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Smoothing effect and asymptotic dynamics of nonautonomous parabolic equations with time-dependent linear operators

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Abstract

In this paper we consider the nonautonomous semilinear parabolic problems with time-dependent linear operators

$$u_t + A(t)u = f(t, u),\ t > \tau; \quad u(\tau) = u_0,$$

in a Banach space X. Under suitable conditions, we obtain regularity results for $u_t(t, x)$ with respect to its spatial variable x and estimates for u_t in stronger spaces (X^α). We then apply those results to a nonautonomous reaction-diffusion equation

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$$u_t - div(a(t,x)\nabla u) + u = f(t,u)$$

with Neumann boundary condition and time-dependent diffusion. From the regularity of u_t we derive the existence of classical solutions and from the estimates for u_t we prove that the variation of the solution u is bounded in the long-time dynamics. We also prove the existence of pullback attractor, as well as the existence of a compact set that contains the long-time dynamics of the derivatives u_t, without requiring any assumption concerning monotonicity or decay in time of $a(t,x)$.
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1. Introduction

Consider the following abstract *singularly nonautonomous* semilinear problem

$$u_t + A(t)u = f(t,u),\ t > \tau; \quad u(\tau) = u_0 \in X^\alpha, \tag{1.1}$$

where X is a Banach space, $A(t)$, $t \in \mathbb{R}$, is a family of closed linear operators defined on a fixed dense subspace D of X and f is a nonlinearity defined in $\mathbb{R} \times X^\alpha$.

The term *singularly nonautonomous* is used to express the fact that the linear operator $A(t)$ is time-dependent, as a counterpart to the semilinear problem where $A(t) = A$, which we refer as *nonsingular*. This terminology, adopted for instance in [8], is not unanimous and, in this case, does not refer to any discontinuity or blow-up in time, meanings that "singularly" can have in other contexts.

We assume that the family $A(t)$, $t \in \mathbb{R}$, satisfies the following properties:

(P.1) The operator $A(t) : D(A(t)) \subset X \to X$ is a closed densely defined linear operator, the domain $D = D(A(t))$ is fixed in time and there are constants $C > 0$ and $\varphi \in (\frac{\pi}{2}, \pi)$ (independent of $t \in \mathbb{R}$) such that

$$\|(\lambda I + A(t))^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda| + 1};\ \forall \lambda \in \Sigma_\varphi \cup \{0\},$$

where $\Sigma_\varphi = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \varphi\}$. We say in this case that the family $A(t)$ is *uniformly sectorial*.

(P.2) There are constants $C > 0$ and $0 < \delta \leq 1$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{\mathcal{L}(X)} \leq C|t - \tau|^\delta. \tag{1.2}$$

To express this fact we say that the function $t \mapsto A(t)A^{-1}(s) \in \mathcal{L}(X)$ is *uniformly Hölder continuous* or δ-*uniformly Hölder continuous* if we seek to emphasize the constant.

As a consequence of (P.2), given any arbitrarily large compact set in $\mathbb{R}^2$, there exists a constant $C > 0$ such that $\|A(t)A^{-1}(\tau)\|_{\mathcal{L}(X)} \leq C$, for all (t,τ) in this compact set. In this case, for

$t, \tau \in [-M, M]$, the norms $\|\cdot\|_{D(A(t))} = \|A(t)\cdot\|_X$ and $\|\cdot\|_{D(A(\tau))} = \|A(\tau)\cdot\|_X$ defined by the operators $A(t)$ and $A(\tau)$, respectively, are equivalent. We shall refer to both norms as $\|\cdot\|_{X^1}$.

Furthermore, since each $A(t)$ is sectorial with $0 \in \rho(A)$, then $A(t)$ is a positive operator, its fractional powers $A(t)^\alpha$ (in the sense of Amann [3]) are well defined and $-A(t)$ generates an analytic C_0-semigroup $T_{-A(t)}(s)$, $s \geq 0$ (see [16, Sections 1.3-1.4]).

We denote by X^α the domain of $A(t)^\alpha$ endowed with the norm $\|\cdot\|_{X^\alpha} = \|A(t)^\alpha \cdot\|_X$. Once again, from the fact that $\|A(t)A^{-1}(\tau)\|_{\mathcal{L}(X)} \leq C$, we can refer to X^α as domain of any operator $A(t)^\alpha$ since they are all equivalent.

Therefore, associated to the family of sectorial operators $A(t)$, $t \in \mathbb{R}$, there exists a scale of fractional spaces $\{X^\alpha\}_{\alpha\in\mathbb{R}}$. This scale will play an essential role in the results we prove in this work. For further details on how to obtain $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ we recommend [3, Chapter V].

For the remaining term in (1.1), $f = f(t, u)$, we require the following property

(NL) The nonlinearity f satisfies: $f : \mathbb{R} \times X^\alpha \to X$, $0 \leq \alpha < 1$, is locally Hölder continuous with exponent $\omega \in (0, 1]$ in the time-variable and locally Lipschitz continuous in X^α. In other words, given any $(t_0, x_0) \in \mathbb{R} \times X^\alpha$ there exist a neighborhood W of this point and a constant $C > 0$ (depending on W) such that

$$\|f(t, u) - f(s, v)\|_X \leq C(|t - s|^\omega + \|u - v\|_{X^\alpha}), \quad \forall (t, u), (s, v) \in W.$$

Those are all the properties we require for the terms in the semilinear problem (1.1). Our goal concerning this type of problem is to prove, for the abstract setting, two properties of $u_t(t, x)$.

The first one is a smoothing effect that the differential equation has on $u_t : [\tau, T) \to X$. This effect can be briefly described as the increase in regularity of the derivative $u_t(t, x)$ in the spatial variable up to $X^{\min\{\delta,\omega\}}$, where δ is the Hölder exponent of the family $A(t)$, $t \in \mathbb{R}$, and ω the Hölder continuity exponent of the nonlinearity f. Once we know that $u_t(t) \in X^\beta$ for $0 \leq \beta < \min\{\delta, \omega\}$, the other property we desire are $\|\cdot\|_{X^\beta}$-estimates for $u_t(t)$ in terms of u, which allow us to evaluate the long-time behavior of the derivative u_t.

We call the attention for a similar analysis performed by Acquistapace and Terreni in [1], where they studied maximal regularity for u and u_t, allowing even more general situations, where the domain of $A(t)$ can change with time or not be dense in X. However, the approach used by them is filled with technicalities and involves the construction of several auxiliary (weighted) spaces which makes difficult a direct application of the results in practical problems.

The results presented in this work, even though they are not as general as the ones presented in [1], follow directly from a careful study of the solution $u : [\tau, T) \to X$ in its mild formulation and use the same language and tools developed in the well-known works of Kato [18,19], Lunardi [21], Sobolevskiĭ [24] and Tanabe [25–27], avoiding the excess of technicality presented in [1]. Moreover, the approach used here extends the one presented by Henry in [16, Chapter 3] for the autonomous case, $A(t) = A$, and the results are stated in a similar manner as the classical theorems found in the work just mentioned ([16, Theorem 3.5.2]). We believe that such presentation of the results facilitates its applicability in practical problems, as we also illustrate in Section 6.

In order to introduce the mild formulation of the solution that will play an essential role in the next sections of this work, some definitions and previous results in the literature must be presented.

If the linear operator did not depend on time, $A(t) = A$, then it would generate a one parameter family of linear operators, called *semigroup*, $T_{-A}(s)$, that plays an essential role in solving the semilinear problem. For a given nonlinearity $f = f(t,u)$, the semilinear problem $u_t + Au = f(t,u)$, $u(\tau) = u_0$, under suitable assumptions on f, is locally solved by

$$u(t) = T_{-A}(t)u_0 + \int_0^t T_{-A}(s) f(s, u(s))ds.$$

We refer to the above expression as *variation of constants formula* and we say that u is a *mild solution* of the semilinear problem.

Hoping to find a replacement for this semigroup family now in the context where $A(t)$ depends on t, several authors searched for a two parameter family of linear operators with similar properties of $T_{-A}(s)$. To be precise, a family with the following properties:

Definition 1.1. Let X be a Banach space. A family $\{U(t,s) \in \mathcal{L}(X); t \geq s\}$ of bounded linear operators is a *linear process associated to* $A(t)$ (or an *evolution system associated to* $A(t)$) if

(1) $U(t,t) = I$ and $U(t,\tau)U(\tau,s) = U(t,s)$, for all $s \leq \tau \leq t$.
(2) $(t,s,x) \mapsto U(t,s)x$ is continuous for $t \geq s$ and for all $x \in X$.
(3) $\|U(t,s)\|_{\mathcal{L}(X)} \leq C$, for all $t \geq s$.
(4) $U(t,s) : X \to D$, $t \mapsto U(t,\tau)x \in X$ is differentiable for each $x \in X$.
(5) The derivative $\partial_t U(t,s)$ is a bounded linear operator in X, $\partial_t U(t,\tau) = -A(t)U(t,\tau)$ and $\|\partial_t U(t,s)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{-1}$.

Kato in [17–19] was the first to prove the existence of this process $\{U(t,\tau); t \geq \tau\}$ for a hyperbolic family of linear operators. The parabolic problem was then studied almost simultaneously by Sobolevskiĭ in [24] and Tanabe in [25–27]. They proved existence of evolution system associated to $A(t)$ satisfying (P.1) and (P.2). Sobolevskiĭ also studied regularity properties of the solution u in the spatial variable x and higher order derivatives in t for u. In his work, existence of local solution for the semilinear problem was obtained ([24, Theorem 7]) and is reproduced in theorem below.

Theorem 1.2. *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) and (P.2) and $f : \mathbb{R} \times X^\alpha \to X$ a nonlinearity satisfying (NL), then there exists $T > \tau$ such that $u : [\tau, T) \to X^\alpha$ given by*

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s) f(s, u(s))ds \tag{1.3}$$

is a strong solution for (1.1) in (τ, T), that is,

(1) $u(\cdot) \in \mathcal{C}([\tau,T), X) \cap \mathcal{C}^1((\tau,T), X)$ and $u(t) \in D$, for $\tau < t < T$;
(2) u satisfies the equation in the usual sense $u_t(t,x) = -A(t)u(t,x) + f(t,u(t,x))$, for all $t \in (\tau, T)$.

Moreover, if $\|u(t)\|_{X^\alpha}$ is bounded in any bounded set $[\tau, t^*]$, then the solution is globally defined in time.

In our paper, we describe some properties of the derivative $u_t(t,x)$ that play an important role in applications and justify several formal calculations. Differentiating (1.3) in t (using the properties described for $U(t,\tau)$ in Definition 1.1), we obtain

$$\begin{aligned} u_t(t) &= -A(t)U(t,\tau)u_0 - A(t)\int_\tau^t U(t,s)f(s,u(s))ds + f(t,u(t)) \\ &= -A(t)U(t,\tau)u_0 - A(t)\int_\tau^t U(t,s)[f(s,u(s)) - f(t,u(t))]ds \\ &\quad - A(t)\int_\tau^t U(t,s)f(t,u(t))ds + f(t,u(t)) \end{aligned} \tag{1.4}$$

but right at this point the singular case presents a major setback if compared to the case $A(t) = A$: it seems that is not possible to “get rid of” $f(t,u(t))$ as it happens when A is independent of time. If $A(t) = A$, expression (1.4) is given by

$$\begin{aligned} u_t(t) &= -AT_{-A}(t-\tau)u_0 - A\int_\tau^t T_{-A}(t-s)[f(s,u(s)) - f(t,u(t))]ds \\ &\quad - A(t)\int_\tau^t T_{-A}(t-s)f(t,u(t))ds + f(t,u(t)) \\ &= -AT_{-A}(t-\tau)u_0 - A\int_\tau^t T_{-A}(t-s)[f(s,u(s)) - f(t,u(t))]ds \\ &\quad - T_{-A}(t-\tau)f(t,u(t)), \end{aligned}$$

where in the last equality we used a result similar to a “Fundamental Theorem of Calculus” for semigroups: $A\left(\int_\tau^t T_{-A}(s)x ds\right) = T_{-A}(t)x - T_{-A}(\tau)x$. This is not available for the process $U(t,s)$ and the properties for $u_t(t)$ seem to be attached to the regularity that f possesses (which is $f(t,u(t)) \in X$). In order to obtain further properties for $u_t(t)$ some authors require more regularity on f, as in Proposition 6.2.5 of [21].

Our abstract results follow from studying the term $A(t)\int_\tau^t U(t,s)f(t,u(t))ds$ and obtaining a suitable characterization for it, one that allows us to dispose the nonlinearity $f(t,u(t))$ in (1.4) and study properties of $u_t(t)$ without requiring any further property on f, besides the ones established in (NL).

We apply the abstract theory to a singular nonautonomous reaction-diffusion equation,

$$u_t - div(a(t,x)\nabla u) + u = f(t,u),$$

in a bounded domain Ω and with Neumann boundary conditions. From the regularity of u_t we derive the existence of classical solutions and from the estimates for u_t we prove that the variations of the solution u is bounded in the long-time dynamics. We also prove the existence of pullback attractor and the existence of a compact set that encloses $u_t(t)$ in the long-time behavior through an iterative procedure that we briefly explain in the sequel.

Since the diffusion term is time-dependent, we are not able to construct a Lyapunov function for the system or to use comparison results for the solutions, which are the usual procedure to study asymptotic dynamics for reaction-diffusion equations.

However, the time-dependence does not pose any problem if we use an iterative procedure inspired in the ideas developed by Moser-Alikakos [2]. We obtain estimates for the norm of the solution in stronger spaces by using estimates in the $L^{2^{k-1}}$-norm in order to obtain estimates in the L^{2^k}-norm, for any $k \in \mathbb{N}$. Those estimates and the variation of constants formula will imply the existence of a compact pullback attracting set. This procedure is quite general and can be applied to other second order parabolic equations with different boundary conditions (such as Dirichlet or mixed boundary conditions).

Moreover, the asymptotic analysis is done without requiring any additional condition concerning monotonicity, decay or asymptotic behavior for the function $a(t,x)$ with respect to the time variable t, which differs from some studies existent in the literature. For example, in [28], in order to study the asymptotic dynamics of singularly nonautonomous parabolic equations $u_t + A(t)u = f(t)$, the author assumed that $A(t)$ approaches an operator $A(\infty)$ as $t \to \infty$, and in this case it was possible to prove an exponential decay for the linear process associated to $A(t)$ and study the asymptotic dynamics.

Other works, like [6,10,13,14], treated a class of singularly nonautonomous damped wave equations in $\mathbb{R}^N$ of the type $u_{tt} - a(t)\Delta u + b(t)u_t = f(u)$. By assuming conditions on the derivative of a, it was possible to obtain an energy function (or Lyapunov function) for the system and derive global existence of solution. We provide a way in which neither asymptotic condition or monotonicity/decay of $a(t,x)$ are necessary.

To attend this agenda, this paper is organized as follows. In Section 2 we present some preliminaries and the abstract result (Theorem 2.5). Section 3 provides several estimates for the linear operators that appear throughout this work. Section 4 and Section 5 are dedicated to prove the smoothing effect of the equation on u_t. Finally, in Section 6, we apply the theory on the singularly nonautonomous reaction-diffusion equation with Neumann boundary conditions.

2. Notations and main abstract result

Before we state the main result, we first establish the notation we use throughout the work and some preliminary results.

(1) X is the phase space and X^α designates the fractional power space, $X^\alpha = D(A(t)^\alpha)$ with the graph norm. The norm in X^α is denoted by $\|\cdot\|_{X^\alpha} = \|A(t)^\alpha \cdot\|_X$.
(2) $X^1 = D(A(t))$ is also denoted by D and does not depend on t.
(3) $\mathcal{L}(X)$ denotes the space of bounded linear operators $T : X \to X$.
(4) $K \subset\subset X$ denotes a compact (or compactly embedded) subset in X.
(5) $T_{-A(t)}(s) \in \mathcal{L}(X)$, $s \geq 0$, represents the analytic C_0- semigroup generated by $-A(t)$.
(6) $U(t,\tau) \in \mathcal{L}(X)$, $t \geq \tau$, denotes the linear process associated to $A(t)$, $t \in \mathbb{R}$.
(7) $u(t,\tau,u_0)$ represents the solution of the semilinear problem (1.1) with initial condition $u(\tau) = u_0$. When the initial conditions do not need to be emphasized, we simply denote $u(t)$.

(8) Let $\gamma, \beta \geq 0$. We denote by $\mathcal{C}^{\gamma}((\tau, T), X^{\beta})$ the space of locally γ-Hölder continuous functions, that is, any element ϕ in $\mathcal{C}^{\gamma}((\tau, T), X^{\beta})$ satisfies (at least for small values of h)

$$\|\phi(t+h) - \phi(t)\|_{X^{\beta}} \leq Ch^{\gamma}.$$

An important subspace of $\mathcal{C}^{\gamma}((\tau, T), X^{\beta})$ that figures in several of our results are the following:

$$\mathcal{C}^{\gamma}_{\theta}((\tau, T), X^{\beta}) = \left\{\phi \in \mathcal{C}^{\gamma}((\tau, T), X^{\beta}) : \sup_{t \in (\tau, T)} (t-\tau)^{\theta} \|\phi(t)\|_{X^{\beta}} < \infty\right\},$$

where $\gamma, \beta, \theta \geq 0$. The elements in this space can be seen as functions such that

$$\|\phi(t+h) - \phi(t)\|_{X^{\beta}} \leq Ch^{\gamma}(t-\tau)^{-\theta}.$$

This notation will be helpful at several moments when integrals need to be estimated. Note that $\int_{\tau}^{t} \|\phi(s)\|_{X^{\beta}} ds$ converges only if $\theta \in [0, 1)$. Therefore, we will carry the subindex θ in order to be aware of the cases for which $\int_{\tau}^{t} \|\phi(s)\|_{X^{\beta}} ds$ makes sense.

2.1. Integral in Banach spaces

The variation of constant formula (1.3) for $u(t)$ and the derivative in t of this expression, (1.4), which provides $u_t(t)$, both present integrals of functions that take values in Banach space, that is, integrals like $\int_{t_1}^{t_2} h(t)dt$, where $h(t) \in X$.

The convergence of $\int_{t_1}^{t_2} h(t)dt$ is strictly connected with the convergence of $\int_{t_1}^{t_2} \|h(t)\|dt$: one will converge if and only if the other does. Therefore, tools on convergence of integrals of real functions will be important, in special the ability of recognizing a *Beta function* whenever it appears in the calculations. *Beta function* is the mapping $\mathcal{B} : (0, \infty) \times (0, \infty) \to \mathbb{R}$ given by

$$\mathcal{B}(a, b) = \int_{0}^{1} u^{a-1}(1-u)^{b-1} du$$

and a simple change of variable turns this integral into a form that shows up frequently:

Lemma 2.1. *If $a, b > 0$ and $\tau < t$, then $\int_{\tau}^{t} (t-s)^{a-1}(s-\tau)^{b-1} ds = (t-\tau)^{a+b-1}\mathcal{B}(a, b)$.*

Integrability properties of $h : (t_1, t_2) \to X$ are listed below and their proofs can be found in [9, Section 2.1].

Proposition 2.2. *If $h \in \mathcal{C}([t_1, t_2], X) \cap \mathcal{C}^1((t_1, t_2), X)$, then $h(t_2) - h(t_1) = \int_{t_1}^{t_2} h'(s)ds$.*

Proposition 2.3. *Let $A : D(A) \subset X \to X$ be a closed linear operator and $h : [\tau, t] \to X$ a continuous function with image in $D(A)$. If $Ah : [\tau, t] \to X$ is also continuous, then $\int_{\tau}^{t} h(s)ds \in D(A)$ and*

$$A \int_{\tau}^{t} h(s)ds = \int_{\tau}^{t} Ah(s)ds.$$

Corollary 2.4. *Let $A : D(A) \subset X \to X$ be a closed linear operator, $h : [\tau, t) \to X$ (or $h : (\tau, t) \to X$) continuous with image in $D(A)$ and $Ah : [\tau, t) \to X$ (or $Ah : (\tau, t) \to X$) also continuous. Assume that $\int_\tau^t h(s)ds$ and $\int_\tau^t Ah(s)ds$ exist. Then, $\int_\tau^t h(s)ds \in D(A)$ and*

$$A\int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds.$$

At this point, it is important to distinguish between existence of $A\int_\tau^t h(s)ds$ and existence of $\int_\tau^t Ah(s)ds$. The first can exist while the second does not.

In other words, if the first term $A\int h$ exists, it does not mean that we can switch the operator with the integral, since $\int Ah$ might not exist.

2.2. Abstract result

We can now state one of the main abstract results of this work, which proof is located at Section 5:

Theorem 2.5. *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) and (P.2) and $f : \mathbb{R} \times X^\alpha \to X$ a nonlinearity satisfying (NL). If $u : [\tau, T) \to X$ is the solution of*

$$u_t(t) + A(t)u = f(t, u), t \in (\tau, T); \quad u(\tau) = u_0 \in X^\alpha,$$

then, for any $0 \le \beta < \min\{\omega, \delta\}$, $u_t(t) \in X^\beta$ and satisfies the estimate

$$\|u_t(t)\|_{X^\beta} \le C(t-\tau)^{-1-\beta+\alpha}\|u_0\|_{X^\alpha} + C(t-\tau)^{-\max\{2\beta,\beta+\alpha\}}.$$

2.3. Preliminaries necessary for the application: pullback attractors

Let X be a Banach space and $\{S(t, \tau) : X \to X; t \ge \tau\}$ a family of operators satisfying:

(1) $S(t, t) = I_X$, for all $t \in \mathbb{R}$.
(2) $S(t, s) = S(t, \tau)S(\tau, s)$, for all $t \ge \tau \ge s$, $s \in \mathbb{R}$.
(3) $(\tau, \infty) \ni t \mapsto S(t, \tau)x$ is continuous for all $x \in X$.

Such family is called a *process in X* and we also denote it by $S(\cdot, \cdot)$. We will usually call it *nonlinear process* to distinguish from the family $U(t, \tau)$ obtained in Definition 1.1. In Section 6, this nonlinear process is obtained trough the solution of the semilinear equation.

We recall in the sequel some basic concepts and results of the theory of pullback attractors. We refer to [7] and references therein for further details.

To compare the distance between two sets in the phase space X, we use the Hausdorff semidistance: given $A, B \subset X$, the *Hausdorff semidistance* between A and B is

$$dist(A, B) = \sup_{a\in A}\inf_{b\in B} d(a, b).$$

Definition 2.6. Let $S(\cdot,\cdot)$ be a process. A family $A(\cdot)=\{A(t)\subset X; t\in\mathbb{R}\}$ *pullback attracts* $B\subset X$ if, for each $t\in\mathbb{R}$,

$$dist(S(t,s)B, A(t)) \overset{s\to-\infty}{\longrightarrow} 0.$$

Definition 2.7. The *pullback attractor* of $S(\cdot,\cdot)$ is a family $\mathcal{A}(\cdot)=\{\mathcal{A}(t)\subset X;\ t\in\mathbb{R}\}$ that satisfies:

(1) $\mathcal{A}(t)$ is compact for all $t\in\mathbb{R}$.
(2) $\mathcal{A}(\cdot)$ is *invariant* by $S(\cdot,\cdot)$, that is, $S(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t)$, for all $t\geq\tau$, $\tau\in\mathbb{R}$.
(3) $\mathcal{A}(\cdot)$ pullback attracts bounded sets of X.
(4) $\mathcal{A}(\cdot)$ is the minimal closed family that satisfies (3).

Theorem 2.8. *[7, Theorem 2.12] Let $S(\cdot,\cdot)$ be a process. The statements below are equivalent:*

(1) $S(\cdot,\cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$.
(2) There exists a family of compact sets $K(\cdot)$ that pullback attracts bounded sets of X.

Corollary 2.9. *If there exists a fixed compact set $K\subset\subset X$ such that, for any bounded set $B\subset X$*

$$dist(S(t,\tau)B, K)\to 0 \text{ when } \tau\to-\infty,$$

then $S(\cdot,\cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$ such that $\cup_{t\in\mathbb{R}}\mathcal{A}(t)\subset K$.

3. Estimates on linear operators

As mentioned in the Introduction, $U(t,\tau)$ is a two parameters family of linear operator associated to $A(t)$ that plays a similar role as the semigroup in the nonsingular case. We provide a brief idea on how to construct this family. For a detailed description, we recommend [22, Chapter 5] or [24].

Suppose $U(t,\tau)\in\mathcal{L}(X)$ is a family satisfying the abstract differential equation, that is, $\partial_t U(t,\tau)=-A(t)U(t,\tau)$. Also, assume that there exists another family $\Phi(t,\tau)\in\mathcal{L}(X)$ such that $U(t,\tau)$ is obtained trough the integral equation

$$U(t,\tau)=T_{-A(\tau)}(t-\tau)+\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds. \tag{3.1}$$

Differentiating in t, adding $A(t)U(t,\tau)$ on both sides and taking into account that $\partial_t U(t,\tau)+A(t)U(t,\tau)=0$, we deduce

$$0=\Phi(t,\tau)-[A(\tau)-A(t)]T_{-A(\tau)}(t-\tau)-\int_\tau^t [A(s)-A(t)]T_{-A(s)}(t-s)\Phi(s,\tau)ds.$$

If we set

$$\varphi_1(t,\tau)=[A(\tau)-A(t)]T_{-A(\tau)}(t-\tau), \tag{3.2}$$

then $\Phi(t,\tau)$ would have to satisfy

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds \tag{3.3}$$

and it would be a fixed point of the map $S(\Psi)(t) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Psi(s)ds$.

If we had a family $\Phi(t,\tau)$ that satisfied (3.3), then we could proceed in the reverse way to obtain $U(t,\tau)$. This is the technique employed to construct the linear process in the parabolic case and the description of $U(t,\tau)$ relies on this auxiliary family $\Phi(t,\tau)$.

Lemma 3.1. *([22, Section 5.6], [24]) Let $A(t), t \in \mathbb{R}$, be a family of uniformly sectorial operators and uniformly δ-Hölder continuous. If $\{\varphi_1(t,\tau) \in \mathcal{L}(X); t \geq \tau\}$ is the family given by (3.2), then:*

(1) $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \varphi_1(t,\tau) \in \mathcal{L}(X)$ is continuous in the uniform topology and

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\delta-1}.$$

(2) There exists a unique family $\{\Phi(t,\tau) \in \mathcal{L}(X); t \geq \tau\}$ that satisfies (3.3). In this case, $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \Phi(t,\tau) \in \mathcal{L}(X)$ is continuous in the uniform topology and

$$\|\Phi(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\delta-1}. \tag{3.4}$$

The family of linear operators $\{U(t,\tau) \in \mathcal{L}(X); t \geq \tau\}$ given by

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds$$

is an evolution process associated to $A(t)$ *and satisfies the conditions in Definition 1.1*

Those four families - $T_{-A(\tau)}(t-\tau)$, $U(t,\tau)$, $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$ - describe the evolution dynamics of the system. In order to obtain the desired results, a good knowledge on estimates of those families in the space $\mathcal{L}(X, X^\beta)$ is necessary.

Most of the estimate results we present in the sequel are already proved in [8] and references therein.

3.1. Estimates for the semigroup $T_{-A(\tau)}(s)$

The first estimate for the semigroup $T_{-A(\tau)}(s)$ generated by a positive sectorial operator is given by

$$\|T_{-A(\tau)}(s)\|_{\mathcal{L}(X)} \leq C, \quad \forall s \geq 0, \tau \in \mathbb{R}.$$

Proposition 3.2. *[8, Proposition 7] There exists $C > 0$, independent of β and t, such that*

$$\|A(t)^{\beta} T_{-A(t)}(\tau)\|_{\mathcal{L}(X)} \leq C\tau^{-\beta}, \quad \forall\, \beta \geq 0, \tau > 0, t \in \mathbb{R},$$
$$\|[T_{-A(t)}(\tau) - I]A(t)^{-\beta}\|_{\mathcal{L}(X)} \leq C\tau^{\beta}, \quad \forall\, 0 \leq \beta \leq 1, \tau > 0, t \in \mathbb{R}.$$

Proposition 3.3. *[8, Proposition 8] For any $\xi \in \mathbb{R}$, $t \leq r$, $\tau > 0$ and $0 \leq \beta \leq 1$*

$$\|A(\xi)^{\beta}[T_{-A(r)}(\tau) - T_{-A(t)}(\tau)]\|_{\mathcal{L}(X)} \leq C\tau^{-\beta}(r-t)^{\delta},$$
$$\|A(\xi)^{\beta}[A(r)T_{-A(r)}(\tau) - A(t)T_{-A(t)}(\tau)]\|_{\mathcal{L}(X)} \leq C\tau^{-\beta-1}(r-t)^{\delta(1-\beta)}.$$

3.2. *Estimates for the families $\varphi_1(t,\tau)$, $\Phi(t,\tau)$*

These families play an important role in the description of the linear process $U(t,\tau)$ and, as expected, estimates on their norm is necessary in the next calculations.

Proposition 3.4. *Let $0 \leq \beta < \delta$. There exists $C > 0$ depending only on β such that, for any $t > \tau$,*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(X,X^{\beta})} \leq C(t-\tau)^{\delta-\beta-1},$$
$$\|\Phi(t,\tau)\|_{\mathcal{L}(X,X^{\beta})} \leq C(t-\tau)^{\delta-\beta-1}.$$

Proof. The statement for the family $\varphi_1(t,\tau)$ follows from (1.2) and the already stated properties for the semigroups

$$\|A(\xi)^{\beta}\varphi_1(t,\tau)\|_{\mathcal{L}(X)} = \|[A(\tau) - A(t)]A(\xi)^{-1}A(\xi)^{1+\beta}T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\delta-1-\beta},$$

whereas the estimate for $\Phi(t,\tau)$ follows from

$$\begin{aligned}
\|\Phi(t,\tau)\|_{\mathcal{L}(X,X^{\beta})} &\leq \|\varphi_1(t,\tau)\|_{\mathcal{L}(X,X^{\beta})} + \int_{\tau}^{t} \|\varphi_1(t,s)\|_{\mathcal{L}(X,X^{\beta})}\|\Phi(s,\tau)\|_{\mathcal{L}(X)}ds \\
&\leq C(t-\tau)^{\delta-\beta-1} + C\int_{\tau}^{t} (t-s)^{\delta-\beta-1}(s-\tau)^{\delta-1}ds \\
&\leq C(t-\tau)^{\delta-\beta-1} + C(t-\tau)^{2\delta-\beta-1} \leq C(t-\tau)^{\delta-\beta-1}. \quad \square
\end{aligned}$$

Proceeding in the same way as it is done in [8, Propositions 3 and 4], we have:

Proposition 3.5. *Let $\tau < \theta < t$. Given any $\beta < \delta$ and $0 \leq \eta < \delta - \beta$,*

$$\|\varphi_1(t,\tau) - \varphi_1(\theta,\tau)\|_{\mathcal{L}(X,X^{\beta})} \leq C(t-\theta)^{\eta}(\theta-\tau)^{(\delta-\eta)-\beta-1}, \tag{3.5}$$
$$\|\Phi(t,\tau) - \Phi(\theta,\tau)\|_{\mathcal{L}(X,X^{\beta})} \leq C(t-\theta)^{\eta}(\theta-\tau)^{(\delta-\eta)-\beta-1}. \tag{3.6}$$

In the same lines of the preceding result, we also need an estimate for the families φ_1 and Φ when both initial and final instant evolve a quantity $h > 0$:

Proposition 3.6. *Let $\tau < t$ and $h > 0$. Then, given any $0 \le \eta < \delta$, there exists a constant C depending only on δ and η such that*

$$\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(X)} \le Ch^{\eta}(t-\tau)^{(\delta-\eta)-1},$$
$$\|\Phi(t+h,\tau+h) - \Phi(t,\tau)\|_{\mathcal{L}(X)} \le Ch^{\eta}(t-\tau)^{(\delta-\eta)-1}.$$

Proof. Note that

$$\begin{aligned}
&\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\\
&\quad = [A(\tau+h) - A(t+h)]T_{-A(\tau+h)}(t-\tau) - [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau)\\
&\quad = [A(\tau+h) - A(t+h) - A(\tau) + A(t)]T_{-A(\tau+h)}(t-\tau)\\
&\qquad + [A(\tau) - A(t)][T_{-A(\tau+h)}(t-\tau) - T_{-A(\tau)}(t-\tau)]\\
&\quad = [A(\tau+h) - A(\tau)]A(\xi)^{-1}A(\xi)T_{-A(\tau+h)}(t-\tau)\\
&\qquad + [A(t) - A(t+h)]A(\xi)^{-1}A(\xi)T_{-A(\tau+h)}(t-\tau)\\
&\qquad + [A(\tau) - A(t)]A(\xi)^{-1}A(\xi)[T_{-A(\tau+h)}(t-\tau) - T_{-A(\tau)}(t-\tau)].
\end{aligned}$$

The terms in last equality above are estimated in $\mathcal{L}(X)$ by $Ch^{\delta}(t-\tau)^{-1}$, $Ch^{\delta}(t-\tau)^{-1}$ and $C(t-\tau)^{\delta}h^{\delta}(t-\tau)^{-1}$, respectively, and the last one follows from Proposition 3.3. Therefore,

$$\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(X)} \le Ch^{\delta}(t-\tau)^{-1}.$$

On the other hand, this difference can be estimated by

$$\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(X)} \le C(t-\tau)^{\delta-1}.$$

Interpolating those two estimates with exponent $\frac{\eta}{\delta}$ and $1-\frac{\eta}{\delta}$, with $0 \le \eta < \delta$, we obtain

$$\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(X)} \le Ch^{\eta}(t-\tau)^{(\delta-\eta)-1}.$$

The last assertion follows from

$$\begin{aligned}
&\|\Phi(t+h,\tau+h) - \Phi(t,\tau)\|_{\mathcal{L}(X)}\\
&\quad \le \left\| \varphi_1(t+h,\tau+h) - \varphi_1(t,\tau) + \int_{\tau+h}^{t+h} \varphi_1(t+h,s)\Phi(s,\tau+h)ds - \int_{\tau}^{t} \varphi_1(t,s)\Phi(s,\tau)ds \right\|_{\mathcal{L}(X)}\\
&\quad \le Ch^{\eta}(t-\tau)^{(\delta-\eta)-1} + \left\| \int_{\tau}^{t} [\varphi_1(t+h,s+h) - \varphi(t,s)]\Phi(s+h,\tau+h)ds \right\|_{\mathcal{L}(X)}\\
&\qquad + \left\| \int_{\tau}^{t} \varphi_1(t,s)[\Phi(s+h,\tau+h) - \Phi(s,\tau)]ds \right\|_{\mathcal{L}(X)}
\end{aligned}$$

$$\leq Ch^{\eta}(t-\tau)^{(\delta-\eta)-1} + C\int_{\tau}^{t} h^{\eta}(t-s)^{(\delta-\eta)-1}(s-\tau)^{\delta-1}ds$$

$$+ C\int_{\tau}^{t} (t-s)^{\delta-1}\|\Phi(s+h,\tau+h)-\Phi(s,\tau)\|_{\mathcal{L}(X)}ds$$

$$\leq Ch^{\eta}(t-\tau)^{(\delta-\eta)-1} + C\int_{\tau}^{t} (t-s)^{\delta-1}\|\Phi(s+h,\tau+h)-\Phi(s,\tau)\|_{\mathcal{L}(X)}ds.$$

Applying Gronwall's inequality [16, p. 190],

$$\|\Phi(t+h,\tau+h)-\Phi(t,\tau)\|_{\mathcal{L}(X)} \leq Ch^{\eta}(t-\tau)^{(\delta-\eta)-1}. \quad \square$$

3.3. *Estimates for the linear process $U(t,\tau)$*

Besides the estimate $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C$ stated in Definition 1.1, we also need the following results:

Proposition 3.7. *[8, Theorem 2.2] Let $\tau < t$ and $0 \leq \gamma \leq \beta < 1+\delta$. Then there exists a constant C depending only on γ and β such that*

$$\|A(t)^{\beta}U(t,\tau)A(\tau)^{-\gamma}\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\gamma-\beta}.$$

Proposition 3.8. *If $\gamma > \beta$ and $0 < \gamma - \beta < 1$, then there exists a constant C depending only on γ and β such that*

$$\|A(t)^{\beta}[U(t,\tau)-I]A(\tau)^{-\gamma}\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\gamma-\beta}.$$

4. A replacement for the fundamental theorem of calculus for semigroups

Both semigroup $T_{-A(\tau)}(t)$ and process $U(t,\tau)$ have their images in D and the operators $A(t)T_{-A(\tau)}(t)$, $A(t)U(t,\tau)$ are well-defined. However, when it comes to integral, we cannot assure whether $\int_{\tau}^{t} A(t)T_{-A(\tau)}(s-\tau)xds$ or $\int_{\tau}^{t} A(t)U(t,s)xds$ even exist, due to the fact that $\|A(t)T_{-A(\tau)}(t-s)\|_{\mathcal{L}(X)}$ and $\|A(t)U(t,s)\|_{\mathcal{L}(X)}$ are bounded by $C(t-s)^{-1}$.

Nevertheless, $A(t)\int_{\tau}^{t} T_{-A(\tau)}(s-\tau)xds$ and $A(t)\int_{\tau}^{t} U(t,s)xds$ exist and they are bounded linear operators in X, as we prove in this section. Moreover, we provide a characterization for $A(t)\int_{\tau}^{t} U(t,s)xds$ that will suit our purpose of studying regularity properties of u_t through its integral formulation.

The fact $A(t)\int_{\tau}^{t} T_{-A(\tau)}(s-\tau)ds \in \mathcal{L}(X)$ has already been proved in [24, Section 1.6] and we present it below.

Lemma 4.1. *For any $x \in X$, $\int_{\tau}^{t} T_{-A(s)}(t-s)wds$ belongs to D and $A(t)\int_{\tau}^{t} T_{-A(s)}(t-s)ds$ is a bounded linear operator satisfying $\left\|A(t)\int_{\tau}^{t} T_{-A(s)}(t-s)ds\right\|_{\mathcal{L}(X)} \leq C$.*

In next lemma, we extend this result to the linear process $U(t,\tau)$.

Lemma 4.2. *For any $x \in X$, $\int_\tau^t U(t,s)x ds$ belongs to D and the following equality holds*

$$A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)xd\xi\right\}ds$$
$$+ A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right\}d\xi$$
$$- A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)xds\right\}d\xi.$$

Furthermore, it satisfies $\left\|A(t)\int_\tau^t U(t,s)ds\right\|_{\mathcal{L}(X)} \leq C$.

Proof. The characterization of the linear process provided in (3.1) and an application of Fubini's Theorem [15, Theorem 2.37] yields

$$\int_\tau^t U(t,s)xds = \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t \left[\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)xd\xi\right]ds$$
$$= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(\xi,s)xds\right]d\xi$$
$$= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi$$
$$+ \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(t,s)xds\right]d\xi$$
$$= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi$$
$$+ \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^t \Phi(t,s)xds\right]d\xi - \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi$$
$$= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(s)}(t-s)\left[\int_\tau^t \Phi(t,\xi)xd\xi\right]d\xi$$

$$+\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x\,ds\right]d\xi$$
$$-\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x\,ds\right]d\xi$$
$$=\int_\tau^t T_{-A(s)}(t-s)\left\{x+\int_\tau^t \Phi(t,\xi)x\,d\xi\right\}ds$$
$$+\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x\,ds\right]d\xi$$
$$-\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x\,ds\right]d\xi$$
$$=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.$$

From Lemma 4.1, estimates on Section 3, and as a consequence of Corollary 2.4, we obtain $\mathcal{I}_1\in D$ and $\|A(t)\mathcal{I}_1\|_X\le C\|x\|_X$. Indeed,

$$\left\|\int_\tau^t A(t)T_{-A(s)}(t-s)\left\{x+\int_\tau^t \Phi(t,\xi)x\,d\xi\right\}ds\right\|_X\le C\|x\|_X+C\left\|\int_\tau^t \Phi(t,\xi)x\,d\xi\right\|_X$$
$$\le C\|x\|_X+C\int_\tau^t (t-\xi)^{\delta-1}d\xi\,\|x\|_X\le C\|x\|_X+C(t-\tau)^\delta\|x\|_X\le C\|x\|_X.$$

It follows from (3.6) with $0\le\eta<\delta$ that $\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi[\Phi(\xi,s)-\Phi(t,s)]x\,ds\right]d\xi$ converges, as we see next

$$\left\|\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]x\,ds\right]d\xi\right\|_X$$
$$\le C\int_\tau^t (t-\xi)^{-1}\left[\int_\tau^\xi [(t-\xi)^\eta(\xi-s)^{\delta-\eta-1}]ds\right]d\xi\,\|x\|_X$$
$$\le C\int_\tau^t (t-\xi)^{\eta-1}(\xi-\tau)^{\delta-\eta}d\xi\,\|x\|_X\le C(t-\tau)^\delta\|x\|_X\le C\|x\|_X.$$

Therefore, Corollary 2.4 implies that $\mathcal{I}_2$ belongs to D and $\|A(t)\mathcal{I}_2\|_X\le C\|x\|_X$.

The fact that $\mathcal{I}_3 \in D$ and $\|A(t)\mathcal{I}_3\|_X \le C\|x\|_X$ follows in the same way:

$$\left\| \int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)x ds\right] d\xi \right\|_X \le C\int_\tau^t (t-\xi)^{-1}\left[\int_\xi^t (t-s)^{\delta-1} ds\right] d\xi \|x\|_X$$

$$\le C\int_\tau^t (t-\xi)^{-1}(t-\xi)^{\delta} d\xi \|x\|_X \le C(t-\tau)^{\delta}\|x\|_X \le C\|x\|_X.$$

Therefore, $\int_\tau^t U(t,s)x ds = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \in D$ and the estimate for $A(t)\left(\int_\tau^t U(t,s)ds\right)$ follows from

$$\left\| A(t)\left(\int_\tau^t U(t,s)x ds\right)\right\|_X \le \|A(t)\mathcal{I}_1\|_X + \|A(t)\mathcal{I}_2\|_X + \|A(t)\mathcal{I}_3\|_X \le C\|x\|_X. \quad \square$$

5. Smoothing effect of the differential equation

5.1. Linear problem

Rather than considering the semilinear problem directly, we first deal with the nonautonomous linear case

$$x_t(t) + A(t)x = g(t),\ t \in (\tau, T); \quad x(\tau) = x_0 \in X,$$

whose solution is given by $x(t) = U(t,\tau)x_0 + \int_\tau^t U(t,s)g(s)ds$.

The characterization obtained in Lemma 4.2 for $A(t)\int_\tau^t U(t,s)ds$ is applied in the expression (1.4) for $x_t(t)$ (with $f(t,u)$ being replaced by $g(t)$), resulting

$$\begin{aligned} x_t(t) &= -A(t)U(t,\tau)x_0 - A(t)\int_\tau^t U(t,s)[g(s)-g(t)]ds - A(t)\int_\tau^t U(t,s)g(t)ds + g(t) \\ &= -A(t)U(t,\tau)x_0 \\ &\quad - A(t)\int_\tau^t U(t,s)[g(s)-g(t)]ds \\ &\quad - \left\{\int_\tau^t \Phi(t,\xi)g(t)d\xi\right\} \\ &\quad + T_{-A(\tau)}(t-\tau)\left\{g(t) + \int_\tau^t \Phi(t,\xi)g(t)d\xi\right\} \end{aligned}$$

$$- \int_{\tau}^{t} [A(t) - A(s)] T_{-A(s)}(t-s) \left\{ g(t) + \int_{\tau}^{t} \Phi(t,\xi) g(t) d\xi \right\} ds$$

$$- A(t) \int_{\tau}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{\tau}^{\xi} [\Phi(\xi,s) - \Phi(t,s)] g(t) ds \right\} d\xi$$

$$+ A(t) \int_{\tau}^{t} T_{-A(\xi)}(t-\xi) \left\{ \int_{\xi}^{t} \Phi(t,s) g(t) ds \right\} d\xi$$

$$= \mathcal{I}_1 x_0 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7.$$

Note that $\mathcal{I}_1$ is a linear operator on X while $\mathcal{I}_2$ to $\mathcal{I}_7$ are elements on X. Moreover, $\mathcal{I}_1$ is the only one depending on the initial condition x_0.

It might seem that equality above would only complicate the analysis. However, the nonlinear term $g(t) \in X$ no longer features in the expression for x_t and all terms (from $\mathcal{I}_1$ to $\mathcal{I}_7$) belongs to a space X^{ξ}, $\xi > 0$, with more regularity, as we see in lemma below.

Lemma 5.1. *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) and (P.2) and $g : (\tau, T) \to X$ is a continuous function in $\mathcal{C}^{\lambda}_{\theta}((\tau, T), X)$, $0 < \lambda \leq 1$ and $0 \leq \theta < 1$, such that, for any $\tau < s < t < T$, there exists a constant $C > 0$ for which*

$$\|g(t) - g(s)\|_X \leq C(t-s)^{\lambda}(s-\tau)^{-\theta}.$$

Given any $0 \leq \beta < \min\{\lambda, \delta\}$, the terms $\mathcal{I}_1$ to $\mathcal{I}_7$ of the above equality belong to X^{β} and satisfy

$\|\mathcal{I}_1 x_0\|_{X^\beta} \leq C(t-\tau)^{-1-\beta}\|x_0\|_X$, $\|\mathcal{I}_2\|_{X^\beta} \leq C(t-\tau)^{(\lambda-\beta)-\theta}$, $\|\mathcal{I}_3\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta}$,
$\|\mathcal{I}_4\|_{X^\beta} \leq C(t-\tau)^{-\beta-\theta}$, $\|\mathcal{I}_5\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta}$, $\|\mathcal{I}_6\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta}$,
$\|\mathcal{I}_7\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta}$,

where the constant C depends on β.

Proof. For each term we estimate its X^{β}-norm, $0 \leq \beta < \min\{\lambda, \delta\}$, proving that it belongs to X^{β}.

Analysis of $\mathcal{I}_1$: It follows from Proposition 3.7 that

$$\|A(t)U(t,\tau)x_0\|_{X^\beta} = \|A(\xi)^{\beta} A(t) U(t,\tau) x_0\|_X \leq C(\beta)(t-\tau)^{-1-\beta}\|x_0\|_X.$$

Analysis of $\mathcal{I}_2$: The fact that $\mathcal{I}_2 \in X^{\beta}$ follows from the estimate

$$\|\mathcal{I}_2\|_{X^\beta} = \left\| A(\xi)^{\beta} A(t) \int_{\tau}^{t} U(t,s)[g(s) - g(t)] ds \right\|_X \leq \int_{\tau}^{t} (t-s)^{-1+(\lambda-\beta)}(s-\tau)^{-\theta} ds$$

$$\overset{\beta<\lambda}{\leq} C(\beta)(t-\tau)^{(\lambda-\beta)-\theta}.$$

Analysis of $\mathcal{I}_3$: From Proposition 3.4, we obtain

$$\|\mathcal{I}_3\|_{X^\beta} = \left\| A(\xi)^\beta \int_\tau^t \Phi(t,\xi)g(t) \right\|_X \le C \int_\tau^t (t-\xi)^{\delta-1-\beta}(t-\tau)^{-\theta} d\xi \le C(\beta)(t-\tau)^{(\delta-\beta)-\theta}.$$

Analysis of $\mathcal{I}_4$: Let $H(t) = g(t) + \int_\tau^t \Phi(t,\xi)g(t)d\xi$. From the properties of g and $\Phi(t,\tau)$, we obtain $\|H(t)\|_X \le C(t-\tau)^{-\theta}$ and

$$\|\mathcal{I}_4\|_{X^\beta} = \left\| A(\xi)^\beta T_{-A(\tau)}(t-\tau)H(t) \right\|_X \le C(t-\tau)^{-\beta-\theta}.$$

Analysis of $\mathcal{I}_5$: Note that once again we have the function $H(t)$ defined while studying the term $\mathcal{I}_4$. Using the estimates for this function, we deduce

$$\begin{aligned}
\|\mathcal{I}_5\|_{X^\beta} &= \left\| A(\xi)^\beta \int_\tau^t [A(t)-A(s)]T_{-A(s)}(t-s)H(t)ds \right\|_X \\
&\le \int_\tau^t \|[A(t)-A(s)]A(\xi)^{-1}A(\xi)^{1+\beta}T_{-A(s)}(t-s)\|_{\mathcal{L}(X)} \|H(t)ds\|_X \\
&\le C(\beta) \int_\tau^t (t-s)^{\delta-\beta-1} ds (t-\tau)^{-\theta} \le C(\beta)(t-\tau)^{(\delta-\beta)-\theta}.
\end{aligned}$$

Analysis of $\mathcal{I}_6$: Applying (3.6), with $\eta \in (\beta,\delta)$, we derive

$$\begin{aligned}
\|\mathcal{I}_6\|_{X^\beta} &= \left\| A(\xi)^\beta \int_\tau^t A(t)T_{-A(\xi)}(t-\xi) \left\{ \int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]g(t)ds \right\} d\xi \right\|_X \\
&\le C \int_\tau^t (t-\xi)^{-1-\beta} \left\{ \int_\tau^\xi (t-\xi)^\eta (\xi-s)^{(\delta-\eta)-1} ds \right\} d\xi (t-\tau)^{-\theta} \\
&\le C \int_\tau^t (t-\xi)^{-1+(\eta-\beta)}(\xi-\tau)^{(\delta-\eta)} d\xi (t-\tau)^{-\theta} \le C(t-\tau)^{(\delta-\beta)-\theta}.
\end{aligned}$$

Analysis of $\mathcal{I}_7$: The last term follows from Proposition 3.2 and the estimate (3.4) for $\Phi(t,\tau)$.

$$\|\mathcal{I}_7\|_{X^\beta} = \left\| A(\xi)^\beta A(t) \int_\tau^t T_{-A(\xi)}(t-\xi) \left\{ \int_\xi^t \Phi(t,s)g(t)ds \right\} d\xi \right\|_X$$

$$\leq \int_{\tau}^{t} (t-\xi)^{-1-\beta} \left\{ \int_{\xi}^{t} (t-s)^{\delta-1} ds \right\} d\xi (t-\tau)^{-\theta} \leq C(t-\tau)^{(\delta-\beta)-\theta}. \quad \square$$

The previous Lemma is a major part in the proof of theorem we state in the sequel.

Theorem 5.2. *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) and (P.2) and assume that $g : (\tau, T) \to X$ a continuous function in $\mathcal{C}^{\lambda}_{\theta}((\tau, T), X)$, $0 \leq \lambda \leq 1$ and $0 \leq \theta < 1$, such that, for any $\tau < s < t < T$, there exists a constant $C > 0$ for which*

$$\|g(t) - g(s)\|_X \leq C(t-s)^{\lambda}(s-\tau)^{-\theta}.$$

If $x : [\tau, T) \to X$ is the solution of

$$x_t(t) + A(t)x = g(t), t \in (\tau, T); \quad x(\tau) = x_0 \in X,$$

then, for any $0 \leq \beta < \min\{\lambda, \delta\}$, $x_t(t)$ is in X^{β} for $t \in (\tau, T)$ and satisfies the estimate $\|x_t(t)\|_{X^{\beta}} \leq C(\beta)(t-\tau)^{-1-\beta}[\|x_0\|_X + 1$. Moreover, if $x_0 \in X^{\alpha}$, the estimate on $x_t(t)$ can be improved to

$$\|x_t(t)\|_{X^{\beta}} \leq C(t-\tau)^{-1-\beta+\alpha}\|x_0\|_{X^{\alpha}} + C(t-\tau)^{-\theta-\beta}.$$

Proof. The result follows from Lemma 5.1, with the exception of the last assertion. This one follows from the fact that $\|\mathcal{I}_1 x_0\|_{X^{\beta}}$ in the previous lemma can be improved if $x_0 \in X^{\alpha}$ by using Theorem 3.7:

$$\|\mathcal{I}_1\|_{X^{\beta}} = \|A(\xi)^{\beta} A(t) U(t,\tau) A(\xi)^{-\alpha} A(\xi)^{\alpha} x_0\|_X \leq C(t-\tau)^{-1-\beta+\alpha}\|x_0\|_{X^{\alpha}}.$$

However, we cannot state which exponent $-1-\beta+\alpha$ and $-\theta-\beta$ is larger than the other. For this reason, we do not group the terms together and we obtain the estimate

$$\|x_t(t)\|_{X^{\beta}} \leq C(t-\tau)^{-1-\beta+\alpha}\|x_0\|_{X^{\alpha}} + C(t-\tau)^{-\theta-\beta}. \quad \square$$

5.2. *Semilinear problem*

Consider now the semilinear case

$$u_t(t) + A(t)u = f(t, u(t)), \ t \in (\tau, T); \quad u(\tau) = u_0 \in X^{\alpha},$$

for $0 \leq \alpha < 1$. Under the properties required for $A(t)$, (P.1) and (P.2), and for the nonlinearity f, expressed in (NL), this problem has a local solution given by

$$u(t) = U(t,\tau)u_0 + \int_{\tau}^{t} U(t,s) f(t, u(s)) ds.$$

If we define $g : (\tau, T) \to X$ as $g(t) := f(t, u(t))$, then u also satisfies

$$u_t + A(t)u = g(t),\ t \in (\tau, T); \quad u(\tau) = u_0 \in X^\alpha.$$

By proving that $g(\cdot) \in \mathcal{C}^\lambda_\theta((\tau, T), X)$ for some $\lambda \in (0, 1]$ and $\theta \in [0, 1)$, then the results on Theorem 5.2 can be translated to the semilinear case. In the next lemma we find values of λ and θ for which the desired result holds. Note that the most regular space X^β for which $u_t(t)$ belongs is limited by the value of λ.

Lemma 5.3. *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) and (P.2) and $f : \mathbb{R} \times X^\alpha \to X$ a nonlinearity satisfying (NL). If $u : [\tau, T) \to X$ is the solution of*

$$u_t(t) + A(t)u = f(t, u(t)),\ t \in (\tau, T); \quad u(\tau) = u_0 \in X^\alpha,$$

then, $g(t) := f(t, u(t))$ belongs to $\mathcal{C}^\eta_{\max\{\alpha,\eta\}}((\tau, T), X)$, for any $\eta \in [0, \min\{\omega, \delta\})$.

Proof. The fact that f is locally Lipschitz implies that, for $t > \tau$ and $h > 0$ small,

$$\|g(t + h) - g(t)\|_X = \|f(t + h, u(t + h)) - f(t, u(t))\|_X \le C(|h|^\omega + \|u(t + h) - u(t)\|_{X^\alpha}).$$

Hence, to obtain Hölder continuity for g, we evaluate the difference $\|u(t + h) - u(t)\|_{X^\alpha}$:

$$\begin{aligned} u(t + h) - u(t) &= U(t + h, \tau)u_0 - U(t, \tau)u_0 + \left(\int_\tau^{\tau+h} + \int_{\tau+h}^{t+h}\right) U(t + h, s)g(s)ds \\ &\quad - \int_\tau^t U(t, s)g(s)ds \\ &= [U(t + h, t) - I]U(t, \tau)u_0 + \int_\tau^{\tau+h} U(t + h, s)g(s)ds \\ &\quad + \int_\tau^t [U(t + h, s + h)g(s + h) - U(t, s)g(s)]ds \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

We estimate each one of the terms above in $\|\cdot\|_{X^\alpha}$. From Propositions 3.7 and 3.8, given any $\eta \in [0, 1)$, we obtain

$$\|\mathcal{I}_1\|_{X^\alpha} = \|[U(t + h, t) - I]A(\xi)^{-\eta}A(\xi)^{\eta+\alpha}U(t, \tau)u_0\|_X \le Ch^\eta(t - \tau)^{-\eta}\|u_0\|_{X^\alpha}.$$

The second term follows the same idea, taking into account that $t \mapsto g(s) \in X$ is continuous and locally bounded.

$$\|\mathcal{I}_2\|_{X^\alpha} \le C \int_\tau^{\tau+h} (t+h-s)^{-\alpha} \|g(s)\|_X ds \le Ch(t-\tau)^{-\alpha}.$$

The last term requires more reasoning and it will be the most restrictive one for the Hölder exponent, as we see below. To perform the necessary calculations, we use formulation (3.1) for the process:

$$\begin{aligned}
\mathcal{I}_3 &= \int_\tau^t U(t+h,s+h)g(s+h) - U(t,s)g(s)ds \\
&= \int_\tau^t \left\{T_{-A(s+h)}(t-s)g(s+h) - T_{-A(s)}(t-s)g(s)\right\} ds \\
&\quad + \int_\tau^t \left\{ \int_{s+h}^{t+h} T_{-A(\xi)}(t+h-\xi)\Phi(\xi,s+h)g(s+h)d\xi \right\} ds \\
&\quad - \int_\tau^t \left\{ \int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)g(s)d\xi \right\} ds.
\end{aligned}$$

Adding and subtracting $T_{-A(s)}(t-s)g(s+h)$ inside the integral on the first term and performing a change of variable in the second integral, we obtain

$$\begin{aligned}
\mathcal{I}_3 &= \int_\tau^t \left\{[T_{-A(s+h)}(t-s) - T_{-A(s)}(t-s)]g(s+h)\right\} ds \\
&\quad + \int_\tau^t \left\{T_{-A(s)}(t-s)[g(s+h)-g(s)]\right\} ds \\
&\quad + \int_\tau^t \left\{ \int_s^t T_{-A(\xi+h)}(t-\xi)\Phi(\xi+h,s+h)g(s+h)d\xi \right\} ds \\
&\quad - \int_\tau^t \left\{ \int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)g(s)d\xi \right\} ds \\
&= \int_\tau^t \left\{[T_{-A(s+h)}(t-s) - T_{-A(s)}(t-s)]g(s+h)\right\} ds \\
&\quad + \int_\tau^t \left\{T_{-A(s)}(t-s)[g(s+h)-g(s)]\right\} ds
\end{aligned}$$

$$+ \int_\tau^t \left\{ \int_s^t [T_{-A(\xi+h)}(t-\xi) - T_{-A(\xi)}(t-\xi)]\Phi(\xi+h, s+h)g(s+h)d\xi \right\} ds$$

$$+ \int_\tau^t \left\{ \int_s^t T_{-A(\xi)}(t-\xi)[\Phi(\xi+h, s+h) - \Phi(\xi, s)]g(s+h)d\xi \right\} ds$$

$$+ \int_\tau^t \left\{ \int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi, s)[g(s) - g(s+h)]d\xi \right\} ds$$

$$= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5.$$

From Proposition 3.3, we deduce

$$\|\mathcal{S}_1\|_{X^\alpha} \le C \int_\tau^t h^\delta (t-s)^{-\alpha} \|g(s+h)\|_X ds \le C h^\delta (t-\tau)^{1-\alpha},$$

$$\|\mathcal{S}_2\|_{X^\alpha} \le \int_\tau^t \|A(\xi)^\alpha T_{-A(s)}(t-s)\| \|g(s+h) - g(s)\|_X ds$$

$$\le C \int_\tau^t (t-s)^{-\alpha} \|g(s+h) - g(s)\|_X ds.$$

Term $\mathcal{S}_3$ also follows from Proposition 3.3 and the estimate for the family $\Phi(\cdot,\cdot)$

$$\|\mathcal{S}_3\|_{X^\alpha} \le C \int_\tau^t \left\{ \int_s^t h^\delta (t-\xi)^{-\alpha} (\xi - s)^{\delta-1} d\xi \right\} ds \le C h^\delta \int_\tau^t (t-s)^{\delta-\alpha} ds \le C h^\delta (t-\tau)^{\delta-\alpha+1}.$$

From Proposition 3.6, given any $0 \le \nu < \delta$,

$$\|\mathcal{S}_4\|_{X^\alpha} \le C \int_\tau^t \left\{ \int_s^t (t-\xi)^{-\alpha} h^\nu (\xi - s)^{(\delta-\nu)-1} d\xi \right\} ds$$

$$\le C h^\nu \int_\tau^t (t-s)^{(\delta-\nu)-\alpha} ds \le C h^\nu (t-\tau)^{1-\alpha+\delta-\nu},$$

$$\|\mathcal{S}_5\|_{X^\alpha} \le C \int_\tau^t \left\{ \int_s^t (t-\xi)^{-\alpha} (\xi - s)^{\delta-1} d\xi \right\} \|g(s) - g(s+h)\|_X ds$$

$$\le C \int_\tau^t (t-s)^{\delta-\alpha} \|g(s+h) - g(s)\|_X ds.$$

Using the results above and estimating by the smallest possible Hölder exponent for h, we obtain

$$
\begin{aligned}
\|\mathcal{I}_3\|_{X^\alpha} &\le Ch^\delta (t-\tau)^{1-\alpha} + Ch^\delta (t-\tau)^{\delta-\alpha+1} + Ch^\nu (t-\tau)^{1-\alpha+\delta-\nu} \\
&\quad + C\int_\tau^t [(t-s)^{-\alpha} + (t-s)^{\delta-\alpha}]\|g(s+h)-g(s)\|_X ds \\
&\le Ch^\nu + C\int_\tau^t (t-s)^{-\alpha}\|g(s+h)-g(s)\|_X ds.
\end{aligned}
$$

From $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ we conclude that, for any $\eta \in [0,1)$ and $\nu \in [0,\delta)$,

$$
\begin{aligned}
\|u(t+h)-u(t)\|_{X^\alpha} &\le Ch^\eta (t-\tau)^{-\eta} + Ch(t-\tau)^{-\alpha} + Ch^\nu \\
&\quad + C\int_\tau^t (t-s)^{-\alpha}\|g(s+h)-g(s)\|_X ds \\
&\le Ch^\nu[(t-\tau)^{-\nu} + (t-\tau)^{-\alpha}] + C\int_\tau^t (t-s)^{-\alpha}\|g(s+h)-g(s)\|_X ds.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
\|g(t+h)-g(t)\|_X &= Ch^\omega + Ch^\nu[(t-\tau)^{-\nu} + (t-\tau)^{-\alpha}] \\
&\quad + C\int_\tau^t (t-s)^{-\alpha}\|g(s+h)-g(s)\|_X ds.
\end{aligned}
\tag{5.1}
$$

An application of Gronwall's inequality yields

$$
\|g(t+h)-g(t)\|_X \le Ch^{\min\{\omega,\nu\}}(t-\tau)^{-\max\{\alpha,\nu\}}
$$

and we conclude that $g(\cdot)$ belongs to $\mathcal{C}^\nu_{\max\{\alpha,\nu\}}((\tau,T),X)$, for any $\nu \in [0,\min\{\omega,\delta\})$. □

We can now prove the abstract theorem stated in Section 2.

5.3. *Proof of Theorem 2.5*

If $u:(\tau,T)\to X^\alpha$, $\alpha\in[0,1)$, is the solution of

$$
u_t(t) + A(t)u = f(t,u(t)),\ t\in(\tau,T);\quad u(\tau)=u_0\in X^\alpha,
$$

then $g(t) = f(t,u(t))$ belongs to $\mathcal{C}^\beta_\theta((\tau,T),X)$ for any $\beta\in[0,\min\{\omega,\delta\})$ and for $\theta = \max\{\alpha,\beta\}$. In this case, Theorem 5.2 states that the solution $x:[\tau,T)\to X$ of

$$x_t(t) + A(t)x = g(t),\ t \in (\tau, T); \quad x(\tau) = u_0 \in X^\alpha,$$

satisfies, for $0 \le \beta < \min\{\omega, \delta\}$, $x_t(t) \in X^\beta$ and

$$\begin{aligned}\|x_t(t)\|_{X^\beta} &\le C(t-\tau)^{-1-\beta+\alpha}\|x_0\|_{X^\alpha} + C(t-\tau)^{-\theta-\beta} \\ &\le C(t-\tau)^{-1-\beta+\alpha}\|x_0\|_{X^\alpha} + C(t-\tau)^{-\max\{2\beta,\beta+\alpha\}}.\end{aligned}$$

From the variation of constants formula,

$$x(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)g(s)ds = U(t,\tau)u_0 + \int_\tau^t U(t,s)f(s,u(s))ds = u(t)$$

and we obtain the desired properties for $u_t(t)$, which proves Theorem 2.5.

Remark 5.4. In [16, Theorem 3.5.2], the author proved the smoothing effect on $u_t(t)$ when $A(t) = A$ and $f(t,u)$ is locally Lipschitz on both variables. In the notation of Theorem 2.5, this case corresponds to $\delta = 1$ and $\omega = 1$. Therefore, for any $0 \le \beta < 1$, $u_t(t) \in X^\beta$ and $\|u_t(t)\|_{X^\beta} \le C(t-\tau)^{-1-\beta+\alpha}$, matching the result found in [16].

Remark 5.5. The fact that $u_t(t) \in X^\beta$ for $\beta \in [0, \min\{\lambda, \omega\})$ is independent of α, as long as $\alpha \in [0, 1)$.

Remark 5.6. There are some works in the literature that deals with the case $\alpha = 1$ in $f : \mathbb{R} \times X^\alpha \to X$, which is called *critical case* (see [4,5] for the nonsingular case and [8] for the singular case). For a class of functions called *ε-regular maps*, the existence of local mild solution can be proved in this situation.

Estimates of item $\mathcal{S}_1 - \mathcal{S}_5$ on Lemma 5.3, which result in the estimate (5.1) for the difference $u(t+h) - u(t)$, would all be impaired if $\alpha = 1$, preventing us to extend those results of regularization and smoothing effect to the ε-regular solution constructed for the problem in the papers just mentioned.

6. Application: singularly nonautonomous reaction-diffusion equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and consider the following singularly nonautonomous reaction-diffusion equation in Ω:

$$\begin{cases} u_t - div(a(t,x)\nabla u) + u = f(t,u), & x \in \Omega,\ t > \tau, \\ \partial_n u = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

We assume that:

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

(A.2) The function $a : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^+$ is continuously differentiable, $a \in \mathcal{C}^1(\mathbb{R} \times \overline{\Omega}, \mathbb{R}^+)$, and has its image in a closed interval $[a_0, a_1] \subset (0, \infty)$. We denote by $b(t,x)$ the gradient function (in x) of $a(t,x)$ and we assume it is bounded, that is, $b(t,x) := \nabla_x a(t,x) \in L^\infty(\Omega)$.

(A.3) Both functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are *Hölder continuous in the first variable* with same Hölder exponent $\delta \in (0,1]$:

$$|a(t,x) - a(s,x)| \le C|t-s|^{\delta}, \quad |b(t,x) - b(s,x)| \le C|t-s|^{\delta}. \tag{6.2}$$

(A.4) The nonlinearity $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a *polynomial growth condition of order* ρ, that is, there exists C and $1 \le \rho < \rho_0$ such that

$$|f(t,\xi) - f(t,\psi)| \le C|\xi - \psi|(1 + |\xi|^{\rho-1} + |\psi|^{\rho-1}),$$
$$|f(t,\xi)| \le C(1 + |\xi|^{\rho}).$$

Moreover, $t \mapsto f(t,\cdot)$ is *locally ω-Hölder continuous* with exponent $\omega \in (0,1]$ and $C > 0$, that is, $|f(t,\cdot) - f(s,\cdot)| \le C|t-s|^{\omega}$.

Remark 6.1. An example of function $a(t,x)$ that satisfies conditions required in (A.2) and (A.3) is $a(t,x) = [c + \sin(t)]\tilde{a}(x)$, where $c > 1$ and $\tilde{a}(\cdot) \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^+)$. In this case, $\delta = 1$ and $a(t,x)$ could represent situations where the diffusion changes periodically with time, such as movement of populations depending on season of the year or time of the day.

As for the nonlinearity $f(t,u)$, a typical representative for functions satisfying (A.4) is $f(t,u) = g(u) = -u|u|^{\rho-1}$, and in this case the estimates for $f(\cdot,\cdot)$ follow easily. More generally, any function $f(t,u) = h(u)$ such that $|h'(u)| \le C(1 + |u|^{\rho-1})$ satisfies the conditions required in (A.4). If $f(t,u) = \xi(t)h(u)$ with ξ locally Hölder continuous, then conditions in (A.4) are also satisfied.

The maximal value for ρ in (A.4), ρ_0, depends on the space chosen to solve the solution, as we will see in the sequel.

The phase space is $X = L^2(\Omega)$ and the linear part of equation (6.1) is given by $A(t) : D(A(t)) \subset L^2(\Omega) \to L^2(\Omega)$ where

$$D = D(A(t)) = \left\{ u \in H^2(\Omega) : \partial_n u = 0 \text{ in } \partial\Omega \right\},$$
$$A(t)u = -div(a(t,x)\nabla u) + u, \text{ for } u \in D.$$

To obtain classical solution, we will consider a bootstrap argument that requires us to work with the realization of $A(t)$ on $L^q(\Omega)$, for $q > 2$. In this case, we denote the domain of the realization as

$$D_q = \left\{ u \in W^{2,q}(\Omega) : \partial_n u = 0 \text{ in } \partial\Omega \right\}$$

and the fractional powers spaces associated to $A(t)$ as X_q^{α}. For the case $q = 2$, we simply omit the subindex.

Lemma 6.2. *If condition (6.2) holds, then, for any $1 < p < \infty$, $\mathbb{R} \ni t \mapsto A(t)A(\tau)^{-1} \in \mathcal{L}(L^p(\Omega))$ is Hölder continuous with exponent δ, that is, $\|[A(t) - A(s)]A(\tau)^{-1}\| \le C|t-s|^{\delta}$, for all $\tau, s, t \in \mathbb{R}$.*

Proof. For $u \in D_p$, we have $A(t)u - A(s)u = -div\,([a(t,x) - a(s,x)]\nabla u)$ and

$$\begin{aligned}
&\int_\Omega |div([a(t,x) - a(s,x)]\nabla u(x))|^p\,dx \\
&\quad = \int_\Omega |\nabla_x([a(t,x) - a(s,x)])\nabla u(x) + [a(t,x) - a(s,x)]\Delta u|^p\,dx \\
&\quad \le (t-s)^{\delta p} \int_\Omega \left\{ \frac{|\nabla_x a(t,x) - \nabla_x a(s,x)|}{|t-s|^\delta} \right\}^p |\nabla u(x)|^p dx \\
&\qquad + (t-s)^{\delta p} \int_\Omega \left\{ \frac{|a(t,x) - a(s,x)|}{|t-s|^\delta} \right\}^p |\Delta u(x)|^p dx \\
&\quad \le C(t-s)^{\delta p} \left\{ \|\nabla u\|^p_{L^p(\Omega)} + \|\Delta u\|^p_{L^p(\Omega)} \right\} \le C(t-s)^{\delta p} \|u\|^p_{W^{2,p}(\Omega)}.
\end{aligned}$$

Therefore, $\|[A(t) - A(s)]u\|^p_{L^p(\Omega)} \le C|t-s|^{p\delta}\|u\|^p_{D_p}$, for all $u \in D_p$. Taking the $p-th$ roots on both sides and replacing u by $A(\tau)^{-1}\tilde{u}$,

$$\|[A(t) - A(s)]A(\tau)^{-1}\tilde{u}\|_{L^p(\Omega)} \le C|t-s|^\delta \|\tilde{u}\|_{L^p(\Omega)}, \quad \forall \tilde{u} \in L^p(\Omega). \quad \square$$

We gather in the sequel some other important properties of the family $A(t)$, $t \in \mathbb{R}$.

Proposition 6.3. *The family of linear operators $A(t)$, $t \in \mathbb{R}$, satisfies:*

(1) $A(t)$, $t \in \mathbb{R}$, is uniformly sectorial.
(2) $A(t)$ has compact resolvent, is positive and self-adjoint operator. Its spectrum consists entirely of isolated eigenvalues, all of them positive and real:

$$\sigma(A(t)) = \{\lambda_i(t) : 1 = \lambda_1(t) \le \lambda_2(t) \le ... \le \lambda_n(t) \le ...\}.$$

(3) The following embeddings hold

$$\begin{aligned}
&\text{If } \tfrac{N}{4} < \alpha \le 1,\ X^\alpha \hookrightarrow C^\nu(\Omega), \quad \text{when } 0 \le \nu < 2\alpha - \tfrac{N}{2}.\\
&\text{If } 0 \le \alpha \le \tfrac{N}{4},\ X^\alpha \hookrightarrow L^r(\Omega), \quad \text{when } 2 \le r < \tfrac{2N}{N-4\alpha}.
\end{aligned}$$

If we consider $A(t)$ acting on $L^q(\Omega)$, the embeddings become

$$\begin{aligned}
&\text{If } \tfrac{N}{2q} < \alpha \le 1,\ X_q^\alpha \hookrightarrow C^\nu(\Omega), \quad \text{when } 0 \le \nu < 2\alpha - \tfrac{N}{q}.\\
&\text{If } 0 \le \alpha \le \tfrac{N}{2q},\ X_q^\alpha \hookrightarrow L^r(\Omega), \quad \text{when } 2 \le q < \tfrac{2N}{N-2\alpha q}.
\end{aligned}$$

(4) If $0 \le \beta < \gamma \le 1$, then X^γ is compactly embedded in X^β.

Statement (1) follows from Theorem 7.3.6 of [22]. Properties in (2) are consequences of [20, Theorem 6.29 and Section V.3.5]. Items (3) and (4) follow from Theorems 1.6.1 and 1.4.8 of [16], respectively.

To fix the ideas we will assume that $\alpha = \frac{1}{2}$ and $N = 3$, but the analysis performed in this section can be extended to any $\alpha \in [0, 1)$ and $N \geq 2$. However, as α changes, so does the growth ρ allowed for f.

Note that from the embeddings in Proposition 6.3, if $\alpha > \frac{3}{4}$ and $N = 3$, f can have polynomial growth of any order ($\rho \in (1, \infty)$) as a consequence of $X^\alpha \hookrightarrow C^\nu(\Omega) \hookrightarrow L^\infty(\Omega)$.

Under those conditions, (6.1) originates the abstract singular semilinear evolution problem:

$$u_t + A(t)u = F(t, u),\ t > \tau; \quad u(\tau) = u_0 \in X^{\frac{1}{2}},$$

where F is the nonlinearity given by $F(t, u)(x) = f(t, u(t, x))$. Since $X^{\frac{1}{2}} \hookrightarrow L^r(\Omega)$, for all $2 \leq r < 6$, the largest growth allowed for F such that $F : \mathbb{R} \times X^{\frac{1}{2}} \to L^2(\Omega)$, is given by $1 \leq \rho < 3$. We have the following properties for F:

Lemma 6.4. *Let $\alpha = \frac{1}{2}$, $N = 3$ and $1 \leq \rho < 3$. Then $F : \mathbb{R} \times X^{\frac{1}{2}} \to L^2(\Omega)$ and satisfies*

$$\begin{aligned}
\|F(t, u) - F(t, \tilde{u})\|_{L^2(\Omega)} &\leq C\|u - \tilde{u}\|_{X^{\frac{1}{2}}}(1 + \|u\|^{\rho-1}_{X^{\frac{1}{2}}} + \|\tilde{u}\|^{\rho-1}_{X^{\frac{1}{2}}}),\\
\|F(t, u)\|_{L^2(\Omega)} &\leq C(1 + \|u\|^\rho_{X^{\frac{1}{2}}}),\\
\|F(t, u) - F(s, u)\|_{L^2(\Omega)} &\leq C|t - s|^\omega.
\end{aligned}$$

Proof. The first one follows from

$$\|F(t, u) - F(t, \tilde{u})\|_{L^2(\Omega)} = \left[\int_\Omega |f(t, u(x)) - f(t, \tilde{u}(x))|^2 dx\right]^{\frac{1}{2}}$$

$$\leq \left[\int_\Omega C^2|u(x) - \tilde{u}(x)|^2(1 + |u(x)|^{2(\rho-1)} + |\tilde{u}(x)|^{2(\rho-1)})dx\right]^{\frac{1}{2}}$$

$$\leq C\left[\left(\int_\Omega |u - \tilde{u}|^{2\rho} dx\right)^{\frac{1}{\rho}}\right]^{\frac{1}{2}}\left[\left(\int_\Omega [1 + |u|^{2(\rho-1)\frac{\rho}{\rho-1}} + |\tilde{u}|^{2(\rho-1)\frac{\rho}{\rho-1}}]dx\right)^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{2}}$$

$$\leq C\|u - \tilde{u}\|_{L^{2\rho}(\Omega)}\left\{1 + \left(\int_\Omega |u|^{2\rho}dx\right)^{\frac{\rho-1}{2\rho}} + \left(\int_\Omega |\tilde{u}|^{2\rho}dx\right)^{\frac{\rho-1}{2\rho}}\right\}$$

$$\leq C\|u - \tilde{u}\|_{L^{2\rho}(\Omega)}\left(1 + \|u\|^{\rho-1}_{L^{2\rho}(\Omega)} + \|\tilde{u}\|^{\rho-1}_{L^{2\rho}(\Omega)}\right).$$

Since $X^{\frac{1}{2}} \hookrightarrow L^{2\rho}(\Omega)$, for all $1 \leq \rho < 3$, it follows that $\|\cdot\|_{L^{2\rho}(\Omega)} \leq \|\cdot\|_{X^{\frac{1}{2}}}$ and we obtain

$$\|F(t,u) - F(t,\tilde{u}))\|_{L^2(\Omega)} \leq C\|u - \tilde{u}\|_{X^{\frac{1}{2}}}(1 + \|u\|^{\rho-1}_{X^{\frac{1}{2}}} + \|\tilde{u}\|^{\rho-1}_{X^{\frac{1}{2}}}).$$

For the second inequality,

$$\|F(t,u)\|_{L^2(\Omega)} \leq \left(\int_\Omega C^2(1+|u|^{2\rho})dx\right)^{\frac{1}{2}} \leq C\left(1 + \left[\int_\Omega |u|^{2\rho}dx\right]^{\frac{1}{2\rho}\rho}\right)$$
$$\leq C(1 + \|u\|^{\rho}_{L^{2\rho}(\Omega)}) \leq C(1 + \|u\|^{\rho}_{X^{\frac{1}{2}}}). \quad \square$$

Remark 6.5. Note that in the proof of Lemma 6.4, we also obtained the following inequality for the nonlinearity F:

$$\|F(t,u)\|_{L^2(\Omega)} \leq C(1 + \|u\|^{\rho}_{L^{2\rho}(\Omega)}). \tag{6.3}$$

6.1. Classical solutions via a bootstrap argument

The conditions required for $a(\cdot,\cdot)$ and $f(\cdot,\cdot)$ ensure that $A(t)$ satisfies properties (P.1) and (P.2), while F satisfies (NL). Therefore, the problem admits local solution $w : [\tau, T] \to X$ given by

$$u(t,\tau,u_0) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(s,u(s))ds$$

such that $u(t) \in D = X^1$ for all $t \in (\tau, T]$, $u_t(t) \in X^\beta$ and

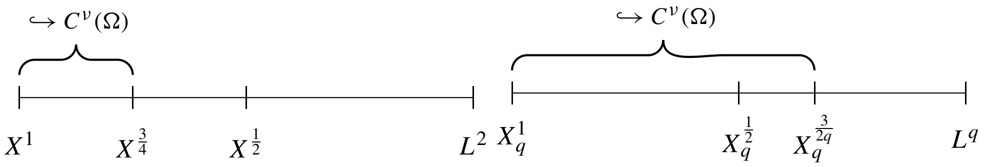
$$\|u_t(t)\|_{X^\beta} \leq C(t-\tau)^{-1-\beta+\alpha}\|u_0\|_{X^\alpha} + C(t-\tau)^{-\max\{2\beta,\beta+\alpha\}},$$

for any $0 \leq \beta < \min\{\omega, \delta\}$ (Theorem 2.5).

The idea to obtain classical solutions, that is, $x \mapsto u(t,x)$ in $C^2(\Omega)$, is to look at the differential equation as

$$A(t)u = F(t,u) - u_t(t). \tag{6.4}$$

If we are able to prove that the right-side belongs to $C(\Omega)$, then the left-side must be $C^2(\Omega)$. This is obtained via a bootstrap argument that we reproduce in the sequel. The embedding results on Proposition 6.3 are summarized in the following illustrative scheme:



On the left side ($q = 2$), a function will be Hölder continuous if it is in a space X^α with $\alpha > \frac{3}{4}$. For $q > 2$, this restriction becomes $\alpha > \frac{3}{2q}$.

First iteration: After any arbitrarily small evolution on time $\tau < t_1$, the solution in t_1, $u(t_1, \tau, u_0)$ belongs to $X^1 \hookrightarrow \mathcal{C}^\nu(\Omega)$ and, consequently,

$$F(t, u(t)) \in \mathcal{C}^\nu(\Omega), \text{ for any } t > t_1.$$

The derivative $u_t(t)$ belongs to X^β, for any $0 \leq \beta < \min\{\delta, \omega\}$, as stated in Theorem 2.5. If there exists $\beta > \frac{3}{4}$ in this interval, then $u_t(t) \in \mathcal{C}^\nu(\Omega)$, for some $\nu > 0$ and for all $t \geq t_1$. In this case, from (6.4),

$$A(t)u(t) \in \mathcal{C}^\nu(\Omega), \text{ implying } u(t) \in \mathcal{C}^{2,\nu}(\Omega), \text{ for all } t \geq t_1,$$

and classical solution is achieved.

If that is not the case, we have $u_t(t) \in X^\beta \hookrightarrow L^r(\Omega)$, where $2 \leq r < \frac{6}{3-2\beta}$. Once again as consequence of (6.4), we obtain

$$A(t)u(t) \in L^r(\Omega), \text{ implying } u(t) \in W^{2,r}(\Omega) \hookrightarrow \mathcal{C}^\nu(\Omega), \text{ for all } t \geq t_1.$$

Second iteration: In case the classical solution was not achieved in the first iteration, we consider the problem now starting at the instant t_1 with initial condition $u(t_1, \tau, u_0) \in \mathcal{C}^\nu(\Omega)$. Since $C^\nu(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 < q < \infty$, we obtain, after any arbitrarily small evolution $t_2 > t_1$, that

$$u_t(t) \in X_q^\beta, \text{ for any } \beta \in [0, \min\{\delta, \omega\}).$$

However, q can assume any value in $(1, \infty)$, and we fix one such that $\frac{3}{2q} < \min\{\delta, \omega\}$, then there exists $\beta > \frac{3}{2q}$ such that $u_t(t) \in X^\beta \hookrightarrow \mathcal{C}^\nu(\Omega)$ for any $t \geq t_2$. Therefore,

$$Au(t) = F(t, u(t)) - u_t(t) \in \mathcal{C}^\nu(\Omega), \text{ and } u(t) \in \mathcal{C}^{2,\nu}(\Omega), \text{ for all } t \geq t_2.$$

6.2. *Estimates for u*

The dependence on t of the linear operator prevents us to construct a Lyapunov function for the problem studied, unless stronger properties on the function $t \mapsto a(t, \cdot)$ are required (for instance, that $t \mapsto a(t, \cdot)$ is non increasing).

To avoid this type of restriction on $a(\cdot, \cdot)$ we apply a different method to obtain estimates for u, which we discuss in the sequel and involves an iteration procedure inspired in a technique developed by Moser-Alikakos (see [2,9,11,12]).

To obtain global well-posedness, we assume that f satisfies a dissipativeness condition:

$$\textbf{(D)} \ \limsup_{|s|\to\infty} \frac{f(t, s)}{s} < 1.$$

Remark 6.6. The value 1 comes from the fact that first eigenvalue of $A(t)$ is $\lambda_1(t) = 1$ (see Proposition 6.3). For other situations, like the operator with Dirichlet boundary conditions, the appropriate dissipativeness would be written in terms of the first eigenvalue of the operator $A(t)$, $\lambda_1(t)$. Since $\lambda_1(t)$ might change with time, a condition that would suit the purpose of ensuring dissipation is $\limsup_{|s|\to\infty} \dfrac{f(t,s)}{s} < \inf\{\lambda_1(t); t \in \mathbb{R}\}$.

A simple use of the definition of *Limsup* allows us to prove the following result:

Lemma 6.7. *Suppose that condition (D) holds, then there exists $\gamma_1 > 0$ such that, for each $\gamma \in (0, \gamma_1)$, there is a constant $M > 0$ such that*

$$f(s)s \le (1-\gamma)s^2 + M, \quad \forall s \in \mathbb{R}. \tag{6.5}$$

We first estimate the L^2-norm for the solution $u(t,\tau,u_0)$. The regularity of the solution allows us to take the inner product in $L^2(\Omega)$ of the equation in (6.1) with u and integrate by parts, obtaining

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} = -\int_\Omega a(t,x)|\nabla u|^2 dx - \|u\|^2_{L^2(\Omega)} + \int_\Omega f(u)u\,dx,$$

and using (A.2), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} \le -\|u\|^2_{L^2(\Omega)} + \int_\Omega f(u)u\,dx. \tag{6.6}$$

Proposition 6.8. *Under the dissipativeness condition (D) and for γ and M as in (6.5), the solution $u(\cdot,\tau,u_0)$ of (6.1) satisfies*

$$\|u(t,\tau,u_0)\|_{L^2(\Omega)} \le 2^{\frac{1}{2}}\left[e^{-\gamma(t-\tau)}\|u_0\|_{L^2(\Omega)} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right],$$

as long as the solution exists.

Proof. Inequality (6.5) can be applied in (6.6) in order to obtain

$$\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} \le \int_\Omega \left[(1-\gamma)u^2 + M\right] dx$$

$$2\gamma\|u\|^2_{L^2(\Omega)} + \frac{d}{dt}\|u\|^2_{L^2(\Omega)} \le 2M|\Omega|$$

$$2\gamma e^{2\gamma(t-\tau)}\|u\|^2_{L^2(\Omega)} + e^{2\gamma(t-\tau)}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} \le e^{2\gamma(t-\tau)}2M|\Omega|$$

$$\frac{d}{dt}\left[e^{2\gamma(t-\tau)}\|u\|^2_{L^2(\Omega)}\right] \le e^{2\gamma(t-\tau)}2M|\Omega|.$$

Integrating from τ to t we derive

$$e^{2\gamma(t-\tau)}\|u(t)\|_{L^2(\Omega)} - \|u_0\|_{L^2(\Omega)} \leq [e^{2\gamma(t-\tau)} - 1]\frac{M}{\gamma}|\Omega|$$

$$\|u(t)\|^2_{L^2(\Omega)} \leq e^{-2\gamma(t-\tau)}\|u_0\|^2_{L^2(\Omega)} + \frac{M}{\gamma}|\Omega|.$$

Taking the square roots on both sides and using the inequality $|a+b|^r \leq 2^r(|a|^r + |b|^r)$ for any $r > 0$, we obtain the desired inequality. $\square$

Once we have L^2- estimates for the solution, we can use the iteration proposed by Moser-Alikakos. This technique consists in obtaining L^{2^k}-estimates of u by using the estimate of the $L^{2^{k-1}}$-norm. In other words, it is an inductive procedure.

Therefore, from the L^2-estimate obtained in the previous proposition, we derive L^4-estimate, then L^8 and so on. This is the procedure applied at the following lemma, but before we establish it, we discuss a convention that we adopt in the next results.

Remark 6.9. Given a bounded set $B \subset X^{\frac{1}{2}}$ such that $\|u_0\|_{X^{\frac{1}{2}}} \leq L$ for $u_0 \in B$, after any evolution in time, the solutions starting with initial conditions in B become bounded in stronger norms. Indeed, from the continuity of the solution, for any $\tau < t^*$, with t^* arbitrarily close to τ, $\|u(t,\tau,u_0)\|_{X^{\frac{1}{2}}} \leq CL$, $\tau \leq t \leq t^*$ and from the variation of constants formula, for $\beta \in [0,1)$,

$$\begin{aligned}\|u(t^*,\tau,u_0)\|_{X^\beta} &\leq \|U(t^*,\tau)u_0\|_{X^\beta} + \int_\tau^{t^*} \|U(t^*,s)F(s,u(s,\tau,u_0))\|_{X^\beta} ds \\ &\leq C(\beta)L + \int_\tau^{t^*} C(t^*-s)^{-\beta}(1 + \|u(s,u_0)\|^\rho_{X^{\frac{1}{2}}})ds \leq C(\beta,\rho,\|u_0\|_{X^{\frac{1}{2}}}),\end{aligned}$$

where $C(\beta,\rho,\|u_0\|_{X^{\frac{1}{2}}})$ denotes a constant that depends on β, ρ and $\|u_0\|_{X^{\frac{1}{2}}}$.

In particular, for $\beta > \frac{3}{4}$ we obtain $\|u(t^*,\tau,u_0)\|_{L^\infty(\Omega)} \leq C(\rho,\|u_0\|_{X^{\frac{1}{2}}})$. In conclusion, given any bounded set B in $X^{\frac{1}{2}}$, after an arbitrary evolution takes place, this set B becomes bounded in $L^\infty(\Omega)$. Since we are interested in the asymptotic dynamics of the problem, whenever we wish to estimate the solution, we will assume that given any bounded set of initial condition B in $X^{\frac{1}{2}}$, this set will also be bounded in $L^\infty(\Omega)$. If that is not the case, we evolve the system any arbitrary time and restart the evolution from this point.

To simplify the notation, we will denote by $\|\cdot\|_{L^p}$ the norm $\|\cdot\|_{L^p(\Omega)}$.

Lemma 6.10. *Suppose that condition (D) holds and γ, M are the constants in (6.5). Let $u(\cdot,\tau,u_0)$ be the solution of (6.1) and assume $u_0 \in L^\infty(\Omega)$. Given any $k \in \mathbb{N}$, there exists constant $c > 0$ independent of k such that, for $t > \tau$,*

$$\|u(t)\|_{L^{2^k}}^{2^k} \leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|u(s)\|_{L^{2^{k-1}}}^{2^k}ds + \left[\frac{M}{\gamma}|\Omega|\right], \tag{6.7}$$

as long as the solution exists.

(N is the dimension of Ω*, which is* 3 *in the case considered. We keep N in the formulation to emphasize the dependence on it and to allow the extension of the result to other situations).*

Proof. Multiplying the equation in (6.1) by u^{2^k-1} and integrating in Ω, we obtain

$$\int_\Omega u_t u^{2^k-1}dx = \int_\Omega div(a(t,x)\nabla u)u^{2^k-1}dx - \int_\Omega u^{2^k}dx + \int_\Omega f(u)uu^{2^k-2}dx.$$

The term on the left side can be replaced by $\frac{1}{2^k}\frac{d}{dt}\int_\Omega u^{2^k}dx = \int_\Omega u_t u^{2^k-1}dx$, whereas from the dissipativeness condition, we deduce

$$\begin{aligned}\int_\Omega (f(u)u)u^{2^k-2}dx &\leq \int_\Omega (1-\gamma)u^{2^k}dx + M\int_\Omega u^{2^k-2}dx\\ &\leq \int_\Omega (1-\gamma)u^{2^k}dx + M\left[\int_\Omega u^{2^k}+1dx\right],\end{aligned}$$

where in the last inequality we used the fact that $a^{2^k-2} < a^{2^k}+1$ for any positive a. Thus,

$$\frac{1}{2^k}\frac{d}{dt}\int_\Omega u^{2^k}dx \leq \int_\Omega div(a(t,x)\nabla u)u^{2^k-1}dx + [M-\gamma]\int_\Omega u^{2^k}dx + M|\Omega|.$$

Integration by parts leads to

$$-\int_\Omega [a(t,x)\nabla u](2^k-1)u^{2^k-2}\nabla u dx = -(2^k-1)\int_\Omega a(t,x)(\nabla u)^2 u^{2^k-2}dx.$$

Note that

$$\nabla\left(u^{2^{k-1}}\right) = 2^{k-1}u^{2^{(k-1)}-1}\nabla u \quad \text{and} \quad \left[\nabla\left(u^{2^{(k-1)}}\right)\right]^2 = 2^{2(k-1)}u^{2^k-2}(\nabla u)^2,$$

so the term $u^{2^k-2}(\nabla u)^2$ can be replaced by $\frac{1}{2^{2(k-1)}}\left[\nabla\left(u^{2^{(k-1)}}\right)\right]^2$. Therefore,

$$-(2^k-1)\int_\Omega a(t,x)(\nabla u)^2u^{2^k-2}dx \leq -a_0(2^k-1)(2^{2-2k})\int_\Omega \left[\nabla\left(u^{2^{(k-1)}}\right)\right]^2dx$$

and the inequality becomes (after multiplying it for 2^k and using $0 < a_0 \leq a(t,x) \leq a_1$),

$$\frac{d}{dt}\int_\Omega u^{2^k}\,dx \le -a_0(2^k-1)(2^{2-k})\int_\Omega \left[\nabla\left(u^{2^{(k-1)}}\right)\right]^2 dx+2^k[M-\gamma]\int_\Omega u^{2^k}\,dx+2^kM|\Omega|. \quad (6.8)$$

An application of Nirenberg-Gagliardo's inequality (Theorem 1.2.2 in [9]) for a certain $v \in W^{1,2}(\Omega)\cap L^1(\Omega)$, with $j=0,\ p=2,\ m=1,\ r=2,\ q=1$ and $\theta=\frac{N}{N+2}$, implies that

$$\|v\|_{L^2(\Omega)} \le C(N,\Omega)\|\nabla v\|_{L^2(\Omega)}^{\frac{N}{N+2}}\|v\|_{L^1(\Omega)}^{\frac{2}{N+2}}.$$

If we also use the Young generalized inequality with conjugated exponents $\xi=\frac{1}{\theta}=\frac{N+2}{N}$ and $\xi'=\frac{N+2}{2}$, we obtain

$$\|v\|_{L^2(\Omega)} \le \varepsilon\|\nabla v\|_{L^2(\Omega)}+\frac{1}{\varepsilon^{\frac{N}{2}}}\|v\|_{L^1(\Omega)}.$$

Taking the square power on both sides (and rearranging ε^2 for ε),

$$(1-\varepsilon)\|v\|^2_{L^2(\Omega)} \le \|v\|^2_{L^2(\Omega)} \le \varepsilon\|\nabla v\|^2_{L^2(\Omega)}+\frac{1}{\varepsilon^{\frac{N}{2}}}\|v\|^2_{L^1(\Omega)}.$$

We apply the above inequality for $v=u^{2^{(k-1)}}$

$$\|u^{2^{(k-1)}}\|^2_{L^2(\Omega)}=\int_\Omega u^{2\left(2^{(k-1)}\right)}dx=\int_\Omega u^{2^k}\,dx,$$

$$\left\|\nabla\left[u^{2^{(k-1)}}\right]\right\|^2_{L^2(\Omega)}=\int_\Omega\left|\nabla\left[u^{2^{(k-1)}}\right]\right|^2dx,$$

$$\|u^{2^{(k-1)}}\|^2_{L^1(\Omega)}=\left(\int_\Omega u^{2^{(k-1)}}\,dx\right)^2,$$

and it becomes

$$-\int_\Omega\left|\nabla\left[u^{2^{(k-1)}}\right]\right|^2dx \le \frac{1}{\varepsilon^{\frac{N}{2}+1}}\left(\int_\Omega u^{2^{(k-1)}}dx\right)^2-\frac{(1-\varepsilon)}{\varepsilon}\int_\Omega u^{2^k}\,dx.$$

We use this inequality with a proper choice of ε and apply it at (6.8) in order to create a negative term multiplying $\int_\Omega u^{2^k}\,dx$

$$\frac{d}{dt}\int_\Omega u^{2^k}\,dx \le -\,a_0\frac{2^k-1}{(2^{k-2})}\frac{(1-\varepsilon)}{\varepsilon}\int_\Omega u^{2^k}\,dx+a_0\frac{2^k-1}{(2^{k-2})}\frac{1}{\varepsilon^{\frac{N}{2}+1}}\left(\int_\Omega u^{2^{(k-1)}}dx\right)^2$$

$$+ 2^k[M-\gamma]\int\limits_\Omega u^{2^k}\,dx + 2^k M|\Omega|.$$

Note that $2 \le \frac{2^k-1}{(2^{k-2})} \le 4$, and we can readjust the preceding inequality to obtain

$$\frac{d}{dt}\int\limits_\Omega u^{2^k}\,dx \le \left[-2a_0\frac{(1-\varepsilon)}{\varepsilon} + 2^k[M-\gamma]\right]\int\limits_\Omega u^{2^k}\,dx + 4a_0\frac{1}{\varepsilon^{\frac{N}{2}+1}}\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^2 + 2^k M|\Omega|.$$

Choosing $\varepsilon = c2^{-k}$, with c small enough to ensure that

$$2a_0\frac{(1-\varepsilon)}{\varepsilon} = 2a_0\frac{1-c2^{-k}}{c2^{-k}} > \frac{2a_0}{c}2^k > 2^k([M-\gamma]+1)$$

(for example, $c = \frac{a_0}{([M-\gamma]+1)}$), we obtain

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega u^{2^k}\,dx &\le -2^k\int\limits_\Omega u^{2^k}\,dx + 4a_0\frac{1}{c^{\frac{N}{2}+1}(2^{-k})^{\frac{N}{2}+1}}\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^2 + 2^k M|\Omega| \\ &= -2^k\int\limits_\Omega u^{2^k}\,dx + c(2^k)^{\frac{N}{2}+1}\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^2 + 2^k M|\Omega|.\end{aligned}$$

We have then achieved the desired differential inequality

$$2^k\int\limits_\Omega u^{2^k}\,dx + \frac{d}{dt}\int\limits_\Omega u^{2^k}\,dx \le c(2^k)^{\frac{N}{2}+1}\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^2 + 2^k M|\Omega|. \tag{6.9}$$

Note that

$$\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^2 = \left[\left(\int\limits_\Omega u^{2^{(k-1)}}\,dx\right)^{\frac{1}{2^{(k-1)}}}\right]^{2^k} = \left[\|u\|_{L^{2^{(k-1)}}(\Omega)}\right]^{2^k}.$$

Inequality (6.9) becomes

$$2^k\|u(t)\|_{L^{2^k}}^{2^k} + \frac{d}{dt}\|u(t)\|_{L^{2^k}}^{2^k} \le c(2^k)^{\frac{N}{2}+1}\|u(t)\|_{L^{2^{k-1}}}^{2^k} + 2^k M|\Omega|,$$

and then

$$\frac{d}{dt}\left[e^{2^k(t-\tau)}\|u(t)\|_{L^{2^k}}^{2^k}\right] \le e^{2^k(t-\tau)}c(2^k)^{\frac{N}{2}+1}\|u(t)\|_{L^{2^k}}^{2^k} + e^{2k(t-\tau)}2^k M|\Omega|.$$

Integrating from τ to t, we obtain

$$e^{2^k(t-\tau)}\|u(t)\|_{L^{2^k}}^{2^k} \le \|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}\int_\tau^t e^{2^k(s-\tau)}\|u(s)\|_{L^{2^{k-1}}}^{2^k}ds + [e^{2^k(t-\tau)}-1]M|\Omega|$$

$$\|u(t)\|_{L^{2^k}}^{2^k} \le e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|u(s)\|_{L^{2^{k-1}}}^{2^k}ds + M|\Omega|$$

and the statement of the lemma follows by considering $\gamma \in (0,1)$ and $M|\Omega| \le \left[\frac{M}{\gamma}|\Omega|\right]$ (this adjustment of constant is just to facilitate future calculus and notations). □

From the recurrence formula obtained in Lemma 6.10, we can derive the next proposition:

Proposition 6.11. *Let M, γ be the constants obtained in (6.5) from the dissipativeness condition (D), and assume $u_0 \in L^\infty(\Omega)$. Given any $k \in \mathbb{N}$, there exists a constant $D = D(N,k,\gamma)$ that depends on k, N and γ, such that, for $t > \tau$,*

$$\|u(t)\|_{L^{2^k}} \le D(N,k,\gamma)\left[e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right],$$

as long as the solution exists.

Proof. We prove for $k=2$ and $k=3$ to see the pattern. The result follows by induction. We will use the inequality $|a+b|^r \le 2^r(|a|^r+|b|^r)$, for any $r>0$, whenever we need to estimate the power of a sum of two terms.

Let $k=2$. As we did for $N=3$, rather then replacing the value of k in the inequalities, we will keep it to help us generalize the inequality for any $k \in \mathbb{N}$. We first estimate the integral that appears in (6.7) using the L^2-bound obtained for the solution in Proposition 6.8:

$$\int_\tau^t e^{2^k(s-\tau)}\|u(s)\|_{L^2}^{2^k}ds \le \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}2^{2^k}\left\{e^{-2^k\gamma(s-\tau)}\|u_0\|_{L^2}^{2^k} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}ds$$

$$\le (2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|u_0\|_{L^2}^{2^k} + (2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2^k}e^{2^k(t-\tau)}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}.$$

Therefore, replacing it in (6.7), we have

$$\|u(t)\|_{L^{2^k}}^{2^k} \le e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2^k(1-\gamma)}e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^2}^{2^k}$$

$$+ c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2^k}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} + \left[\frac{M}{\gamma}|\Omega|\right]$$

$$\leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2^k(1-\gamma)}e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^2}^{2^k}$$
$$+ c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{2}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}$$
$$\leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^2}^{2^k}$$
$$+ c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}$$

and we assumed that $c(2^k)^{\frac{N}{2}+1}$ and $\frac{M}{\gamma}|\Omega|$ are larger or equal than 1 (we can increase c, M if that is not the case). Moreover, let $d_{k-1,k}$ denotes the embedding constant of $L^{2^k}(\Omega) \hookrightarrow L^{2^{k-1}}(\Omega)$. Then $\|u_0\|_{L^2} \leq d_{k-1,k}\|u_0\|_{L^{2^k}}$ and equality above can be written as

$$\|u(t)\|_{L^{2^k}}^{2^k} \leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + c(2^k)^{\frac{N}{2}+1}\frac{d_{k-1,k}^{2^k}}{1-\gamma}(2^{\frac{1}{2}})^{2^k}2^{2^k}e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k}$$
$$+ c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}2^{2^k}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}$$
$$\leq c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d_{k-1,k}^{2^k})(2^{\frac{1}{2}})^{2^k}2^{2^k}\left\{e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.$$

Extracting the 2^k-root and denoting $C(k,N,\gamma) = \left\{c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d_{k-1,k}^{2^k})\right\}^{\frac{1}{2^k}}$, we obtain

$$\|u(t)\|_{L^{2^k}} \leq C(k,N,\gamma)2(2^{\frac{1}{2}})2^{\frac{1}{2^k}}\left\{e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\}$$
$$= D(2,N,\gamma)\left\{e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\},$$

where $D(k,N,\gamma) = 2^{k-1}\left(\prod_{i=1}^{k}2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^{k}C(i,N,\gamma)\right)$.

For $k=3$ the calculation is analogous. The previous information about $\|u(t)\|_{L^4}$ allows us to obtain now estimates on $\|u(t)\|_{L^{2^3}}$. First, the integral on (6.7) satisfies

$$\int_\tau^t e^{2^k(s-\tau)}\|u(s)\|_{L^4}^{2^k}$$
$$\leq D(2,N,\gamma)^{2^k}2^{2^k}\int_\tau^t e^{2^k(s-\tau)}\left\{e^{-2^k\gamma(s-\tau)}\|u_0\|_{L^4}^{2^k} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}ds$$

$$\leq D(2,N,\gamma)^{2^k}2^{2^k}\left\{\frac{1}{2^k(1-\gamma)}e^{2^k(1-\gamma)(t-\tau)}\|u_0\|_{L^4}^{2^k}+\frac{1}{2^k}e^{2^k(t-\tau)}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.$$

Replacing this expression in inequality (6.7), we obtain

$$\begin{aligned}\|u(t)\|_{L^{2^k}}^{2^k} &\leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k}+D(2,N,\gamma)^{2^k}2^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{2^k}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^4}^{2^k}\\ &\quad+D(2,N,\gamma)^{2^k}2^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{2^k}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}+\left[\frac{M}{\gamma}|\Omega|\right]\\ &\leq e^{-2^k(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k}+D(2,N,\gamma)^{2^k}2^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}d_{k-1,k}^{2^k}\|u_0\|_{L^{2^k}}^{2^k}\\ &\quad+D(2,N,\gamma)^{2^k}2^{2^k}c(2^k)^{\frac{N}{2}+1}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\\ &\leq D(2,N,\gamma)^{2^k}2^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d_{k-1,k}^{2^k})\left\{e^{-2^k\gamma(t-\tau)}\|u_0\|_{L^{2^k}}^{2^k}+\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.\end{aligned}$$

Extracting the 2^k-root and using $\left\{c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d_{k-1,k}^{2^k})\right\}^{\frac{1}{2^k}}=C(k,N,\gamma)$, we obtain

$$\begin{aligned}\|u(t)\|_{L^{2^k}} &\leq D(2,N,\gamma)C(3,N,\gamma)(2)(2^{\frac{1}{2^k}})\left[e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}}+\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right]\\ &=D(3,N,\gamma)\left[e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}}+\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right].\end{aligned}$$

In general,

$$\|u(t)\|_{L^{2^k}}\leq D(k,N,\gamma)\left[e^{-\gamma(t-\tau)}\|u_0\|_{L^{2^k}}+\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right],$$

where $C(i,N,\gamma)=\left\{c(2^i)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d_{i-1,i}^{2^i})\right\}^{\frac{1}{2^i}}$ and

$$D(k,N,\gamma)=2^{k-1}\left(\prod_{i=1}^{k}2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^{k}C(i,N,\gamma)\right). \quad \square$$

We can use the estimates of the solution in the L^{2^k}-norms, alongside with the formula of variation of constants, to obtain estimates in better spaces. From the growth condition $1\leq\rho<3$ on f and inequality (6.3), we obtain (adjusting the constant)

$$\|F(t,u(t))\|_{L^2(\Omega)} \le C(1+\|u(t)\|^{\rho}_{L^{2\rho}(\Omega)}) \le C(1+\|u(t)\|^{\rho}_{L^{2^3}(\Omega)}).$$

This fact and the results obtained earlier imply:

Proposition 6.12. *Suppose that condition (D) holds and γ, M are the constants in (6.5). Let $0 \le \beta < 1$ and $u_0 \in L^\infty(\Omega)$. There exist constants E_1 and E_2 depending on Ω, β, ρ, N, k ($N = 3 = k$ in the case considered), M and γ, such that, for $\tau < t-1 < t$,*

$$\|u(t,\tau,u_0)\|_{X^\beta} \le E_1 e^{-\gamma(t-\tau)}\left(\|u_0\|_{L^\infty} + \|u_0\|^{\rho}_{L^\infty}\right) + E_2,$$

as long as the solution exists.

Proof. From the variation of constants formula and the usual $L^{p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega)$ embeddings whenever $p_2 \le p_1$, we obtain (adjusting constants when needed)

$$\begin{aligned}
&\|u(t,\tau,u_0)\|_{X^\beta} = \|u(t,t-1,u(t-1,\tau,u_0))\|_{X^\beta} \\
&\le \|U(t,t-1)u(t-1,\tau,u_0)\|_{X^\beta} + \int_{t-1}^{t} \|U(t,s)F(s,u(s,\tau,u_0))\|_{X^\beta} ds \\
&\le C(\beta)(1)^{-\beta}\|u(t-1,\tau,u_0)\|_{L^2} + \int_{t-1}^{t} C(\beta)(t-s)^{-\beta}\|F(s,u(s,\tau,u_0))\|_{L^2(\Omega)} ds \\
&\le C(\beta)(2^{\frac{1}{2}})\left\{e^{-\gamma(t-1-\tau)}\|u_0\|_{L^2} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\} + C(\beta)\int_{t-1}^{t}(t-s)^{-\beta}\left(1+\|u(t)\|^{\rho}_{L^{2^3}}\right) ds \\
&\le C(\beta)(2^{\frac{1}{2}})\left\{e^{-\gamma(t-1-\tau)}\|u_0\|_{L^2} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\} \\
&\quad + C(\beta)\int_{t-1}^{t}(t-s)^{-\beta}\left(1 + D(3,N,\gamma)^{\rho}(2)^{\rho}\left\{e^{-\gamma\rho(s-\tau)}\|u_0\|^{\rho}_{L^{2^3}} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}\right\}\right) ds \\
&\le C(\beta)(2^{\frac{1}{2}})e^{-\gamma(t-\tau)}\|u_0\|_{L^\infty}|\Omega|^{\frac{1}{2}} + C(\beta)(2^{\frac{1}{2}})\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}} \\
&\quad + C(\beta) + C(\beta)D(3,N,\gamma)^{\rho}(2)^{\rho}\left\{e^{-\gamma\rho(t-1-\tau)}\|u_0\|^{\rho}_{L^\infty}|\Omega|^{\frac{\rho}{2^3}} + \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}\right\} \\
&\le C(\beta)(2^{\frac{1}{2}})e^{-\gamma(t-\tau)}\|u_0\|_{L^\infty}|\Omega|^{\frac{1}{2}} + C(\beta)D(3,N,\gamma)^{\rho}(2)^{\rho}e^{-\gamma\rho(t-\tau)}\|u_0\|^{\rho}_{L^\infty}|\Omega|^{\frac{\rho}{2^3}} \\
&\quad + C(\beta) + C(\beta)(2^{\frac{1}{2}})\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}} + C(\beta)D(k,N,\gamma)^{\rho}(2)^{\rho}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}} \\
&\le E_1 e^{-\gamma(t-\tau)}\left(\|u_0\|_{L^\infty} + \|u_0\|^{\rho}_{L^\infty}\right) + E_2,
\end{aligned}$$

where

$$E_1 = C(\beta)(2)^\rho D(3, N, \gamma)^\rho (|\Omega|^{\frac{1}{2}} + |\Omega|^{\frac{\rho}{2^3}}),$$
$$E_2 = 3C(\beta)D(3, N, \gamma)^\rho (2)^\rho \left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}. \quad \square \tag{6.10}$$

6.3. Global well-posedness and pullback attractor for the equation in $X^{\frac{1}{2}}$

The X^β-estimate obtained in Proposition 6.12 for $0 \le \beta < 1$ implies, in particular, that $\|u(t,\tau,u_0)\|_{X^{\frac{1}{2}}}$ is bounded in each bounded interval $[\tau, T]$. Therefore, we can define a nonlinear process $S(t,\tau) : X^{\frac{1}{2}} \to X^{\frac{1}{2}}$ given by the solution $u(t,\tau,u_0)$, that is,

$$S(t,\tau)u_0 = u(t,\tau,u_0) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(s,u(s))ds.$$

We prove this result in the sequel.

Theorem 6.13. *Let M, γ the constants in (6.5) obtained from the dissipativeness condition (D). If $u(\cdot,\tau,u_0)$ is the solution of (6.1), then it is globally defined and associated to it there is a nonlinear process $S(t,\tau)$ in $X^{\frac{1}{2}}$ given by $S(t,\tau)u_0 = u(t,\tau,u_0)$, for all $t \ge \tau$.*

Moreover, the closed ball in X^β centered in zero and with radius E_2, $B_{X^\beta}[0, E_2]$, is a pullback attracting set for the process $S(t,\tau)$ in the topology of X^β, where E_2 is given in (6.10) and depends on β, ρ, N, k, M and γ.

Proof. Let $u_0 \in X^{\frac{1}{2}}$. It follows from Remark 6.9 that after any arbitrarily small evolution in time, $u^* = u(t^*,\tau,u_0) \in L^\infty(\Omega)$. Therefore, if we start the evolution at instant t^* and at the point u^*, then Proposition 6.12 implies that $\|u(t,t^*,u^*)\|_{X^\beta}$ is finite in any bounded interval $[t^*, T]$. For $\beta \ge \frac{1}{2}$, this boundedness implies global existence of the solution (Theorem 1.2).

Moreover, let $B \subset X^{\frac{1}{2}}$ be a bounded set such that $\|u_0\|_{X^{\frac{1}{2}}} \le L$ for any $u_0 \in B$. It also follows from Remark 6.9 that after an arbitrarily small evolution in time, $t^* > \tau$, the elements $u^* = u(t^*,\tau,u_0)$ are bounded in $L^\infty(\Omega)$, that is, $\|u^*\|_{L^\infty(\Omega)} \le \tilde{L}$. Therefore,

$$\|u(t,t^*,u^*)\|_{X^\beta} \le E_1 e^{-\gamma(t-t^*)}(\tilde{L} + \tilde{L}^\rho) + E_2,$$

and $dist(S(t,t^*)S(t^*,\tau)u_0, E_2) = dist(u(t,t^*,u^*), B_{X^\beta}[0,E_2]) \overset{t-t^*\to\infty}{\longrightarrow} 0$, uniformly for $u_0 \in B$. $\square$

The existence of pullback attractor is now a simple consequence of the previous result.

Theorem 6.14. *Assume that $N = 3$, $1 \le \rho < 3$, $a : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}^+$ satisfies (A.2) and (A.3) and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (A.4) and (D).*

The solution $u(t,\tau,u_0)$ for the equation (6.1) in Ω defines a nonlinear process $S(t,\tau)$ in $X^{\frac{1}{2}}$ which has a pullback attractor $\{\mathcal{A}(t); t\in\mathbb{R}\}$ in $X^{\frac{1}{2}}$. Moreover, $\cup_{t\in\mathbb{R}}\mathcal{A}(t)\subset\mathcal{C}^{\eta}(\Omega)$ for some $\eta>0$ and pullback attracts bounded sets of $X^{\frac{1}{2}}$ in the topology of $\mathcal{C}^{\eta}(\Omega)$.

Proof. In Theorem 6.13 we proved the existence of a pullback attracting bounded set in X^{β} for any $0\le\beta<1$. Since X^{β} is compactly embedded in $X^{\frac{1}{2}}$ for $\beta>\frac{1}{2}$ (Proposition 6.3), we conclude that $B_{X^{\beta}}[0,E_2]$ is a compact pullback attracting set for the process $S(t,\tau)$ in $X^{\frac{1}{2}}$.

Therefore, from Corollary 2.9, there exists a pullback attractor

$$\mathcal{A}(t)\subset B_{X^{\beta}}[0,E_2]\subset\subset X^{\frac{1}{2}},\quad \forall t\in\mathbb{R},$$

that attracts bounded sets of $X^{\frac{1}{2}}$ in the topology of X^{β}. Moreover, if $\beta>\frac{N}{4}=\frac{3}{4}$, then $X^{\beta}\hookrightarrow\mathcal{C}^{\eta}(\Omega)$ and the last statement follows. □

6.4. *Properties of u_t*

For a nonsingular reaction-diffusion equation ($A(t)=A$), the construction of a Lyapunov function for the system is usually available. This provides further information on the long time behavior of the solution. For instance, if f is time-independent, the equation is autonomous and, under suitable conditions, it has an associated semigroup $T(\cdot)$.

If $\mathcal{E}=\{y : T(t)y=y \text{ for all } t\ge 0\}$ denotes the set of equilibrium point for $T(\cdot)$ (which we assumed discrete), then all solutions converge to an equilibrium point. In other words, all the solutions converge to a constant function in the long-time dynamics, the derivative in time will approach zero and the solution will be close to a solution of the associated elliptic equation $Au=f(u)$. This allows a better description of the attractor in terms of equilibria and heteroclinic orbits connecting them (see [7, Chapter 12] or [23, Chapter 10]).

For the singularly nonautonomous case this situation changes, especially due to the fact that the elliptic operator itself changes with time and the associated elliptic equation is $A(t)u=f(u)$. There are no reasons then to say that the solution approaches a constant value (an equilibrium) as the dynamics evolves. The derivative in time for the solution does not vanish in the long-time. However, we are able to prove that after a certain time, those derivatives are enclosed in a compact set of the phase space and the variation of the solution in the long-time is somehow controlled. That is essentially the content of next proposition.

Proposition 6.15. *Suppose that condition (D) holds and γ, M are the constants in (6.5). Let $0\le\beta<\min\{\delta,\omega\}$ and $u_0\in L^{\infty}(\Omega)$. There exist constants F_1 and F_2 depending on Ω, β, ρ, N, k ($N=3=k$ in the case considered), M and γ, such that, for $\tau<t-1<t$,*

$$\|u_t(t,\tau,u_0)\|_{X^{\beta}}\le F_1e^{-\gamma(t-\tau)}\left(\|u_0\|_{L^{\infty}}+\|u_0\|_{L^{\infty}}^{\rho}\right)+F_2.$$

Moreover, for any $\varepsilon>0$, $B_{X^{\beta}}[0,F_2+\varepsilon]$ is a pullback absorbing bounded set for $u_t(t,\tau,u_0)$ in the topology of X^{β}. If $\min\{\delta,\omega\}>\frac{1}{2}$, then $u_t(t)\in X^{\frac{1}{2}}$ and

$$B_{X^{\beta}}[0,F_2+\varepsilon]\subset\subset X^{\frac{1}{2}}$$

is a compact set in $X^{\frac{1}{2}}$ *that encloses* $u_t(t)$ *as the evolution takes place. If* $\min\{\delta, \omega\} > \frac{N}{4} = \frac{3}{4}$, *than* $B_{X^\beta}[0, F_2 + \varepsilon] \subset\subset C^\nu(\Omega)$, *for some* $\nu > 0$.

Proof. From Theorem 2.5 and Proposition 6.12

$$\begin{aligned}\|u_t(t,\tau,u_0)\|_{X^\beta} &\le C(\beta)(t-(t-1))^{-1-\beta+\frac{1}{2}}\|u(t-1,\tau,u_0)\|_{X^{\frac{1}{2}}} + C(\beta)(t-(t-1))^{-\max\{2\beta,1\}} \\ &\le C(\beta)[\|u(t-1,\tau,u_0)\|_{X^{\frac{1}{2}}} + 1] \\ &\le C(\beta)E_1 e^{-\gamma(t-\tau)}\left(\|u_0\|_{L^\infty} + \|u_0\|^\rho_{L^\infty}\right) + C(\beta)[E_2+1],\end{aligned}$$

and taking $F_1 = C(\beta)E_1$, $F_2 = C(\beta)[E_2+1]$, we obtain the desired inequality. The other statements follow from Proposition 6.3. □

The last assertion on Proposition 6.15 can be read as $t \mapsto |u_t(t,x)| \in \mathbb{R}$ being bounded for each $x \in \overline{\Omega}$. In this case, the solution $t \mapsto |u(t,x)| \in \mathbb{R}$ can increase/decrease in the long-time dynamics, but those variations are somehow controlled and limited. Therefore, even though $u(t)$ does not approach an equilibrium state, its variation in the long-time dynamics are limited.

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