On initial value and terminal value problems for subdiffusive stochastic Rayleigh-Stokes equation

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Abstract

In this paper, we study two stochastic problems for time-fractional Rayleigh-Stokes equation including the initial value problem and the terminal value problem. Here, two problems are perturbed by Wiener process, the fractional derivative are taken in the sense of Riemann-Liouville, the source function and the time-spatial noise are nonlinear and satisfy the globally Lipschitz conditions. We attempt to give some existence results and regularity properties for the mild solution of each problem.

1 Introduction

Stochastic partial differential equations (SPDEs) play an important role in the modeling of many phenomena in various fields, such as fluid mechanics, physics, astrophysics, hydrodynamics, biology, etc. [8, 20, 22, 35, 36]. In addition, fractional differential equations (FDEs) have received much attention recently with successful applications in various sciences such as engineering, physics, biology, etc. [12, 21, 23]. For some impressive works in recent time on stochastic fractional differential equations (SFDEs), the readers can refer to [9, 13, 14, 19, 25, 31, 32, 33, 34].

Let $\mathcal{X} \subset \mathbb{R}^N, N \geq 1$, be a bounded domain (with sufficiently smooth boundary $\partial\mathcal{X}$ for $N \geq 2$). In this paper, we study two following problems for an SFDE named Rayleigh-Stokes equation (RSE).

• **The initial value problem:** Consider the following time-fractional Rayleigh-Stokes stochastic equation

$$\left(\frac{\partial}{\partial t} - \Delta - \varkappa \frac{\partial^\alpha}{\partial t^\alpha}\Delta\right) u = F(t,u) + \sigma(t,u)\dot{\mathcal{W}}(t), \quad \text{on} \quad J \times \mathcal{X}, \tag{1}$$

where $J = (0,T)$, $\varkappa$ is a positive constant and $0 < \alpha < 1$, Δ is the Laplacian operator. This equation is subjected to the Dirichlet boundary condition

$$u(t,x) = 0, \quad (t,x) \in J \times \partial\mathcal{X}, \tag{2}$$

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30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

and the initial value condition

$$u(0,x) = \psi(x), \quad x \in \mathcal{X}. \tag{3}$$

Here, $\partial^\alpha/\partial t^\alpha$ stands for the Riemann-Liouville fractional derivative

$$\frac{\partial^\alpha}{\partial t^\alpha} v(t) := \frac{\partial}{\partial t}\int_0^t \mu_{1-\alpha}(s) v(t-s,x)ds, \qquad \mu_\beta(s) := \frac{1}{\Gamma(\beta)} s^{\beta-1}, \quad (\beta > 0), \tag{4}$$

whereupon Γ is the Gamma function, see [21]. $\{\mathcal{W}(t,.)\}_{t\in\overline{\mathcal{J}}}$ represents a standard Wiener process (see Subsection 2.2), the generalized derivative $\dot{\mathcal{W}}(t) = \frac{\partial}{\partial t}\mathcal{W}(t)$ describes a white noise, the nonlinear term F and the time-spatial-noise σ will be specified later. The problem of finding u satisfying (1),(2),(3) is called an initial value problem.

• **The terminal value problem:** Let the initial condition (3) be replaced by the final value condition

$$u(T,x) = \varphi(x), \quad x \in \mathcal{X}. \tag{5}$$

Then, the problem of finding u satisfying (1),(2),(5) is called a final value problem.

The Rayleigh-Stokes problem arises in modeling the behavior of some non-Newtonian fluids. The Riemann-Liouville derivative in this model has been found to be flexible in describing viscoelastic behaviors of the flow [1, 7, 24]. Let us propose some details on the history of two problems we are interested in and some related studies. We begin with the deterministic model of the initial value problem (when the term $\sigma(t,u)\dot{\mathcal{W}}(t)$ disappears). For the homogeneous problem, the Sobolev regularities for both smooth and nonsmooth initial data was established in [1]. For the linear case, Shen [24] used fractional Laplace transform to contribute the exact solution. In [30], the exact solutions was constructed by using the eigenfunction expansion on a rectangular domain. The readers can refer to some other studies on the exact solutions for the Rayleigh-Stokes problem in [10],[27]. In [29], Zaky extended and developed Legendre-tau algorithms for solving one-dimensional and two-dimensional fractional Rayleigh–Stokes problem for a heated generalized second grade fluid. Some other works [4, 5, 6, 15] used different numerical methods to study the Rayleigh-Stokes problem with a heated generalized second grade fluid. In contract to the initial value problem, the study of the terminal value problem is still limited. We can list here some papers concerned with the deterministic model of this kind of problem [17, 18, 26].

To the best of our knowledge, both the initial value problem and terminal value problem for the stochastic Rayleigh-Stokes equation driven by Wiener process have not been investigated in the literature. This motivated us to study the existence, uniqueness and regularity of each of two stochastic problems. It is the fact that the analysis technique used in the deterministic case cannot be applied in the stochastic case. The problems of finding suitable spaces for the solutions in the stochastic case is our challenges since the solutions are more complex and it is required to cleverly improve the estimates for the solution operators.

The main contributions of this paper are as follows. In Section 2, we provide some preliminary results including some solution spaces, fractional calculus and stochastic analysis techniques. The existence, uniqueness result and some regularity properties of the mild solution to the initial value problem (1),(2),(3) is investigated in Section 3. In Section 4, we continue to study the existence, uniqueness, regularity of the mild solution to the terminal value problem (1),(2),(5).

2 Preliminaries

2.1 Hilbert scale spaces

For convenience, we denote by $L_t^\infty := L^\infty(0,T)$, $L_t^p := L^p(0,T)$, $p \ge 1$. By $L_x^r := L^r(\mathcal{X})$ $(r \ge 1)$, $H_x^\xi := H^\xi(\mathcal{X})$ $(\xi \ge 0)$, we denote the Lebesgue and the Sobolev-Slobodeckij spaces respectively, and $H_{0,x}^1$ the closure of $C_c^\infty(\mathcal{X})$ in H_x^1. Suppose that the operator $A = -\Delta$ is defined on the domain $D(A) = H_{0,x}^1 \cap H_x^2$. Since $\mathcal{X}$ is a bounded domain with sufficiently smooth boundary and A is self-adjoint on L_x^2, it is well-known there exists a sequence of eigenfunctions $\phi_j \in D(A)$ and a sequence of corresponding eigenvalues λ_j such that $A\phi_j(x) = \lambda_j\phi_j(x)$ on $\mathcal{X}$. Additionally, the sequence $\{\lambda_j\}_{j=1,2,...}$ is positive, non-decreasing, and tends to infinity as $j \to \infty$, $\{\phi_j\}_{j\ge1}$ forms an orthonormal basis of L_x^2.

Next, we introduce the Hilbert scale spaces which will be defined by basing on $\{\phi_j\}_{j\ge1}$. For a non-negative number ξ, we define by $\dot{H}_x^\xi$ the space of all functions $v \in L_x^2$ such that

$$\|v\|_{\dot{H}_x^\xi} := \Big(\sum_{j\ge1} \lambda_j^{2\xi} |\langle v, e_j\rangle|^2\Big)^{\frac{1}{2}} < \infty,$$

where $\langle .,. \rangle$ is the usual inner product of L_x^2. If $\xi = 0$, one can see $\dot{H}_x^0 \equiv L_x^2$. Let $\dot{H}^{-\xi}$ be the dual space of $\dot{H}^\xi$. The fractional operator (see [3]) $A^\xi : \dot{H}^{\xi/2} \to \dot{H}^{-\xi/2}$ can be defined as

$$A^\xi v := \sum_{j\ge1} \lambda_j^\xi \langle v, e_j\rangle e_j, \quad v \in \dot{H}^{\xi/2}.$$

2.2 Stochastic processes on the Hilbert scale spaces

Let $\big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\overline{J}}\big)$ be a filtered complete probability space, which satisfies that the filtration $\{\mathcal{F}_t\}_{t\in\overline{J}}$ is a right continuous increasing family and that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Throughout this paper, for each $\xi \ge 0$, notation $\{\mathcal{W}_\xi(t,.)\}_{t\in\overline{J}}$ describes a $\dot{H}_x^\xi$-valued Wiener process which is defined on $\big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\overline{J}}\big)$ with a finite trace covariance operator $\mathcal{Q}_\xi$ on $\dot{H}_x^\xi$. Let us denote $\{\beta_j^{(\xi)}\}_{k=1}^\infty$ the spectrum of $\mathcal{Q}_\xi$, i.e. $\mathcal{Q}_\xi\phi_j^{(\xi)} = \beta_j^{(\xi)}\phi_j^{(\xi)}$, then the finite trace of $\mathcal{Q}_\xi$ is $\mathrm{Tr}(\mathcal{Q}_\xi) = \sum_{j=1}^\infty \beta_j^{(\xi)} < \infty$. According to the above setting, $\{\mathcal{W}_\xi(t,.)\}_{t\in\overline{J}}$ can be expanded in the form

$$\mathcal{W}_\xi(t,x) = \mathcal{Q}_\xi^{\frac{1}{2}} \sum_{j=1}^\infty \vartheta_j^{(\xi)}(t)\phi_j^{(\xi)}(x),$$

whereupon $\{\vartheta_j^{(\xi)}(t)\}_{j=1}^\infty$ is a sequence of mutually independent one-dimension standard Wiener processes.

Let us recall the definition of the expectation. For a random variable $\chi : \Omega \to \dot{H}_x^\xi$ and a real number $q \ge 2$, we note that $\mathbb{E}\|\chi\|_{\dot{H}_x^\xi}^q := \displaystyle\int_\Omega \|\chi(\omega)\|_{\dot{H}_x^\xi}^q d\mathbb{P}(\omega)$. Notation $L_\omega^q \dot{H}_x^\xi := L^q(\Omega; \dot{H}_x^\xi)$ denotes the space of all $\dot{H}_x^\xi$-valued random variables endowed with the norm $\|v\|_{L_\omega^q \dot{H}_x^\xi}^q := \mathbb{E}\|v\|_{\dot{H}_x^\xi}^p < \infty$, for all $v \in L_\omega^q \dot{H}_x^\xi$. Let us define by $L_0^2(\dot{H}_x^\xi, \dot{H}_x^{\xi'})$

the space of all linear bounded operators Ψ from $\mathcal{Q}_\xi^{\frac{1}{2}}\dot{H}_x^\xi$ to $\dot{H}_x^{\xi'}$ such that $\Psi\mathcal{Q}_\xi^{\frac{1}{2}}$ is a Hillbert-Schmidt operator from $\dot{H}_x^\xi$ to $\dot{H}_x^{\xi'}$ where its norm is given by

$$\|\Psi\|_{L_0^2(\dot{H}_x^\xi,\dot{H}_x^{\xi'})} := \left\{\sum_{j=1}^\infty \left\|\Psi\mathcal{Q}_\xi^{\frac{1}{2}}\phi_j^{(\xi)}\right\|_{\dot{H}_x^{\xi'}}^2\right\}^{\frac{1}{2}} < \infty, \quad \text{for } \Psi \in L_0^2(\dot{H}_x^\xi, \dot{H}_x^{\xi'}).$$

In the case $\xi = \xi'$, we will write $L_0^2(\dot{H}_x^\xi)$ in stead of $L_0^2(\dot{H}_x^\xi, \dot{H}_x^{\xi'})$, and just L_0^2 if $\xi = 0$.

The following lemma is useful to estimate the stochastic integrals, and is a straightforward consequence of the Burkholder-Davis-Gundy inequality proved in Da Prato and Zabczyk [2, Lemma 7.2, page 182].

Lemma 2.1 (Burkholder-Davis-Gundy-type inequality). *Given $q \geq 2$, $t_1, t_2 \in \overline{J}$, and let $\{\Phi(t)\}_{t_1\leq t\leq t_2}$ be a $L_0^2(\dot{H}_x^\xi, \dot{H}_x^{\xi'})$-valued predictable stochastic process satisfying*

$$\mathbb{E}\left[\left(\int_{t_1}^{t_2} \|\Phi(s)\|_{L_0^2(\dot{H}_x^\xi)}^2\, ds\right)^{q/2}\right] < \infty.$$

Then, it holds that

$$\mathbb{E}\left[\left\|\int_{t_1}^{t_2} \Phi(s)d\mathcal{W}_\xi(s,.)\right\|_{\dot{H}_x^\xi}^q\right] \leq C(q)\mathbb{E}\left[\left(\int_{t_1}^{t_2} \|\Phi(s)\|_{L_0^2(\dot{H}_x^\xi)}^2\, ds\right)^{q/2}\right],$$

where the constant $C(q)$ is equal to $\left(\frac{q(q-1)}{2}\right)^{1/2}\left(\frac{q}{q-1}\right)^{(q-2)/2}$.

2.3 Solution spaces and assumptions on nonlinearities

Let $\xi \geq 0$ and $q \geq 2$. By $L_t^\infty L_\omega^q \dot{H}_x^\xi := L^\infty(0,T; L^q(\Omega, \dot{H}_x^\xi))$, we denote the space of all $\mathcal{F}_t$-adapted measurable processes w from $\overline{J}$ to $L_\omega^q\dot{H}_x^\xi$ such that

$$\|w\|_{L_t^\infty L_\omega^q\dot{H}_x^\xi} := \operatorname*{ess\,sup}_{t\in J} \|w(t)\|_{L_\omega^q\dot{H}_x^\xi} < \infty.$$

For $\rho > 0$, we introduce the following space

$$L_{t,\rho}^\infty L_\omega^q\dot{H}_x^\xi := \Big\{w \in L_t^\infty L_\omega^q\dot{H}_x^\xi : \quad \|w\|_{L_{t,\rho}^\infty L_\omega^q\dot{H}_x^\xi} := \sup_{t\in\overline{J}_*} e^{-\rho t}\|w(t,.)\|_{L_\omega^q\dot{H}_x^\xi} < \infty\Big\},$$

where $\overline{J}_* = \overline{J}\setminus\{0\}$.

We define by $C_tL_\omega^q\dot{H}_x^\xi := C(\overline{J}; L_\omega^q\dot{H}_x^\xi)$ the space of all continuous $\mathcal{F}_t$-adapted measurable processes w from $\overline{J}$ to $L_\omega^q\dot{H}_x^\xi$ such that

$$\|w\|_{C_tL_\omega^q\dot{H}_x^\xi} := \sup_{t\in\overline{J}} \|w(t)\|_{L_\omega^q\dot{H}_x^\xi} < \infty.$$

Similarly, the notation $C_{t,*}L_\omega^q\dot{H}_x^\xi := C(\overline{J}_*; L_\omega^q\dot{H}_x^\xi)$, presents the set of all continuous $\mathcal{F}_t$-adapted measurable processes w from $\overline{J}_*$ to $L_\omega^q\dot{H}_x^\xi$. By $C_{t,*}^\theta L_\omega^q\dot{H}_x^\xi := C^\theta(\overline{J}_*; L_\omega^q\dot{H}_x^\xi)$, we introduce the following Hölder continuous space of exponent θ

$$C_{t,*}^\theta L_\omega^q\dot{H}_x^\xi = \left\{w \in C_{t,*}L_\omega^q\dot{H}_x^\xi : \quad \|w\|_{C_{t,*}^\theta L_\omega^q\dot{H}_x^\xi} := \sup_{r,t\in\overline{J}_*} \frac{\|w(r,.)-w(t,.)\|_{L_\omega^q\dot{H}_x^\xi}}{|r-t|^\theta} < \infty\right\}.$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

By $\mathcal{Z}^{\theta}_{t,*}L^q_\omega \dot{H}^\xi_x := \mathcal{Z}^{\theta}(\overline{J}_*; L^q_\omega \dot{H}^\xi_x)$, we denote the space of all processes w in the space $C_{t,*}L^q_\omega \dot{H}^\xi_x$ such that $\lim_{t\to 0^+} t^{q\theta}\mathbb{E}\,\|w(t,.)\|^q_{L^q_\omega \dot{H}^\xi_x}$ exists finitely and

$$\|w\|_{\mathcal{Z}^{\theta}_{t,*}L^q_\omega \dot{H}^\xi_x} := \sup_{t\in\overline{J}_*} t^\theta \|w(t,.)\|_{L^q_\omega \dot{H}^\xi_x} = \left(\sup_{t\in\overline{J}_*} t^{q\theta}\|w(t,.)\|^q_{L^q_\omega \dot{H}^\xi_x}\right)^{\frac{1}{q}} < \infty.$$

Next, let us introduce the following definitions for the nonlinearity and the time-spatial noise.

Definition 2.1 (*Globally Lipschitz nonlinearity*)**.** Let $q \geq 2$ and $\xi \geq 0$. The nonlinearity F is said to be globally Lipschitz and written by $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$ if $F : \overline{J}\times\mathcal{X}\times L^q_\omega \dot{H}^\xi_x \longrightarrow L^q_\omega \dot{H}^\xi_x$ and there exists a positive function $K_1 := K_1(t)$ on $\overline{J}$ such that

$$\mathbb{E}\left\|F(t,\cdot,w^\dagger(t,\cdot)) - F(t,\cdot,w^\ddagger(t,\cdot))\right\|^q_{\dot{H}^\xi_x} \leq K_1^q(t)\,\mathbb{E}\left\|w^\dagger(t,.) - w^\ddagger(t,.)\right\|^q_{\dot{H}^\xi_x},$$

for all $w^\dagger$, $w^\ddagger$ in $L^q_\omega \dot{H}^\xi_x$ and $t \in \overline{J}$.

Definition 2.2 (*Globally Lipschitz time-spatial-noise*)**.** Let $q \geq 2$ and $\xi \geq 0$. The time-spatial-noise σ is said to be globally Lipschitz and written by $\sigma \in \mathcal{L}_{glo}(L^q_\omega \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$ if $\sigma : \overline{J}\times L^q_\omega \dot{H}^\xi_x \longrightarrow L^2_0(\dot{H}^\xi_x)$ and there exists a positive function $K_2 := K_2(t)$ on $\overline{J}$ satisfying

$$\mathbb{E}\left\|\sigma(t,w^\dagger) - \sigma(t,w^\ddagger)\right\|^q_{L^2_0(\dot{H}^\xi_x)} \leq K_2^q(t)\,\mathbb{E}\left\|w^\dagger - w^\ddagger\right\|^q_{\dot{H}^\xi_x},$$

for all $w^\dagger$, $w^\ddagger$ in $L^q_\omega \dot{H}^\xi_x$ and $t \in \overline{J}$.

3 Existence, regularity of the solution of the initial value problem

3.1 Mild solution for initial value problem

In this subsection, we give the definition of mild solution to Problem (1)-(3). To do this, let us first consider the following deterministic problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta - \varkappa\frac{\partial^\alpha}{\partial t^\alpha}\Delta\right) u = F(t,u), & (t,x)\in J\times\mathcal{X}, \\ u(t,x) = 0, & (t,x)\in J\times\partial\mathcal{X}, \\ u(0,x) = \psi(x), & x\in\mathcal{X}, \end{cases} \tag{6}$$

We now express $u(t,.)$ in the form $u(t,x) = \sum_{j=1}^\infty u_j(t)\phi_j^{(\xi)}(x)$, where we denote $u_j(t) := \langle u(t,.),\phi_j^{(\xi)}\rangle_{\dot{H}^\xi_x}$. Due to the arguments in [1], the coefficients $u_j(t)$ have the following expression

$$u_j(t) = \mathcal{R}_{\alpha,j}(t)\psi_j + \int_0^t \mathcal{R}_{\alpha,j}(t-s)F_j(s,.,u(s,.))ds, \tag{7}$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

whereupon

$$\mathcal{R}_{\alpha,j}(t) = \int_0^\infty e^{-st}\mathcal{K}_{\alpha,j}(s)ds$$
$$\mathcal{K}_{\alpha,j}(s) = \frac{\varkappa\lambda_j s^\alpha \sin\alpha\pi}{\pi\left(-s+\varkappa\lambda_j s^\alpha\cos\alpha\pi+\lambda_j\right)^2+\pi\left(\varkappa\lambda_j s^\alpha\sin\alpha\pi\right)^2}. \tag{8}$$

For fixed s, $F_j(s,.,u(s,.))$ denotes by the j-th coefficient of $x\mapsto F(s,x,u(s,x))$ in $\dot{H}^\xi_x$. For the sake of convenience, for $t\in\overline{J}$, we define

$$\mathcal{G}_\alpha(t) := \sum_{j=1}^\infty \langle\cdot,\phi_j^{(\xi)}\rangle_{\dot{H}^\xi_x}\mathcal{R}_{\alpha,j}(t)\phi_j^{(\xi)}, \tag{9}$$

Then, we have the following representation for the solution to Problem (6)

$$u(t,x) = \mathcal{G}_\alpha(t)\psi(x) + \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds. \tag{10}$$

Motivated by (10), we state the definitions of mild solution to the initial value problem (1)-(3) as follows:

Definition 3.1. An $\mathcal{F}_t$-adapted process $\{u(t,.)\}_{t\in\overline{J}}$ in $C^\theta_{t,*}L^q_\omega\dot{H}^\xi_x$ with some $\theta\in(0,1)$, $q\geq 2$, $\xi\geq 0$, is called a mild solution of Problem (1) if there holds that

$$u(t,x) = \mathcal{G}_\alpha(t)\psi(x) + \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds$$
$$+\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,u(s,x))d\mathcal{W}_\xi(s,x). \tag{11}$$

for almost all ω for fixed t.

3.2 Properties of solution operators

We begin with the following lemma which shows basic bounds for the functions $\mathcal{R}_{\alpha,j}$, $j\geq 1$.

Lemma 3.1. *Given $\alpha\in(0,1)$, $\delta\in[0,2]$, $\epsilon>0$ and $j\in\mathbb{Z}^+$. Then*

a) $\mathcal{K}_{\alpha,j}(s)\leq c_{\alpha,1}\lambda_j^{-1}s^{-\alpha}$, for all $s>0$. Furthermore, there exists $s_0>0$ such that

$$\mathcal{K}_{\alpha,j}(s)\leq \mathcal{C}_{\alpha,1}\lambda_j^{\delta-1}s^{\alpha-\delta},\ \text{for all } s\geq s_0, \tag{12}$$

where $\mathcal{C}_{\alpha,1}$, $c_{\alpha,1}$ are two positive constants.

b) $\mathcal{R}_{\alpha,j}(0)=1$ and $0<\mathcal{R}_{\alpha,j}(t)\leq \dfrac{c_{\alpha,2}}{1+\lambda_j t^{1-\alpha}}$, for $t\in\overline{J}_*$, where $c_{\alpha,2}$ is a positive constant.

c) The classical derivative of $\mathcal{R}_{\alpha,j}(t)$ finitely exists for $t\in\overline{J}_*$ and is bounded as follows

$$\frac{d}{dt}\mathcal{R}_{\alpha,j}(t)\leq c_{\alpha,3}\big(\lambda_j^{-1}+\lambda_j^{\delta-1}t^{-(2+\alpha-\delta+\epsilon)}\big), \qquad t\in\overline{J}_*, \tag{13}$$

where $c_{\alpha,3}$ is a positive constant.

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

Proof. a) Consider $\mathcal{K}_{\alpha,j}(s)$ as defined in (8). Then,

$$\begin{aligned}
&(-s+\kappa\lambda_j s^\alpha \cos\alpha\pi+\lambda_j)^2+(\kappa\lambda_j s^\alpha \sin\alpha\pi)^2\\
&= s^2+\kappa^2\lambda_j^2 s^{2\alpha}\left(\cos^2\alpha\pi+\sin^2\alpha\pi\right)+\lambda_j^2-2\lambda_j s-2\kappa\lambda_j s^{\alpha+1}\cos\alpha\pi+2\kappa\lambda_j^2 s^\alpha\cos\alpha\pi\\
&= s^2+\lambda_j^2+\lambda_j^2\kappa s^\alpha(\kappa s^\alpha-2\cos\alpha\pi)+2\lambda_j s(\kappa s^\alpha\cos\alpha\pi-1),
\end{aligned}$$

which show that there exists $s_0>0$ such that

$$(-s+\kappa\lambda_j s^\alpha \cos\alpha\pi+\lambda_j)^2+(\kappa\lambda_j s^\alpha \sin\alpha\pi)^2 \geq s^2+\lambda_j^2, \quad \text{for all } s\geq s_0.$$

We now estimate $s^2+\lambda_j^2$ by applying the Young inequality. For $\delta\in(0,2)$, we set $p:=p(\delta)=\frac{2}{\delta}$, $q:=q(\delta)=\frac{2}{2-\delta}$. Since $\frac{1}{p}+\frac{1}{q}=1$, the Young inequality yields $\frac{s^p}{p}+\frac{\lambda_j^q}{q}\geq s\lambda_j$. It follows that there exists a positive constant $\mathcal{C}_0$ such that

$$s^2+\lambda_j^2\geq \mathcal{C}_0 s^\delta\lambda_j^{2-\delta}, \quad \text{for } \delta\in(0,2). \tag{14}$$

On the other hand, it is clear that (14) holds for $\delta=0$ or $\delta=2$. Hence, there exists a positive constant $\mathcal{C}_{\alpha,1}$ such that

$$\mathcal{K}_{\alpha,j}(s)\leq \frac{\varkappa\lambda_j s^\alpha\sin\alpha\pi}{\pi(s^2+\lambda_j^2)}\leq \mathcal{C}_{\alpha,1}\lambda_j^{\delta-1}s^{\alpha-\delta}, \quad \text{for all } s\geq s_0,\ \delta\in[0,2]. \tag{15}$$

In addition, since $(-s+\kappa\lambda_j s^\alpha\cos\alpha\pi+\lambda_j)^2+(\kappa\lambda_j s^\alpha\sin\alpha\pi)^2\geq(\kappa\lambda_j s^\alpha\sin\alpha\pi)^2$, there exists $c_{\alpha,1}>0$ such that

$$\mathcal{K}_{\alpha,j}(s)\leq \frac{\varkappa\lambda_j s^\alpha\sin\alpha\pi}{\pi(\varkappa\lambda_j s^\alpha\sin\alpha\pi)^2}\leq c_{\alpha,1}\lambda_j^{-1}s^{-\alpha}, \quad \text{for all } s>0. \tag{16}$$

b) The proof of part b) can be found in [16].
c) For $t,r\in\overline{J}_*$ such that $r>t$, we will estimate $|\mathcal{R}_{\alpha,j}(r)-\mathcal{R}_{\alpha,j}(t)|$. We have

$$\begin{aligned}
|\mathcal{R}_{\alpha,j}(r)-\mathcal{R}_{\alpha,j}(t)| &\leq \int_0^{s_0}|e^{-sr}-e^{-st}|\mathcal{K}_{\alpha,j}(s)ds+\int_{s_0}^\infty|e^{-sr}-e^{-st}|\mathcal{K}_{\alpha,j}(s)ds\\
&=: \mathcal{I}_1+\mathcal{I}_2.
\end{aligned} \tag{17}$$

Noting that $|e^{-sr}-e^{-st}|=|e^{-st}\left(1-e^{-s(r-t)}\right)|\leq se^{-st}(r-t)$. For $\mathcal{I}_1$, one can see from (16) that

$$\mathcal{I}_1\lesssim\int_0^{s_0}se^{-st}(r-t)\mathcal{K}_{\alpha,j}(s)ds\leq c_{\alpha,1}(r-t)\lambda_j^{-1}\int_0^{s_0}e^{-st}s^{1-\alpha}ds. \tag{18}$$

For $\mathcal{I}_2$, it follows from (15) that

$$\mathcal{I}_2\leq\mathcal{C}_{\alpha,1}\int_{s_0}^\infty se^{-st}(r-t)\lambda_j^{\delta-1}s^{\alpha-\delta}ds\leq\mathcal{C}_{\alpha,1}(r-t)\lambda_j^{\delta-1}\int_{s_0}^\infty e^{-st}s^{1+\alpha-\delta}ds, \quad \delta\in[0,2].$$

Noting that there always exists $\mathcal{C}_{\alpha,2}>0$ such that $e^{-st}\leq\mathcal{C}_{\alpha,2}(st)^{-(2+\alpha-\delta+\epsilon)}$, for all $\epsilon>0$, then

$$\begin{aligned}
\mathcal{I}_2 &\leq \mathcal{C}_{\alpha,1}\mathcal{C}_{\alpha,2}(r-t)\lambda_j^{\delta-1}\int_{s_0}^\infty t^{-(2+\alpha-\delta+\epsilon)}s^{-1-\epsilon}ds\\
&\leq \mathcal{C}_{\alpha,1}\mathcal{C}_{\alpha,2}(r-t)\lambda_j^{\delta-1}t^{-(2+\alpha-\delta+\epsilon)}\int_{s_0}^\infty s^{-1-\epsilon}ds.
\end{aligned} \tag{19}$$

From (17)-(19), we deduce that there exists $c_{\alpha,3} > 0$ such that

$$|\mathcal{R}_{\alpha,j}(r) - \mathcal{R}_{\alpha,j}(t)| \le c_{\alpha,3}(r-t)\left(\lambda_j^{-1} + \lambda_j^{\delta-1} t^{-(2+\alpha-\delta+\epsilon)}\right).$$

This implies that the classical derivative of $\mathcal{R}_{\alpha,j}(t)$ finitely exists for $t \in \overline{J}_*$ and is bounded as

$$\frac{d}{dt}\mathcal{R}_{\alpha,j}(t) \le c_{\alpha,3}\big(\lambda_j^{-1} + \lambda_j^{\delta-1} t^{-(2+\alpha-\delta+\epsilon)}\big), \qquad t \in \overline{J}_*.$$

This completes the proof. □

The following lemma gives the appropriate estimates for the solution operators. From now on, if an operator $\mathcal{G} : L^q_\omega \dot{H}^{\xi'} \to L^q_\omega \dot{H}^{\xi}$, for $\xi, \xi' \ge 0$, satisfies

$$\|\mathcal{G}v\|_{L^q_\omega \dot{H}^\xi} \le C\, \|v\|_{L^q_\omega \dot{H}^{\xi'}}, \qquad v \in L^q_\omega \dot{H}^{\xi'}, \tag{20}$$

we will write $\|\mathcal{G}\|_{L^q_\omega \dot{H}^{\xi'} \to L^q_\omega \dot{H}^\xi}$ for short.

Lemma 3.2. *Given $t \in \overline{J}$, $\xi \ge 0$ and $q \ge 2$. Then, there holds*

$$\|\mathcal{G}_\alpha(t)\|_{L^q_\omega \dot{H}^\xi_x \to L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2}.$$

Proof. Let $v \in L^q_\omega \dot{H}^\xi_x$ and $v_j := \langle v, \phi_j \rangle$. By applying Lemma 3.1 and the fact that $1 + \lambda_j t^{1-\alpha} \ge 1$, it is obvious that $\mathcal{R}_{\alpha,j}(t) \le c_{\alpha,2}$, for $t \in \overline{J}$. It follows that

$$\|\mathcal{G}_\alpha(t)v\|_{L^q_\omega \dot{H}^\xi_x} = \left[\mathbb{E}\left(\sum_{j=1}^{\infty} \mathcal{R}^2_{\alpha,j}(t)\lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \le c_{\alpha,2}\|v\|_{L^q_\omega \dot{H}^\xi_x}.$$

This completes the proof. □

Lemma 3.3. *Consider $\xi \ge 0$ and $q \ge 2$. Let ϵ_1, ϵ_2 be two positive constants satisfying $\alpha + \epsilon_1 < 1$ and $\epsilon_2 < \min\{\alpha + \epsilon_1, 1 - (\alpha + \epsilon_1)\}$. There exists $c_{\alpha,4}, c_{\alpha,5} > 0$ such that, for $t, r \in \overline{J}_*$ satisfying $0 < r - t < 1$, it holds*

$$\big\|\mathcal{G}_\alpha(r) - \mathcal{G}_\alpha(t)\big\|_{L^q_\omega \dot{H}^{\xi+1}_x \to L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,4}(r-t)^{1-(\alpha+\epsilon_1)}, \tag{21}$$

$$\big\|\mathcal{G}_\alpha(r) - \mathcal{G}_\alpha(t)\big\|_{L^q_\omega \dot{H}^{\xi}_x \to L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,5}(r-t)^{\epsilon_2} t^{-(\alpha+\epsilon_1+\epsilon_2)}. \tag{22}$$

Proof. Let $v \in L^q_\omega \dot{H}^\xi_x$ and $\mathcal{P}_{\alpha,j}(t) := \frac{d}{dt}\mathcal{R}_{\alpha,j}(t)$. Applying Part c of Lemma 3.1 implies that

$$\begin{aligned}
\big\|\mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v\big\|_{L^q_\omega \dot{H}^\xi_x} &= \left[\mathbb{E}\left(\sum_{j=1}^{\infty} |\mathcal{R}_{\alpha,j}(r) - R_{\alpha,j}(t)|^2\, \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\
&\le \left[\mathbb{E}\left(\sum_{j=1}^{\infty} \left|\int_t^r \mathcal{P}_{\alpha,j}(s)ds\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\
&\le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty} \left|\int_t^r \left(\lambda_j^{-1} + \lambda_j^{\delta-1} s^{-(2+\alpha-\delta+\epsilon)}\right) ds\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q},
\end{aligned}$$

where $\delta \in [0, 2]$ and $\epsilon > 0$.

• Firstly, we prove that (21) holds. By choosing $\delta = 2$ and $\epsilon = \epsilon_1$, one arrives at

$$\begin{aligned}
&\left\|\mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v\right\|_{L^q_\omega \dot{H}^\xi_x} \\
&\quad \le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\lambda_j^{-1}(r-t) + \lambda_j \frac{r^{1-(\alpha+\epsilon_1)} - t^{1-(\alpha+\epsilon_1)}}{1-(\alpha+\epsilon_1)}\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\
&\quad \le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\lambda_1^{-2}(r-t) + \frac{r^{1-(\alpha+\epsilon_1)} - t^{1-(\alpha+\epsilon_1)}}{1-(\alpha+\epsilon_1)}\right|^2 \lambda_j^{2(\xi+1)} v_j^2\right)^{q/2}\right]^{1/q}.
\end{aligned}$$

It follows that there exists $c_{\alpha,4} > 0$ such that

$$\begin{aligned}
\left\|\mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v\right\|_{L^q_\omega \dot{H}^\xi_x} &\le c_{\alpha,4}(r-t)^{1-(\alpha+\epsilon_1)}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\lambda_j^{2(\xi+1)} v_j^2\right)^{q/2}\right]^{1/q} \\
&\le c_{\alpha,4}(r-t)^{1-(\alpha+\epsilon_1)}\left\|v\right\|_{L^q_\omega \dot{H}^{\xi+1}_x}.
\end{aligned}$$

• Next, we prove that (22) holds. By choosing $\delta = 1$, $\epsilon = \epsilon_1$, one also have

$$\begin{aligned}
&\left\|\mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v\right\|_{L^q_\omega \dot{H}^\xi_x} \\
&\quad \le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\int_t^r \left(\lambda_j^{-1} + s^{-(1+\alpha+\epsilon_1)}\right) ds\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\
&\quad \le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\lambda_j^{-1}(r-t) + \frac{t^{-(\alpha+\epsilon_1)} - r^{-(\alpha+\epsilon_1)}}{\alpha+\epsilon_1}\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\
&\quad \le c_{\alpha,3}\left[\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\lambda_1^{-1}(r-t) + \frac{r^{\alpha+\epsilon_1} - t^{\alpha+\epsilon_1}}{(\alpha+\epsilon_1)t^{\alpha+\epsilon_1} r^{\alpha+\epsilon_1}}\right|^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q}.
\end{aligned}$$

Since $r^{\alpha+\epsilon_1} \ge (r-t)^{\alpha+\epsilon_1-\epsilon_2} t^{\epsilon_2}$, one can see

$$\frac{r^{\alpha+\epsilon_1} - t^{\alpha+\epsilon_1}}{t^{\alpha+\epsilon_1} r^{\alpha+\epsilon_1}} \le \frac{(r-t)^{\alpha+\epsilon_1}}{t^{\alpha+\epsilon_1+\epsilon_2}(r-t)^{\alpha+\epsilon_1-\epsilon_2}} = (r-t)^{\epsilon_2} t^{-(\alpha+\epsilon_1+\epsilon_2)}.$$

By two latter estimates, we deduce that there exists $c_{\alpha,5} > 0$ such that

$$\left\|\mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v\right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,5}(r-t)^{\epsilon_2} t^{-(\alpha+\epsilon_1+\epsilon_2)} \left\|v\right\|_{L^q_\omega \dot{H}^\xi_x}.$$

The proof is completed. □

3.3 The existence and uniqueness of mild solution to Problem (1),(2),(3)

In this part, we state the main results for Problem (1),(2),(3). The first theorem states the existence and uniqueness of the solution to Problem (1)-(3) in the space $L^\infty_t L^q_\omega \dot{H}^\xi_x$. The second theorem investigates the regularity of the solution on the space $C^\gamma_{t,*} L^q_\omega \dot{H}^\xi_x$, for any positive constant γ satisfying $\gamma < \min\{1-\alpha, \frac{1}{q}\}$.

Theorem 3.1. *Let $\xi \ge 0$ and $q \ge 2$. Assume that $\psi \in L^q_\omega \dot{H}^\xi_x$, $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$, $F(\cdot,\cdot,0) = 0$, and $\sigma \in \mathcal{L}_{glo}(L^q_\omega \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$, $\sigma(\cdot,0) = 0$. Assume further that $K_1, K_2 \in L^\infty_t$. Then, there exists $\overline{\rho} > 0$ such that Problem (1)-(3) has a unique solution $u \in L^\infty_{t,\overline{\rho}} L^q_\omega \dot{H}^\xi_x$.*

Proof. Let ρ be a positive constant and $\Phi: L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x \to L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x$ be defined as follows

$$\Phi w(t,x) = \mathcal{G}_\alpha(t)\psi(x) + \int_0^t \mathcal{G}_\alpha(t-s) F(s,x,w(s,x))ds + \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,x))d\mathcal{W}_\xi(s,x).$$

The proof is divided into two steps. In the first step, we show that Φ is well-defined on $L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x$. In the second step, we apply Banach fixed point theorem to show that there exists $\overline{\rho} > 0$ such that Problem (1)-(3) has a unique solution $u \in L^\infty_{t,\overline{\rho}} L^q_\omega \dot{H}^\xi_x$.

Step 1. Firstly, it is required to show that Φ is well-defined on $L^\infty_t L^q_\omega \dot{H}^\xi_x$. The first term $\mathcal{G}_\alpha(t)\psi(x)$ can be estimated by using Lemma 3.2 as $\|\mathcal{G}_\alpha(t)\psi\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|\psi\|_{L^q_\omega \dot{H}^\xi_x}$ for $t \in \overline{J}$. This leads to

$$e^{-\rho t} \|\mathcal{G}_\alpha(t)\psi\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} e^{-\rho t} \|\psi\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|\psi\|_{L^q_\omega \dot{H}^\xi_x},$$

which implies that $\|\mathcal{G}_\alpha(t)\psi\|_{L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|\psi\|_{L^q_\omega \dot{H}^\xi_x}$. For the second term, by applying Lemma 3.2, $F(\cdot,\cdot,0) = 0$, and $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$, one can see

$$\begin{aligned}\left\| \int_0^t \mathcal{G}_\alpha(t-s) F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} &\le \int_0^t \|\mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))\|_{L^q_\omega \dot{H}^\xi_x} ds \\ &\le c_{\alpha,2} \int_0^t \left(\mathbb{E} \|F(s,\cdot,w(s,\cdot))\|^q_{\dot{H}^\xi_x} \right)^{\frac{1}{q}} ds \\ &\le c_{\alpha,2} \|K_1\|_{L^\infty_t} \int_0^t \left(\mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x} \right)^{\frac{1}{q}} ds, \end{aligned} \tag{23}$$

where we have used $\mathbb{E}\|F(s,\cdot,w(s,\cdot))\|^q_{\dot{H}^\xi_x} \le \|K_1\|^q_{L^\infty_t} \mathbb{E}\|w(s,.)\|^q_{\dot{H}^\xi_x}$. It follows that

$$e^{-\rho t} \left\| \int_0^t \mathcal{G}_\alpha(t-s) F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|K_1\|_{L^\infty_t} \int_0^t e^{-\rho(t-s)} e^{-\rho s} \|w(s,\cdot)\|_{L^q_\omega \dot{H}^\xi_x} ds.$$

Using the fact that $\int_0^t e^{-\rho(t-s)} ds = \rho^{-1}(1 - e^{-\rho t}) = \rho^{-1}$, we deduce that

$$\left\| \int_0^t \mathcal{G}_\alpha(t-s) F(s,\cdot,w(s,\cdot))ds \right\|_{L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|K_1\|_{L^\infty_t} \rho^{-1} \|w\|_{L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x}.$$

We continue to estimate the last term by considering two following cases.

- If $q = 2$, the Itô isometry yields

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \\ &= \left(\mathbb{E} \int_0^t \|\mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} ds \right)^{\frac{1}{q}} \\ &\le \left(\int_0^t \|\mathcal{G}_\alpha(t-s)\|^q_{\dot{H}^\xi_x \to \dot{H}^\xi_x} \mathbb{E} \|\sigma(s,w(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} ds \right)^{\frac{1}{q}}. \end{aligned} \tag{24}$$

Lemma 3.2, $\psi(\cdot,0)=0$, and $\psi\in\mathcal{L}_{glo}(L^q_\omega\dot{H}^\xi_x;L^2_0(\dot{H}^\xi_x))$ allow that

$$\begin{aligned}
&e^{-\rho t}\left\|\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega\dot{H}^\xi_x}\\
&\quad\le c_{\alpha,2}e^{-\rho t}\left(\int_0^t K_2^q(s)\mathbb{E}\left\|w(s,.)\right\|^q_{\dot{H}^\xi_x}ds\right)^{\frac{1}{q}}\\
&\quad\le c_{\alpha,2}\left\|K_2\right\|_{L^\infty_t}\left(\int_0^t e^{-q\rho(t-s)}e^{-q\rho s}\left\|w(s,\cdot)\right\|^q_{L^q_\omega\dot{H}^\xi_x}ds\right)^{\frac{1}{q}}\\
&\quad\le c_{\alpha,2}\left\|K_2\right\|_{L^\infty_t}\left\|w\right\|_{L^\infty_{t,\rho}L^q_\omega\dot{H}^\xi_x}\left(\int_0^t e^{-q\rho(t-s)}ds\right)^{\frac{1}{q}}\\
&\quad\le c_{\alpha,2}\left\|K_2\right\|_{L^\infty_t}\left\|w\right\|_{L^\infty_{t,\rho}L^q_\omega\dot{H}^\xi_x}q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}. \qquad (25)
\end{aligned}$$

where we have used the fact that $\mathbb{E}\left\|\sigma(s,w(s,\cdot))\right\|^q_{L^2_0(\dot{H}^\xi_x)}\le K_2^q(s)\mathbb{E}\left\|w(s,\cdot)\right\|^q_{\dot{H}^\xi_x}$.

- If $q>2$, the consequence of Burkholder-Davis-Gundy's inequality yields

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega\dot{H}^\xi_x}\\
&\quad\le C(q)\left[\mathbb{E}\left(\int_0^t\left\|\mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))\right\|^2_{L^2_0(\dot{H}^\xi_x)}ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\\
&\quad\le C(q)\left[\mathbb{E}\left(\int_0^t\left\|\mathcal{G}_\alpha(t-s)\right\|^2_{\dot{H}^\xi_x\to\dot{H}^\xi_x}\left\|\sigma(s,w(s,\cdot))\right\|^2_{L^2_0(\dot{H}^\xi_x)}ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\\
&\quad\le c_{\alpha,2}C(q)\left[\mathbb{E}\left(\int_0^t\left\|\sigma(s,w(s,\cdot))\right\|^2_{L^2_0(\dot{H}^\xi_x)}ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}. \qquad (26)
\end{aligned}$$

From the Hölder inequality we deduce

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega\dot{H}^\xi_x}\\
&\le c_{\alpha,2}C(q)\left(\int_0^t ds\right)^{(q-2)/(2q)}\left(\int_0^t\mathbb{E}\left\|\sigma(s,w(s,\cdot))\right\|^q_{L^2_0(\dot{H}^\xi_x)}ds\right)^{1/q}\\
&\le c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\left\|K_2\right\|_{L^\infty_t}\left(\int_0^t\mathbb{E}\left\|w(s,\cdot)\right\|^q_{\dot{H}^\xi_x}ds\right)^{1/q}, \qquad (27)
\end{aligned}$$

where we recall that $\sigma(\cdot,0)=0$, $\sigma\in\mathcal{L}_{glo}(L^q_\omega\dot{H}^\xi_x;L^2_0(\dot{H}^\xi_x))$. Hence, we derive the estimate

$$\begin{aligned}
&e^{-\rho t}\left\|\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega\dot{H}^\xi_x}\\
&\le c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\left\|K_2\right\|_{L^\infty_t}\left(\int_0^t e^{-q\rho(t-s)}e^{-q\rho s}\left\|w(s,\cdot)\right\|^q_{L^q_\omega\dot{H}^\xi_x}ds\right)^{1/q}\\
&\le c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\left\|K_2\right\|_{L^\infty_t}\left\|w\right\|_{L^\infty_{t,\rho}L^q_\omega\dot{H}^\xi_x}q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}. \qquad (28)
\end{aligned}$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

By the above arguments, one can see

$$\|\Phi w\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x} \le c_{\alpha,2}\|\psi\|_{L^q_\omega \dot H^\xi_x} + \left(c_{\alpha,2}T\|K_1\|_{L^\infty_t}\rho^{-1} + c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\|K_2\|_{L^\infty_t} q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}\right)\|w\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x},$$

which implies that Φ is well-defined on $L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x$.

Step 2. Next, we will apply Banach's fixed point theorem to show that Φ has a unique fixed point. For $w \in L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x$, we define

$$\Phi_1 w(t,x) := \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,w(s,x))ds,$$
$$\Phi_2 w(t,x) := \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,x))d\mathcal{W}_\xi(s,x).$$

For $w^\dagger, w^\ddagger \in L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x$, by a similar way as in (23), it is obvious that

$$\begin{aligned} & e^{-\rho t}\left\|\Phi_1 w^\dagger(t,\cdot) - \Phi_1 w^\ddagger(t,\cdot)\right\|_{L^q_\omega \dot H^\xi} \\ & \le e^{-\rho t}\int_0^t \left\|\mathcal{G}_\alpha(t-s)\left(F(s,\cdot,w^\dagger(s,\cdot)) - F(s,\cdot,w^\ddagger(s,\cdot))\right)\right\|_{L^q_\omega \dot H^\xi} ds \\ & \le c_{\alpha,2}\int_0^t e^{-\rho(t-s)}e^{-\rho s}\left(\mathbb{E}\left\|w^\dagger(s,\cdot) - w^\ddagger(s,\cdot)\right\|^q_{\dot H^\xi}\right)^{\frac{1}{q}} ds \\ & \le c_{\alpha,2}\|K_1\|_{L^\infty_t}\left\|w^\dagger - w^\ddagger\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x}\rho^{-1}. \end{aligned} \tag{29}$$

We continue with the second operator Φ_2 in the two following cases.

• If $q = 2$, similarly to (24)-(25), one arrives at

$$e^{-\rho t}\left\|\Phi_2 w^\dagger(t,\cdot) - \Phi_2 w^\ddagger(t,\cdot)\right\|_{L^q_\omega \dot H^\xi} \le c_{\alpha,2}\|K_2\|_{L^\infty_t}\left\|w^\dagger - w^\ddagger\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x} q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}. \tag{30}$$

• If $q > 2$, by a similar way as in (26)-(28), one arrives at

$$e^{-\rho t}\left\|\Phi_2 w^\dagger(t,\cdot) - \Phi_2 w^\ddagger(t,\cdot)\right\|_{L^q_\omega \dot H^\xi} \le c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\|K_2\|_{L^\infty_t}\left\|w^\dagger - w^\ddagger\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi_x} q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}. \tag{31}$$

Combining (29), (30), (31), we deduce that

$$\begin{aligned} & \left\|\Phi w^\dagger(t,\cdot) - \Phi w^\ddagger(t,\cdot)\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi} \\ & \le \left\|\Phi_1 w^\dagger(t,\cdot) - \Phi_1 w^\ddagger(t,\cdot)\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi} + \left\|\Phi_2 w^\dagger(t,\cdot) - \Phi_2 w^\ddagger(t,\cdot)\right\|_{L^\infty_{t,\rho}L^q_\omega \dot H^\xi} \\ & \le \left(c_{\alpha,2}T\|K_1\|_{L^\infty_t}\rho^{-1} + c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\|K_2\|_{L^\infty_t} q^{-\frac{1}{q}}\rho^{-\frac{1}{q}}\right)\left\|w^\dagger - w^\ddagger\right\|_{L^\infty_t L^q_\omega \dot H^\xi_x}. \end{aligned} \tag{32}$$

By choosing $\rho = \overline{\rho}$ with $\overline{\rho}$ is large enough such that

$$c_{\alpha,2}T\|K_1\|_{L^\infty_t}\overline{\rho}^{-1} + c_{\alpha,2}C(q)T^{\frac{q-2}{2q}}\|K_2\|_{L^\infty_t} q^{-\frac{1}{q}}\overline{\rho}^{-\frac{1}{q}} < 1,$$

the estimate (32) shows that Φ is a contraction on $L^\infty_{t,\overline{\rho}}L^q_\omega \dot H^\xi_x$. Hence, we conclude that Problem (1)-(3) has a unique solution $u \in L^\infty_{t,\overline{\rho}}L^q_\omega \dot H^\xi_x$. □

Remark 3.1. *In Theorem 3.1, the function σ is defined by Definition 2.2. Here, we can find a constant C_ρ such that $C_\rho \to 0$ as $\rho \to \infty$, and $e^{-\rho t}\|\Phi_2 w^\dagger(t,\cdot) - \Phi_2 w^\ddagger(t,\cdot)\|_{L^q_\omega \dot{H}^\xi}$ is bounded by $C_\rho \|w^\dagger - w^\ddagger\|_{L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x}$ (see (30)-(31)). Therewith, the global existence has been obtained consequently. However, we note that the above result does not hold true if σ is locally Lipschitz continuous. For example, let us consider $q = 2$ for simplicity and assume that σ satisfies the following locally Lipschitz continuity*

$$\|\sigma(t,w^\dagger) - \sigma(t,w^\ddagger)\|_{L^2_\omega L^2_0(\dot{H}^\xi_x)} \leq \left(\|w^\dagger\|_{L^2_\omega \dot{H}^\xi_x} + \|w^\ddagger\|_{L^2_\omega \dot{H}^\xi_x}\right) \|w^\dagger - w^\ddagger\|_{L^2_\omega \dot{H}^\xi_x}. \quad (33)$$

Let U_R be the closed ball of center at zero and radius R in $L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x$. For $w^\dagger$, $w^\ddagger$ taken in this ball, the following chain can be obtained by making uses of Lemma 3.2 and the condition (33)

$$\begin{aligned} &e^{-\rho t}\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^2_\omega \dot{H}^\xi_x} \\ &\quad \leq e^{-\rho t}\left(\int_0^t \|\mathcal{G}_\alpha(t-s)\|^2_{\dot{H}^\xi_x \to \dot{H}^\xi_x} \mathbb{E}\,\|\sigma(s,w(s,\cdot))\|^2_{L^2_0(\dot{H}^\xi_x)}\, ds\right)^{1/2} \\ &\quad \leq c_{\alpha,2} e^{-\rho t}\left(\int_0^t \left(\|w^\dagger(s,\cdot)\|_{L^2_\omega \dot{H}^\xi_x} + \|w^\ddagger(s,\cdot)\|_{L^2_\omega \dot{H}^\xi_x}\right)^2 \|w^\dagger(s,\cdot) - w^\ddagger(s,\cdot)\|^2_{L^2_\omega \dot{H}^\xi_x} ds\right)^{1/2} \\ &\quad \leq c_{\alpha,2} e^{-\rho t}\left(\int_0^t 4R^2 e^{4\rho s}\left(e^{-\rho s}\|w^\dagger(s,\cdot) - w^\ddagger(s,\cdot)\|_{L^2_\omega \dot{H}^\xi_x}\right)^2 ds\right)^{1/2}. \end{aligned}$$

By taking the essential supremum of $e^{-\rho s}\|w^\dagger(s,\cdot) - w^\ddagger(s,\cdot)\|_{L^2_\omega \dot{H}^\xi_x}$ on $(0,T)$, we imply from the above chain that

$$\begin{aligned} &e^{-\rho t}\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^2_\omega \dot{H}^\xi_x} \\ &\qquad\qquad \leq 2Rc_{\alpha,2}\left(\int_0^t e^{4\rho s - 2\rho t} ds\right)^{1/2} \|w^\dagger - w^\ddagger\|_{L^\infty_{t,\rho} L^q_\omega \dot{H}^\xi_x}. \end{aligned}$$

A direct computation shows that $e^{-2\rho t}\int_0^t e^{4\rho s} ds$ equals $(e^{2\rho t} - e^{-2\rho t})/(4\rho)$, and so it tends to positive infinity as ρ approaches positive infinity. Therefore, we cannot derive the contraction of the mapping Φ analogously Proof of Theorem 3.1.

It is well-known that, constructing global existence of problems that including locally Lipschitz continuous nonlinearity and noise is not an easy task. In the future, we will study global existence of RSE with more a general assumption than (33).

Now, we study the regularity of the solution in the space $L^\infty_t L^q_\omega \dot{H}^\xi_x$. From now on, we use $a \lesssim b$ to describe that there exists a positive constant $\mathcal{C}$ such that $a \leq \mathcal{C} b$.

Theorem 3.2. *Consider $\xi \geq 0$ and $q \geq 2$. Assume that ψ, F, σ satisfy the conditions of Theorem 3.1. Then, the following regularity estimate holds*

$$\|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x} \lesssim \|\psi\|_{L^q_\omega \dot{H}^\xi_x}.$$

Proof. We aim to estimate $\|u(t,\cdot)\|_{L^q_\omega \dot{H}^\xi_x}$ firstly by using some materials which have been proved in the proof of Theorem 3.1. Recall that $\|\mathcal{G}_\alpha(t)\psi\|_{L^q_\omega \dot{H}^\xi_x} \leq c_{\alpha,2}\|\psi\|_{L^q_\omega \dot{H}^\xi_x}$ for

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

$t \in \overline{J}$. From (23), we have

$$\left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,u(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \|K_1\|_{L^\infty_t} \int_0^t \|u(s,\cdot)\|_{L^q_\omega \dot{H}^\xi_x} ds$$
$$\le c_{\alpha,2} \|K_1\|_{L^\infty_t} T^{\frac{q-1}{q}} \left(\int_0^t \|u(s,.)\|^q_{L^q_\omega \dot{H}^\xi_x} \right)^{\frac{1}{q}} ds.$$

• If $q = 2$, estimate (24) leads to

$$\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,u(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \left(\int_0^t K_2^q(s) \mathbb{E} \|u(s,\cdot)\|^q_{\dot{H}^\xi_x} ds \right)^{\frac{1}{q}}$$
$$\le c_{\alpha,2} \|K_2\|_\infty \left(\int_0^t \|u(s,\cdot)\|^q_{L^q_\omega \dot{H}^\xi_x} ds \right)^{\frac{1}{q}}. \quad (34)$$

• If $q > 2$, estimate (27) implies

$$\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,u(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x}$$
$$\le c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \|K_2\|_{L^\infty_t} \left(\int_0^t \|u(s,\cdot)\|^q_{L^q_\omega \dot{H}^\xi_x} ds \right)^{\frac{1}{q}}. \quad (35)$$

By the above arguments, one can see there exists $\mathcal{M}_{q,\alpha,1}, \mathcal{M}_{q,\alpha,2}(T) > 0$ such that

$$\|u(t,\cdot)\|^q_{L^q_\omega \dot{H}^\xi_x} \le \mathcal{M}_{q,\alpha,1} \|\psi\|^q_{L^q_\omega \dot{H}^\xi_x} + \mathcal{M}_{q,\alpha,2}(T) \int_0^t \|u(s,\cdot)\|^q_{L^q_\omega \dot{H}^\xi_x} ds.$$

By applying the Gronwall inequality (see [28]),

$$\|u(t,\cdot)\|^q_{L^q_\omega \dot{H}^\xi_x} \le \mathcal{M}_{q,\alpha,1} \|\psi\|^q_{L^q_\omega \dot{H}^\xi_x} \exp\left(t\mathcal{M}_{q,\alpha,2}(T)\right) \le \mathcal{M}_{q,\alpha,1} \|\psi\|^q_{L^q_\omega \dot{H}^\xi_x} \exp\left(T\mathcal{M}_{q,\alpha,2}(T)\right),$$

which implies that $\|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x} \lesssim \|\psi\|_{L^q_\omega \dot{H}^\xi_x}$. □

Next, we prove a regularity result for the solution in the space $C^{\epsilon_2}_{t,*} L^q_\omega \dot{H}^\xi_x$. For $b \in (0,1)$ and $q \ge 1$, we define

$$\mathbb{K}^{q,b}_{t,*} := \left\{ f : \sup_{t \in \overline{J}_*} \int_0^t (t-s)^{-qb} |f(s)|^q ds < \infty \right\}. \quad (36)$$

Theorem 3.3. *Consider $\xi \ge 0$ and $\epsilon_1, \epsilon_2 > 0$ such that $\alpha + \epsilon_1 < 1$ and $\epsilon_2 < \min\{\alpha + \epsilon_1, 1 - (\alpha + \epsilon_1)\}$. Let q be a positive constant satisfying $q(\alpha + \epsilon_1 + \epsilon_2) < 1$. Assume that $\psi \in L^q_\omega \dot{H}^{\xi+1}_x$, F, σ satisfy conditions in Theorem 3.1. Assume further that $K_1 \in \mathbb{K}^{1,\alpha+\epsilon_1+\epsilon_2}_{t,*}$ and $K_2 \in \mathbb{K}^{q,\alpha+\epsilon_1+\epsilon_2}_{t,*}$. Then, $u \in C^{\epsilon_2}_{t,*} L^q_\omega \dot{H}^\xi_x$ and satisfies*

$$\|u(r,\cdot) - u(t,\cdot)\|_{L^q_\omega \dot{H}^\xi_x} \lesssim (r-t)^{\epsilon_2} \left(\|\psi\|_{L^q_\omega \dot{H}^{\xi+1}_x} + \|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x} \right), \quad \text{for } t, r \in \overline{J}_*. \quad (37)$$

Corollary 3.1. *Consider the case $\xi = 0$. If all assumptions in Theorem 3.3 are satisfied, then u belongs to $\in C^{\epsilon_2}_{t,*} L^q_\omega L^2_x$ and satisfies*

$$\|u(r,\cdot) - u(t,\cdot)\|_{L^q_\omega L^2_x} \lesssim (r-t)^{\epsilon_2} \left(\|\psi\|_{L^q_\omega \dot{H}^1_x} + \|u\|_{L^\infty_t L^q_\omega L^2_x} \right), \quad \text{for } t, r \in \overline{J}_*.$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

Corollary 3.2. *If K_1, K_2 are two positive constants (independent of t), then $K_1 \in \mathbb{K}_{t,*}^{1,\alpha+\epsilon_1+\epsilon_2}$ and $K_2 \in \mathbb{K}_{t,*}^{q,\alpha+\epsilon_1+\epsilon_2}$. Hence, the result* (37) *holds in this case.*

Proof. From the integral equation (11), it is obvious that

$$
\begin{aligned}
\left\|u(r,\cdot)-u(t,\cdot)\right\|_{L^q_\omega \dot{H}^\xi_x} \leq& \left\|\left(\mathcal{G}_\alpha(r)-\mathcal{G}_\alpha(t)\right)\psi\right\|_{L^q_\omega \dot{H}^\xi_x} + \left\|\int_t^r \mathcal{G}_\alpha(r-s)F(s,\cdot,u(s,\cdot))ds\right\|_{L^q_\omega \dot{H}^\xi_x}\\
&+\left\|\int_0^t \big(\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s)\big)F(s,\cdot,u(s,\cdot))ds\right\|_{L^q_\omega \dot{H}^\xi_x}\\
&+\left\|\int_t^r \mathcal{G}_\alpha(r-s)\sigma(s,u(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega \dot{H}^\xi_x}\\
&+\left\|\int_0^t \big(\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s)\big)\sigma(s,u(s,\cdot))d\mathcal{W}_\xi(s,\cdot)\right\|_{L^q_\omega \dot{H}^\xi_x}\\
&=: \mathcal{E}_1(r,t)+\mathcal{E}_2(r,t)+\mathcal{E}_3(r,t)+\mathcal{E}_4(r,t)+\mathcal{E}_5(r,t). \qquad (38)
\end{aligned}
$$

By Lemma 3.3, we directly obtain an upper bound for $\mathcal{E}_1(r,t)$

$$
\mathcal{E}_1(r,t) \leq \left\|\mathcal{G}_\alpha(r)-\mathcal{G}_\alpha(t)\right\|_{L^q_\omega \dot{H}^{\xi+1}_x \to L^q_\omega \dot{H}^\xi_x} \|v\|_{L^q_\omega \dot{H}^{\xi+1}_x} \lesssim (r-t)^{1-(\alpha+\epsilon_1)} \|v\|_{L^q_\omega \dot{H}^{\xi+1}_x}. \quad (39)
$$

For $\mathcal{E}_2(r,t)$, we can bound it by using Lemma 3.2 and the condition $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$, $F(\cdot,\cdot,0)=0$ as

$$
\begin{aligned}
\mathcal{E}_2(r,t) &\leq \int_t^r \|\mathcal{G}_\alpha(r-s)F(s,\cdot,u(s,\cdot))\|_{L^q_\omega \dot{H}^\xi_x} ds\\
&\lesssim \int_t^r \left(\mathbb{E}\|u(s,\cdot)\|^q_{\dot{H}^\xi_x}\right)^{\frac{1}{q}} ds \leq (r-t)\|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x}.
\end{aligned}
$$

For $\mathcal{E}_3(r,t)$, we can bound it by using Lemma 3.3 and the condition $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$, $F(\cdot,\cdot,0)=0$ as

$$
\begin{aligned}
\mathcal{E}_3(r,t) &\leq \int_0^t \|\left(\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s)\right)F(s,\cdot,u(s,\cdot))\|_{L^q_\omega \dot{H}^\xi_x} ds\\
&\lesssim (r-t)^{\epsilon_2}\int_0^t (t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}\|F(s,\cdot,u(s,\cdot))\|_{L^q_\omega \dot{H}^\xi_x} ds\\
&\lesssim (r-t)^{\epsilon_2}\int_0^t (t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}K_1(s)\left(\mathbb{E}\|u(s,\cdot)\|^q_{\dot{H}^\xi_x}\right)^{\frac{1}{q}} ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathcal{E}_3(r,t) &\lesssim (r-t)^{\epsilon_2}\left(\sup_{t\in\overline{J}_*}\int_0^t (t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}K_1(s)ds\right)\|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x}\\
&\lesssim (r-t)^{\epsilon_2}\|u\|_{L^\infty_t L^q_\omega \dot{H}^\xi_x}. \qquad (40)
\end{aligned}
$$

We continue with the last two terms $\mathcal{E}_4(r,t)$, $\mathcal{E}_5(r,t)$ by considering them in two cases of q.

- If $q=2$, we can bound $\mathcal{E}_4(r,t)$ by using a similar way as in (24)-(25)

$$
\mathcal{E}_4(r,t) = \left(\mathbb{E}\int_t^r \|\mathcal{G}_\alpha(r-s)\sigma(s,u(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} ds\right)^{\frac{1}{q}} \lesssim \left(\int_t^r K_2^q(s)\mathbb{E}\|u(s,\cdot)\|^q_{L^2_0(\dot{H}^\xi_x)} ds\right)^{\frac{1}{q}}.
$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

It follows that

$$\mathcal{E}_4(r,t) \lesssim (r-t)^{\frac{1}{q}} \|u\|_{L_t^\infty L_\omega^q \dot{H}_x^\xi}. \tag{41}$$

For $\mathcal{E}_5(r,t)$, the Itô isometry and Lemma 3.3 allow that

$$\begin{aligned}
\mathcal{E}_5(r,t) &= \left(\mathbb{E}\int_0^t \|(\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s))\,\sigma(s,u(s,\cdot))\|^q_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{1}{q}} \\
&\le \left(\mathbb{E}\int_0^t \|\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s)\|^q_{\dot{H}_x^\xi\to\dot{H}_x^\xi}\,\|\sigma(s,u(s,\cdot))\|^q_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{1}{q}} \\
&\lesssim (r-t)^{\epsilon_2}\left(\mathbb{E}\int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}\,\|\sigma(s,u(s,\cdot))\|^q_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\sigma(\cdot,0)=0$, $\sigma\in\mathcal{L}_{glo}(L_\omega^q\dot{H}_x^\xi;L_0^2(\dot{H}_x^\xi))$, we deduce that

$$\begin{aligned}
\mathcal{E}_5(r,t) &\lesssim (r-t)^{\epsilon_2}\left(\int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}K_2^q(s)\mathbb{E}\,\|u(s,\cdot)\|^q_{\dot{H}_x^\xi}\,ds\right)^{\frac{1}{q}} \\
&\lesssim (r-t)^{\epsilon_2}\left(\sup_{t\in\overline{J}_*}\int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}K_2^q(s)ds\right)^{\frac{1}{q}}\|u\|_{L_t^\infty L_\omega^q\dot{H}_x^\xi} \\
&\lesssim (r-t)^{\epsilon_2}\,\|u\|_{L_t^\infty L_\omega^q\dot{H}_x^\xi}.
\end{aligned} \tag{42}$$

• If $q>2$, we can bound $\mathcal{E}_4(r,t)$ by using a similar way as in (26)-(28)

$$\begin{aligned}
\mathcal{E}_4(r,t) &\le C(q)\left[\mathbb{E}\left(\int_t^r \|\mathcal{G}_\alpha(r-s)\sigma(s,u(s,\cdot))\|^2_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \\
&\lesssim \left(\int_t^r ds\right)^{(q-2)/(2q)}\left(\int_t^r \mathbb{E}\,\|\sigma(s,u(s,\cdot))\|^q_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{1/q} \\
&\lesssim (r-t)^{\frac{q-2}{2q}}\left(\int_t^r K_2^q(s)\mathbb{E}\,\|u(s,\cdot)\|^q_{\dot{H}_x^\xi}\,ds\right)^{1/q}.
\end{aligned}$$

This leads to

$$\mathcal{E}_4(r,t) \lesssim (r-t)^{\frac{1}{2}} \|u\|_{L_t^\infty L_\omega^q \dot{H}_x^\xi}. \tag{43}$$

For $\mathcal{E}_5(r,t)$, the consequence of Burkholder-Davis-Gundy's inequality and Lemma 3.3 yield

$$\begin{aligned}
\mathcal{E}_5(r,t) &\le C(q)\left[\mathbb{E}\left(\int_0^t \|(\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s))\,\sigma(s,u(s,.))\|^2_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \\
&\lesssim \left[\mathbb{E}\left(\int_0^t \|\mathcal{G}_\alpha(r-s)-\mathcal{G}_\alpha(t-s)\|^2_{\dot{H}_x^\xi\to\dot{H}_x^\xi}\,\|\sigma(s,u(s,.))\|^2_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \\
&\lesssim (r-t)^{\epsilon_2}\left[\mathbb{E}\left(\int_0^t (t-s)^{-2(\alpha+\epsilon_1+\epsilon_2)}\,\|\sigma(s,u(s,.))\|^2_{L_0^2(\dot{H}_x^\xi)}\,ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Applying the Holder inequality,

$$\begin{aligned}
\mathcal{E}_5(r,t) &\lesssim (r-t)^{\epsilon_2}\left(\int_0^t ds\right)^{(q-2)/(2q)}\left(\int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}\mathbb{E}\left\|\sigma(s,u(s,.))\right\|^q_{L_0^2(\dot{H}_x^\xi)}ds\right)^{1/q}\\
&\lesssim (r-t)^{\epsilon_2}\left(\sup_{t\in\overline{J}_*} t^{(q-2)/(2q)}\int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}K_2^q(s)ds\right)^{\frac{1}{q}}\|u\|_{L_t^\infty L_\omega^q\dot{H}_x^\xi}\\
&\lesssim (r-t)^{\epsilon_2}\|u\|_{L_t^\infty L_\omega^q\dot{H}_x^\xi}\,. \qquad (44)
\end{aligned}$$

Now, combining (38)-(44), we conclude that $u\in C^{\epsilon_2}_{t,*}L^q_\omega\dot{H}^\xi_x$ and satisfies

$$\|u(r,.)-u(t,.)\|_{L^q_\omega\dot{H}^\xi_x}\lesssim (r-t)^{\epsilon_2}\left(\|\psi\|_{L^q_\omega\dot{H}^{\xi+1}_x}+\|u\|_{L_t^\infty L^q_\omega\dot{H}^\xi_x}\right),\quad \text{for } t,r\in\overline{J}_*,$$

where we recall that $\epsilon_2<1-(\alpha+\epsilon_1)$. This completes the proof. □

Remark 3.2. *Now, let us explain the reason why we need the estimate in part a) of Lemma 3.1 instead of the overestimate*

$$\mathcal{K}_{\alpha,j}(s)\le c_{\alpha,1}\lambda_j^{-1}s^{-\alpha},\quad \textit{for all } s>0, \qquad (45)$$

which was used in [16]. Recall that the estimate in part a) of Lemma 3.1 helps us to obtain the result $u\in C^{\epsilon_2}_{t,*}L^q_\omega\dot{H}^\xi_x$. In contrast to this, by using the overestimate (45), we could not obtain $u\in C^{\theta}_{t,*}L^q_\omega\dot{H}^\xi_x$, with some $\theta>0$. We can explain the reason as follows. Look at the terms $\int_0^t(t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}K_1(s)ds$ in (40) and $\int_0^t(t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)}K_2(s)ds$ in (44). If the overestimate (45) is used instead of the estimate in part a) of Lemma 3.1, there would be a trouble because the two terms we have mentioned would be changed by $\int_0^t(t-s)^{-(2-\alpha)}K_1(s)ds$ and $\int_0^t(t-s)^{-q(2-\alpha)}K_2(s)ds$ respectively. Since two latter integrals are not convergent, we could not obtain $u\in C^{\theta}_{t,*}L^q_\omega\dot{H}^\xi_x$, with some $\theta>0$. Hence, it is required to improve the overestimate (45) by a new one as in part a) of Lemma 3.1.

4 Existence, regularity of the solution of the terminal value problem

4.1 Mild solution for terminal value problem

Consider the following deterministic problem

$$\begin{cases}\left(\frac{\partial}{\partial t}-\Delta-\varkappa\frac{\partial^\alpha}{\partial t^\alpha}\Delta\right)u=F(t,u), & (t,x)\in J\times\mathcal{X},\\ u(t,x)=0, & (t,x)\in J\times\partial\mathcal{X},\\ u(T,x)=\varphi(x), & x\in\mathcal{X},\end{cases} \qquad (46)$$

From equation (7), by substituting $t=T$, we have

$$u_j(0)=\mathcal{R}^{-1}_{\alpha,j}(T)\varphi_j-\int_0^T\mathcal{R}^{-1}_{\alpha,j}(T)\mathcal{R}_{\alpha,j}(T-s)F_j(s,.,u(s,.))ds, \qquad (47)$$

where $\varphi_j := \langle \varphi$ and $\phi_j^{(\xi)}\rangle_{\dot{H}_x^\xi}$, $\mathcal{R}_{\alpha,j}$ is given by (8). Combining (7) and (47), we obtain

$$u_j(t) = \mathcal{R}_{\alpha,j}^{-1}(T)\mathcal{R}_{\alpha,j}(t)\varphi_j + \int_0^t \mathcal{R}_{\alpha,j}(t-s)F_j(s,.,u(s,.))ds$$
$$- \int_0^T \mathcal{R}_{\alpha,j}^{-1}(T)\mathcal{R}_{\alpha,j}(t)\mathcal{R}_{\alpha,j}(T-s)F_j(s,.,u(s,.))ds.$$

Defining the operator $\mathcal{G}_\alpha(t)$ as in (9) and

$$\mathcal{G}_{\alpha,1}^T(t) := \mathcal{G}_\alpha^{-1}(T)\mathcal{G}_\alpha(t), \quad \mathcal{G}_{\alpha,2}^T(t,r) := \mathcal{G}_{\alpha,1}^T(t)\mathcal{G}_\alpha(r) = \mathcal{G}_\alpha^{-1}(T)\mathcal{G}_\alpha(t)\mathcal{G}_\alpha(r),$$

we have the following representation for the solution to Problem (46)

$$\begin{aligned} u(t,x) = \mathcal{G}_{\alpha,1}^T(t)\varphi(x) &+ \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds \\ &- \int_0^T \mathcal{G}_{\alpha,2}^T(t,T-s)F(s,x,u(s,x))ds. \end{aligned} \tag{48}$$

Motivated by (48), we give the definitions of mild solution to the terminal value problem (1),(2),(5) as follows:

Definition 4.1. An $\mathcal{F}_t$-adapted process $\{u(t,.)\}_{t\in\overline{J}}$ in $\mathcal{Z}_{t,*}^\theta L_\omega^q \dot{H}_x^\xi$ with some $\theta \in (0,1)$, $q \geq 2$, $\xi \geq 0$, is called a mild solution of Problem (1) if there holds that

$$\begin{aligned} u(t,x) = \mathcal{G}_{\alpha,1}^T(t)\varphi(x) &+ \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds \\ &- \int_0^T \mathcal{G}_{\alpha,2}^T(t,T-s)F(s,x,u(s,x))ds \\ &+ \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,u(s,x))d\mathcal{W}_\xi(s,x) \\ &- \int_0^T \mathcal{G}_{\alpha,2}^T(t,T-s)\sigma(s,u(s,x))d\mathcal{W}_\xi(s,x), \end{aligned} \tag{49}$$

for almost all ω for fixed t.

4.2 Properties of solution operators

Lemma 4.1. *Given t, r in $\overline{J}_*$, $q \geq 2$ and $\xi \geq 0$. Then*

$$\left\|\mathcal{G}_{\alpha,1}^T(t)\right\|_{L_\omega^q \dot{H}_x^\xi \to L_\omega^q \dot{H}_x^\xi} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-(1-\alpha)}.$$

Proof. Let $v \in L_\omega^q \dot{H}_x^\xi$. By using Lemma 3.1 and the fact that $\mathcal{R}_{\alpha,j}(T) \geq c_{\alpha,1}\lambda_j^{-1}$ (see [16]), we deduce that $0 \leq \frac{\mathcal{R}_{\alpha,j}(t)}{\mathcal{R}_{\alpha,j}(T)} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \frac{\lambda_j}{1+\lambda_j t^{1-\alpha}} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-(1-\alpha)}$. It follows that

$$\begin{aligned} \left\|\mathcal{G}_{\alpha,1}^T(t)v\right\|_{L_\omega^q \dot{H}_x^\xi} &= \left\|\mathcal{G}_\alpha^{-1}(T)\mathcal{G}_\alpha(t)v\right\|_{L_\omega^q \dot{H}_x^\xi} = \left[\mathbb{E}\left(\sum_{j=1}^\infty \left(\frac{\mathcal{R}_{\alpha,j}(t)}{\mathcal{R}_{\alpha,j}(T)}\right)^2 \lambda_j^{2\xi} v_j^2\right)^{q/2}\right]^{1/q} \\ &\leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-(1-\alpha)} \|v\|_{L_\omega^q \dot{H}_x^\xi}. \end{aligned}$$

This completes the proof. ∎

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

Lemma 4.2. *Let $t, r, \tau \in \overline{J}_*$ such that $r > t$. For any real number ξ, there holds*

$$\left\| \frac{d}{dt}\mathcal{G}_\alpha(t) \right\|_{L^q_\omega \dot{H}^\xi_x \to L^q_\omega \dot{H}^\xi_x} + \left\| \frac{d}{dt}\mathcal{G}^T_{\alpha,1}(t) \right\|_{L^q_\omega \dot{H}^\xi_x \to L^q_\omega \dot{H}^\xi_x} \lesssim t^{-(2-\alpha)}. \tag{50}$$

Proof. Since $\mathcal{K}_{\alpha,j}(s) \lesssim \lambda_j^{-1} s^{-\alpha}$, thanks to the fundamental theorem of calculus, one deduces

$$\begin{aligned} |\mathcal{R}_{\alpha,j}(r) - \mathcal{R}_{\alpha,j}(t)| &= \int_0^\infty |e^{-sr} - e^{-st}| \mathcal{K}_{\alpha,j}(s) ds \\ &\lesssim \int_0^\infty \frac{|e^{-sr} - e^{-st}|}{\lambda_j s^\alpha} ds \le \frac{r-t}{\lambda_j} \frac{\Gamma(2-\alpha)}{t^{2-\alpha}}. \end{aligned} \tag{51}$$

Hence, the classical derivative of $\mathcal{R}_{\alpha,j}(t)$ finitely exists for $t \in \overline{J}^*$, and is bounded as follows

$$\mathcal{P}_{\alpha,j}(t) = \frac{d}{dt}\mathcal{R}_{\alpha,j}(t) \lesssim \lambda_j^{-1} t^{-(2-\alpha)}.$$

Let $v \in L^q_\omega \dot{H}^\xi_x$. Applying the latter estimate,

$$\begin{aligned} \left\| \mathcal{G}_\alpha(r)v - \mathcal{G}_\alpha(t)v \right\|_{L^q_\omega \dot{H}^\xi_x} &= \left[\mathbb{E}\left(\sum_{j=1}^\infty |\mathcal{R}_{\alpha,j}(r) - R_{\alpha,j}(t)|^2 \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \\ &\lesssim \left[\mathbb{E}\left(\sum_{j=1}^\infty \left| \int_t^r \mathcal{P}_{\alpha,j}(s) ds \right|^2 \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \\ &\lesssim (r-t) t^{-(2-\alpha)} \left[\mathbb{E}\left(\sum_{j=1}^\infty \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \lesssim (r-t) t^{-(2-\alpha)} \|v\|_{L^q_\omega \dot{H}^\xi_x}, \end{aligned}$$

where we have used $\lambda_j^{-1} \le \lambda_1^{-1}$. Similarly, we have

$$\begin{aligned} \left\| \mathcal{G}^T_{\alpha,1}(r)v - \mathcal{G}^T_{\alpha,1}(t)v \right\|_{L^q_\omega \dot{H}^\xi_x} &= \left[\mathbb{E}\left(\sum_{j=1}^\infty \mathcal{R}^{-2}_{\alpha,j}(T) |\mathcal{R}_{\alpha,j}(r) - R_{\alpha,j}(t)|^2 \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \\ &\lesssim \left[\mathbb{E}\left(\sum_{j=1}^\infty \mathcal{R}^{-2}_{\alpha,j}(T) \left| \int_t^r \mathcal{P}_{\alpha,j}(s) ds \right|^2 \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \\ &\lesssim (r-t) t^{-(2-\alpha)} \left[\mathbb{E}\left(\sum_{j=1}^\infty \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \lesssim (r-t) t^{-(2-\alpha)} \|v\|_{L^q_\omega \dot{H}^\xi_x}. \end{aligned}$$

From two latter estimates, we conclude that (50) holds. □

4.3 The existence and uniqueness of mild solution to Problem (1),(2),(5)

For $q \ge 2$, we denote $\mathbf{K}_b(L^q_t) := \left\{ f : t^{b-1} f(t) \in L^q_t \right\}$. This subsection is aimed to contribute the existence and uniqueness of mild solution to Problem (1),(2),(5). To this end, we prepare some useful estimates for the terms appearing in equation (48). Firstly, the following lemma shows upper bounds for two first terms in $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$.

Lemma 4.3. *Given $q \ge 2$ and $w \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$ with $\xi \ge 0$.*

a) Assume that $\varphi \in L^q_\omega \dot{H}^\xi_x$*. Then,* $\mathcal{G}^T_{\alpha,1}(t)\varphi \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$ *and*

$$\left\|\mathcal{G}^T_{\alpha,1}(t)\varphi\right\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x} \le \frac{c_{\alpha,2}}{c_{\alpha,1}} \|\varphi\|_{L^q_\omega \dot{H}^\xi_x}.$$

b) Assume that F *is continuous in time,* $F(\cdot,\cdot,0) = 0$*,* $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$ *such that* $K_1 \in \mathbf{K}_\alpha(L^1_t)$*, then* $\int_0^t \mathcal{G}_\alpha(t-s)F(s,x,w(s,x))ds \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$*, and*

$$t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \widetilde{K}_{1,q\ge 2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x},$$

where $\widetilde{K}_{1,q\ge 2} := \sup_{t \in \overline{J}_*} t^{1-\alpha} \int_0^t s^{-(1-\alpha)} K_1(s) ds.$

Proof. a) By Lemma 4.2, we can see that $\frac{d}{dt}\mathcal{G}^T_{\alpha,1}(t)\varphi$ finitely exists in $L^q_\omega \dot{H}^\xi_x$, and so that $\mathcal{G}^T_{\alpha,1}(t)\varphi$ belongs to $C_{t,*} L^q_\omega \dot{H}^\xi_x$. In addition, by Lemma 4.1, one can see that

$$t^{(1-\alpha)} \left(\mathbb{E} \left\| \mathcal{G}^T_{\alpha,1}(t)\varphi \right\|^q_{\dot{H}^\xi_x} \right)^{1/q} \le \frac{c_{\alpha,2}}{c_{\alpha,1}} \left(\mathbb{E} \|\varphi\|^q_{\dot{H}^\xi_x} \right)^{1/q},$$

which shows $\mathcal{G}^T_{\alpha,1}(t)\varphi \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$.

b) According to Lemma 3.2, Lemma 4.2 and noting that F is continuous with respect to the first variable, it is obvious that the quantity $\int_0^t \mathcal{G}_\alpha(t-s)F(s,x,w(s,x))ds$ belongs to the space $C_{t,*} L^q_\omega \dot{H}^\xi_x$. Since $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$ and $F(\cdot,\cdot,0) = 0$, we have the following estimate

$$\mathbb{E}\|F(s,\cdot,w(s,\cdot))\|^q_{\dot{H}^\xi_x} \le K^q_1(s) \mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x}.$$

Therefore,

$$\begin{aligned} \left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} &\le \int_0^t \|\mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))\|_{L^q_\omega \dot{H}^\xi_x} ds \\ &\le c_{\alpha,2} \int_0^t K_1(s) \left(\mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x} \right)^{1/q} ds, \end{aligned}$$

which associates with $\left(\mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x} \right)^{1/q} \le s^{-(1-\alpha)} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}$ (since $w \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$) to obtain that

$$\begin{aligned} & t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \\ & \le c_{\alpha,2} \left(\sup_{t \in \overline{J}_*} t^{1-\alpha} \int_0^t s^{-(1-\alpha)} K_1(s) ds \right) \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}. \end{aligned}$$

This implies that $\int_0^t \mathcal{G}_\alpha(t-s)F(s,x,w(s,x))ds \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$. □

Next, the following lemma states an upper bound for the forth term in $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$.

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

Lemma 4.4. *Consider $q \geq 2$ satisfying $q(1-\alpha) < 1$ and $w \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$ with $\xi \geq 0$. Assume that σ is continuous in time, $\sigma(\cdot, 0) = 0$, $\sigma \in \mathcal{L}_{glo}(L^q_\omega \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$ such that $K_2 \in \mathbf{K}_\alpha(L^q_t)$.*

a) If $q = 2$, then $\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,x) \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$, and

$$t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \leq c_{\alpha,2} \widetilde{K}^{1/q}_{2,q=2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}.$$

b) If $q > 2$, then $\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,x) \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$, and

$$t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \leq c_{\alpha,2} C(q) \widetilde{K}^{1/q}_{2,q=2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}.$$

Here, $\widetilde{K}_{2,q=2} := \sup_{t \in \overline{J}_*} t^{(1-\alpha)q} \int_0^t s^{-(1-\alpha)q} K^q_2(s) ds$, $\widetilde{K}_{2,q>2} := \sup_{t \in \overline{J}_*} t^{\frac{q-2\alpha}{2}} \int_0^t K^q_2(s) s^{-(1-\alpha)q} ds.$

Proof. Lemma 3.2, Lemma 4.2 combined with the continuity of σ with respect to the first variable allow that the integral $\int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,x) \in C_{t,*} L^q_\omega \dot{H}^\xi_x$. Now, let us consider the two following cases for q.

• If $q = 2$, then the Itô isometry yields that

$$\begin{aligned}
\mathbb{E} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|^q_{\dot{H}^\xi_x} &= \mathbb{E} \int_0^t \|\mathcal{G}_\alpha(t-s)\sigma(s, w(s,.))\|^q_{L^2_0(\dot{H}^\xi_x)} ds \\
&\leq \int_0^t \|\mathcal{G}_\alpha(t-s)\|^q_{\dot{H}^\xi_x \to \dot{H}^\xi_x} \mathbb{E} \|\sigma(s, w(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} ds \\
&\leq c^q_{\alpha,2} \int_0^t \mathbb{E} \|\sigma(s, w(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} ds.
\end{aligned}$$

Since $\sigma \in \mathcal{L}_{glo}(L^q_\omega \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$ and $\sigma(\cdot, 0) = 0$, we immediately deduce that

$$\mathbb{E} \|\sigma(s, w(s,\cdot))\|^q_{L^2_0(\dot{H}^\xi_x)} \leq K^q_2(s) \mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x}.$$

Furthermore, it follows from $w \in \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$ that $\mathbb{E} \|w(s,\cdot)\|^q_{\dot{H}^\xi_x} \leq s^{-(1-\alpha)q} \|w\|^q_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}$. Henceforth,

$$\mathbb{E} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|^q_{\dot{H}^\xi_x} \leq c^q_{\alpha,2} \int_0^t s^{-(1-\alpha)q} K^q_2(s) ds \|w\|^q_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x},$$

which consequently implies

$$\begin{aligned}
&t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s, w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \\
&\leq c_{\alpha,2} \left(\sup_{t \in \overline{J}_*} t^{(1-\alpha)q} \int_0^t s^{-(1-\alpha)q} K^q_2(s) ds \right)^{1/q} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}.
\end{aligned}$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

• If $q > 2$, then using again the consequence of the Burkholder-Davis-Gundy inequality and the same techniques as the previous case,

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{\dot{H}_x^\xi}^q \\
&\le C^q(q)\mathbb{E}\left[\left(\int_0^t \|\mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))\|_{L_0^2(\dot{H}_x^\xi)}^2 ds \right)^{q/2} \right] \\
&\le C^q(q)c_{\alpha,2}^q\mathbb{E}\left[\left(\int_0^t \|\sigma(s,w(s,\cdot))\|_{L_0^2(\dot{H}_x^\xi)}^2 ds \right)^{q/2} \right].
\end{aligned}$$

Applying the Hölder inequality,

$$\begin{aligned}
&\left[\mathbb{E}\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{\dot{H}_x^\xi}^q \right]^{1/q} \\
&\le c_{\alpha,2}C(q)\left(\int_0^t ds \right)^{(q-2)/(2q)} \left(\int_0^t \mathbb{E}\left\| \sigma(s,w(s,\cdot)) \right\|_{L_0^2(\dot{H}_x^\xi)}^q ds \right)^{1/q} \\
&\le c_{\alpha,2}C(q)t^{\frac{q-2}{2q}} \left(\int_0^t K_2^q(s)\mathbb{E}\left\| w(s,\cdot) \right\|_{\dot{H}_x^\xi}^q ds \right)^{1/q}
\end{aligned}$$

where we recall that $\sigma(\cdot,0)=0$, $\sigma \in \mathcal{L}_{glo}(L_\omega^q\dot{H}_x^\xi; L_0^2(\dot{H}_x^\xi))$. Multiplying both sides of the latter estimates with $t^{1-\alpha}$,

$$\begin{aligned}
&t^{1-\alpha}\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L_\omega^q\dot{H}_x^\xi} \\
&\le c_{\alpha,2}C(q)\left(\sup_{t\in\overline{J}_*} t^{\frac{q-2\alpha}{2}} \int_0^t K_2^q(s)s^{-(1-\alpha)q}ds \right)^{1/q} \|w\|_{\mathcal{Z}_{t,*}^{1-\alpha}L_\omega^q\dot{H}_x^\xi}.
\end{aligned}$$

This completes the proof. □

Using two previous lemmas, we now aim to establish the existence and uniqueness of the mild solution of the terminal value problem (1),(3),(5). We denote the constants

$$\begin{aligned}
\Pi_1 &:= c_{\alpha,2}\left(1+\frac{c_{\alpha,2}}{c_{\alpha,1}}T^{\alpha-1}\right)\left(\widetilde{K}_{1,q\ge2}+\widetilde{K}_{2,q=2}^{1/q}\right), \\
\Pi_2 &:= c_{\alpha,2}\left(1+\frac{c_{\alpha,2}}{c_{\alpha,1}}T^{\alpha-1}\right)\left(\widetilde{K}_{1,q\ge2}+C(q)\widetilde{K}_{2,q>2}^{1/q}\right).
\end{aligned}$$

Theorem 4.1. *Assume that φ, F, K_1 satisfy hypothese in Lemma 4.3, and σ, K_2 satisfy the ones in Lemma 4.4. If the condition $\Pi_1\mathbf{1}_{q=2}+\Pi_2\mathbf{1}_{q>2}<1$ holds, then Problem* (1),(3),(5) *possesses a unique solution*

$$u \in \mathcal{Z}_{t,*}^{\alpha-1}L_\omega^q\dot{H}_x^\xi. \tag{52}$$

Furthermore, there holds that

$$\|u(t,\cdot)\|_{L_\omega^q\dot{H}_x^\xi} \lesssim t^{\alpha-1}, \quad t\in\overline{J}_*. \tag{53}$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

Corollary 4.1. *Consider the case $\xi = 0$. If all assumptions in Theorem 4.1 are satisfied, then u belongs to $\in \mathcal{Z}^{\alpha-1}_{t,*} L^q_\omega L^2_x$.*

Corollary 4.2. *If K_1, K_2 are two positive constants (independent of t), then conditions $K_1 \in \mathbf{K}_\alpha(L^1_t)$ and $K_2 \in \mathbf{K}_\alpha(L^q_t)$ hold. Hence, the results (52)-(53) hold in this case.*

Proof. Let us first consider the case $q = 2$ and $\Pi_1 < 1$. To prove the existence of the mild solution in $\mathcal{Z}^{1-\alpha}_{t,*} L^2_\omega \dot{H}^\xi_x$, we will use the Banach contraction principle. After proving this mapping is well-defined, we will prove it has a unique fixed point in a ball of this space. For the sake of convenience, we naturally divide the proof into the following steps.

Step 1. *Constructing a well-defined mapping on $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$.* Let us denote

$$\begin{aligned}\mathcal{S}w(t,x) :=\ & \mathcal{G}^T_{\alpha,1}(t)\varphi(x) + \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,w(s,x))ds \\ & - \int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)F(s,x,w(s,x))ds + \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,x) \\ & - \int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,x).\end{aligned}$$

By applying Lemma 4.3 and Lemma 4.4, we deduce that the two first and forth terms of the right-hand-side belong to $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$. Let us consider the third and last terms. Recall that we have proved in Part b of Lemma 4.3 and to Part a of Lemma 4.4 that

$$t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \widetilde{K}_{1,q\ge 2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}, \tag{54}$$

$$t^{1-\alpha} \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \le c_{\alpha,2} \widetilde{K}^{1/q}_{2,q=2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}. \tag{55}$$

for each $t \in \overline{J}_*$. This together with Part a of Lemma 4.3 implies that

$$\int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)F(s,x,w(s,x))ds = \mathcal{G}^T_{\alpha,1}(t) \int_0^T \mathcal{G}_\alpha(T-s)F(s,x,w(s,x))ds \tag{56}$$

and

$$\int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,x) = \mathcal{G}^T_{\alpha,1}(t) \int_0^T \mathcal{G}_\alpha(T-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,x) \tag{57}$$

which belong to the space $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$. Thus, we conclude that $\mathcal{S}w$ belongs to $\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$.

Step 2. *Finding a ball $\overline{B}(0;R) \subset \mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x$ of center 0 and radius R such that $\mathcal{S}$ is a contraction mapping on.* By taking supremum on $t \in \overline{J}_*$, two inequalities yield that

$$\begin{aligned}\left\| \int_0^t \mathcal{G}_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x} &\le c_{\alpha,2} \widetilde{K}_{1,q\ge 2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}. \\ \left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x} &\le c_{\alpha,2} \widetilde{K}^{1/q}_{2,q=2} \|w\|_{\mathcal{Z}^{1-\alpha}_{t,*} L^q_\omega \dot{H}^\xi_x}.\end{aligned}$$

30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

By combining Part a and Part b of Lemma 4.3, one can obtain the following estimate

$$\begin{aligned}\left\| \int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)F(s,\cdot,w(s,\cdot))ds \right\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x} &\leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \left\| \int_0^T \mathcal{G}_\alpha(T-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega \dot{H}^\xi_x} \\ &\leq \frac{c^2_{\alpha,2}}{c_{\alpha,1}} T^{\alpha-1}\widetilde{K}_{1,q\geq 2}\|w\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x}.\end{aligned}$$

Similarly, combining Part a of Lemma 4.3 and Part a of Lemma 4.4 implies

$$\begin{aligned}&\left\| \int_0^T \mathcal{G}^T_{\alpha,2}(t,T-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x} \\ &\leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \left\| \int_0^T \mathcal{G}_\alpha(T-s)\sigma(s,w(s,\cdot))d\mathcal{W}_\xi(s,\cdot) \right\|_{L^q_\omega \dot{H}^\xi_x} \\ &\leq \frac{c^2_{\alpha,2}}{c_{\alpha,1}} T^{\alpha-1}\widetilde{K}^{1/q}_{2,q=2}\|w\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x}. \end{aligned} \tag{58}$$

In the above arguments, we have used the estimates for the four last term of $\mathcal{S}w$. Meanwhile, since φ is defined by Lemma 4.3, the first term of $\mathcal{S}w$ is estimated by $c_{\alpha,2}c^{-1}_{\alpha,1}\|\varphi\|_{L^q_\omega \dot{H}^\xi_x}$. All these estimates can be taken together to allow that

$$\|\mathcal{S}w\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x} \leq c_{\alpha,2}c^{-1}_{\alpha,1}\|\varphi\|_{L^q_\omega \dot{H}^\xi_x} + \Pi_1\|w\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x}.$$

Since $\Pi_1 < 1$, there always exists a positive number R which satisfies the equation $c_{\alpha,2}c^{-1}_{\alpha,1}\|\varphi\|_{L^q_\omega \dot{H}^\xi_x} + \Pi_1 R = R$. Therefore, for all $w \in \overline{B}(0;R)$, we have $\|\mathcal{S}w\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x} \leq R$, i.e., Sw belongs to $\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x$. Moreover, the above arguments show that

$$\left\|\mathcal{S}w^\dagger - \mathcal{S}w^\ddagger\right\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x} \leq \mathcal{C}_1\|w^\dagger - w^\ddagger\|_{\mathcal{Z}^{1-\alpha}_{t,*}L^q_\omega \dot{H}^\xi_x},$$

where we recall that $F \in L_{glo}(L^q_\omega \dot{H}^\xi_x)$ and $\sigma \in \mathcal{L}_{glo}(L^q_\omega \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$.

Finally, the existence and uniqueness of a mild solution u in $\mathcal{Z}^{1-\alpha}_{t,*}L^2_\omega \dot{H}^\xi_x$ is obtained by the Banach contraction principle. For the case $q > 2$, $\Pi_2 < 1$, we note that the proof can be similarly based on Part a where it is necessary to estimate the Itô integral (58) in the case $q > 2$ (instead of $q = 2$) by using Part b of Lemma 4.4. □

5 Conclusion

In this study, we consider the initial value problem and the terminal value problem for a stochastic time-fractional Rayleigh-Stokes equation, where the source function and the time-spatial noise are nonlinear. By using some useful stochastic analysis techniques and fractional calculus, we obtain the existence, uniqueness of the mild solution of each problem. Furthermore, some regularity properties and continuity results for the solutions are also proposed.

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30 Jun 2020 00:17:17 PDT
200526-Tran Version 2 - Submitted to Discrete Contin. Dyn. Syst. Ser. B

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