_a \circ s_2)|_{\text{hook+col}} = (s_a \circ s_2)|_{\text{hook+col}}^{\gamma=0} = \sum_{k<a} s_{(2a-2k, 2^k)},$$

we want to look at the following problem: $(s_2 \circ s_a \circ s_b)|_{\text{hook+col}}^{\gamma=0}$. Note 4.3 guarantees that the analysis will be identical for any $b \geq 2$, so we suppose $b = 2$ without loss of generality. Decomposing $2 \cdot s_2 = p_2 + p_{1,1}$ as usual, we get

$$\begin{aligned} 2 \cdot (s_2 \circ s_a \circ s_2)|_{\text{hook+col}}^{\gamma=0} &= \sum_{k<a} \left(p_2 \circ s_{(2a-2k, 2^k)}\right)|_{\text{hook+col}}^{\gamma=0} \\ &+ \left(\left(\sum_{k<a} s_{(2a-2k, 2^k)} \right)^2 \right) \Bigg|_{\text{hook+col}}^{\gamma=0}. \end{aligned} \tag{2}$$

Squaring the second summation gives

$$\sum_{k<a} s^2_{(2a-2k, 2^k)} + 2 \sum_{i<j<a} s_{(2a-2i, 2^i)} \cdot s_{(2a-2j, 2^j)}.$$

Lemma 4.20 now allows for a simplification of our original equation,

$$(s_2 \circ s_a \circ s_2)|_{\text{hook+col}}^{\gamma=0} = \left(\sum_{i \leq j < a} s_{(2a-2i, 2^i)} \cdot s_{(2a-2j, 2^j)} \right) \Bigg|_{\text{hook+col}}^{\gamma=0}.$$

We already know how to explicitly compute these products from Lemma 4.19. For each term in the second sum, we get the resulting hook+column partitions λ with $m_1(\lambda) = 0$ and whose first row verifies

$$\lambda_1 \in \{4a - 2(i + j), \ 4a - 2(i + j + 1)\}.$$

Consequently, the hook+column partitions that will appear in the sum are those whose first row is in the set $\{2, 4, 6, ..., 4a\}$. The question now is how many times does each one appear. Let us start by giving the answer:

Lemma 4.21. *For any* $a \geq 2$*,*

$$(s_2 \circ s_a \circ s_2)|^{\gamma=0}_{\text{hook+col}} = \sum_{k=1}^{2a} \min\{k, 2a-k+1\} \cdot s_{(2k, 2^{2a-k})}.$$

Equivalently, the hook+column sequence of $(s_2 \circ s_a \circ s_2)$ *for* $\gamma = 0$ *is*

$$(1, 2, ..., a-1, a, a, a-1, ..., 2, 1).$$

Example 4.22. This is a really nice result. Recall example 4.8,

$$(s_2 \circ s_4 \circ s_2)|^{\gamma=0}_{\text{hook+col}} = 1s_{(16)} + 2s_{(14,2)} + 3s_{(12,2^2)} + 4s_{(10,2^3)} + 4s_{(8,2^4)} + 3s_{(6,2^5)} + 2s_{(4,2^6)} + 1s_{(2,2^7)}.$$

Proof. Take $a \in \mathbb{N}$ and $k \in \{1, 2, ..., 2a\}$. We ask how many integer pairs (i, j) are there in the polytope $\Delta := \{0 \leq i \leq j < a\}$, which are solutions to *either* of these two equations:

$$\begin{cases} 4a - 2(i+j) = 2k, \\ 4a - 2(i+j+1) = 2k, \end{cases} \quad \text{or, equivalently,} \quad \begin{cases} i + j = 2a - k, \\ i + j = 2a - k - 1. \end{cases} \tag{3}$$

Refer to Figure 16 for a graphical representation.

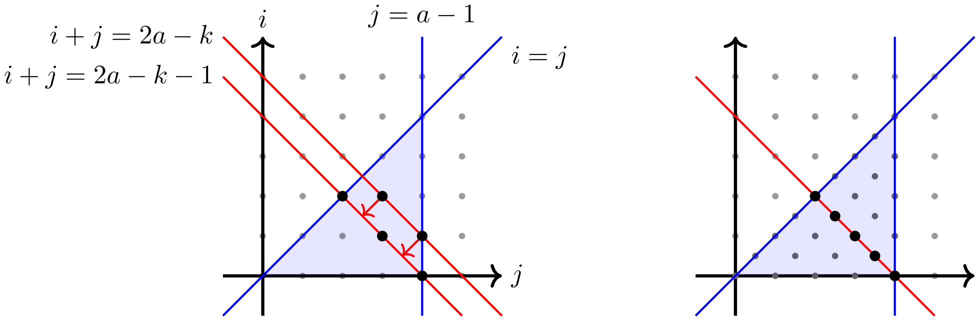


Figure 16: We let $a = 5$ and $k = 5$. The polytope Δ is shaded in blue. Each black dot represents a valid pair. On the right, we illustrate the result of the described projection.

To help count the black dots, we'll project them orthogonally from one of the lines to the other one, in such a way that all the black dots remain inside Δ. More precisely, if $k \geq a$ then project onto $\{i + j = 2a - k\}$ and vice versa.

One can easily see now what the coefficients are going to look like. Noting that the biggest line is counted twice (because we change the projection mid way), results in the desired integer sequence. ∎

If we now want to know which hook+column partitions ν with $m_1(\nu) \neq 0$ subsets appear in $(s_2 \circ s_a \circ s_2)$, we have to go back to our previous equation (2).

A close inspection of Lemma 4.19 for the square of a hook+column λ verifying $m_1(\lambda) = 0$, reveals that only one such ν emerges, namely $(2\lambda_1 - 1, 2^{2m_2(\lambda)}, 1^\bullet)$. At the same time, this is the only ν emerging from $p_2 \circ s_\lambda$, and with a minus sign, cancelling itself in our previous equation.

$$\left(p_2 \circ s_\lambda + (s_\lambda)^2\right)\big|_{\text{hook+col}} = (\cdots)|^{\gamma=0}_{\text{hook+col}} + s_{(2\lambda_1-1, 2^{2m_2(\lambda)}, 1^\bullet)} - s_{(2\lambda_1-1, 2^{2m_2(\lambda)}, 1^\bullet)} = (\cdots)|^{\gamma=0}_{\text{hook+col}}.$$

Thereby, we get the following equality

$$(s_2 \circ s_a \circ s_2)|_{\text{hook+col}}^{\gamma\neq 0} = \left(\sum_{i<j<a} s_{(2a-2i,2^i)} \cdot s_{(2a-2j,2^j)} \right) \Bigg|_{\text{hook+col}}^{\gamma\neq 0}.$$

Once again, for each of those products, only one such partition emerges, which is $(4a-2i-2j-1, 2^{i+j}, 1)$. To discuss the multiplicity, we ask, for a given $k \in \{1, 2, ..., 2a\}$, how many pairs $0 \leq i < j < a$ are there such that $4a-2i-2j-1 = 2k-1$ or, equivalently, $i+j = 2a-k$. It's a similar question to the previous one (maybe just *half* the question), and the answer is $\min\left\{\left\lfloor \frac{k-2}{2} \right\rfloor, \left\lfloor \frac{2a-k}{2} \right\rfloor\right\}$. All things considered, we arrive at our final expression:

Theorem 4.23. *For any $a, b \geq 2$,*

$$(s_2 \circ s_a \circ s_b)|_{\text{hook+col}} = \sum_{k=1}^{2a} \min\{k, 2a-k+1\} \cdot s_{\left(2(a(b-2)+k),\ 2^{2a-k}\right)}$$

$$+ \sum_{k=1}^{2a} \min\left\{\left\lfloor \frac{k-2}{2} \right\rfloor, \left\lfloor \frac{2a-k}{2} \right\rfloor\right\} \cdot s_{\left(2(a(b-2)+k)-1,\ 2^{2a-k},\ 1\right)}.$$

We found another proof of 4.23 that is identical in nature, and albeit it offers less possibilities for future generalizations, its beauty deserves an honorable mention in this work. We are again letting $b = 2$ and asking, for $a \in \mathbb{N}$ and $k \in \{1, 2, ..., 2a\}$, how many (i, j) are there in $\Delta = \{0 \leq i \leq j < a\}$ which are solutions to either of the equations of (3). By a change of variables $I = a - i$, $J = a - j$, we are equivalently looking for pairs $(I, J) \in \{1 \leq I \leq J \leq a\}$ which are solutions to either of these equations:

$$\begin{cases} I + J = k, \\ I + J = k + 1. \end{cases}$$

Start by placing the points $1, 2, ..., a$ in a line. Now join any two of those points I and J with a semicircle over the line if they satisfy the first equation. Similarly, join them with a semicircle under the line if they satisfy the second equation. Refer to Figure 17 for an example.

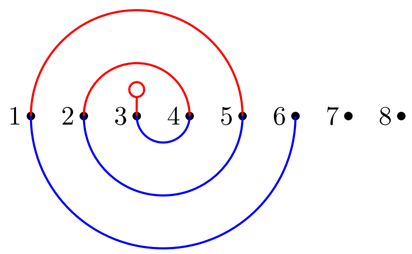


Figure 17: Let $a = 8$, $k = 6$. There are $\left\lfloor \frac{k}{2} \right\rfloor = 3$ red semicircles over the line and $\left\lfloor \frac{k+1}{2} \right\rfloor = 3$ blue semicircles under the line. In total, $k = 6$ semicircles.

Now count the number of total semicircles. We can now see that the resulting sequence will be symmetric in k, as the two conditions $1 \leq I$ and $J \leq a$ have a similar effect on the diagram. Letting k be sufficiently small, we can remove the condition $J \leq a$ and do the counting. There will be $\left\lfloor \frac{k}{2} \right\rfloor$ semicircles over the line and $\left\lfloor \frac{k+1}{2} \right\rfloor$ under the line. Summing up these two values gives k, and letting k vary yields the sequence $(1, 2, ..., a, a, ..., 2, 1)$.

A similar argument can be made for the second part of the proof, with semicircles only over the line and without letting (I, I) be a valid semicircle.

4.3 A symmetry result

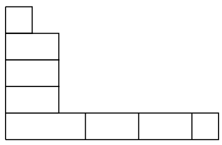
As we have seen in the first iterations, the hook+column sequences of $s_2^{\circ k}$ (and other similar functions) appear to have symmetric behaviours. In particular, every previous example show that the last non-vanishing coefficient coincides with the first non-vanishing coefficient; the second-to-last non-vanishing coefficient with the second one, and so on.

$$(s_2 \circ s_4 \circ s_2)|_{\text{hook+col}}^{\gamma=0} = 1s_{(16)} + 2s_{(14,2)} + 3s_{(12,2^2)} + 4s_{(10,2^3)} + 4s_{(8,2^4)} + 3s_{(6,2^5)} + 2s_{(4,2^6)} + 1s_{(2^8)}.$$

In fact, this holds with more generality. In order to prove this phenomenon, we will need to define some new concepts to precisely state what are we going to show.

Definition 4.24. A **brick** is a horizontal $1 \times m$ subdiagram of a Young diagram. A **brick diagram** of shape λ a Young diagram of λ which is tiled with bricks. The partition arising from the ordering of the brick sizes is called the type.

In particular, every Young diagram is a brick diagram of type $(1, ..., 1)$. Note, however, that any domino diagram is not a brick diagram of type $(2, ..., 2)$, as we would need some "bricks" to be vertically placed. A more general example of a brick diagram is given in Figure 18.



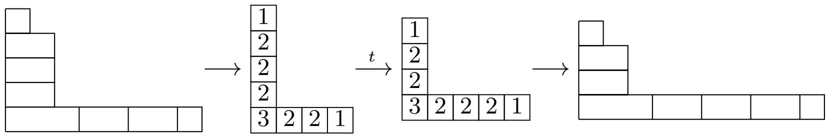
$$; \quad \begin{cases} \text{Shape } \lambda = (8, 2^3, 1), \\ \text{Type } \mu = (3, 2^5, 1^2). \end{cases}$$

Figure 18: A brick diagram of a hook+column partition.

Definition 4.25 (r-flip). Let $\lambda = (\lambda_1, 2^\beta, 1^\gamma)$ be a hook+column partition. Let $r < \lambda_1 - \gamma$ be a fixed natural number, such that $\lambda_1 - \gamma - r$ is even. We define λ^r (sometimes $r(\lambda)$) as the resulting partition of the following process:

1. Start with the Young diagram for λ.
2. Divide the diagram into bricks such that:
   - row one is made of one brick of size r, followed by some bricks of size 2 and exactly γ bricks of size 1.
   - every other row is made of one brick.
3. Collapse every brick into a single cell with a coefficient denoting the size of the brick. We should now have a (not necessarily semi standard) tableau T.
4. Transpose T.
5. Expand every cell into a brick with size the coefficient of the cell.
6. The shape of our resulting brick diagram is λ^r.

Example 4.26. To illustrate this process, let $\lambda = (8, 2^3, 1)$ and let $r = 3$.



Then λ^r is $(10, 2^2, 1)$.

Definition 4.27 (r-symmetry). Let $f \in \Lambda_n$ be an homogeneous symmetric function of degree n, expressed as $\sum d_\lambda s_\lambda$ over the Schur's basis. We say that f is r-symmetric for a given r if $d_\lambda = d_{\lambda^r}$ for every hook+column partition λ.

In a more general language, we defined f as r-symmetric if the Schur-polynomials $(f)|^{\gamma}_{\text{hook+col}}$ are symmetric every fixed γ. We are now ready to state our theorem:

Theorem 4.28. *Let $f \in \Lambda_n$ be r-symmetric for a fixed r, and such that $[\mu]f$ is non-negative for every hook+column partition μ. Then, $s_2 \circ f \in \Lambda_{2n}$ is R-symmetric for $R = 2r - 2$, and every resulting coefficient is non-negative.*

Once this theorem is proven, we would have shown that $s_2^{\circ k} \circ s_a \circ s_b$ is r-symmetric for every k, a, b for some r. Indeed, $s_a \circ s_b$ is trivially r-symmetric for every a, b from Theorem 4.2, with $r = ab - 2(a-1)$, and the previous theorem tells us that s_2 preserves this symmetry.

But before giving the proof, let us discuss some notation. As always, express s_2 as $\frac{1}{2}(p_{1,1} + p_2)$. From Lemma 4.12, we can derive an explicit formula for $[\lambda](s_2 \circ f)$. If λ_1 is odd, then

$$[\lambda](p_2 \circ f) = \operatorname{sgn}_2(\lambda) \cdot \left[\left(r + \delta + \frac{\gamma - 1}{2},\ 2^{\frac{\beta}{2}},\ 1^{\frac{\gamma-1}{2}}\right)\right] f,$$

whereas if λ_1 is even, then

$$[\lambda](p_2 \circ f) = \sum_{m_2 \le \frac{\beta}{2}} \operatorname{sgn}_2(\lambda) \left[\left(r + \delta + \frac{\gamma}{2} - 1,\ 2^{m_2},\ 1^{\beta + \frac{\gamma}{2} - 2m_2}\right)\right] f \quad +$$

$$+ \sum_{m_2 \le \frac{\beta-1}{2}} \operatorname{sgn}_2(\lambda) \left[\left(r + \delta + \frac{\gamma}{2},\ 2^{m_2},\ 1^{\beta + \frac{\gamma}{2} - 2m_2 - 1}\right)\right] f.$$

Here, the multiplicity of the 1s were computed by adjusting the sizes of partitions to be exactly $\frac{|\lambda|}{2}$. Let $\mathtt{D}^2_\lambda$ be the multiset of partitions μ such that $[\lambda](p_2 \circ s_\mu)$ doesn't vanish and with multiplicities $[\mu]f$. Then, the above formulas translate to one simple equation (in terms of one complex multiset),

$$[\lambda](p_2 \circ f) = \operatorname{sgn}_2(\lambda) \cdot \#\mathtt{D}^2_\lambda.$$

When λ_1 is odd, the multiset only has one distinct element, and when λ_1 is even, we can split the multiset into two parts,

$$\mathtt{D}^2_\lambda = \left\{\mu \ : \ {\scriptstyle 2\mu_1 = \lambda_1 \atop m_2(\mu) \le \beta/2}\right\} \cup \left\{\mu \ : \ {\scriptstyle 2\mu_1 = \lambda_1 + 2 \atop m_2(\mu) \le \frac{\beta-1}{2}}\right\} =: {}^0\mathtt{D}^2_\lambda \ \cup \ {}^2\mathtt{D}^2_\lambda.$$

We omitted the condition $|\mu| = \frac{|\lambda|}{2}$ for clarity's sake.

Similarly, we can achieve some formulas for $p_{1,1}$, using Lemma 4.19. Letting m_2 be $m_2(\mu,\nu) = m_2(\mu)+m_2(\nu)$ and m_1 be $m_1(\mu,\nu) = \min\{m_1(\mu), m_1(\nu)\}$ for the sake of simplification, we get

$$[\lambda](p_{1,1}\circ f) = \sum_{\substack{\mu_1+\nu_1=\lambda_1\\ m_2\le\beta\le m_2+m_1}} [\mu]f\cdot[\nu]f +$$

$$+ \sum_{\substack{\mu_1+\nu_1=\lambda_1+1\\ m_2\le\beta\le 1+m_2+m_1}} \chi^{\beta}_{(\mu,\nu)}\cdot[\mu]f\cdot[\nu]f + \sum_{\substack{\mu_1+\nu_1=\lambda_1+2\\ 1+m_2\le\beta\le 1+m_2+m_1}} [\mu]f\cdot[\nu]f,$$

where $\chi^{\beta}_{(\mu,\nu)}$ is 1 if any extreme on β is attained, and 2 otherwise. Note also that the condition $|\mu|+|\nu| = |\lambda|$ was omitted. This leads to the definition of another multiset, $\mathtt{D}^{1,1}_{\lambda}$, made of the *ordered*[10] pairs (μ,ν) such that $[\lambda](s_\mu s_\nu)$ doesn't vanish and with multiplicities $[\mu]f\cdot[\nu]f$ or $\chi^{\beta}_{(\mu,\nu)}\cdot[\mu]f\cdot[\nu]f$ accordingly. This time,

$$[\lambda](p_{1,1}\circ f) = \#\mathtt{D}^{1,1}_{\lambda},$$

and the multiset will be naturally split into three multisets,

$$\mathtt{D}^{1,1}_{\lambda} = \left\{(\mu,\nu)\ :\ {}^{\mu_1+\nu_1=\lambda_1}_{m_2\le\beta\le m_2+m_1}\right\} \cup \left\{(\mu,\nu)\ :\ {}^{\mu_1+\nu_1=\lambda_1+1}_{m_2\le\beta\le 1+m_2+m_1}\right\} \cup \left\{(\mu,\nu)\ :\ {}^{\mu_1+\nu_1=\lambda_1+2}_{1+m_2\le\beta\le 1+m_2+m_1}\right\},$$

which we will call ${}^0\mathtt{D}^{1,1}_{\lambda}$, ${}^1\mathtt{D}^{1,1}_{\lambda}$ and ${}^2\mathtt{D}^{1,1}_{\lambda}$ respectively.

Note 4.29. These multisets, or rather their underlying sets, can be interpreted as some integer polytopes in $\mathbb{Z}^6$, by identifying a hook+column pair $(\mu,\nu) = \big((\mu_1, 2^{m_2(\mu)}, 1^{m_1(\mu)}), (\nu_1, 2^{m_2(\nu)}, 1^{m_1(\nu)})\big)$ with the point $\big(\mu_1, m_2(\mu), m_1(\mu), \nu_1, m_2(\nu), m_1(\nu)\big)$. Now, the equalities and inequalities that define our sets are viewed as the restriction to certain hyperplanes and regions of the space. Once this work is done, the integer points inside the intersection of those hyperplanes and regions form the announced polytope.

Proof of 4.28. Let $(f)|_{\text{hook+col}} = \sum([\mu]f)\cdot s_\mu$ and $s_2\circ(f)|_{\text{hook+col}} = \sum d_\lambda s_\lambda$. By hypothesis, $[\mu]f = [\mu^r]f$ for every μ. We want to show that $d_\lambda = d_{\lambda^R}$ for any fixed $\lambda = (\lambda_1, 2^\beta, 1^\gamma)\vdash 2n$, where $R = 2r-2$. I claim[11] we can find a $\delta\ge 0$ such that $\lambda_1 = R+2\delta+\gamma$. We now have a direct way of expressing the R-flip on λ — we think of it as interchanging β and δ (see Figure 19).

As usual, we will express s_2 in the powersum base, to get

$$2d_\lambda = \underbrace{[\lambda](p_2\circ f)}_{\mathrm{sgn}_2(\lambda)\#\mathtt{D}^2_\lambda} + \underbrace{[\lambda](p_{1,1}\circ f)}_{\#\mathtt{D}^{1,1}_\lambda} \quad ; \quad 2d_{\lambda^R} = \underbrace{[\lambda^R](p_2\circ f)}_{\mathrm{sgn}_2(\lambda^R)\#\mathtt{D}^2_{\lambda^R}} + \underbrace{[\lambda^R](p_{1,1}\circ f)}_{\#\mathtt{D}^{1,1}_{\lambda^R}}.$$

Now, the non-negativity of the resulting coefficients will fall from 4.30, as any given coefficient is either product and sum of positive coefficients, or a number of type $([\mu]f)^2-[\mu]f$, or a number of type $2([\mu]f)^2-[\mu]f$. In any case, it is a non-negative integer.

Also, recall from its definition that the sign depends solely on the size of the 1's part of our partitions, which is invariant under the r-flip. So proving $\mathtt{D}^2_\lambda = \mathtt{D}^2_{\lambda^R}$ and $\mathtt{D}^{1,1}_\lambda = \mathtt{D}^{1,1}_{\lambda^R}$ will suffice to prove r-symmetry. We will let the followings lemmas contain all the technical details:

[10] That is, if $\mu\ne\nu$, then both (μ,ν) and (ν,μ) are in the set with the same multiplicity. If we were to consider the *unordered* pairs, then (μ,ν) appears with multiplicity $2\cdot[\mu]f\cdot[\nu]f$.

[11] As Lemma 4.30 establishes, we can suppose that the λ arises from the product of μ and ν. By hypothesis, the r-flip must be defined for μ, ν, and thus $r<\mu_1-m_1(\mu)$, $r<\nu_1-m_1(\nu)$. Adding up these two inequalities, we get $R<(\mu_1+\nu_1-2)-(m_1(\mu)+m_1(\nu))\le\lambda_1-\gamma$, from a close inspection of Lemma 4.19.

The parity of $2n, R$, and γ gives $\lambda_1-R-\gamma$ even.

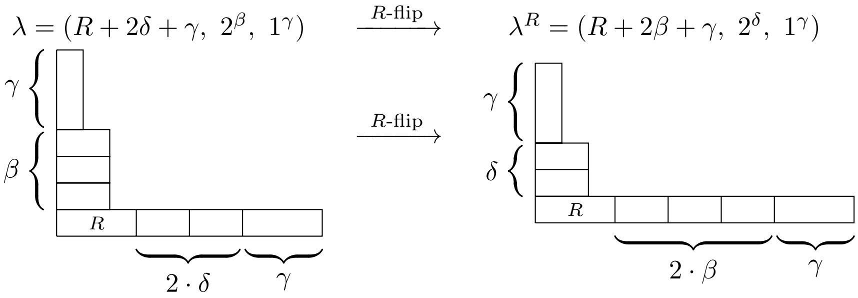


Figure 19: A direct way of expressing the R-flip on λ.

- If λ_1 is odd, then $\mathtt{D}^2_\lambda = \mathtt{D}^2_{\lambda^R}$ by 4.31.
- If λ_1 is even, then ${}^0\mathtt{D}^2_\lambda = {}^2\mathtt{D}^2_{\lambda^R}$ by 4.32. Relabelling λ for λ^R gives ${}^0\mathtt{D}^2_{\lambda^R} = {}^2\mathtt{D}^2_\lambda$.
- By 4.33, ${}^0\mathtt{D}^{1,1}_\lambda = {}^2\mathtt{D}^{1,1}_{\lambda^R}$, and relabelling λ for λ^R gives ${}^0\mathtt{D}^{1,1}_{\lambda^R} = {}^2\mathtt{D}^{1,1}_\lambda$.
- Finally, ${}^1\mathtt{D}^{1,1}_\lambda = {}^1\mathtt{D}^{1,1}_{\lambda^R}$ by 4.34.

And thus $\mathtt{D}^2_\lambda = \mathtt{D}^2_{\lambda^R}$ and $\mathtt{D}^{1,1}_\lambda = \mathtt{D}^{1,1}_{\lambda^R}$, proving that r-symmetry is preserved under $(s_2 \circ \cdot)$. ■

Lemma 4.30. *Under the hypotheses and notations of 4.28, $\mu \in \mathtt{D}^2_\lambda$ implies $(\mu, \mu) \in \mathtt{D}^{1,1}_\lambda$.*

Proof. First of all, if λ_1 is odd then $\mathtt{D}^2_\lambda$ has only one element, namely $\mu = \left(\frac{\lambda_1+1}{2}, 2^{\frac{\beta}{2}}, 1^{\frac{\gamma-1}{2}}\right)$. One can easily verify that μ does indeed verify the equations

$$2\mu_1 = \lambda_1 + 1 \quad ; \quad 2m_2(\mu) \leq \beta \leq 2m_2(\mu) + m_1(\mu),$$

and thus $\mu \in \mathtt{D}^{1,1}_\lambda$.

If now λ_1 is even, suppose that μ verifies

$$\mu_1 = \frac{\lambda_1}{2} \quad ; \quad m_2(\mu) \leq \frac{\beta}{2}.$$

We get $2\mu_1 = \lambda_1$ and $2m_2(\mu) \leq \beta$. It remains to show that $\beta \leq 2m_2(\mu) + m_1(\mu)$. But by contradiction, if $\beta > 2m_2(\mu) + m_1(\mu)$, then from the equality

$$2n = \lambda_1 + 2\beta + \gamma = 2\mu_1 + 4m_2(\mu) + 2m_1(\mu) = 2 \cdot n,$$

we conclude

$$\lambda_1 + \gamma < 2\mu_1 = \lambda_1.$$

This is obviously a contradiction. If now μ verifies the other set of equations, that is

$$\mu_1 = \frac{\lambda_1}{2} + 1 \quad ; \quad m_2(\mu) \leq \frac{\beta - 1}{2},$$

then $2\mu_1 = \lambda_1 + 2$ and $2m_2(\mu) + 1 \leq \beta$. It remains to show $\beta \leq 2m_2(\mu) + m_1(\mu) + 1$, which we do similarly to the previous case. ■

Lemma 4.31. *Under the hypotheses and notations of 4.28, if λ_1 is odd then $\mathtt{D}^2_\lambda = \mathtt{D}^2_{\lambda^R}$.*

Proof. If $\lambda_1 = R + 2\delta + \gamma$ is odd, then so is $\lambda_1^R = R + 2\beta + \gamma$. The only element in $\mathtt{D}^2_\lambda$ is

$$\mu = \left(r + \delta + \frac{\gamma - 1}{2},\ 2^{\frac{\beta}{2}},\ 1^{\frac{\gamma-1}{2}}\right).$$

Under the r-flip, the image of said partition is

$$\left(r + \beta + \frac{\gamma - 1}{2},\ 2^{\frac{\delta}{2}},\ 1^{\frac{\gamma-1}{2}}\right) = \mu^r.$$

This is the only partition appearing in $\mathtt{D}^2_{\lambda^R}$. By hypotheses, both these partitions appear with the same multiplicity $[\mu]f = [\mu^r]f$ in their corresponding multisets. This ends the proof. ■

The rest of lemmas will follow the same spirit as the previous one, but will not be as trivial to prove.

Lemma 4.32. *Under the hypotheses and notations of 4.28, if λ_1 is even then ${}^0\mathtt{D}^2_\lambda = {}^2\mathtt{D}^2_{\lambda^R}$.*

Proof. If λ_1 is now even, then

$$\mathtt{D}^2_\lambda = \left\{\mu \ :\ \substack{2\mu_1 = \lambda_1 \\ m_2(\mu) \le \beta/2}\right\} \cup \left\{\mu \ :\ \substack{2\mu_1 = \lambda_1 + 2 \\ m_2(\mu) \le \frac{\beta - 1}{2}}\right\} =: {}^0\mathtt{D}^2_\lambda \cup {}^2\mathtt{D}^2_\lambda.$$

Explicitly, we can write (up to multiplicity)

$${}^0\mathtt{D}^2_\lambda = \left\{\left(r + \delta + \frac{\gamma}{2} - 1,\ 2^{m_2},\ 1^{\beta + \frac{\gamma}{2} - 2m_2}\right) \quad : \quad m_2 \le \frac{\beta}{2}\right\},$$

where the size of the 1's part is computed adjusting the size of the partitions to be $n = \frac{|\lambda|}{2}$. Similarly,

$${}^2\mathtt{D}^2_{\lambda^R} = \left\{\left(r + \beta + \frac{\gamma}{2},\ 2^{m_2'},\ 1^{\bullet}\right) \quad : \quad m_2' \le \frac{\delta - 1}{2}\right\}.$$

We will now apply the r-flip on every element of ${}^0\mathtt{D}^2_\lambda$. Following the r-flip algorithm, the first part of said partitions are made of one r-brick, some 2-bricks and $\beta + \frac{\gamma}{2} - 2m_2$ extra 1-bricks. So we can solve for the number of 2-bricks which is $m_2' := \frac{\delta + 2m_2 - \beta - 1}{2}$. And thus applying the r-flip on the multisets yields

$$r({}^0\mathtt{D}^2_\lambda) = \left\{\left(r + \beta + \frac{\gamma}{2},\ 2^{m_2'},\ 1^{\beta + \frac{\gamma}{2} - 2m_2}\right) \quad : \quad m_2 \le \frac{\beta}{2}\right\}.$$

We aim to identify $r({}^0\mathtt{D}^2_\lambda)$ with ${}^2\mathtt{D}^2_{\lambda^R}$. The only thing that remains to show is that

$$m_2 \le \frac{\beta}{2} \quad \Leftrightarrow \quad m_2' \le \frac{\delta - 1}{2}.$$

From the expression of m_2', the inequality $m_2' \le \frac{\delta - 1}{2}$ simplifies to $\frac{\delta - 1 + 2m_2 - \beta}{2} \le \frac{\delta - 1}{2}$, and it is now clear that this is equivalent to the inequality $m_2 \le \frac{\beta}{2}$. ■

Lemma 4.33. *Under the hypotheses and notations of 4.28, we get ${}^0\mathtt{D}^{1,1}_\lambda = {}^2\mathtt{D}^{1,1}_{\lambda^R}$.*

Proof. Note that the first parts of μ and ν sum to the first part of λ if and only if the rest of μ and ν sum to the rest of λ. This is expressed as

$$2m_2(\mu) + m_1(\mu) + 2m_2(\nu) + m_1(\nu) = 2\beta + \gamma.$$

Adding $R = 2r - 2$ to both sides of the equation gives

$$\underbrace{r + 2m_2(\mu) + m_1(\mu)}_{\mu_1^r} + \underbrace{r + 2m_2(\nu) + m_1(\nu)}_{\nu_1^r} - 2 = \underbrace{R + 2\beta + \gamma}_{\lambda_1^R}.$$

So $\mu_1 + \nu_1 = \lambda_1$ if and only if $\mu_1^r + \nu_1^r = \lambda_1^R + 2$.

On the other hand, immediately from the Remmel and Whitney product rule [24], we can say similar things about the presented inequalities. More precisely, $m_2 \leq \beta$ if and only if $m_1(\mu) + m_1(\nu) \geq \gamma$, as β is "made of" cells coming from $m_2(\mu)$, $m_2(\nu)$, $m_1(\mu)$ and $m_1(\nu)$, with the "excess" cells forming γ.

Knowing that $\mu_1 + \nu_1 = \lambda_1$, we write

$$\big(r + 2m_2(\mu^r) + m_1(\mu)\big) + \big(r + 2m_2(\nu^r) + m_1(\nu)\big) = R + 2\delta + \gamma = 2r - 2 + 2\delta + \gamma.$$

And so,

$$m_2 \leq \beta \quad \Leftrightarrow \quad m_1(\mu) + m_1(\nu) \geq \gamma \quad \Leftrightarrow \quad \underbrace{m_2(\mu^r) + m_2(\nu^r)}_{=:\, m_2'} + 1 \leq \delta.$$

In a similar fashion, one can show

$$\beta \leq m_2 + m_1 \quad \Leftrightarrow \quad \gamma \geq |m_1(\mu) - m_1(\nu)| \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \delta \leq m_2' + 1 + \underbrace{\frac{1}{2}\big(m_1(\mu) + m_1(\nu) - |m_1(\mu) - m_1(\nu)|\big)}_{=\ \min\{m_1(\mu), m_1(\nu)\}\ =\ m_1\ =\ m_1'} = m_2' + 1 + m_1'.$$

Summing up, we have proved that a pair (μ, ν) is in ${}^0\mathtt{D}_\lambda^{1,1}$ if and only if (μ^r, ν^r) is in ${}^2\mathtt{D}_{\lambda^R}^{1,1}$, and thus ${}^0\mathtt{D}_\lambda^{1,1} = {}^2\mathtt{D}_{\lambda^R}^{1,1}$ by r-symmetry hypothesis on f. ■

Lemma 4.34. *Under the hypotheses and notations of 4.28, we get* ${}^1\mathtt{D}_\lambda^{1,1} = {}^1\mathtt{D}_{\lambda^R}^{1,1}$.

Proof. Quite similarly to the previous lemma, one can proof

$$\mu_1 + \nu_1 = \lambda_1 + 1 \quad \Leftrightarrow \quad \mu_1^r + \nu_1^r = \lambda_1^R + 1.$$

For the second part, the Remmel and Whitney product rule says that

$$m_2(\mu) + m_2(\nu) \leq \beta \quad \Leftrightarrow \quad m_1(\mu) + m_1(\nu) \geq \gamma - 1,$$

and following the same scheme of proof, we convert that second inequality to $m_2'(\mu) + m_2'(\nu) \leq \delta$. The last inequality comes once again from the Remmel and Whitney rule, which now tells us that

$$\beta \leq 1 + m_2(\mu) + m_2(\nu) + \min\{m_1(\mu), m_1(\nu)\} \quad \Leftrightarrow \quad \gamma + 1 \geq |m_1(\mu) - m_1(\nu)|,$$

from where the result arises. Following in each steps the case in which the equalities are attained, we also proved that $\chi^\beta_{(\mu,\nu)} = \chi^\delta_{(\mu^r,\nu^r)}$. ■

5 Final remarks on some experimental results

In the previous section, we showed that the hook+column sequences of $s_2^{\circ k} \circ s_a \circ s_b$ are always symmetric for any γ. Judging from our recollected data, it seems that this is true for any function of the form $s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}$. We establish the first of three conjectures about structural behaviours of the forementioned sequences:

Conjecture 1. *The hook+column sequences of $s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}$ for any fixed γ are symmetric.*

An avid reader might have already spotted some additional structures in the example sequences discussed in section 4. A finite sequence is said to be unimodal (or concave, monotonic) if it is weakly increasing until a certain point, and weakly decreasing from there on. That is, if we can write the sequence as follows: $a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_{n-1} \geq a_n$. Theorem 4.23 is a prime example of unimodality. Recall the example that illustrated said result:

Example 5.1. The hook+column sequence of $s_2 \circ s_4 \circ s_2$ for $\gamma = 0$ is $(1, 2, 3, 4, 4, 3, 2, 1)$.

The available data would imply the following:

Conjecture 2. *The hook+column sequences of $s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}$ for any fixed γ are unimodal.*

We haven't found a proof of this yet, but we would like to briefly showcase the progress we have made.

Many different methods can be used to prove unimodality in combinatorial sequences [31]. However, no single method appears to be successful in a general setting. This makes proving such phenomenon an arduous task, and there are many instances of sequences which were believed to be unimodal for ages, but were only recently proved to be so. Thankfully, it seems as if symmetric sequences are easier to work with, so hopefully showing that $s_2^{\circ k} \circ s_a \circ s_b$ is unimodal is a more achievable goal, given that we have already proven Theorem 4.28. Other structural behaviours also help simplify the problem. For instance, if certain polynomials arising from the sequences are real-rooted, then the sequences are log-concave[12], which in turn implies unimodality. This approach[13] fails to hold generality in our case (one can check that $s_2^{\circ 5}$ and $\gamma = 0$ do not give a log-concave sequence).

Perhaps the most naive way of proving unimodality of symmetric combinatorial sequences (that is, sequences of number which *count* combinatorial objects) would be to establish an injective map from the set of objects counted by $a_{\beta-1}$ to the set of objects counted by a_β provided that both numbers are in the increasing part of the sequence. For our case — that is, the functions $s_2^{\circ k} \circ s_a \circ s_b$ —, that would mean establishing an injection from $\mathtt{D}_{\nu_{\beta-1}}$ to $\mathtt{D}_{\nu_\beta}$ provided that $\beta \trianglelefteq \beta^R$. Even though $(\nu_\theta, \nu_\pi) \mapsto (\nu_{\theta+1}, \nu_\pi)$ appears to be such injection, we have checked that it does not work well in the extreme cases, and does not lead us in the correct direction.

Another obvious — although not necessarily easy — approach to proving unimodality of combinatorial sequences is to work with the object themselves and try and understand why there would be less or more of one type than another type. This is ultimately the goal of a pure combinatorialist. However, the nature of our work does not let us work in this manner, as a combinatorial interpretation of plethysm is indeed *the* question we want to some day answer. Relatively recently, a very celebrated proof of this kind was found by Kathleen O'Hara of the unimodality of q-binomial

[12] A finite sequence is said to be log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for every i.

[13] Two examples of this approach are found in [6, 11]. One shall note, however, that the ultimate goal of those articles is not to show unimodality, but asymptotic normality, which we will later discuss.

coefficients [21]. We are very thankful to Emmanuel Briand for pointing out that this problem is related to ours. From Corollary 4.5 and Lemma 4.6, our problem is reduced to studying whether

$$\frac{s_{n_1} \circ s_{n_2} \circ \cdots \circ s_{n_k}[1-x-y] \cdot (x-y)}{(-1)^{\gamma}(1-x)(1-y)(x^{\gamma+1}y^{\gamma+1})} = \sum_{\beta} a_{\beta}(xy)^{\beta}$$

is a unimodal polynomial — a polynomial whose coefficients form an unimodal sequence — in the variable (xy) for every fixed γ. The following equality,

$$(s_a \circ s_b)[1+q] = s_a[(b+1)_q] = \binom{a+b}{a}_q = \sum_m b_m q^m,$$

where $(n)_q$ is the q-analogue of n and $\binom{n}{k}_q$ is the q-binomial coefficient, indicates that the famous proof by O'Hara can be rephrased in a similar manner; whether $(s_a \circ s_b)[1+q]$ is a unimodal polynomial in the variable q.

Asymptotic normality is another structural phenomenon commonly found in combinatorial sequences. Experimental evidence suggest that the hook+column sequences of $(s_2)^{\circ k}$ for any fixed γ are asymptotically normal when letting k tend to infinity, as Figure 20 illustrates.

Function	γ	Hook+column sequence
$s_2^{\circ 2}$	0	(1, 1)
$s_2^{\circ 3}$	0	(1, 2, 2, 1)
$s_2^{\circ 4}$	0	(1, 3, 8, 13, 13, 8, 3, 1)
$s_2^{\circ 5}$	0	(1, 4, 20, 72, 205, 446, 756, 986, 986, 756, 446, 205, 72, 20, 4, 1)

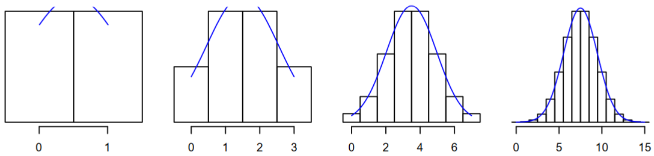


Figure 20: On top, a table showing the hook+column sequence of $s_2^{\circ k}$ for $\gamma = 0$ up to $k = 5$. Below, plots of the aforementioned sequences, the x axis being β, and represented as the normalized histogram whose frequencies read $(a_0, a_1, ..., a_\beta, ...)$. They appear overlaid with Gaussian curves of adjusted mean and variance.

Moreover, a χ^2 normality test returns the p-values shown in Table 5, which are extremely big[14], indicating that the Gaussian curve *perfectly* fits our sequences, even for small values of k.

Conjecture 3. *For each fixed γ, the hook+column sequence of $s_2^{\circ k}$ is asymptotically normal, in the sense that their relative sums approach a Gaussian curve when k tends to infinity.*

See [3, 6, 11] for more details in asymptotic normality of combinatorial integer sequences. As previously mentioned, the methods presented in [6, 11] cannot be used to solve our problem, as our sequences are not always log-concave.

[14] We adopt the usual convention of accepting our sequence as normal if the p-value is bigger than 0.05.

$\gamma =$	0	1	2	3	4	5	6	7
$s_2^{\circ 2}$	1							
$s_2^{\circ 3}$	1	1						
$s_2^{\circ 4}$	1	0.92	1	1				
$s_2^{\circ 5}$	1	1	1	0.99	0.98	1	1	1

Table 5: The p-values resulting from the χ^2 normality test on the hook+column sequences of $s_2^{\circ k}$ up to $k = 5$ and for every possible γ.

We have used sagemath to compute some data supporting these conjectures. For instance, our data for $f_{n,m} := s_n \circ s_m \circ s_2$ suggest that both the limit when n tends to infinity and the limit when m tends to infinity of $f_{n,m}$ yields asymptotic normality of the corresponding hook+column sequences (see Figure 21).

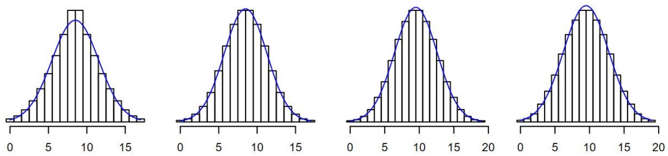


Figure 21: In order, the histogram plots for the hook+column sequences associated to $s_9 \circ s_2 \circ s_2$, $s_6 \circ s_3 \circ s_2$, $s_5 \circ s_4 \circ s_2$ and $s_4 \circ s_5 \circ s_2$, always for $\gamma = 0$. They appear overlaid with Gaussian curves of adjusted mean and variance.

Furthermore, we were able to make some reasonable guesses about what the hook+column sequences of $f_{n,m}$ approach to.

- Let $m = 2$. The hook+column sequence for $f_{n,2} = s_n \circ s_2 \circ s_2$, $\gamma = 0$ seems to be

  $$(1, 2, 4, 7, \ldots, 1 + T_n,\ 1 + T_n, \ldots, 7, 4, 2, 1)\,,$$

  where T_n is the nth triangular number (OEIS A000124, A000217). This has been verified until $n = 9$.

- Let $m = 3$. The hook+column sequences of $f_{n,3} = s_n \circ s_3 \circ s_2$ and $\gamma = 0$ up to $n = 6$ are shown in Table 6.

  Unlike the previous examples, each sequence is not simply a longer version of the previous ones. However, they tend to stabilize. Their stable limits seems to be

  $$(1, 2, 5, 10, 19, 33, 57, 92, 147, 227, \ldots),$$

  the number of partitions with two kinds of 1s, 2s, and 3s (OEIS A000098).

Function	γ	Hook+column sequence
$s_1 \circ s_3 \circ s_2$	0	(1, 1, 1)
$s_2 \circ s_3 \circ s_2$	0	(1, 2, 3, 3, 2, 1)
$s_3 \circ s_3 \circ s_2$	0	(1, 2, 5, 7, 8, 7, 5, 2, 1)
$s_4 \circ s_3 \circ s_2$	0	(1, 2, 5, 10, 15, 18, 18, 15, 10, 5, 2, 1)
$s_5 \circ s_3 \circ s_2$	0	(1, 2, 5, 10, 15, 19, 28, 36, 38, 36, 28, 19, 10, 5, 2, 1)
$s_6 \circ s_3 \circ s_2$	0	(1, 2, 5, 10, 19, 33, 49, 63, 72, 72, 63, 49, 33, 19, 10, 5, 2, 1)

Table 6: The hook+column sequence for $s_n \circ s_3 \circ s_2$ and $\gamma = 0$, up to $n = 6$.

- Let $m = 4$. The hook+column sequences of $f_{n,4} = s_n \circ s_4 \circ s_2$ and $\gamma = 0$ up to $n = 5$ are shown in Table 7.

Function	γ	Hook+column sequence
$s_1 \circ s_4 \circ s_2$	0	(1, 1, 1, 1)
$s_2 \circ s_4 \circ s_2$	0	(1, 2, 3, 4, 4, 3, 2, 1)
$s_3 \circ s_4 \circ s_2$	0	(1, 2, 5, 8, 11, 13, 13, 11, 8, 5, 2, 1)
$s_4 \circ s_4 \circ s_2$	0	(1, 2, 5, 11, 18, 26, 34, 38, 38, 34, 26, 18, 11, 5, 2, 1)
$s_5 \circ s_4 \circ s_2$	0	(1, 2, 5, 11, 22, 36, 55, 74, 90, 100, 100, 90, 74, 55, 36, 22, 11, 5, 2, 1)
$s_6 \circ s_4 \circ s_2$	0	(1, 2, 5, 11, 22, 41, 68, 103, 144, 184, 217, 236, 236, 217, 184, 144, 103, 68, 41, 22, 11, 5, 2, 1)

Table 7: The hook+column sequence for $s_n \circ s_4 \circ s_2$ and $\gamma = 0$, up to $n = 6$.

Again, the coefficients tend to stabilize. This time, their stable limits seems to be

$$(1, 2, 5, 11, 22, 42, 77, 135, 231...),$$

the number of partitions of $2n$ (OEIS A058696).

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