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Dynamics of stochastic nonlocal partial differential equations

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Abstract This paper is concerned with the asymptotic behavior of solutions to nonlocal stochastic partial differential equations with multiplicative and additive noise driven by a standard Brownian motion, respectively. First of all, the stochastic nonlocal differential equations are transformed into their associated conjugated random differential equations, we then construct the dynamical systems to the original problems via the properties of conjugation. Next, in the case of multiplicative noise, we establish the existence of the random attractor when it absorbs every bounded deterministic set. Particularly, it is shown the pullback random attractor, which is also forward attracting, becomes a singleton when the external forcing term vanishes at zero. Eventually, in the case of additive noise, two approaches are applied to prove the existence of pullback random attractors with the help of energy estimations. Actually, these two attractors turn out to be the same one.

1 Introduction

The study of nonlocal problems modeled by partial differential equations has been receiving much attention recently, as the published literature confirm (see, for instance, [1,5,6,13,16–18,26,27,29,30] and the references cited therein). Just to explain the interest of this type of models, let us describe two situations in which nonlocal equations are fully justified: one is related to population dynamics and the other to physical problems involving heat transfer. On the one hand, a migrating population in some habitat (domain for our mathematical model), and a problem of heat transfer in a conductor, on the other hand. Then, it is sensible to assume that the velocity at which the motions (or the heat transfer) take place is in accordance with the Fourier Law

$$\mathbf{v}(x,t) = -a\nabla u(x,t),$$

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where $u(x,t)$ denotes the density of population at time t in the point $x \in \mathcal{O}$ of the domain (or the temperature of the conductor at time t in the point $x \in \mathcal{O}$), and the constant a depends on the type of phenomenon. However, considering that a is constant is a simplistic approximation of the complex biology (or physics) which is behind the scene. When we consider the migration of bacteria in a container, it is clear that the environment is of crucial importance, and one can easily realize that it is sensible to assume that the constant a can be of the form

$$a = a\left(\int_{\mathcal{O}'} u(x,t)\,dx\right),$$

where $\mathcal{O}' \subset \mathcal{O}$ is a subdomain. For instance, if the bacteria population tends to leave crowded areas, it is natural to assume that a is an increasing function, or if the population is attracted by that population growing in $\mathcal{O}'$, then a may be supposed to decrease.

In the case of heat transfer, it is clear that the measurements are not made pointwise, in general, but via some local average. Taking these comments into account, both problems could be modeled by a reaction-diffusion problem with initial value and some appropriate boundary conditions, of the form:

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u), & \text{in } \mathcal{O} \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \text{in } \mathcal{O}, \end{cases} \tag{1}$$

where l is an appropriate linear mapping (more general than the one described above for our examples) and f represents some kind of external force or action on the system (for instance, related to control problems pursuing that some goals can be achieved in our real phenomenon).

However, a very important and determining fact for every real-world happening is the effect that some randomness or stochasticity may produce on our problem. Reality is subjected to randomness and uncertainties, what suggests that our model can be more realistic if we consider some kind of noise in the mathematical formulation. There are many possibilities to introduce these noisy terms in our model, and each one yields to a different model which may need different techniques to handle it. For instance, the noise can appear as another source of external force acting on the system, which implies that we could analyze the following model,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + g(u)\frac{dW(t)}{dt}, & \text{in } \mathcal{O} \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \text{in } \mathcal{O}, \end{cases} \tag{2}$$

where, for simplicity, $W(t)$ is a standard Brownian motion (other kinds of noise are also considered in the literature).

Another possibility is that we have a deterministic equation as in (1) but the initial values are random, since measurements of these are always subjected to random errors. A third one is to consider that some of the parameters (or functions) in the model are affected by a random parameter ω in a probability space $(\Omega, \mathcal{F}, P)$. In this case, the mathematical system becomes

$$\begin{cases} \frac{\partial u}{\partial t} - a(\omega, l(u))\Delta u = f(\omega, u), & \text{in } \mathcal{O} \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \text{in } \mathcal{O}. \end{cases} \tag{3}$$

This kind of problem in the local framework has been analyzed by J. C. Cortés and his collaborators (see [7,8,15]) obtaining relevant results for the moments of solutions as well as the probability density functions (PDFs).

Nowadays, the theory of random dynamical systems has very well developed (see, e.g., [2–4,9,10,12,19,21,25,31,38,39] and the references cited therein). Therefore, our interest in this paper is to exploit the well-posedness and long-time behavior of some nonlocal stochastic problems, which includes, in particular, the motivating examples described above. Applying the theory of random dynamical systems to equation (2) requires that the noise must have a special structure. Recall that in a finite-dimensional situation of stochastic ordinary equations, (2) generates a random dynamical system provided g is locally Lipschitz (and f satisfies similar assumptions), while in the case of partial differential equations, this has not been proved in general, but only in the cases that $g(u)$ is linear (multiplicative noise) or constant (additive noise). In fact, these will be the cases we consider in our analysis.

The main idea to deal with random dynamical systems is to transform the original stochastic problems into random differential equations, which generate random dynamical systems, via a conjugation. The advantages of the transformed random systems are that they can be studied by using the tools of the deterministic theory of partial differential equations and exploiting the ergodic properties of an auxiliary process used for the transformation, the so-called Ornstein-Uhlenbeck process. Once the random problem has been proved to possess well-posedness and that generates a random dynamical system, we can transfer the same properties to the original stochastic problem.

Therefore, this paper is devoted to study the long-time behavior of the following two stochastic nonlocal partial differential equations with multiplicative and additive noise, respectively,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \sigma u \circ \frac{dW(t)}{dt}, & \text{in } \mathcal{O} \times (s, \infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times (s, \infty), \\ u(s) = u_0, & \text{in } \mathcal{O}, \end{cases} \tag{4}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \phi \frac{dW(t)}{dt}, & \text{in } \mathcal{O} \times (s, \infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times (s, \infty), \\ u(s) = u_0, & \text{in } \mathcal{O}, \end{cases} \tag{5}$$

where, as we mentioned, $W(t)$ is a standard Brownian motion, and $\circ$ denotes the Stratonovich sense in the stochastic term.

In this manuscript, we assume $\mathcal{O} \subset \mathbb{R}^N$ is a bounded open set, $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$, $a \in C(\mathbb{R}; \mathbb{R}^+)$ and there exists a positive constant m, such that

$$0 < m \le a(k), \qquad \forall k \in \mathbb{R}. \tag{6}$$

In addition, we suppose a is locally Lipschitz, i.e., for every $R > 0$ there exits a constant L_R such that

$$|a(k) - a(r)| \le L_R |k - r|, \qquad \forall k, r \in \mathbb{R},\ |k|, |r| \le R. \tag{7}$$

Also, $f \in C(\mathbb{R})$ and there exist constants $\eta > 0$ and $C_f > 0$ such that,

$$|f(k)| \le C_f(1 + |k|), \qquad \forall k \in \mathbb{R}, \tag{8}$$

$$(f(k) - f(r))(k - r) \le \eta (k - r)^2, \qquad \forall k, r \in \mathbb{R}. \tag{9}$$

Moreover, we denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(\mathcal{O})$, $\|\cdot\|$ and $((\cdot,\cdot))$ the norm and the inner product of $H_0^1(\mathcal{O})$, separately. Recall that for every $v \in H_0^1(\mathcal{O})$, the Poincaré inequality

$$\lambda_1(\mathcal{O})|v|^2 \le \|v\|^2$$

holds (see [33]), where $\lambda_1(\mathcal{O})$ is related to domain $\mathcal{O}$ and is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. In the sequel, unless otherwise specified, we write λ_1 instead of $\lambda_1(\mathcal{O})$.

The content of this paper is as follows. In Sect. 2, we recall some preliminaries needed for our analysis. First, we recall the basic concepts from the theory of random dynamical systems: random sets, absorption, attraction, asymptotic compactness, conjugation of random dynamical systems and random attractors. Next, we include some ergodic properties of the Ornstein–Uhlenbeck process. In Sect. 3, we analyze the case of multiplicative noise, by performing a conjugation and obtain a random dynamical system generated by the transformed random partial differential equations. Then we prove the existence of absorbing random compact sets ensuring the existence of random attractors. Also, we show it is possible to obtain additional information about the structure of this random attractor. In fact, when it becomes a singleton and this is exponentially stable as solution of our initial problem. Moreover, the case in which the noise is interpreted in the Itô sense is also considered, showing that the random attractor exists always provided the noise is large enough (i.e., without imposing any restriction on the smallness of f). Finally, in Sect. 4, we analyze the additive noise case. On this occasion, we first describe a more detailed proof of the well-posedness of the solutions, since there appears a new nonlocal term in the right-hand side of the transformed equation. Then we use two techniques to prove the existence of random attractors, one proves that there exists a compact random set absorbing the deterministic bounded sets of the phase space, while the other approach is applied to obtain the existence of a random tempered set which absorbs the tempered sets that are not only the bounded deterministic ones.

2 Preliminaries

We recall some basic concepts related to random dynamical systems and properties of Ornstein–Uhlenbeck processes which will be used throughout this paper. Although the content of this section can be found in several published works, we prefer to include them in our paper to make it more readable and as much self-contained as possible.

2.1 Random dynamical systems

In this section, we will introduce some basic concepts related to random dynamical systems and the concept of random attractor, for more details, see [2,3,21,23] and references therein.

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space with Borel σ-algebra $\mathcal{B}(\mathbb{X})$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1 $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system, if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2 A continuous random dynamical system (RDS) on $\mathbb{X}$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}, \qquad (t, \omega, x) \to \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}))$-measurable and satisfies, for almost all (a.a.) $\omega \in \Omega$,

(i) $\varphi(0, \omega, \cdot)$ is the identity on $\mathbb{X}$;
(ii) $\varphi(t+s, \omega, \cdot) = \varphi(t, \theta_s\omega, \cdot) \circ \varphi(s, \omega, \cdot)$, for all $t, s \in \mathbb{R}^+$;
(iii) $\varphi(t, \omega, \cdot) : \mathbb{X} \to \mathbb{X}$ is continuous for all $t \in \mathbb{R}^+$.

When a random dynamical system is generated by a stochastic differential equation in the Itô sense, and driven by an m-dimensional two-sided Wiener process W_t, the probability space can be identified with the canonical space of continuous mappings $\Omega = C_0(\mathbb{R}; \mathbb{R}^m)$, i.e., every event $\omega \in \Omega$ is a continuous functions $\omega : \mathbb{R} \to \mathbb{R}^m$, such that $\omega(0) = 0$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \qquad t \in \mathbb{R}. \tag{10}$$

Moreover, we can identify $W_t(\omega) = \omega(t)$ for every $\omega \in \Omega$.

It is well-known that finite-dimensional stochastic differential equations generate random dynamical systems (cf. [28]), but this is not true in general for infinite-dimensional equations. However, for the particular kind of noise, as will be in our case, we can apply the following lemma to obtain a random dynamical system [10].

Lemma 1 *Let ψ be a random dynamical system. Suppose that the mapping $T : \Omega \times \mathbb{X} \to \mathbb{X}$ possesses the following properties: for every fixed $\omega \in \Omega$, the mapping $T(\omega, \cdot)$ is a homeomorphism on $\mathbb{X}$, and for fixed $x \in \mathbb{X}$, mappings $T(\cdot, x)$, $T^{-1}(\cdot, x)$ are measurable. Then the mapping*

$$(t, \omega, x) \to \varphi(t, \omega, x) := T^{-1}(\theta_t\omega, \psi(t, \omega, T(\omega, x)))$$

is a (conjugated) random dynamical system.

Notice that, the measurability of φ follows from the properties of T in Lemma 1. Later on, we will transform our stochastic evolution equation containing a noise term into an evolution equation without noise but random coefficients.

Before stating the definition of random attractors, we first give the definition of random sets via the following lemma.

Definition 3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random set C on $\mathbb{X}$ is a measurable subset of $\mathbb{X} \times \Omega$ with respect to the product σ-algebra of the Borel σ-algebra of $\mathbb{X}$ and $\mathcal{F}$.

A random set, satisfying some measurability properties, can be regarded as a family of sets parameterized by the random parameter ω. More precisely, a random set C can be identified with the family of its ω-fibers $C(\omega)$, defined by

$$C(\omega) = \{x \in \mathbb{X} : (x, \omega) \in C\}, \quad \omega \in \Omega.$$

When a random set $C \subset \mathbb{X} \times \Omega$ has closed fibers, it is said to be a closed random set, if and only if for every $x \in \mathbb{X}$, the mapping

$$\omega \in \Omega \to d(x, C(\omega)) \in [0, +\infty)$$

is measurable. Similarly, when the fibers of C are compact, C is said to be a compact random set. For more details, see [14,22].

Definition 4 A random bounded set $B(\omega) \subset \mathbb{X}$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$, if for a.a. $\omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0, \qquad \forall \beta > 0, \tag{11}$$

where $d(B) = \sup_{x\in B} \|x\|_{\mathbb{X}}$.

Definition 5 Let $\mathcal{D}$ be a universe, i.e., a collection of random sets in $\mathbb{X}$, and let $\{K(\omega)\}_{\omega\in\Omega}$ be another random set (not necessarily in $\mathcal{D}$). It is said that $\{K(\omega)\}_{\omega\in\Omega}$ is an absorbing set for φ with respect to $\mathcal{D}$, if for all $B \in \mathcal{D}$ and a.a. $\omega \in \Omega$, there exists $t_B(\omega) > 0$, such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \qquad \text{for all} \quad t \geq t_B(\omega).$$

Definition 6 Let $\mathcal{D}$ be a universe in $\mathbb{X}$. Then, a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of $\mathbb{X}$ is called a global random $\mathcal{D}$–attractor (pullback $\mathcal{D}$–attractor) for φ, if, for a.a. $\omega \in \Omega$, the following conditions are satisfied:

(i) $\mathcal{A}(\omega)$ is a compact set;
(ii) $\mathcal{A}(\omega)$ is strictly invariant, i.e.,

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \qquad \text{for all} \quad t \geq 0;$$

(iii) $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, i.e., for all $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, we have

$$\lim_{t\to+\infty} d(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where

$$d(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_{\mathbb{X}}$$

is the Hausdorff semi-metric (here, $A, B \subset \mathbb{X}$).

The following existence result of random attractors to a continuous random dynamical system is a slight generalization of Theorem 3.5 in [24].

Theorem 1 *Let $K \in \mathcal{D}$ be a closed random absorbing set for the continuous random dynamical system $(\varphi(t))_{t\geq 0}$, which satisfies the following asymptotic compactness condition: for a.a. $\omega \in \Omega$, each sequence $x_n \in \varphi(t_n, \theta_{-t_n}, K(\theta_{-t_n}\omega))$ with $t_n \to +\infty$ has a convergent subsequence in $\mathbb{X}$. Then the RDS φ has a unique global random attractor*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq t_K(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Note that Theorem 1 is valid for every universe $\mathcal{D}$, in particular when it is formed by random tempered sets. In this respect, sometimes it is sufficient to consider a smaller universe to obtain the same random attractor provided by the tempered universe (or others). Indeed, it is sufficient to state another theorem taking into account the bounded deterministic sets, which provides the same attractor based on the work developed by Crauel and Flandoli [20, Theorem 3.11].

Theorem 2 *Suppose there exists a random compact set $D(\omega)$ which absorbs every bounded deterministic set $B \subset \mathbb{X}$. Then, the set*

$$\mathcal{A}(\omega) = \overline{\bigcup_{B\subset\mathbb{X}} \Lambda_B(\omega)}$$

is a random attractor for φ, where the union is taken over all $B \subset \mathbb{X}$ bounded, and $\Lambda_B(\omega)$ is the omega-limit set of B given by

$$\Lambda_B(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta_{-t}\omega, B)}.$$

Moreover (cf. [19]), this random attractor is unique and, under the ergodicity assumption on θ_t, there exists a compact set $K \subset \mathbb{X}$, such that the random attractor is the omega-limit set of K, that is,

$$\mathcal{A}(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta_{-t}\omega, K)}, \quad \textit{for a.a. } \omega \in \Omega.$$

2.2 Ornstein–Uhlenbeck process

Let us consider the following initial value problem for a one-dimensional stochastic differential equation,

$$\begin{cases} dz = -zdt + dW_t, & t \geq t_0, \\ z(t_0) = z_0. \end{cases} \tag{12}$$

The solution of problem (12) has the following form,

$$\begin{aligned} z(t; t_0, z_0) &= e^{-(t-t_0)z_0} + \int_{t_0}^{t} e^{-(t-s)} dW_s \\ &= e^{-(t-t_0)} z_0 + W_t - e^{-(t-t_0)} W_{t_0} - \int_{t_0}^{t} e^{-(t-s)} W_s ds. \end{aligned}$$

It is well-known, Eq.(12) has a random fixed point in the sense of random dynamical system generating a stationary solution, namely, the stationary Ornstein–Uhlenbeck process (see [10] for more details).

Now, let us take $t_0 \to -\infty$ in the above equality,

$$\lim_{t_0\to -\infty} z(t; t_0, z_0) = W_t - \int_{-\infty}^{t} e^{-(t-s)} W_s(\omega) ds := z^*(\theta_t\omega),$$

where $z^*(\omega) = -\int_{-\infty}^{0} e^s W_s(\omega) ds$. Therefore, it is straightforward to prove that $z(t, \omega) := z^*(\theta_t\omega)$ is a stationary solution of the Langevin equation in (12). In addition, the random variable z^* satisfies the following properties for all $\omega \in \bar{\Omega}$, where $\bar{\Omega} \subset \Omega$ and $P(\bar{\Omega}) = 1$:

$$\begin{aligned} &\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{|t|} = 0, \\ &\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t z^*(\theta_\tau\omega) d\tau = 0, \\ &\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_\tau\omega)| d\tau = \mathbb{E}|z^*| < \infty. \end{aligned} \tag{13}$$

Remark 1 Consider θ defined in (10) on $\bar{\Omega}$ instead of Ω, although we keep using the notation Ω instead of $\bar{\Omega}$. This mapping possesses the same properties as the original one, if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\bar{\Omega}$ denoted also by $\mathcal{F}$.

3 Nonlocal partial differential equations on a bounded domain with multiplicative noise

In this section, we consider the following nonlocal partial differential equation with multiplicative noise in the sense of Stratonovich,

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \sigma u \circ \frac{dW(t)}{dt}, \qquad (x,t) \in \mathcal{O} \times (s,\infty), \tag{14}$$

with the boundary and initial value conditions,

$$u(x,t) = 0, \quad (x,t) \in \partial\mathcal{O} \times (s,\infty), \qquad \text{and} \qquad u(s) = u_0, \quad x \in \mathcal{O}, \tag{15}$$

respectively. Here, $\mathcal{O}$ is a bounded open set in $\mathbb{R}^N$, σ is a constant, $W(t)$ is a two-side standard Brownian motion, and $\circ$ denotes the Stratonovich sense in the stochastic term.

To prove the well-posedness of problem (14)–(15), we will perform a conjugation given by a transformation involving Ornstein–Uhlenbeck process, that allows us to obtain a random partial differential equation. Observe that, it is easy to prove the well-posedness of the latter case, thanks to the conjugation, so does the original problem.

To start off, we denote by $u(\cdot) := u(t; s, \omega, u_0)$ the solution to problem (14)–(15). Now, we do the change of variable $v(t) = u(t)e^{-\sigma z^*(\theta_t\omega)}$. By formal computations (which can be justified rigorously by the Itô formula applied to the equivalent Itô equation to problem (14)), it follows the process $v(\cdot) := v(\cdot; s, \omega, v_0)$ with initial value $v(s) := v_0 = u_0 e^{-\sigma z^*(\theta_s\omega)}$ satisfying,

$$\begin{aligned} dv(t) &= [(a(l(u))\Delta u + f(u))dt + \sigma u \circ dW_t]e^{-\sigma z^*(\theta_t\omega)} \\ &\quad + u\sigma z^*(\theta_t\omega)dt - e^{-\sigma z^*(\theta_t\omega)} u\sigma \circ dW_t \\ &= [a(l(u))\Delta u + f(u)]dt e^{-\sigma z^*(\theta_t\omega)} + u\sigma z^*(\theta_t\omega)e^{-\sigma z^*(\theta_t\omega)}dt \\ &= a(e^{\sigma z^*(\theta_t\omega)}(l(v)))\Delta v dt + e^{-\sigma z^*(\theta_t\omega)} f(e^{\sigma z^*(\theta_t\omega)}v)dt + v\sigma z^*(\theta_t\omega)dt. \end{aligned}$$

The above transformation can be written as the following abstract form,

$$\begin{cases} \frac{\partial v}{\partial t} - \hat{a}(\theta_t\omega, l(v))\Delta v = F(\theta_t\omega, v), & \text{in } \mathcal{O} \times (s,\infty), \\ v(x,t) = 0, & \text{on } \partial\mathcal{O} \times (s,\infty), \\ v(s) = v_0, & \text{in } \mathcal{O}, \end{cases} \tag{16}$$

where

$$\hat{a}(\omega, k) = a(e^{\sigma z^*(\omega)}k), \ k \in \mathbb{R}, \qquad \text{and} \qquad F(\omega, v) = e^{-\sigma z^*(\omega)} f(e^{\sigma z^*(\omega)}v) + v\sigma z^*(\omega).$$

Proposition 1 *Based on assumptions (6)–(9), the following conditions hold true to problem (16). For almost all $\omega \in \Omega$, function $\hat{a}(\omega, \cdot) \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies (6)–(7) while slightly modifying L_R in (7) by $e^{\sigma z^*(\omega)} L_R$. Furthermore, there exist a constant $C_F := C_F(\omega, \sigma, C_f)$, and η, which is the same as (9), such that,*

$$|F(\omega, s)| \le C_F(1 + |s|) \quad \textit{and} \quad (F(\omega, s) - F(\omega, r))(s - r) \le \eta |s - r|^2.$$

Proof For almost all $\omega \in \Omega$ and for all $s \in \mathbb{R}$, by (8), we have

$$\begin{aligned}|F(\omega,s)| &\le |e^{-\sigma z^*(\omega)} f(e^{\sigma z^*(\omega)} s)| + |\sigma z^*(\omega)s| \\ &\le e^{-\sigma z^*(\omega)} C_f (1 + |e^{\sigma z^*(\omega)} s|) + |\sigma z^*(\omega)||s| \\ &\le C_f e^{-\sigma z^*(\omega)} + (C_f + |\sigma z^*(\omega)|)|s| \\ &\le C_F(1+|s|),\end{aligned} \tag{17}$$

where $C_F = \max\{C_f e^{-\sigma z^*(\omega)}, C_f + |\sigma z^*(\omega)|\}$. Besides, for almost all $\omega \in \Omega$, for all s, $r \in \mathbb{R}$, by (9) we obtain,

$$\begin{aligned}&(F(\omega,s) - F(\omega,r))(s-r) \\ &= e^{-2\sigma z^*(\omega)} |f(e^{\sigma z^*(\omega)} s) - f(e^{\sigma z^*(\omega)} r)||e^{\sigma z^*(\omega)} s - e^{\sigma z^*(\omega)} r| \\ &\le \eta e^{-2\sigma z^*(\omega)} \left(e^{\sigma z^*(\omega)} s - e^{\sigma z^*(\omega)} r\right)^2 \\ &\le \eta |s-r|^2.\end{aligned} \tag{18}$$

The proof is finished. □

3.1 Well-posedness of problem (14)–(15)

We will prove the existence and uniqueness of solution to problem (16) in the following sense.

Definition 7 Let the initial value $v_0 \in L^2(\mathcal{O})$. A weak solution to problem (16) is a function $v(\cdot) = v(\cdot; s, \omega, v_0)$, which belongs to $L^2([s,T]; H_0^1(\mathcal{O})) \cap L^\infty([s,T]; L^2(\mathcal{O}))$ for almost all $\omega \in \Omega$, such that for all $T \ge s$,

$$\frac{d}{dt}(v(t), \vartheta) + \hat{a}(\theta_t\omega, l(v))((v, \vartheta)) = (F(\theta_t\omega, v), \vartheta), \qquad \forall \vartheta \in H_0^1(\mathcal{O}), \tag{19}$$

where equation (19) must be understood in the sense of $\mathcal{D}'(s, \infty)$.

Theorem 3 *Suppose that function $a \in C(\mathbb{R}; \mathbb{R}^+)$ fulfills (6)–(7), function $f \in C(\mathbb{R})$ satisfies (8)–(9) and $l \in L^2(\mathcal{O})$. Then, for each initial datum $v_0 \in L^2(\mathcal{O})$, for almost all $\omega \in \Omega$, there exists a unique weak solution to problem (16). In addition, this solution behaves continuously in $L^2(\mathcal{O})$ with respect to the initial data.*

Proof The existence of at least one solution to the random nonlocal partial differential equation (16) with initial value v_0 follows straightforwardly from [13] by adapting the Galerkin method and energy estimations for each ω fixed. Indeed, thanks to Proposition 1, i.e., the facts that $\hat{a}(\omega, \cdot)$ fulfils (6) and function $F(\omega, \cdot)$ satisfies (17), together with $l \in L^2(\mathcal{O})$, we can apply the result in [13] to prove that equation (16) possesses a unique solution $v(\cdot; s, \omega, v_0) \in C([s,T]; L^2(\mathcal{O})) \cap L^2([s,T]; H_0^1(\mathcal{O}))$ with $v(s; s, \omega, v_0) = v_0$, for every $T > s$. In addition, since $\hat{a}(\omega, \cdot)$ is locally Lipschitz with Lipschitz constant $L_a e^{\sigma z^*(\omega)}$, and $F(\omega, \cdot)$ fulfills (18), by a standard reasoning, the uniqueness of weak solution and the continuity with respect to the initial data can be proved analogously. □

At this point, thanks to the transformation $v(t) = u(t) e^{-\sigma z^*(\theta_t\omega)}$, the following result holds.

Theorem 4 *Assume the conditions of Theorem 3 hold. Then, for each initial datum $u_0 \in L^2(\mathcal{O})$ and for almost all $\omega \in \Omega$, there exists a unique weak solution to problem (14)–(15). In addition, this solution behaves continuously in $L^2(\mathcal{O})$ with respect to the initial data.*

3.2 Existence of random attractors to problem (14)–(15)

In this subsection, we will prove the hypotheses in Theorem 2 hold with the purpose of proving the existence of a random attractor $\mathcal{A}(\omega)$ associated with problem (14)–(15). To this end, we transform the stochastic equation with multiplicative noise into the deterministic equation with a random parameter. To show problem (14)–(15) generates a random dynamical system, define the mapping $T(\omega) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by $T(\omega)u_0 = e^{\sigma z^*(\omega)}u_0$, which is, obviously, a homeomorphism. Thus, by means of the following change of variable,

$$v(t) = T^{-1}(\theta_t\omega)u(t) = u(t)e^{-\sigma z^*(\theta_t\omega)},$$

where u is the solution of problem (14)–(15), it follows,

$$\frac{\partial v}{\partial t} - a(e^{\sigma z^*(\theta_t\omega)}l(v))\Delta v = e^{-\sigma z^*(\theta_t\omega)}f(e^{\sigma z^*(\theta_t\omega)}v) + v\sigma z^*(\theta_t\omega), \tag{20}$$

which is exactly (16). Since $T(\omega)$ is a homeomorphism, if equation (16) generates a random dynamical system, so does (14)–(15). This property will be presented below.

Lemma 2 *Equation (16) generates a continuous random dynamical system* $(\psi(t))_{t\geq 0}$ *over* $(\Omega, \mathcal{F}_0, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$*, where*

$$\psi(t, \omega, v_0) = v(t; 0, \omega, v_0), \quad \forall v_0 \in L^2(\mathcal{O}), \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.e.}$$

Moreover, if we define Π *by*

$$\Pi(t, \omega, v_0) = T(\theta_t\omega)\psi(t, \omega, T^{-1}(\omega)v_0),$$

then, Π *is another random dynamical system for the process,*

$$(t, \omega, v_0) \to \Pi(t, \omega, v_0),$$

which solves (14)–(15) for any initial value $v_0 \in L^2(\mathcal{O})$ *at the initial time* $s = 0$.

Proof Note that, when we have a random differential equation

$$\frac{dv(t)}{dt} = G(\theta_t\omega, v(t)), \tag{21}$$

with G regular enough to ensure well-posedness of this problem. If we denote by $v(t; s, \omega, v_0)$ the solution of (21) with initial value $v(s) = v_0$, then it is straightforward to check that

$$v(t, s, \omega, v_0) = v(t - s, 0, \theta_s\omega, v_0), \quad \forall t \geq s.$$

Defining

$$\varphi(t, \omega, v_0) = v(t; 0, \omega, v_0),$$

it follows that φ is a random dynamical system. Indeed, $\varphi(0, \omega, v_0) = v(0; 0, \omega, v_0) = v_0$ holds obviously. As for the cocycle property, i.e.,

$$\varphi(t, \theta_r\omega, \varphi(r, \omega, \cdot)) = \varphi(t + r, \omega, \cdot), \quad r, t \geq 0,$$

observe that,

$$\varphi(t, \theta_r\omega, \varphi(r, \omega, v_0)) = \varphi(t, \theta_r\omega, v(r, 0, \omega, v_0)) = v(t, 0, \theta_r\omega, v(r, 0, \omega, v_0)) = v_1(t),$$

while

$$\varphi(t + r, \omega, v_0) = v(t + r, 0, \omega, v_0) = v_2(t).$$

Now, it is easy to check that v_1 and v_2 are both solutions of the following initial value problem (IVP),

$$\begin{cases} \frac{dv}{dt} = G(\theta_t\omega, v), \\ v(s) = v_0. \end{cases}$$

Then $v_1 \equiv v_2$ holds by the uniqueness of the above IVP, which implies the cocycle property holds true. As (20) satisfies the abstract form (21), the first statement follows.

Besides, thanks to Lemma 1, the process Π is a conjugation RDS since $T(\omega)$ is a homeomorphism. □

In the following lines, we will focus on the existence of random attractors for the dynamical system Π related to the stochastic nonlocal partial differential equation (14)–(15).

Theorem 5 *Assume that function $a \in C(\mathbb{R}; \mathbb{R}^+)$ fulfills (6)–(7), function $f \in C(\mathbb{R})$ satisfies (8)–(9) and $l \in L^2(\mathcal{O})$. In addition, let $m\lambda_1 > 3C_f$. Then there exists a unique random attractor $\mathcal{A}(\omega)$ for the dynamical system $\Pi(t, \omega, u_0)$ associated to equation (14)–(15).*

Proof In order to prove this result by using Theorem 2, it is necessary to find a random compact absorbing set $K(\omega)$ (which will be given by the ball of center 0 and radius $r_2(\omega)$ in $H_0^1(\mathcal{O})$) absorbing every bounded non-random set $D \subset L^2(\mathcal{O})$. In this way, the natural compact embedding $H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ is essential.

We first derive the boundedness of $v(\cdot) = v(\cdot; t_0, \omega, v_0)$ in $L^2(\mathcal{O})$ for all $t \in [t_0, -1]$ with $t_0 \leq -1$, where $v_0 = e^{-\sigma z^*(\theta_{t_0}\omega)}u_0$ and $u_0 \in D$. Multiply (20) by $v(t)$ in $L^2(\mathcal{O})$, thanks to (8) and the Young inequality, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v(t)|^2 &+ a(e^{\sigma z^*(\theta_t\omega)}l(v))\|v(t)\|^2 \\ &= e^{-\sigma z^*(\theta_t\omega)}(f(e^{\sigma z^*(\theta_t\omega)}v), v) + \sigma z^*(\theta_t\omega)|v(t)|^2 \\ &\leq e^{-\sigma z^*(\theta_t\omega)}C_f\int_{\mathcal{O}}|v(t)|dx + C_f|v(t)|^2 + \sigma z^*(\theta_t\omega)|v(t)|^2 \\ &\leq \frac{1}{2}e^{-2\sigma z^*(\theta_t\omega)}C_f|\mathcal{O}| + \left(\frac{3C_f}{2} + \sigma z^*(\theta_t\omega)\right)|v(t)|^2, \end{aligned}$$

thanks to the Poincaré inequality and (6), we have

$$\frac{d}{dt}|v(t)|^2 + m\|v(t)\|^2 \leq (-m\lambda_1 + 3C_f + 2\sigma z^*(\theta_t\omega))|v(t)|^2 + e^{-2\sigma z^*(\theta_t\omega)}C_f|\mathcal{O}|. \quad (22)$$

Integrating (22) between t_0 and -1, it follows

$$\begin{aligned} |v(-1)|^2 &\leq e^{\int_{t_0}^{-1}(-m\lambda_1+3C_f+2\sigma z^*(\theta_s\omega))ds} \\ &\quad\times\left(\int_{t_0}^{-1}C_f|\mathcal{O}|e^{-2\sigma z^*(\theta_s\omega)}e^{\int_{t_0}^{s}(m\lambda_1-3C_f-2\sigma z^*(\theta_\tau\omega))d\tau}ds + |v(t_0)|^2\right) \\ &\leq e^{-(m\lambda_1-3C_f)(-1-t_0)+\int_{t_0}^{-1}2\sigma z^*(\theta_s\omega)ds}|v(t_0)|^2 \\ &\quad + C_f|\mathcal{O}|\int_{t_0}^{-1}e^{-2\sigma z^*(\theta_s\omega)}e^{-(m\lambda_1-3C_f)(-1-s)+\int_s^{-1}2\sigma z^*(\theta_\tau\omega)d\tau}ds \\ &\leq e^{(m\lambda_1-3C_f)}\left[e^{(m\lambda_1-3C_f)t_0+\int_{t_0}^{-1}2\sigma z^*(\theta_s\omega)ds}|v(t_0)|^2\right. \\ &\quad \left.+ C_f|\mathcal{O}|\int_{t_0}^{-1}e^{-2\sigma z^*(\theta_s\omega)}e^{(m\lambda_1-3C_f)s+\int_s^{-1}2\sigma z^*(\theta_\tau\omega)d\tau}ds\right]. \end{aligned}$$

Consequently, for a given deterministic bounded set $D \subset L^2(\mathcal{O})$, there exist a constant $\rho > 0$ and $T(\omega, \rho) \le -1$, $\mathbb{P}$-a.e., such that, for any $u_0 \in D \subset B(0, \rho)$, for all $t_0 \le T(\omega, \rho)$, we have

$$|v(-1; t_0, \omega, e^{-\sigma z^*(\theta_{t_0}\omega)} u_0)|^2 \le r_1^2(\omega),$$

with

$$r_1^2(\omega) = e^{(m\lambda_1 - 3C_f)} \left(1 + C_f |\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma z^*(\theta_s \omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma z^*(\theta_\tau \omega) d\tau} ds \right).$$

Indeed, by means of the properties of Ornstein–Uhlenbeck process (13) and assumption $m\lambda_1 > 3C_f$, it is possible to choose $t_0 \le T(\omega, \rho)$, such that

$$\begin{aligned} & e^{(m\lambda_1 - 3C_f)t_0 + \int_{t_0}^{-1} 2\sigma z^*(\theta_s \omega) ds} |e^{-\sigma z^*(\theta_{t_0}\omega)} u_0|^2 \\ & \le e^{(m\lambda_1 - 3C_f)t_0 + \int_{t_0}^{-1} 2\sigma z^*(\theta_s \omega) ds + 2\sigma z^*(\theta_{t_0}\omega)} |\rho|^2 \\ & \le e^{t_0 \left[(m\lambda_1 - 3C_f) + 2\sigma \frac{1}{t_0} \int_{t_0}^{0} z^*(\theta_s \omega) ds - 2\sigma \frac{z^*(\theta_{t_0}\omega)}{t_0} \right]} |\rho|^2 \\ & \le 1. \end{aligned}$$

Secondly, we will prove $v \in L^\infty([-1, t]; L^2(\mathcal{O})) \cap L^2([-1, t]; H_0^1(\mathcal{O}))$ with $t \in [-1, 0]$ by energy estimations. From (22), we know

$$\frac{d}{dt} |v(t)|^2 + m\|v(t)\|^2 \le (-m\lambda_1 + 3C_f + 2\sigma z^*(\theta_t \omega)) |v(t)|^2 + C_f |\mathcal{O}| e^{-2\sigma z^*(\theta_t \omega)},$$

integrating the above inequality from -1 to t with $t \in [-1, 0]$, we obtain

$$\begin{aligned} |v(t)|^2 & \le e^{\int_{-1}^{t} (-m\lambda_1 + 3C_f + 2\sigma z^*(\theta_s \omega)) ds} \left(\int_{-1}^{t} \left(C_f |\mathcal{O}| e^{-2\sigma z^*(\theta_s \omega)} - m\|v(s)\|^2 \right) \right. \\ & \quad \left. \times e^{\int_{-1}^{s} (m\lambda_1 - 3C_f - 2\sigma z^*(\theta_\tau \omega)) d\tau} ds + |v(-1)|^2 \right) \\ & \le e^{-(m\lambda_1 - 3C_f)(t+1) + \int_{-1}^{t} 2\sigma z^*(\theta_s \omega) ds} |v(-1)|^2 \\ & \quad + C_f |\mathcal{O}| \int_{-1}^{t} e^{-2\sigma z^*(\theta_s \omega) + (3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma z^*(\theta_\tau \omega) d\tau} ds \\ & \quad - m \int_{-1}^{t} e^{(3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma z^*(\theta_\tau \omega) d\tau} \|v(s)\|^2 ds. \end{aligned} \tag{23}$$

Therefore, from (23), we obtain

$$\begin{aligned} |v(t)|^2 & \le e^{-(m\lambda_1 - 3C_f)(t+1) + \int_{-1}^{t} 2\sigma z^*(\theta_s \omega) ds} |v(-1)|^2 \\ & \quad + C_f |\mathcal{O}| \int_{-1}^{t} e^{-2\sigma z^*(\theta_s \omega) + (3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma z^*(\theta_\tau \omega) d\tau} ds, \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^{0} e^{(m\lambda_1 - 3C_f)s + \int_s^0 2\sigma z^*(\theta_r \omega) dr} \|v(s)\|^2 ds & \le \frac{1}{m} e^{-(m\lambda_1 - 3C_f) + \int_{-1}^{0} 2\sigma z^*(\theta_s \omega) ds} |v(-1)|^2 \\ & \quad + \frac{C_f |\mathcal{O}|}{m} \int_{-1}^{0} e^{-2\sigma z^*(\theta_s \omega) + (m\lambda_1 - 3C_f)s + \int_s^0 2\sigma z^*(\theta_\tau \omega) d\tau} ds. \end{aligned}$$

Thus, by the similar arguments, we conclude that for a given deterministic subset $D \subset B(0,\rho) \subset L^2(\mathcal{O})$, there exists $T(\omega,\rho) \le -1$, $\mathbb{P}$-a.e., such that for all $t_0 \le T(\omega,\rho)$, for all $u_0 \in D$, we have

$$|v(t)|^2 \le e^{-(m\lambda_1 - 3C_f)(t+1)+\int_{-1}^t 2\sigma z^*(\theta_s\omega)ds} r_1^2(\omega) \\ + C_f|\mathcal{O}| \int_{-1}^t e^{-2\sigma z^*(\theta_s\omega)+(3C_f - m\lambda_1)(t-s)+\int_s^t 2\sigma z^*(\theta_\tau\omega)d\tau} ds,$$

and

$$\int_{-1}^0 e^{(m\lambda_1-3C_f)s+\int_s^0 2\sigma z^*(\theta_\tau\omega)d\tau}\|v(s)\|^2 ds \le \frac{1}{m} e^{-(m\lambda_1-3C_f)+\int_{-1}^0 2\sigma z^*(\theta_s\omega)ds} r_1^2(\omega) \\ + \frac{C_f|\mathcal{O}|}{m}\int_{-1}^0 e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma z^*(\theta_\tau\omega)d\tau} ds. \tag{24}$$

Thirdly, the boundedness of $v(\cdot)$ in $H_0^1(\mathcal{O})$ for all $t \in [-1,0]$ and compact embedding $H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ ensure us to prove the existence of a compact absorbing ball in $L^2(\mathcal{O})$. To obtain a bound in $H_0^1(\mathcal{O})$, multiply (20) by $-\Delta v(t)$, with the help of (9) and the Young inequality, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + a(e^{\sigma z^*(\theta_t\omega)} l(v))|-\Delta v(t)|^2 \\ &= e^{-\sigma z^*(\theta_t\omega)}(f(e^{\sigma z^*(\theta_t\omega)}v), -\Delta v) + \sigma z^*(\theta_t\omega)(v, -\Delta v) \\ &\le \frac{1}{m} e^{-2\sigma z^*(\theta_t\omega)} C_f^2|\mathcal{O}| + \frac{C_f^2}{m}|v(t)|^2 + \frac{m}{2}|\Delta v(t)|^2 + \sigma z^*(\theta_t\omega)\|v(t)\|^2. \end{aligned} \tag{25}$$

Using the Poincaré inequality, (25) can be bounded by

$$\begin{aligned} \frac{d}{dt}\|v(t)\|^2 &\le -m|\Delta v(t)|^2 + \frac{2}{m}C_f^2|\mathcal{O}|e^{-2\sigma z^*(\theta_t\omega)} + \frac{2C_f^2}{m}|v(t)|^2 + 2\sigma z^*(\theta_t\omega)\|v(t)\|^2 \\ &\le \left(-m\lambda_1 + \frac{2C_f^2}{m\lambda_1} + 2\sigma z^*(\theta_t\omega)\right)\|v(t)\|^2 + \frac{2}{m}C_f^2|\mathcal{O}|e^{-2\sigma z^*(\theta_t\omega)}. \end{aligned} \tag{26}$$

Integrating (26) between s and 0 with $s \in [-1,0]$, we obtain

$$\begin{aligned} \|v(0)\|^2 &\le e^{\int_s^0 (2C_f^2/m\lambda_1 - m\lambda_1 + 2\sigma z^*(\theta_\tau\omega))d\tau} \\ &\quad \times \left(\int_s^0 \frac{2}{m}C_f^2|\mathcal{O}|e^{-2\sigma z^*(\theta_\tau\omega)} e^{\int_s^\tau (m\lambda_1 - 2C_f^2/m\lambda_1 - 2\sigma z^*(\theta_t\omega))dt} d\tau + \|v(s)\|^2\right) \\ &\le e^{(m\lambda_1 - 2C_f^2/m\lambda_1)s + \int_s^0 2\sigma z^*(\theta_\tau\omega)d\tau}\|v(s)\|^2 \\ &\quad + \frac{2}{m}C_f^2|\mathcal{O}|\int_s^0 e^{-2\sigma z^*(\theta_\tau\omega)+(m\lambda_1-2C_f^2/m\lambda_1)\tau+\int_\tau^0 2\sigma z^*(\theta_t\omega)dt} d\tau. \end{aligned}$$

Integrating the above inequality again in $[-1,0]$, we have

$$\|v(0)\|^2 \le \int_{-1}^0 e^{(m\lambda_1-2C_f^2/m\lambda_1)s+\int_s^0 2\sigma z^*(\theta_\tau\omega)d\tau}\|v(s)\|^2 ds \\ + \frac{2}{m}C_f^2|\mathcal{O}|\int_{-1}^0 e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-2C_f^2/m\lambda_1)s+\int_s^0 2\sigma z^*(\theta_r\omega)dr} ds.$$

Thanks to the assumption $3C_f < m\lambda_1$, it is easy to check $m\lambda_1 - 3C_f < m\lambda_1 - \frac{2C_f^2}{m\lambda_1}$, together with (24), we have

$$\begin{aligned}\|v(0)\|^2 &\le \int_{-1}^{0} e^{(m\lambda_1-3C_f)s+\int_{-1}^{0}2\sigma z^*(\theta_\tau\omega)d\tau}\|v(s)\|^2 ds \\ &\quad + \frac{2}{m}C_f^2|\mathcal{O}|\int_{-1}^{0} e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma z^*(\theta_r\omega)dr}ds \\ &\le \frac{1}{m}e^{-(m\lambda_1-3C_f)+\int_{-1}^{0}2\sigma z^*(\theta_s\omega)ds}r_1^2(\omega) \\ &\quad + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma z^*(\theta_r\omega)dr}ds.\end{aligned}$$

Therefore, it is straightforward that

$$\begin{aligned}\|u(0)\|^2 &= \|v(0)e^{\sigma z^*(\omega)}\|^2 \\ &\le \frac{1}{m}e^{-(m\lambda_1-3C_f)+2\sigma z^*(\omega)+\int_{-1}^{0}2\sigma z^*(\theta_s\omega)ds}r_1^2(\omega) \\ &\quad + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma z^*(\theta_s\omega)+2\sigma z^*(\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma z^*(\theta_r\omega)dr}ds.\end{aligned}$$

Consequently, there exists $r_2(\omega)$ such that for a given $\rho > 0$, there exists $\tilde{T}(\omega,\rho) \le -1$ satisfying, for all $t_0 \le \tilde{T}(\omega,\rho)$ and $u_0 \in L^2(\mathcal{O})$ with $|u_0| \le \rho$,

$$\|u(0;t_0,\omega,u_0)\|^2 \le r_2^2(\omega),$$

where

$$\begin{aligned}r_2^2(\omega) &= \frac{1}{m}e^{\int_{-1}^{0}2\sigma z^*(\theta_s\omega)ds+2\sigma z^*(\omega)}\left(1 + C_f|\mathcal{O}|\int_{-\infty}^{-1} e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1}2\sigma z^*(\theta_\tau\omega)d\tau}ds\right) \\ &\quad + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma z^*(\theta_s\omega)+(m\lambda_1-3C_f)s+2\sigma z^*(\omega)+\int_s^0 2\sigma z^*(\theta_r\omega)dr}ds,\end{aligned}$$

which is well-defined. Thus, we conclude from Theorem 2 that there exists a unique random attractor $\mathcal{A}(\omega)$ to problem (14)–(15). □

It is interesting now to provide more information about the structure of the attractor. We will start with a particular case in which this becomes just a singleton {0}. In future, we can discuss other options, singleton but not zero, finite dimensionality, etc.

In what follows, we return to equation (20) assuming the external forcing term satisfies $f(0) = 0$. At this point, we are in a position to prove the random attractor to problem (14)–(15) becomes a singleton {0}.

Theorem 6 *In addition to assumptions of Theorem 5, suppose that $m\lambda_1 > \eta$ and f(0)=0. Then the random attractor of problem (14)–(15) becomes* {0}. *In fact, it is an exponentially stable solution. Moreover, the random attractor is also forward attracting.*

Proof (i) Exponential stability. Multiply (20) with $v(t)$, by (6), (8) and the Poincaré inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v(t)|^2+m\lambda_1|v(t)|^2\\&\le e^{-2\sigma z^*(\theta_t\omega)}(f(e^{\sigma z^*(\theta_t\omega)}v(t)),e^{\sigma z^*(\theta_t\omega)}v(t))+\sigma z^*(\theta_t\omega)|v(t)|^2\\&\le(\eta+\sigma z^*(\theta_t\omega))|v(t)|^2.\end{aligned}$$

Consequently,

$$|v(t)|^2\le e^{\int_0^t 2(-m\lambda_1+\eta+\sigma z^*(\theta_s\omega))ds}|v(0)|^2,$$

by conjugation of operator $T(\omega)$ defined in Lemma 2, we know

$$\begin{aligned}|u(t;0,\omega,u_0)|^2&=|e^{\sigma z^*(\theta_t\omega)}v(t;0,\omega,e^{-\sigma z^*(\omega)}u_0)|^2\\&\le e^{2\sigma z^*(\theta_t\omega)-2\sigma z^*(\omega)+2(-m\lambda_1+\eta)t+\int_0^t 2\sigma z^*(\theta_s\omega)ds)}|u_0|^2\\&\le e^{t\left(\frac{2|\sigma z^*(\theta_t\omega)|+2|\sigma z^*(\omega)|}{t}+2(-m\lambda_1+\eta)+\frac{1}{t}\int_0^t 2\sigma z^*(\theta_s\omega)ds\right)}|u_0|^2.\end{aligned}\tag{27}$$

(13) allows us to conclude that

$$\lim_{t\to\pm\infty}\frac{2|\sigma z^*(\theta_t\omega)|+2|\sigma z^*(\omega)|}{t}=0\quad\text{and}\quad\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t 2\sigma z^*(\theta_s\omega)ds=0,$$

with the help of assumption $m\lambda_1>\eta$, we deduce there exists $T(\omega)>0$, such that for all $t\ge T(\omega)$,

$$\frac{2|\sigma z^*(\theta_t\omega)|+2|\sigma z^*(\omega)|}{t}+\frac{1}{t}\int_0^t 2\sigma z^*(\theta_s\omega)ds+2(-m\lambda_1+\eta)<0.$$

It implies the exponential asymptotic stability of $u\equiv 0$ to problem (20).

(ii) Random attractor is singleton. By replacing ω by $\theta_{-t}\omega$ in (27), for all $t\ge 0$, we have

$$\begin{aligned}|u(t;0,\theta_{-t}\omega,u_0)|^2&=|e^{\sigma z^*(\omega)}v(t;0,\theta_{-t}\omega,e^{-\sigma z^*(\theta_{-t}\omega)}u_0)|^2\\&\le e^{2\sigma z^*(\omega)-2\sigma z^*(\theta_{-t}\omega)+2(-m\lambda_1+\eta)t+\int_0^t 2\sigma z^*(\theta_{s-t}\omega)ds}|u_0|^2\\&\le e^{t\left(\frac{2|\sigma z^*(\omega)|+2|\sigma z^*(\theta_{-t}\omega)|}{t}+2(-m\lambda_1+\eta)+\frac{1}{t}\int_{-t}^0 2\sigma z^*(\theta_s\omega)ds\right)}|u_0|^2,\end{aligned}$$

which shows that

$$dist(\Pi(t,\theta_{-t}\omega,u_0),\{0\})\to 0\quad\text{as}\quad t\to+\infty.$$

(iii) Singleton random attractor is also forward attracting. From (27), it follows that

$$dist(\Pi(t,\omega,u_0),\{0\})\to 0\quad\text{as}\quad t\to+\infty.$$

Therefore, the random attractor $\{0\}$ is also forward attracting. The proof of this theorem is complete. □

3.3 A comment on nonlocal stochastic partial differential equations with Itô noise

We are now interested in studying the following nonlocal stochastic partial differential equation,

$$\frac{\partial u}{\partial t}-a(l(u))\Delta u=f(u)+\sigma u\cdot\frac{dW(t)}{dt},\qquad(x,t)\in\mathcal{O}\times(s,\infty)\tag{28}$$

with the boundary and initial value conditions,

$$u(x,t)=0, \quad (x,t)\in\partial\mathcal{O}\times(s,\infty), \qquad \text{and} \qquad u(s)=u_0, \quad x\in\mathcal{O}, \tag{29}$$

respectively. Here, we denote by $\cdot$ the Itô sense of the stochastic term. It is well-known (28) is equivalent to

$$\frac{\partial u}{\partial t}-a(l(u))\Delta u+\frac{\sigma^2}{2}u=f(u)+\sigma u\circ\frac{dW(t)}{dt}. \tag{30}$$

Observe that once the equivalence between equations (28) and (30) is presented, along with the results we have proved in previous sections for (14)–(15) about Stratonovich integral, the well-posedness of equation (28) follows in the same way as in the proof of Theorem 4. As for the conclusions about random attractors, we can proceed likewise as in the proofs of theorems 5 and 6 without imposing $m\lambda_1>3C_f$ and $m\lambda_1>\eta$, respectively. Instead, we assume large Itô noise in problem (28) (i.e., σ is large enough). This can be done by simple energy estimations.

4 Nonlocal partial differential equations on a bounded domain with additive noise

In this section, we will investigate well-posedness and asymptotic behavior of solutions to the following stochastic nonlocal partial differential equation with additive noise,

$$\begin{cases}\frac{\partial u}{\partial t}-a(l(u))\Delta u=f(u)+\phi\frac{dW(t)}{dt}, & \text{in } \mathcal{O}\times(s,\infty),\\ u(x,t)=0, & \text{on } \partial\mathcal{O}\times(s,\infty),\\ u(s)=u_0, & \text{in } \mathcal{O},\end{cases} \tag{31}$$

where $\phi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$. Function f satisfies the same assumptions as in Sect. 3 (cf. (8)–(9)) and $l\in L^2(\mathcal{O})$. Moreover, let $a\in C(\mathbb{R};\mathbb{R}^+)$ be locally Lipschitz (cf. (7)), and there exist two positive constants m and M, such that

$$0<m\le a(k)\le M, \qquad \forall k\in\mathbb{R}. \tag{32}$$

Let us denote by $u(\cdot):=u(\cdot;s,\omega,u_0)$ the solution of equation (31). By similar arguments as in the multiplicative case, but using now a different conjugation operator $S(\omega):L^2(\mathcal{O})\to L^2(\mathcal{O})$ given by $S(\omega)u_0=u_0-\phi z^*(\omega)$. On account of the change of variable as below,

$$v(t)=S(\theta_t\omega)u(t)=u(t)-\phi z^*(\theta_t\omega),$$

It follows, for almost all $\omega\in\Omega$, $v(\cdot):=v(\cdot;s,\omega,v_0)$ satisfies,

$$\begin{cases}\frac{\partial v}{\partial t}=a(l(v)+z^*(\theta_t\omega)l(\phi))\Delta v(t)+f(v+\phi z^*(\theta_t\omega))\\ \qquad +\phi z^*(\theta_t\omega)+a(l(v)+z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)\Delta\phi, & \text{in } \mathcal{O}\times\mathbb{R}^+,\\ v(x,t)=0, & \text{on } \partial\mathcal{O}\times\mathbb{R}^+,\\ v(s)=u_0-\phi z^*(\theta_s\omega):=v_0, & \text{in } \mathcal{O}.\end{cases} \tag{33}$$

We begin with the definitions of weak and strong solutions to problem (33).

Definition 8 Let $\phi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$ and the initial value $v_0\in L^2(\mathcal{O})$. Then a weak solution to equation (33) is a function $v(\cdot):=v(\cdot;s,\omega,v_0)$, that belongs to $L^2([s,T];H_0^1(\mathcal{O}))\cap$

$L^\infty([s,T]; L^2(\mathcal{O}))$ for almost all $\omega \in \Omega$, such that for all $T \geq s$,

$$\begin{aligned}&\frac{d}{dt}(v(t),\vartheta) + a(l(v)+z^*(\theta_t\omega)l(\phi))((v,\vartheta)) + a(l(v)+z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)((\phi,\vartheta))\\ &\quad = (f(v+\phi z^*(\theta_t\omega)),\vartheta) + z^*(\theta_t\omega)(\phi,\vartheta), \qquad \forall\vartheta \in H_0^1(\mathcal{O}),\end{aligned} \tag{34}$$

where equation (33) must be understood in the sense of $\mathcal{D}'(s,\infty)$.

Definition 9 A strong solution to (33) is a weak solution v that also satisfies that $v \in L^2([s,T]; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})) \cap L^\infty([s,T]; H_0^1(\mathcal{O}))$ for all $T \geq s$.

4.1 Well-posedness of problem (31)

The existence and uniqueness of problem (31) can be proved by Galerkin method similarly to problem (14)–(15). The random nonlocal equation (33) is more complicated since there is an extra nonlocal term $a(l(v)+z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)\Delta\phi$. Therefore, we prefer to include rough details of the proof of the well-posedness of solutions to make our paper more readable. We emphasize that the upper bound M on nonlocal operator a (cf. (32)) is imposed to handle this extra term.

Using spectral theory, there exits a sequence $\{w_i\}_{i\geq 1}$ which is a Hilbert basis of $L^2(\mathcal{O})$ composed by the eigenfunction of $-\Delta$ in $H_0^1(\mathcal{O})$. Firstly, we consider the function $v_n(t) := v_n(t; s, \omega, v_0) = \sum_{j=1}^n \varphi_{nj}(t) w_j$ for all $n \geq 1$, the unique local solution to

$$\begin{cases}\frac{d}{dt}(v_n(t),w_j) + a(l(v_n)+z^*(\theta_t\omega)l(\phi))((v_n,w_j)) + a(l(v_n)+z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)((\phi,w_j))\\ = (f(v_n+\phi z^*(\theta_t\omega)),w_j) + z^*(\theta_t\omega)(\phi,w_j),\\ (v_n(s),w_j) = (v_0,w_j), \qquad j=1,2,\cdots,n.\end{cases}$$

Notice that the above equation is a Cauchy problem for the following ordinary differential system in $\mathbb{R}^n$,

$$\begin{aligned}&\varphi_{nj}'(t) + \lambda_j a(l(v_n)+z^*(\theta_t\omega)l(\phi))\varphi_{nj}(t) + a(l(v_n)+z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)((\phi,w_j))\\ &\quad = (f(v_n+\phi z^*(\theta_t\omega)),w_j) + z^*(\theta_t\omega)(\phi,w_j), \qquad j=1,2,\cdots,n,\end{aligned} \tag{35}$$

where $t \geq s$, λ_j is the eigenvalue associated to the eigenfunction w_j, the vector $(\varphi_{n1},\varphi_{n2},\cdots,\varphi_{nn})$ is unknown.

Proposition 2 *Suppose $a \in C(\mathbb{R};\mathbb{R}^+)$ fulfills (32), $f \in C(\mathbb{R})$ verifies (8), $\phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and $l \in L^2(\mathcal{O})$. Then, there exists at least a local solution $(\varphi_{n1},\varphi_{n2},\cdots,\varphi_{nn})$ to the ordinary differential system (35) defined on some interval $[s,t_n)$, for almost all fixed $\omega \in \Omega$, and for each initial value $v_0 \in L^2(\mathcal{O})$. Moreover, if a is locally Lipschitz (cf.(6)) and f satisfies (9), the uniqueness of local solution is ensured.*

Proof The proof follows the lines of the proof of Proposition 2.3 in [27] by using a generalization of Peano's Theorem. We omit the details here. □

Theorem 7 *Suppose that a is locally Lipschitz (cf.(6)) and fulfills (32), $f \in C(\mathbb{R})$ satisfies (8)–(9), $\phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and $l \in L^2(\mathcal{O})$. Then, there exists a unique weak solution to problem (33), for almost all $\omega \in \Omega$ and initial datum $v_0 \in L^2(\mathcal{O})$. In addition, this solution behaves continuously in $L^2(\mathcal{O})$ with respect to the initial data.*

Proof The existence of weak solution to problem (33). Multiplying by φ_{nj} in (35), summing from $j = 1$ to n and using (32), we obtain

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 + m\|v_n(t)\|^2 + a(l(v_n) + z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)((\phi, v_n(t)))$$
$$\leq (f(v_n + \phi z^*(\theta_t\omega)), v_n(t)) + z^*(\theta_t\omega)(\phi, v_n(t)), \qquad a.e.\ t \in [0, t_n),$$

where $[0, t_n)$ is the interval of existence of maximal solution. By (8), the Poincaré and Young inequalities, we derive

$$\begin{aligned}
\frac{d}{dt}|v_n(t)|^2 + 2m\|v_n(t)\|^2 \leq\ & \frac{4C_f^2|\mathcal{O}|}{m\lambda_1} + \frac{m\lambda_1}{4}|v_n(t)|^2 + 2C_f|v_n(t)|^2 + \frac{4C_f^2|z^*(\theta_t\omega)|^2}{m\lambda_1}|\phi|^2 \\
& + \frac{m\lambda_1}{4}|v_n(t)|^2 + \frac{2|z^*(\theta_t\omega)|}{\sqrt{\lambda_1}}|\phi|\|v_n(t)\| \\
& -2a(l(v_n) + z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)((\phi, v_n(t))) \\
\leq\ & \frac{4C_f^2|\mathcal{O}|}{m\lambda_1} + \frac{m\lambda_1}{4}|v_n(t)|^2 + 2C_f|v_n(t)|^2 + \frac{4C_f^2|z^*(\theta_t\omega)|^2}{m\lambda_1^2}\|\phi\|^2 \\
& + \frac{m\lambda_1}{4}|v_n(t)|^2 + \frac{4|z^*(\theta_t\omega)|^2}{\lambda_1^2 m}\|\phi\|^2 + \frac{m}{4}\|v_n(t)\|^2 \\
& + \frac{4M^2}{m}|z^*(\theta_t\omega)|^2\|\phi\|^2 + \frac{m}{4}\|v_n(t)\|^2,
\end{aligned}$$

it implies

$$\frac{d}{dt}|v_n(t)|^2 + m\|v_n(t)\|^2 \leq 2C_f|v_n(t)|^2 + \frac{4C_f^2|\mathcal{O}|}{m\lambda_1} + \left[\frac{(4C_f^2+4)}{m\lambda_1^2} + \frac{4M^2}{m}\right]|z^*(\theta_t\omega)|^2\|\phi\|^2. \quad (36)$$

Integrating (36) between s and t with $s \leq t < t_n$, we have

$$\begin{aligned}
|v_n(t)|^2 + m\int_s^t \|v_n(r)\|^2 dr \leq\ & |v_n(s)|^2 + \frac{4C_f^2|\mathcal{O}|(T-s)}{m\lambda_1} \\
& + \left[\frac{(4C_f^2+4)}{m\lambda_1^2} + \frac{4M^2}{m}\right]\|\phi\|^2 \int_s^t |z^*(\theta_r\omega)|^2 dr \\
& + 2C_f\int_s^t |v_n(r)|^2 dr.
\end{aligned}$$

Therefore, $\{v_n(\cdot)\}_{n=1}^{\infty}$ is well defined on (s, t_n) thanks to the Gronwall lemma. Actually, for all $T > s$ and for almost all $\omega \in \Omega$, it is bounded in $L^\infty([s, T]; L^2(\mathcal{O})) \cap L^2([s, T]; H_0^1(\mathcal{O}))$. Additionally, by assumption (32), it is obvious that

$$a(l(v_n) + z^*(\theta_t\omega)l(\phi)) \leq M, \qquad \forall t \in [s, T],\ \ \forall n \geq 1.$$

Therefore, there exists a positive constant C such that

$$\int_s^T |a(l(v_n) + z^*(\theta_t\omega)l(\phi))|\| - \Delta v_n(t)\|^2_{H^{-1}(\mathcal{O})} dt \leq C\int_s^T \|v_n(t)\|^2 dt, \quad (37)$$

and

$$\int_s^T |a(l(v_n) + z^*(\theta_t\omega)l(\phi))|\| - \Delta\phi\|^2_{H^{-1}(\mathcal{O})} dt \leq C\|\phi\|^2|T-s|. \quad (38)$$

Taking into account $\{v_n\}_{n=1}^{\infty}$ is bounded in $L^2([s,T];H_0^1(\mathcal{O}))$ and $\phi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$, we deduce that the sequences $\{-a(l(v_n)+z^*(\theta_t\omega)l(\phi))\Delta v_n\}$ and $\{-a(l(v_n)+z^*(\theta_t\omega)l(\phi))\Delta\phi\}$ are bounded in $L^2([s,T];H^{-1}(\mathcal{O}))$.

On the other hand, using (8), we have

$$\begin{aligned}\int_s^T\int_{\mathcal{O}}|f(v_n(t)+z^*(\theta_t\omega)\phi)|^2dxdt &\le 2C_f^2\int_s^T\int_{\mathcal{O}}(1+|v_n(t)+z^*(\theta_t\omega)\phi|^2)dxdt\\ &\le 2C_f^2(T-s)|\mathcal{O}|+4C_f^2\int_s^T|v_n(t)|^2dt \\ &\quad+4C_f^2|\phi|^2\int_s^T|z^*(\theta_t\omega)|^2dt.\end{aligned}\tag{39}$$

Since $\{v_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}([s,T];L^2(\mathcal{O}))$, $z^*(\theta.\omega)$ is continuous in $[s,T]$ and $\phi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$, we have $\{f(v_n(t)+z^*(\theta_t\omega)\phi)\}$ is bounded in $L^2([s,T];L^2(\mathcal{O}))$.

To prove the sequence $\{v_n'\})_{n=1}^{\infty}$ is bounded in $L^2([s,T];H^{-1}(\mathcal{O}))$, we define the projector $P_n:H^{-1}(\mathcal{O})\to H^{-1}(\mathcal{O})$ (cf. [27, Theorem 2.4]). Then by (37)-(39), we derive

$$\begin{aligned}\int_s^T\|v_n'(t)\|^2_{H^{-1}(\mathcal{O})}dt &\le\int_s^T\|a(l(v_n(t))+z^*(\theta_t\omega)l(\phi))(\Delta v_n(t)+\Delta\phi)\\ &\quad+P_nf(v_n(t)+z^*(\theta_t\omega)\phi)+z^*(\theta_t\omega)\phi\|^2_{H^{-1}(\mathcal{O})}dt\\ &\le\left(4C+\frac{16C_f^2}{\lambda_1^2}\right)\int_s^T\|v_n(t)\|^2dt+\frac{8C_f^2(T-s)|\mathcal{O}|}{\lambda_1}\\ &\quad+4C\|\phi\|^2(T-s)+\left(\frac{4}{\lambda_1^2}+\frac{16C_f^2}{\lambda_1^2}\right)\|\phi\|^2\int_s^T|z^*(\theta_t\omega)|^2dt.\end{aligned}$$

Therefore, by means of compactness arguments and the Aubin–Lions lemma, there exists a subsequence of $\{v_n\}_{n=1}^{\infty}$ (relabeled the same) and $v\in L^{\infty}([s,T];L^2(\mathcal{O}))\cap L^2([s,T];H_0^1(\mathcal{O}))$ with $\{v_n'\}_{n=1}^{\infty}\in L^2([s,T];H^{-1}(\mathcal{O}))$. It is easy to check v is the weak solution to problem (32) owing the same reason as [27, Theorem 2.4].

The uniqueness and continuity with respect to initial data. Assume that there exist two weak solutions, $v_1(\cdot;s,\omega,v_0^1)$ and $v_2(\cdot;s,\omega,v_0^2)$ to equation (32). For short, we will denote $v_i(\cdot)=v_i(\cdot;s,\omega,v_0^i)$ for $i=1,2$. From the energy equality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_1(t)-v_2(t)|^2+a(l(v_1(t))+z^*(\theta_t\omega)l(\phi))\|v_1(t)-v_2(t)\|^2\\ &\quad=[a(l(v_2(t))+z^*(\theta_t\omega)l(\phi))-a(l(v_1(t))+z^*(\theta_t\omega)l(\phi))]((v_2(t),v_1(t)-v_2(t)))\\ &\qquad+(f(v_1(t)+\phi z^*(\theta_t\omega))-f(v_2(t)+\phi z^*(\theta_t\omega)),v_1(t)-v_2(t)),\end{aligned}$$

a.e. $t\in[s,T]$.

Since $v_i\in C([s,T];L^2(\mathcal{O}))$, there exists a bounded set $\mathcal{S}\in L^2(\mathcal{O})$ such that $\{v_i(t)\}_{t\in[s,T]}\subset\mathcal{S}$. Besides, taking into account that $l\in L^2(\mathcal{O})$, $\phi\in H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$ and $z^*(\theta_t\omega)$ is continuous with respect to t, there exists a constant $R>0$, such that $\{l(v_i(t)+z^*(\theta_t\omega)l(\phi))\}_{t\in[s,T]}\subset[-R,R]$. Then, by means of (32), (7) and (9), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_1(t)-v_2(t)|^2+m\|v_1(t)-v_2(t)\|^2\\ &\le L_a(R)|l||v_1(t)-v_2(t)|\|v_2(t)\|\|v_1(t)-v_2(t)\|+\eta|v_1(t)-v_2(t)|^2.\end{aligned}\tag{40}$$

Applying the Poincaré inequality to (40), we have

$$\frac{d}{dt}|v_1(t) - v_2(t)|^2 \le C(t)|v_1(t) - v_2(t)|^2, \qquad a.e.\ t \in [s, T], \quad \mathbb{P}\text{-a.e.},$$

where

$$C(t) = \frac{L_a^2(R)|l|^2\|v_2(t)\|^2 + 4m\eta}{2m}.$$

Thus, both results, the uniqueness of solution and the continuity with respect to the initial data to problem (32), follow immediately by the Gronwall lemma. □

Theorem 8 *Under the assumptions of Theorem 7, for every $\varepsilon > 0$ and $T > s + \varepsilon$, the weak solution v belongs to $C((s, T]; H_0^1(\mathcal{O})) \cap L^2([s+\varepsilon, T]; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))$. In fact, if the initial datum $v_0 \in H_0^1(\mathcal{O})$, then the function $v \in C([s, T]; H_0^1(\mathcal{O})) \cap L^2([s, T]; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))$ for every $T > s$.*

Proof Since we are working on a deterministic problem with random parameters, the proof of this theorem follows the standard energy estimations, see [13, Theorem 2.5, Chapter 2], we omit the details here. □

4.2 Existence of random attractors to (31): the first approach

As in the multiplicative case (cf. Sect. 3), the solution $v(\cdot; 0, \omega, v_0)$ of (33) generates a random dynamical system $\Xi : L^2(\mathcal{O}) \to L^2(\mathcal{O})$, defined by

$$\Xi(t, \omega, v_0) = v(t; 0, \omega, v_0), \qquad \forall v_0 \in L^2(\mathcal{O}), \quad \forall \omega \in \Omega.$$

Thanks to the conjugation and Theorem 7, there is a mapping $\Psi : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ such that

$$\Psi(t, \omega, u_0) = u(t; 0, \omega, u_0) = v(t; 0, \omega, u_0 - \phi z^*(\omega)) + \phi z^*(\theta_t\omega), \quad \forall v_0 \in L^2(\mathcal{O}), \forall \omega \in \Omega,$$

which exactly is the random dynamical system generated by (31).

Theorem 9 *In addition to hypotheses of Theorem 7, let $m\lambda_1 > 4C_f$. Then, there exists a random $\mathcal{D}_F$-attractor $\mathcal{A}_F(\omega)$ (where $\mathcal{D}_F$ is the universe of fixed bounded sets) for the dynamical system $\Psi(t, \omega, u_0)$ associated to equation (31).*

Proof The idea to prove the existence of random $\mathcal{D}_F$-attractor to (31) is the same as Theorem 5 by using Theorem 2. Namely, looking for a random compact absorbing set $K(\omega)$ (which will be given by the ball of center 0 and radius $r_4(\omega)$ in $H_0^1(\mathcal{O})$) absorbing every bounded deterministic set $D \subset L^2(\mathcal{O})$, together with the fact compact embedding $H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, we achieve the goal. Firstly, multiplying (33) by $v(t) := v(t; s, \omega, v_0)$ in $L^2(\mathcal{O})$, by (32), we obtain

$$\begin{aligned}\frac{d}{dt}|v(t)|^2 \quad &+ 2m\|v(t)\|^2 \le 2(f(v(t) + \phi z^*(\theta_t\omega)), v(t)) \\ &+ 2z^*(\theta_t\omega)(\phi, v(t)) + 2M\|\phi\|\|v(t)\|,\end{aligned}$$

with the help of (8), the Young and Poincaré inequalities, we have

$$\begin{aligned}\frac{d}{dt}|v(t)|^2 + m\|v(t)\|^2 &\le (-m\lambda_1 + 2C_f(\alpha_1 + 1) + \alpha_2)|v(t)|^2 + \frac{C_f|\mathcal{O}|}{\alpha_1} \\ &+ \left(\frac{C_f}{\alpha_1\lambda_1} + \frac{1}{\alpha_2\lambda_1}\right)|z^*(\theta_t\omega)|^2\|\phi\|^2 + \frac{M^2}{\alpha_3}\|\phi\|^2 + \alpha_3\|v(t)\|^2.\end{aligned} \tag{41}$$

Choosing $\alpha_1 = \frac{1}{2}, \alpha_2 = C_f$ and $\alpha_3 = \frac{m}{2}$ in (41), we derive,

$$\begin{aligned}\frac{d}{dt}|v(t)|^2 &\le -(m\lambda_1 - 4C_f)|v(t)|^2 + 2C_f|\mathcal{O}| \\ &\quad + \left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 - \frac{m}{2}\|v(t)\|^2.\end{aligned} \tag{42}$$

Ignoring the last term of (42), integrating in $[t_0, -1]$ with $t_0 \le -1$, we have

$$\begin{aligned}|v(-1)|^2 &\le e^{-(m\lambda_1-4C_f)(-1-t_0)}\Bigg[\int_{t_0}^{-1}\left(2C_f|\mathcal{O}| + \left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2\right) \\ &\quad \times e^{(m\lambda_1-4C_f)(t-t_0)}dt + |v(t_0)|^2\Bigg] \\ &\le e^{-(m\lambda_1-4C_f)(-1-t_0)}|v(t_0)|^2 \\ &\quad + \int_{t_0}^{-1} e^{-(m\lambda_1-4C_f)(-t-1)}\left(2C_f|\mathcal{O}| + \left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2\right)dt \\ &\le e^{(m\lambda_1-4C_f)}\Bigg[e^{(m\lambda_1-4C_f)t_0}|v(t_0)|^2 \\ &\quad + \int_{t_0}^{-1} e^{(m\lambda_1-4C_f)t}\left(2C_f|\mathcal{O}| + \left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2\right)dt\Bigg].\end{aligned}$$

Consequently, for a given $B(0,\rho) \subset L^2(\mathcal{O})$, there exists $T(\omega,\rho) \le -1$, such that for all $t_0 \le T(\omega,\rho)$ and for all $u_0 \in B(0,\rho)$,

$$|v(-1; t_0, \omega, u(t_0) - \phi z^*(\theta_{t_0}\omega))|^2 \le r_3^2(\omega),$$

with

$$\begin{aligned}r_3^2(\omega) = \ & 1 + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} \\ & + \int_{-\infty}^{-1} e^{(m\lambda_1-4C_f)(t+1)}\left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 dt,\end{aligned}$$

which is well defined. Indeed, it is enough to choose $T(\omega,\rho)$ such that, for any $t_0 \le T(\omega,\rho)$, we have

$$\begin{aligned}e^{(m\lambda_1-4C_f)(t_0+1)}|v(t_0)|^2 &= e^{(m\lambda_1-4C_f)(t_0+1)}|u(t_0) - \phi z^*(\theta_{t_0}\omega)|^2 \\ &\le 2e^{(m\lambda_1-4C_f)(t_0+1)}(\rho^2 + |\phi|^2|z^*(\theta_{t_0}\omega)|^2) \\ &\le 1.\end{aligned}$$

From (42), for $t \in [-1, 0]$, we have

$$\begin{aligned}|v(t)|^2 &\le e^{-(m\lambda_1-4C_f)(t+1)}\Bigg[\int_{-1}^{t}\Bigg(2C_f|\mathcal{O}| + \left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 \\ &\quad - \frac{m}{2}\|v(s)\|^2\Bigg)e^{(m\lambda_1-4C_f)(s+1)}ds + |v(-1)|^2\Bigg].\end{aligned}$$

Therefore,

$$\begin{aligned}|v(t)|^2 &\le e^{-(m\lambda_1-4C_f)(t+1)}|v(-1)|^2 + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} \\ &\quad + \int_{-1}^{t} e^{-(m\lambda_1-4C_f)(t-s)}\left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 ds,\end{aligned}$$

and

$$\int_{-1}^{0} e^{(m\lambda_1-4C_f)s}\|v(s)\|^2 ds \le \frac{2}{m}e^{-(m\lambda_1-4C_f)}|v(-1)|^2 + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1-4C_f)} + \frac{2}{m}\int_{-1}^{0} e^{(m\lambda_1-4C_f)s}\left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 ds. \tag{43}$$

Thus, we conclude for a given $B(0,\rho) \subset L^2(\mathcal{O})$, there exists $T(\omega,\rho) \le -1$, such that for all $t_0 \le T(\omega,\rho)$ and for all $u_0 \in B(0,\rho)$,

$$|v(t)|^2 \le e^{-(m\lambda_1-4C_f)(t+1)} r_3^2(\omega) + \frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f} + \int_{-1}^{t} e^{-(m\lambda_1-4C_f)(t-s)}\left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 ds,$$

and

$$\int_{-1}^{0} e^{(m\lambda_1-4C_f)s}\|v(s)\|^2 ds \le \frac{2}{m}e^{-(m\lambda_1-4C_f)} r_3^2(\omega) + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1-4C_f)} + \frac{2}{m}\int_{-1}^{0} e^{(m\lambda_1-4C_f)s}\left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 ds. \tag{44}$$

To obtain a bounded absorbing set in $H_0^1(\mathcal{O})$, multiplying (33) by $-\Delta v(t)$, making use of (8), (32), the Poincaré and Young inequalities, we have

$$\frac{d}{dt}\|v(t)\|^2 \le -(m\lambda_1-4C_f)\|v(t)\|^2 + \lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v(t)|^2 + \left(C_f\lambda_1 + \frac{\lambda_1}{C_f}\right)|z^*(\theta_t\omega)|^2|\phi|^2 + \frac{M^2}{m}|\Delta\phi|^2.$$

Integrating the above inequality between s and 0, where $s \in [-1, 0]$, we have

$$\|v(0)\|^2 \le e^{(m\lambda_1-4C_f)s}\|v(s)\|^2 + \int_{s}^{0}\Big(\lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v(t)|^2 + (C_f\lambda_1 + \lambda_1 C_f^{-1})|z^*(\theta_t\omega)|^2|\phi|^2 + \frac{M^2}{m}|\Delta\phi|^2\Big)e^{(m\lambda_1-4C_f)t} dt.$$

Integrating again the above inequality in $[-1, 0]$, together with (44), it follows

$$\|v(0)\|^2 \le \frac{2}{m}e^{-(m\lambda_1-4C_f)} r_3^2(\omega) + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1-4C_f)} + \frac{2}{m}\int_{-1}^{0} e^{(m\lambda_1-4C_f)s} \times \left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m}\right)\|\phi\|^2 ds + \int_{-1}^{0} e^{(m\lambda_1-4C_f)t} \times \left(\lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v(t)|^2 + (C_f\lambda_1 + \lambda_1 C_f^{-1})|z^*(\theta_t\omega)|^2|\phi|^2 + \frac{M^2}{m}|\Delta\phi|^2\right)dt.$$

Consequently, there exists $r_4(\omega)$ satisfying for a given $\rho > 0$, there exists $T(\omega,\rho) \le -1$, such that for all $t_0 \le T(\omega,\rho)$ and $|u_0| \le \rho$,

$$\|u(0; t_0, \omega, u_0)\|^2 := \|v(0; t_0, \omega, u_0 - \phi z^*(\theta_{t_0}\omega)) + \phi z^*(\omega)\|^2 \le r_4^2(\omega),$$

where

$$r_4^2(\omega) = 2\|\phi\|^2|z^*(\omega)|^2 + \left(4m^{-1} + 2\lambda_1 C_f\right) r_3^2(\omega) + \frac{8C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{4\lambda_1 C_f^2|\mathcal{O}|}{(m\lambda_1 - 4C_f)^2}$$
$$+ \left(4m^{-1} + 2\lambda_1 C_f\right) \int_{-1}^{0} e^{(m\lambda_1 - 4C_f)s} \left(\frac{|z^*(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_s\omega)|^2}{\lambda_1} + \frac{2M^2}{m} \right) \|\phi\|^2 ds$$
$$+ 2\int_{-1}^{0} e^{(m\lambda_1 - 4C_f)t} \left(\lambda_1 C_f|\mathcal{O}| + (C_f\lambda_1 + \lambda_1 C_f^{-1})|z^*(\theta_t\omega)|^2|\phi|^2 + \frac{M^2}{m}|\Delta\phi|^2 \right) dt.$$

Thus, we conclude from Theorem 2 that there exists a unique random attractor $\mathcal{A}_F(\omega)$ to equation (31) with respect to deterministic bounded sets. □

Remark 2 Notice that even if the restriction involving $m\lambda_1$ and $4C_f$ in Theorem 9 could be weakened to

$$m\lambda_1 > 2C_f,$$

this gap does not affect the results we have proved. Since we only need to pick up sufficiently small α_1 and α_2 in (41), such that $-m\lambda_1 + 2C_f(1 + \alpha_1) + \alpha_2$ keeps being negative.

4.3 Existence of attractors to (31): the second approach

In this section, we establish the basic result about the existence of a (pullback) random attractor $\mathcal{A}(\omega)$ of problem (31), for the case where the attracted universe is not composed of fixed bounded sets but of families of sets depending on ω. However, in the end we will show both attractors derived from the two approaches, $\mathcal{A}_F(\omega)$ and $\mathcal{A}(\omega)$, are the same.

From now on, we assume that function f satisfies

$$|f(s)| \le C_f(1 + |s|), \qquad \forall s \in \mathbb{R}, \tag{45}$$

where $C_f \in [0, m\lambda_1/4)$. The next lemma shows estimations of weak solution v of equation (33).

Lemma 3 *Suppose a satisfies (32) and (7), $f \in C(\mathbb{R})$ fulfills (45), $\phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$, $l \in L^2(\mathcal{O})$ and $v_0 \in L^2(\mathcal{O})$. Then the solution $v(t) := v(t; s, \omega, v_0)$ to problem (33) satisfies*

$$|v(t)|^2 \le \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + e^{-(m\lambda_1 - 4C_f)(t-s)}|v_0|^2$$
$$+ e^{-(m\lambda_1 - 4C_f)t} \int_s^t e^{(m\lambda_1 - 4C_f)\tau} \left(\frac{(2C_f + 1/C_f)}{\lambda_1} + \frac{M^2}{m} \right) |z^*(\theta_\tau\omega)|^2 \|\phi\|^2 d\tau.$$

Proof From energy equality, by (32), (45), the Poincaré and Young inequalities, we have

$$\frac{d}{dt}|v(t)|^2 + m\lambda_1|v(t)|^2 + m\|v(t)\|^2 \le (2C_f(1 + \beta_1) + \beta_2)|v(t)|^2 + \frac{C_f|\mathcal{O}|}{\beta_1}$$
$$+ \left(\frac{C_f}{\beta_1} + \frac{1}{\beta_2} \right) |z^*(\theta_t\omega)|^2|\phi|^2 + \beta_3\|v(t)\|^2 + \frac{M^2}{\beta_3}\|\phi\|^2|z^*(\theta_t\omega)|^2.$$

Picking up $\beta_1 = \frac{1}{2}$, $\beta_2 = C_f$ and $\beta_3 = m$, which implies

$$\frac{d}{dt}|v(t)|^2 \le (-m\lambda_1 + 4C_f)|v(t)|^2 + 2C_f|\mathcal{O}| + \left(\frac{(2C_f + 1/C_f)}{\lambda_1} + \frac{M^2}{m} \right) |z^*(\theta_t\omega)|^2 \|\phi\|^2.$$

Finally, multiplying by $e^{(m\lambda_1-4C_f)t}$ and integrating between s and t to above inequality, the result of this lemma holds. □

Remark 3 Analogously, let β_1 and β_2 in the proof of Lemma 3 are small enough as Remark 2. We can release the assumption of Lemma 3 to $m\lambda_1 > 2C_f$.

Thanks to Lemma 3, now we are able to use the universe of tempered sets, denoted by $\mathcal{D}$, to show the existence of a (pullback) random $\mathcal{D}$-attractor to equation (31) by Theorem 1. The next lemma presents that Ψ and Ξ have random absorbing sets, respectively.

Lemma 4 *Suppose the conditions of Lemma 3 hold. Then there exist* $\{K^{\Psi}(\omega)\}_{\omega\in\Omega}$ *and* $\{K^{\Xi}(\omega)\}_{\omega\in\Omega}$ *that both belong to* $\mathcal{D}$*, such that* $\{K^{\Psi}(\omega)\}_{\omega\in\Omega}$ *and* $\{K^{\Xi}(\omega)\}_{\omega\in\Omega}$ *are random absorbing sets for* Ψ *and* Ξ *in* $\mathcal{D}$*, respectively. Namely, for any* $\hat{B}=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$*, there exist* $T(\hat{B})>0$ *and* $\tilde{T}(\hat{B})>0$*, such that*

$$\Psi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset K^{\Psi}(\omega),\qquad \textit{for all}\quad t\geq T(\hat{B}),$$

and

$$\Xi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset K^{\Xi}(\omega),\qquad \textit{for all}\quad t\geq \tilde{T}(\hat{B}),$$

are true, separately.

Proof Existence of random absorbing set $K^{\Psi}(\omega)$. From Lemma 3, we have for $v(t)=v(t;0,\omega,v_0)$

$$|v(t)|^2\leq\frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f}+e^{-(m\lambda_1-4C_f)t}|v_0|^2$$
$$+e^{-(m\lambda_1-4C_f)t}\int_0^t e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2d\tau.$$

Substituting ω by $\theta_{-t}\omega$ and v_0 by $u_0-\phi z^*(\omega)$ in the expression of Ξ, respectively, we obtain

$$\begin{aligned}
&|\Xi(t,\theta_{-t}\omega,u_0-\phi z^*(\theta_{-t}\omega))|^2\\
&\leq\frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f}+e^{-(m\lambda_1-4C_f)t}|u_0-\phi z^*(\theta_{-t}\omega)|^2\\
&\quad+\int_0^t e^{-(m\lambda_1-4C_f)(t-\tau)}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_{\tau-t}\omega)|^2\|\phi\|^2d\tau\\
&\leq\frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f}+2e^{-(m\lambda_1-4C_f)t}|u_0|^2+2e^{-(m\lambda_1-4C_f)t}|z^*(\theta_{-t}\omega)|^2|\phi|^2\\
&\quad+\int_{-t}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2d\tau\\
&\leq\frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f}+2e^{-(m\lambda_1-4C_f)t}|u_0|^2+2e^{-(m\lambda_1-4C_f)t}|z^*(\theta_{-t}\omega)|^2|\phi|^2\\
&\quad+\int_{-\infty}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2d\tau.
\end{aligned}$$

Notice that, thanks to the properties of Ornstein–Uhlenbeck process z^* (cf. (13)), it follows that

$$\int_{-\infty}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2d\tau<\infty,$$

and

$$\lim_{t\to+\infty} e^{-(m\lambda_1-4C_f)t}|z^*(\theta_{-t}\omega)|^2|\phi|^2 = 0.$$

Taking into account for any $u_0 \in B(\theta_{-t}\omega)$,

$$\Psi(t,\theta_{-t}\omega,u_0) = \varXi(t,\theta_{-t}\omega,u_0-\phi z^*(\theta_{-t}\omega)) + \phi z^*(\omega),$$

it arrives,

$$\begin{aligned}
&|\Psi(t,\theta_{-t}\omega,u_0)|^2 \le 2|\varXi(t,\theta_{-t}\omega,u_0-\phi z^*(\theta_{-t}\omega))|^2 + 2|z^*(\omega)|^2|\phi|^2\\
&\le \frac{4C_f|\mathcal{O}|}{m\lambda_1-4C_f} + 4e^{-(m\lambda_1-4C_f)t}|d(B(\theta_{-t}\omega))|^2 + 4e^{-(m\lambda_1-4C_f)t}|z^*(\theta_{-t}\omega)|^2|\phi|^2\\
&\quad + 2\int_{-\infty}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2 d\tau + 2|z^*(\omega)|^2|\phi|^2.
\end{aligned}$$

Denoting, for all $\omega \in \Omega$,

$$\begin{aligned}
R^2_\Psi(\omega) = &\frac{4C_f|\mathcal{O}|}{m\lambda_1-4C_f} + 2|z^*(\omega)|^2|\phi|^2\\
&+ 2\int_{-\infty}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)}{\lambda_1}+\frac{M^2}{m}\right)|z^*(\theta_\tau\omega)|^2\|\phi\|^2 d\tau + 1.
\end{aligned} \tag{46}$$

Accordingly,

$$\begin{aligned}
R^2_\varXi(\omega) = &\frac{2C_f|\mathcal{O}|}{m\lambda_1-4C_f}\\
&+ 2\int_{-\infty}^0 e^{(m\lambda_1-4C_f)\tau}\left(\frac{(2C_f+1/C_f)|z^*(\theta_\tau\omega)|^2}{\lambda_1}+\frac{M^2}{m}\right)\|\phi\|^2 d\tau + 1.
\end{aligned} \tag{47}$$

Similarly, by means of the properties of Ornstein–Uhlenbeck process z^*, we derive

$$\lim_{t\to\infty} e^{-(m\lambda_1-4C_f)t}|d(B(\theta_{-t}\omega))|^2 = 0, \quad \lim_{t\to\infty} e^{-(m\lambda_1-4C_f)t}|\phi|^2|z^*(\theta_{-t}\omega)|^2 = 0.$$

Consequently,

$$K^\Psi(\omega) = \overline{B_{L^2(\mathcal{O})}(0,R_\Psi(\omega))} \quad \text{and} \quad K^\varXi(\omega) = \overline{B_{L^2(\mathcal{O})}(0,R_\varXi(\omega))}, \tag{48}$$

are absorbing closed random sets for Ψ and $\varXi$, respectively.

Check $K^\Psi(\omega) \in \mathcal{D}$. For this purpose, we only need to show

$$\lim_{t\to\infty} e^{-\beta t}R_\Psi(\theta_{-t}\omega) = 0, \qquad \forall\beta > 0.$$

To verify it goes to zero when t is sufficiently large, just observe for any $\beta > 0$,

$$
\begin{aligned}
e^{-\beta t} R_{\Psi}^2(\theta_{-t}\omega) &= e^{-\beta t} \frac{4C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + 2e^{-\beta t}|z^*(\theta_{-t}\omega)|^2|\phi|^2 + e^{-\beta t} \\
&\quad + 2e^{-\beta t} \int_{-\infty}^{0} e^{(m\lambda_1 - 4C_f)\tau} \left(\frac{(2C_f + 1/C_f)}{\lambda_1} + \frac{M^2}{m} \right) |z^*(\theta_{\tau - t}\omega)| \|\phi\|^2 d\tau \\
&= \underbrace{e^{-\beta t} \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + 2e^{-\beta t}|z^*(\theta_{-t}\omega)|^2|\phi|^2 + e^{-\beta t}}_{\longrightarrow 0 \text{ as } t \to \infty} \\
&\quad + 2\underbrace{e^{-\beta t} \int_{-\infty}^{-t} e^{(m\lambda_1 - 4C_f)(r+t)} \left(\frac{(2C_f + 1/C_f)}{\lambda_1} + \frac{M^2}{m} \right) |z^*(\theta_r\omega)| \|\phi\|^2 dr.}_{\longrightarrow 0 \text{ as } t \to \infty}
\end{aligned}
$$

The results for the other random dynamical system Ξ are proved by similar arguments. We omit the details. □

Our objective now is to prove the existence of (pullback) random $\mathcal{D}$-attractor of (31) in $L^2(\mathcal{O})$ by using Theorem 1. To this end, it is sufficient to prove that each sequence $u_n \in \Psi(t_n, \theta_{-t_n}(\omega), K^{\Psi}(\theta_{-t_n}\omega))$ with $t_n \to +\infty$ has a convergent subsequence in $L^2(\mathcal{O})$. Initially, we establish some estimations for random dynamical system Ξ by energy equality (see also [34–36]).

Lemma 5 *Under assumptions of Lemma 3, for any $t \in \mathbb{R}$, there exists $T(K^{\Xi}(\omega), t) < t - 2$, such that for all $s \leq T(K^{\Xi}(\omega), t)$ and $v_0 \in K^{\Xi}(\theta_s\omega)$, it fulfills*

$$
\begin{cases}
|v(r; s, \theta_s\omega, v_0)|^2 \leq \rho_1(\omega, t), & \forall r \in [t-2, t], \\
\displaystyle\int_{r-1}^{r} \|v(\tau; s, \theta_s\omega, v_0)\|^2 d\tau \leq \rho_2(\omega, t), & \forall r \in [t-1, t], \\
\displaystyle\int_{r-1}^{r} \|v'(\tau; s, \theta_s\omega, v_0)\|_{H^{-1}(\mathcal{O})}^2 d\tau \leq \rho_3(\omega, t), & \forall r \in [t-1, t],
\end{cases}
\tag{49}
$$

where $K^{\Xi}(\omega)$ is given in (48), and

$$
\begin{aligned}
\rho_1(\omega, t) &= 1 + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + e^{-(m\lambda_1 - 4C_f)(t-2)} \int_{-\infty}^{t} e^{(m\lambda_1 - 4C_f)\tau} \\
&\quad \times \left(\frac{(2C_f + 1/C_f)}{\lambda_1} + \frac{M^2}{m} \right) |z^*(\theta_\tau\omega)|^2 \|\phi\|^2 d\tau; \\
\rho_2(\omega, t) &= \frac{\lambda_1}{m\lambda_1 - 4C_f} \Bigg(\rho_1^2(t, \omega) + 2C_f|\mathcal{O}| + \frac{M^2}{m}\|\phi\|^2 \\
&\quad + (2C_f + 1/C_f)|\phi|^2 \max_{r \in [t-1, t]} \int_{r-1}^{r} |z^*(\theta_s\omega)|^2 ds \Bigg); \\
\rho_3(\omega, t) &= 4M^2 C \rho_2(\omega, t) + \frac{8C_f}{\lambda_1}(1 + 2\rho_1(\omega, t)) \\
&\quad + \left(\frac{4 + 16C_f}{\lambda_1^2} + 4CM^2 \right) \|\phi\|^2 \max_{r \in [t-1, t]} \int_{r-1}^{r} |z^*(\theta_s\omega)|^2 ds.
\end{aligned}
$$

Proof Let $T(K^{\Xi}(\omega), t) < t - 2$, such that

$$
e^{-(m\lambda_1 - 4C_f)(t-2)} e^{(m\lambda_1 - 4C_f)s} |v_0|^2 \leq 1, \qquad \forall v_0 \in K^{\Xi}(\theta_s\omega), \ \ \forall s \leq T(K^{\Xi}(\omega), t).
$$

Then the first statement in (49) follows directly from Lemma 3, using the increasing character of the exponential.

In the following lines, we obtain similar estimations for the other two inequalities in (49) by means the Galerkin approximations. In the sequel, with the help of compactness arguments, we will obtain the same ones for the solutions. Observe that the first estimation in (49) holds true due to the Galerkin approximations.

From the energy equality for the Galerkin approximation, by (32), it follows

$$\begin{aligned}\frac{d}{dt}|v_n(t)|^2 \ &+2m\|v_n(t)\|^2 \le 2(f(v_n+\sigma z^*(\theta_t\omega)), v_n(t)) + 2|z^*(\theta_t\omega)|(\phi, v_n(t))\\ &+2M\|\phi\|\|v_n(t)\|.\end{aligned}$$

Applying the Poincaré and Young inequalities, using (45), we deduce

$$\begin{aligned}\frac{d}{dt}|v_n(t)|^2 \ &+(m-4C_f/\lambda_1)\|v_n(t)\|^2 \le 2C_f|\mathcal{O}| + \frac{M^2}{m}\|\phi\|^2\\ &+(2C_f+1/C_f)|z^*(\theta_t\omega)|^2|\phi|^2. \end{aligned} \tag{50}$$

Integrating (50) between $r-1$ and r, where $r\in[t-1,t]$, we have for all $n\in\mathbb{N}$,

$$\begin{aligned}\int_{r-1}^{r}\|v_n(s)\|^2ds \le \frac{\lambda_1}{m\lambda_1-4C_f}\Bigg(&|v_n(r-1)|^2+2C_f|\mathcal{O}|+\frac{M^2}{m}\|\phi\|^2\\ &+(2C_f+1/C_f)|\phi|^2\int_{r-1}^{r}|z^*(\theta_s\omega)|^2ds\Bigg) \le \rho_2(t,\omega),\end{aligned} \tag{51}$$

where $\rho_2(\omega,t)$ is given in the statement, thanks to the first inequality in (49) for v_n. Taking inferior limit in (51) and using the well-known fact that $v_n(\cdot;s,\theta_s\omega,v_0)$ converges to $v(\cdot;s,\theta_s\omega,v_0)$ weakly in $L^2([r-1,r];H_0^1(\mathcal{O}))$ for all $r\in[t-1,t]$, the second inequality in (49) holds.

Finally, by (33), we derive

$$\begin{aligned}\|v_n'(t)\|^2_{H^{-1}(\mathcal{O})} &\le 4|a(l(v_n)+z^*(\theta_t\omega)l(\phi))|^2\|\Delta v_n\|^2_{H^{-1}(\mathcal{O})} + \frac{4}{\lambda_1}|f(v_n+\phi z^*(\theta_t\omega))|^2\\ &+\frac{4|z^*(\theta_t\omega)|^2}{\lambda_1}|\phi|^2 + 4|a(l(v_n)+z^*(\theta_t\omega)l(\phi))|^2|z^*(\theta_t\omega)|^2\|\Delta\phi\|^2_{H^{-1}(\mathcal{O})},\end{aligned}$$

a.e. $t>s$. By assumption (32), we have

$$|a(l(v_n(t))+z^*(\theta_t\omega)l(\phi))|^2 \le M^2.$$

Together with the facts that f satisfies (45), $-\Delta$ is the isometric isomorphism from $H_0^1(\mathcal{O})$ into $H^{-1}(\mathcal{O})$, and the two estimations we have proved already for v_n in (49), we obtain that

$$\int_{r-1}^{r}\|v_n'(\tau)\|^2_{H^{-1}(\mathcal{O})}d\tau \le \rho_3(\omega,t), \qquad \forall r\in[t-1,t],\ \ \forall n\in\mathbb{N}, \tag{52}$$

where $\rho_3(\omega,t)$ is the expression given in the statement. Now, taking inferior limit in (52) and bearing in mind that $v_n'(\cdot;s,\theta_s\omega,v_0)$ converges to $v'(\cdot;s,\theta_s\omega,v_0)$ weakly in $L^2([r-1,r];H^{-1}(\mathcal{O}))$ for all $r\in[t-1,t]$, the third estimate in (49) holds. □

This section is concluded with the following proposition showing the random dynamical system $\varXi$ is pullback asymptotically compact. Taking into account the relationship between Ψ and $\varXi$ below,

$$\Psi(t,\theta_{-t}\omega,u_0) = \varXi(t,\theta_{-t}\omega,u_0-\phi z^*(\theta_{-t}\omega)) + \phi z^*(\omega), \tag{53}$$

we will end up to proving the random dynamical system Ψ is also pullback asymptotically compact. For this purpose, we apply an energy method with continuous functions.

Proposition 3 *Suppose the conditions of Lemma 5 are true, then the random dynamical system Ξ is $\mathcal{D}$-pullback asymptotically compact. That is, each sequence*

$$v(0; -t_n, \omega, v_0) \in \Xi(t_n, \theta_{-t_n}\omega, K^{\Xi}(\theta_{-t_n}\omega))$$

with $t_n \to +\infty$ has a convergent subsequence in $L^2(\mathcal{O})$, where $K^{\Xi}(\omega)$ is given in (48).

Proof Consider $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to +\infty$ as $n \to \infty$. Let $\{v_0^n\}_{n=1}^{\infty} \in K^{\Xi}(\theta_{-t_n}\omega)$, our aim is to prove the sequence $v(0; -t_n, \omega, v_0^n) = \Xi(t_n, \theta_{-t_n}\omega, v_0^n)$ is relatively compact in $L^2(\mathcal{O})$. For short, we will denote $v^n(\cdot) := v(\cdot; -t_n, \omega, v_0^n)$.

By means of Lemma 5, the continuity of functions a and $z^*(\theta_t\omega)$, $l \in L^2(\mathcal{O})$ and $\phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$, we know there exist $T(K^{\Xi}(\omega), t) < t-2$ and $n_1 \geq 1$, such that $-t_n \leq T(K^{\Xi}(\omega), t)$ for all $n \geq n_1$, $\{v^n\}_{n\geq n_1}$ is bounded in $L^{\infty}([t-2, t]; L^2(\mathcal{O})) \cap L^2([t-2, t]; H_0^1(\mathcal{O}))$, $\{f(v^n + \phi z^*(\theta.\omega))\}_{n\geq n_1}$ is bounded in $L^2([t-2, t]; L^2(\mathcal{O}))$, the sequences $\{-a(l(v^n) + \phi z^*(\theta.\omega))\Delta v^n\}$ and $\{(v^n)'\}_{n\geq n_1}$ are bounded in $L^2([t-2, t]; H^{-1}(\mathcal{O}))$. Then using the Aubin–Lions Lemma, there exists $v \in L^{\infty}([t-2, t]; L^2(\mathcal{O})) \cap L^2([t-2, t]; H_0^1(\mathcal{O}))$ with $v' \in L^2([t-2, t]; H^{-1}(\mathcal{O}))$, such that for a subsequence (relabeled the same), it holds

$$\begin{cases} v^n \to v \ \text{ weak-star in } \ L^{\infty}([t-2, t]; L^2(\mathcal{O})); \\ v^n \to v \ \text{ weakly in } \ L^2([t-2, t]; H_0^1(\mathcal{O})); \\ (v^n)' \to v' \ \text{ weakly in } \ L^2([t-2, t]; H^{-1}(\mathcal{O})); \\ v^n \to v \ \text{ strongly in } \ L^2([t-2, t]; L^2(\mathcal{O})); \\ v^n(x,\tau) \to v(x,\tau) \ \text{ a.e. } (x,\tau) \in \mathcal{O} \times [t-2, t]; \\ v^n(\tau) \to v(\tau) \ \text{ strongly in } \ L^2(\mathcal{O}), \ \text{ a.e. } \ \tau \in [t-2, t], \end{cases} \tag{54}$$

$$f(v^n + \phi z^*(\theta.\omega)) \to f(v + \phi z^*(\theta.\omega)) \ \text{ weakly in } \ L^2([t-2, t]; L^2(\mathcal{O})), \tag{55}$$

and

$$-a(l(v^n + \phi z^*(\theta.\omega)))\Delta v^n \to -a(l(v + \phi z^*(\theta.\omega)))\Delta v \ \text{ weakly in } \ L^2([t-2, t]; H^{-1}(\mathcal{O})). \tag{56}$$

Furthermore, $v \in C([t-2, t]; L^2(\mathcal{O}))$ and using the convergence, it is not difficult to prove that v is a weak solution of (32). Since $\{(v^n)'\}_{n\geq n_1}$ is bounded in $L^2([t-2, t]; H^{-1}(\mathcal{O}))$, we have that $\{v^n\}_{n\geq n_1}$ is equicontinuous in $H^{-1}(\mathcal{O})$ on $[t-2, t]$. Namely, for fixed $\varepsilon > 0$ and for each fixed $\omega \in \Omega$, $\mathbb{P}$-a.e., consider $\tau_1, \tau_2 \in [t-2, t]$ with $|\tau_1 - \tau_2| < \delta_{\varepsilon}$, then

$$\begin{aligned} \|v^n(\tau_2) - v^n(\tau_1)\|_{H^{-1}(\mathcal{O})}^2 &\leq \left(\sup_{v\in H_0^1(\mathcal{O})/\|v\|=1} \left| < \int_{\tau_1}^{\tau_2} (v^n(r)'dr, v > \right| \right)^2 \\ &\leq \left(\int_{\tau_1}^{\tau_2} \|(v^n(r))'\|_{H^{-1}(\mathcal{O})} dr \right)^2 \\ &\leq \rho_3(\omega, t)|\tau_1 - \tau_2|. \end{aligned}$$

The result is true by simply taking $\delta_{\varepsilon} = \min\{\varepsilon^2/\rho_3(t, \omega), 1\}$. In addition, as $\{v^n\}_{n\geq n_1}$ is bounded in $C([t-2, t]; L^2(\mathcal{O}))$, combined with the embedding $L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$, the

Arzelà-Ascoli theorem yields (for another subsequence, relabeled again the same),

$$v^n \to v \qquad \text{strongly in} \quad C([t-2,t]; H^{-1}(\mathcal{O})). \tag{57}$$

Now, consider a sequence $\{\tau_n\} \subset [t-2,t]$ which converges to τ_*. Since $\{v^n\}_{n \geq n_1}$ is bounded in $C([t-2,t]; L^2(\mathcal{O}))$, there exists a subsequence of $\{v^n(\tau_n)\}_{n\geq n_1}$ (relabeled the same) and $\zeta \in L^2(\mathcal{O})$, such that

$$v^n(\tau_n) \to \zeta \qquad \text{weakly in} \quad L^2(\mathcal{O}). \tag{58}$$

Let us prove that $\zeta = v(\tau_*)$. For any fixed $\varepsilon > 0$, from (57) we deduce that there exists $n_\varepsilon \in \mathbb{N}$ such that,

$$\|v^n(\tau) - v(\tau)\|_{H^{-1}(\mathcal{O})} \leq \frac{\varepsilon}{2}, \qquad \forall n \geq n_\varepsilon(\omega), \ \ \forall \tau \in [t-2,t].$$

From this and the fact $v \in C([t-2,t]; H^{-1}(\mathcal{O}))$, we deduce

$$v^n(\tau_n) \to v(\tau_*) \qquad \text{strongly in} \quad H^{-1}(\mathcal{O}). \tag{59}$$

Observe (58)–(59), by the uniqueness of limit we obtain,

$$v^n(\tau_n) \to v(\tau_*) \qquad \text{weakly in} \quad L^2(\mathcal{O}). \tag{60}$$

Notice, if we prove

$$v^n \to v \qquad \text{strongly in} \quad C([t-2,t]; L^2(\mathcal{O})), \tag{61}$$

in particular the sequence $\{v(t; -t_n, \omega, u_0^n - \phi z^*(\theta_{-t_n}\omega))\}$ will be relatively compact in $L^2(\mathcal{O})$. We establish (61) by contradiction, suppose that there exists $\varepsilon > 0$, a sequence $\{\tilde{t}_n\} \subset [t-2,t]$, without loss of generality converging to some t_* with

$$|v^n(\tilde{t}_n) - v(t_*)| \geq \varepsilon, \qquad \forall n \geq 1. \tag{62}$$

On the other hand, making use of the energy equality (6) and (45), the Poincaré and Young inequalities, the estimation

$$\begin{aligned} |g(\tau)|^2 \leq \ & |g(r)|^2 + 2C_f|\mathcal{O}||\tau - r| + \frac{M^2}{m}\|\phi\|^2|\tau - r| \\ & + \left(2C_f + 1/C_f\right)|\phi|^2 \int_r^\tau |z^*(\theta_s\omega)|^2 ds. \end{aligned}$$

holds with g replaced by v or any v^n.

Now we define the functions,

$$J_n(\tau) = |v^n(\tau,\omega)|^2 - 2C_f|\mathcal{O}|\tau - \frac{M^2}{m}\|\phi\|^2\tau - (2C_f + 1/C_f)|\phi|^2 \int_{t-2}^\tau |z^*(\theta_s\omega)|^2 ds,$$

$$J(\tau) = |v(\tau,\omega)|^2 - 2C_f|\mathcal{O}|\tau - \frac{M^2}{m}\|\phi\|^2\tau - (2C_f + 1/C_f)|\phi|^2 \int_{t-2}^\tau |z^*(\theta_s\omega)|^2 ds.$$

From the regularity of v and v^n, together with the above equalities, it makes sure the functions J and J_n are continuous and non-increasing on $[t-2,t]$. In addition, we have

$$J_n(\tau) \to J(\tau), \qquad \tau \in [t-2,t].$$

Hence, there exists a sequence $\{\tilde{t}_k\} \subset (t-2, \tau_*)$, such that $\tilde{t}_k \to \tau_*$ as $k \to +\infty$ and

$$\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k), \qquad \forall k \geq 1.$$

For any fixed arbitrary $\varepsilon > 0$, from the continuity of J on $[t-2, t]$, there exists $k(\varepsilon) > 1$ such that

$$|J(\tilde{t}_k) - J(t_*)| \le \frac{\varepsilon}{2}, \qquad \forall k \ge k(\varepsilon).$$

Now consider $n(\varepsilon) \ge 1$, such that $\tilde{t}_n \ge \tilde{t}_{k(\varepsilon)}$ and

$$|J_n(\tilde{t}_{k(\varepsilon)}) - J(\tilde{t}_{k(\varepsilon)})| \le \frac{\varepsilon}{2}, \quad \forall n \ge n(\varepsilon).$$

Since all J_n are non-increasing functions, we deduce for all $n \ge n(\varepsilon)$,

$$\begin{aligned} J_n(\tilde{t}_n) - J(t_k) &\le J_n(\tilde{t}_{k(\varepsilon)}) - J(t_k) \le |J_n(\tilde{t}_{k(\varepsilon)}) - J(t_k)| \\ &\le |J_n(\tilde{t}_{k(\varepsilon)}) - J(\tilde{t}_{k(\varepsilon)})| + |J(\tilde{t}_{k(\varepsilon)}) - J(t_*)| \\ &\le \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, from above we deduce,

$$\limsup_{n \to +\infty} J_n(\tilde{t}_n) \le J(t_*).$$

Thus,

$$\limsup_{n \to +\infty} |v^n(\tilde{t}_n)| \le |v(t_*)|.$$

From this, (60) applied to the sequence $\{\tilde{t}_n\}$, it satisfies that the sequence $\{v^n(\tilde{t}_n)\}$ converges to $v(t_*)$ strongly in $L^2(\mathcal{O})$, which is contradictory with (62). Therefore, (61) is proved.

Particularly, we take $t = 0$ in the above analyses. The conclusion that sequence $v^n(0; -t_n, \omega, v_0^n) = \Xi(t_n, \theta_{-t_n}, v_0^n)$, where $v_0^n \in K^{\Xi}(\theta_{-t_n}\omega)$, is relatively compact in $L^2(\mathcal{O})$ holds immediately. □

Theorem 10 *Suppose the assumptions of Lemma 5 hold. Then the random dynamical system Ψ has a $\mathcal{D}$-random attractor in $L^2(\mathcal{O})$.*

Proof As a consequence of the results of Proposition 3, from Theorem 1, we obtain the existence of the $\mathcal{D}$-random attractor for the cocycle Ξ in $L^2(\mathcal{O})$. At last, based on the relationship between Ψ and Ξ (cf. (53)), $\phi z^*(\omega)$ is a constant in $L^2(\mathcal{O})$, the existence of a $\mathcal{D}$-random attractor for Ψ is proved. □

4.4 A comment on attractors $\mathcal{A}_F(\omega)$ and $A(\omega)$.

This section is concluded with one comment concerning the relationship between two different attractors $\mathcal{A}_F(\omega)$ and $\mathcal{A}(\omega)$ obtained in Sects. 4.2 and 4.3, separately. Let us first recall the following result which can be found in [22].

Theorem 11 *Suppose that (θ, φ) is an RDS on a Polish space X, such that there exists a compact attracting set for the family of all compacts deterministic subsets of X. Then there exists a random pullback attractor A, and this attractor is unique in the sense that whenever A' is a random pullback attractor for every compact deterministic set then $A = A'$, $\mathbb{P}$-a.s. Furthermore, if $\mathcal{B}$ is an arbitrary collection of random sets with a random pullback attractor $A_{\mathcal{B}}$, then $A_{\mathcal{B}} \subset A$, $\mathbb{P}$-a.e. Additionally, if $\mathcal{B}$ contains every compact deterministic set, then $A_{\mathcal{B}} = A$, $\mathbb{P}$-a.s.*

The following result gives us a relation of the two derived random attractors.

Theorem 12 *The random attractors $\mathcal{A}_F(\omega)$ obtained in Sect. 4.2 and $\mathcal{A}(\omega)$ derived in Sect. 4.3 for problem (31) are the same.*

Proof It is worth mentioning the existence of random attractor $\mathcal{A}_F(\omega)$ proved in Sect. 4.2 is a compact attracting set of the family of all compacts deterministic subsets of $L^2(\mathcal{O})$ (cf. $B(0, \rho)$). While the random attractor $\mathcal{A}(\omega)$ derived in Sect. sec:4.3 is a compact attracting set of the family of all tempered sets. Theorem 11 ensures that $\mathcal{A}(\omega) \subset \mathcal{A}_F(\omega)$. Furthermore, it is clear that tempered sets contains compact deterministic set, then $\mathcal{A}(\omega) = \mathcal{A}_F(\omega)$. □

Conclusions and final remarks

We have successfully analyzed the asymptotic behavior of solutions to nonlocal stochastic partial differential equations with multiplicative and additive noise, driven by a standard Brownian motion, by means of an appropriate change of variable which is the standard way to proceed in the frameworks of random dynamical systems and random attractors.

Recently, B. X. Wang and his collaborators (see, e.g., [37] and the references therein) have been studying some stochastic PDE models driven by colored noise thanks to the Wong–Zakai approximations. Motived by their work, it is reasonable to study the dynamics of stochastic nonlocal differential equations driven by colored noise to obtain similar but interesting results to the ones in this paper and which can be helpful to analyze stochastic equations with nonlinear noise.

Also, notice that the methods provided in this manuscript to handle stochastic nonlocal partial differential equations are only valid for those which are equivalent to random ones, i.e., linear multiplicative and additive noise. When the nonlocal partial differential equation is driven by a nonlinear stochastic term, such as

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + g(t, u)\frac{dW(t)}{dt},$$

we need to adopt a different method to solve this problem. This will be the objective of our forthcoming work.

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