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On the Exponential Stability of Stochastic Perturbed Singular Systems in Mean Square

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Abstract

The approach of Lyapunov functions is one of the most efficient ones for the investigation of the stability of stochastic systems, in particular, of singular stochastic systems. The main objective of the paper is the analysis of the stability of stochastic perturbed singular systems by using Lyapunov techniques under the assumption that the initial conditions are consistent. The uniform exponential stability in mean square and the practical uniform exponential stability in mean square of solutions of stochastic perturbed singular systems based on Lyapunov techniques are investigated. Moreover, we study the problem of stability and stabilization of some classes of stochastic singular systems. Finally, an illustrative example is given to illustrate the effectiveness of the proposed results.

Keywords Stochastic perturbed singular systems · Consistent initial conditions · Lyapunov techniques · Itô formula · Brownian motion · Nontrivial solution · Practical exponential stability in mean square · Stabilization

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1 Introduction

Singular systems are those whose dynamics are overseen by a mixture of algebraic and differential equations. In that sense, the algebraic equations describe the constraints to the solution of the differential part. These systems are also referred to as descriptor systems, differential–algebraic systems, semi-state systems, or generalized state-space systems.

In [22], Rosenbrock introduced singular systems and tackled the transformation of linear descriptor systems. Since then, differential–algebraic equations are becoming increasingly essential in many applications such as electrical engineering, aircraft dynamics, economics, optimization problems, chemical, biology, robotics, etc.

What's more, singular systems have been one of the major fundamental and representative research fields of control theory because of their extensive applications.

As it is well known, environmental noise exists and cannot neglect it in many dynamical systems. Taking into consideration noise, the mathematical model leads to stochastic differential–algebraic equations (SDAEs). This kind of equation can be viewed as a generalization of differential–algebraic equations (DAEs) and stochastic differential equations (SDEs).

Stability of stochastic differential equations (SDEs) has become a very prevalent issue of the newest research in Mathematics and its applications. Consequently, the stability analysis of singular stochastic systems became one of the critical topics in many research studies. Different authors handled the problem of stability and feedback stabilization of singular stochastic systems. Several researchers have produced extensive results, we would like to mention here the references [3,4,8,25–28], among others. It has been recognized that noise can be used to destabilize a given stable system. Nevertheless, it can also be used to stabilize a given unstable system or to make a given stable system even more stable, see [14,19,23,24,26].

In the present paper, we consider the combination of Itô stochastic representation and singular form to define a vast class of systems that modeled stochastic differential–algebraic equations (SDAEs). Our motivation is that there are not many works about the stability of this particular class of systems.

The method of Lyapunov functions is one of the most powerful means to investigate the stability of stochastic dynamical systems. Lyapunov stability of stochastic dynamical systems has attracted the attention of diverse authors. We would like to mention here the references [9,10,12,16,18], among others. This paper aims to study the stability of differential–algebraic stochastic equations.

Numerous crucial variants to Lyapunov's original concepts were proposed in [1,2,11]. In the case when the origin is not inevitably an equilibrium point, we can investigate the stability of the SDAEs in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball. This property is defined as practical stability. To the best of our knowledge, no work has been brought about the practical stability of stochastic singular systems in the literature.

In the present paper, our goal is to investigate the notion of practical stability of stochastic singular systems by using the Lyapunov techniques.

This paper is structured in the following way: Sect. 2 presents some notations, definitions, and preliminary lemmas which will be used in the next sections. In Sect. 3, we discuss the problem of existence and uniqueness of our system, as well as establish sufficient conditions for uniform exponential stability in mean square and practical uniform exponential stability in mean square of stochastic perturbed singular systems by means of Lyapunov functions. In Sect. 4, we state sufficient conditions for uniform exponential and uniform practical exponential stabilization in the mean square of stochastic singular systems. In Sect. 5, we display an illustrative example to show the applicability of our abstract theory. Eventually, in Sect. 6, some conclusions are given.

2 Preliminary Results

In this section, we introduce some notations and preliminary results.

Notations

$\mathbb{R}$: Real vector space.

$\mathbb{C}$: Complex vector space.

$\mathbb{I}$: Identity matrix.

$F = (f_{ij}) \in \mathbb{R}^{n\times n}$: Real matrix.

F^T : Transpose of matrix F.

F^D : Drazin inverse of matrix F.

$F > 0$: Positive definite matrix.

$F \geq 0$: Positive semi definite matrix.

$\aleph$: Null space (kernel) of matrix F.

$\text{im}(F)$: The image of matrix F.

$\lambda(F)$: Eigenvalue of matrix F.

$\lambda_{\max}(F)$ $(\lambda_{\min}(F))$: The maximum (minimum) eigenvalue of a symmetric matrix F.

$||F|| = \sqrt{\lambda_{\max}(F^T F)}$: Euclidean matrix norm of F.

$||x|| = \sqrt{x^T x}$: Euclidean norm of $x \in \mathbb{R}^n$.

Now, let consider the following linear continuous singular system:

$$E\dot{x}(t) = Ax(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $x(t_0) = x_0 \in \mathbb{R}^n$ is the initial condition.

$E, A \in \mathbb{R}^{n\times n}$ are constant matrices, with E is a singular matrix and $rank(E) = r < n$.

Systems like the one in (2.1) are known as linear time-invariant singular systems. They arise naturally in many applications such as networks, aircraft, and robot dynamics, economics as well as optimization problems, biology, etc. Different authors dealt with the question of existence and uniqueness of solutions to linear time-invariant singular systems [6,13]. Under the assumption that the initial conditions were consistent, the stability of the system (2.1) was evaluated, and sufficient conditions for the stability of linear time-invariant systems introduced. The survey of the fundamental theory of the stability of linear time-invariant singular systems can be found in [5,15,20].

For the system (2.1), we need the following definitions and assumptions.

Definition 2.1 The pair (E, A) is said to be

1. Regular, if $det(zE - A)$ is not identically zero for some $z \in \mathbb{C}$.
2. Impulse-free, if $\text{degree}[\det(zE - A)] = \text{rank}(E)$.

Lemma 2.1 [13] *We suppose that the pair (E, A) is regular and impulse-free. Then, the solution to system (2.1) exists and is impulse-free and unique on $\mathbb{R}_+$.*

2.1 Consistent Initial Conditions

The singularity of matrix E will ensure that solutions of equation (2.1) exist only for exceptional choices of x_0. The problem of consistent initial conditions is not characteristic of the systems in the classical form. Nevertheless, it is a crucial one for the singular systems. We will claim that an initial condition $x_0 \in \mathbb{R}^n$ is consistent if there exists a differentiable continuous solution of equation (2.1). The generation and investigation of consistent initial conditions have received very much attention in the literature [6,20]. Some of these, most essential results are presented here.

Campbell [6] proved that x_0 is a consistent initial condition for (2.1) if and only if:

$$\left(\mathbb{I} - \widehat{E}\widehat{E}^D\right) x_0 = 0,$$

or, in analog notation:

$$\mathcal{W}_{K^*} = \aleph\left(\mathbb{I} - \widehat{E}\widehat{E}^D\right),$$

where $\widehat{E}^D$ is the Drazin inverse of matrix $\widehat{E}$ and $\widehat{E} = (\lambda E + A)^{-1}_{|\lambda=0} .E$.

The essential geometric apparatus in the characterization of the subspace of $\mathcal{W}_{K^*}$ of consistent initial conditions is the subspace sequence :

$$\mathcal{W}_0 \in \mathbb{R}^n$$
$$\vdots$$
$$\mathcal{W}_{i+1} = A^{-1}(E\mathcal{W}_i), \quad i \geq 0.$$

Ownes and Debeljkovic [20] established the following lemma:

Lemma 2.2 *The subsequence $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \ldots\}$ is nested in the sense that*

$$\mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_3 \supset \ldots$$

Moreover,

$$\aleph(A) \subset \mathcal{W}_i, \quad \forall i \geq 0,$$

and there exists an integer $K \geq 0$, *such that*

$$\mathcal{W}_{K+1} = \mathcal{W}_K.$$

Then, it is obvious that

$$\mathcal{W}_{K+i} = \mathcal{W}_K, \quad \forall i \geq 1.$$

If K^* *is the smallest integer with this property, then*

$$\mathcal{W}_K \cap \aleph(E) = \{0\}, \quad K \geq K^*, \tag{2.2}$$

provided that $(\lambda E - A)$ *is invertible for some* $\lambda \in \mathbb{R}$.

Notice that, if $x_0 \in \mathcal{W}_{K^*}$, *then* $x(t) \in \mathcal{W}_{K^*}, \quad \forall t \geq 0$.

3 Stability Analysis for a Class of Stochastic Perturbed Singular Systems

It is evident that (2.1) is a deterministic linear singular system. Nevertheless, some parameters may be excited or perturbed by some environmental noise (Oksendal [17], Mao [18]).

In this case, (2.1) becomes:

$$Edx(t) = Ax(t)dt + (\text{Perturbation term })\text{“noise”}. \tag{3.1}$$

Having the motivations stated above and encouraged by these pioneering works, we intend to study the problem of stability of stochastic perturbed singular systems (3.1) with respect to the solutions of the unperturbed system (2.1), under the assumption that the initial conditions are consistent.

The results presented in this section provide some sufficient conditions for the stability of some classes of stochastic perturbed singular systems in the sense of Lyapunov.

Assume that some parameters of the linear continuous singular system (2.1) are excited or perturbed by a standard Brownian motion, and the stochastic perturbed singular system has the following form:

$$Edx(t) = Ax(t)dt + \Pi g(t, x(t))dB_t, \tag{3.2}$$

where $x(t_0) = x_0 \in \mathbb{R}^n$ is the initial condition of the system, $B_t \in \mathbb{R}$ is a standard Brownian motion defined on a complete probability space (Ω, F, P) with $B_0 = 0$.

$E, A \in \mathbb{R}^{n \times n}$ are constant matrices, with E is a singular matrix and $rank(E) = r < n$. $\Pi \in \mathbb{R}^{n \times n}$ is a constant matrix, such that $\text{im}\Pi = \mathcal{W}_{K^*}$, and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Remark 3.1 The term Πg is a structured perturbation that ensures that the algebraic equation in the system is not affected by the stochastic perturbation. This approach is well studied [5]. It is necessary to restrict the set of allowable perturbations to guarantee consistency with the SDAEs (3.2). Indeed, Πg is a structured perturbation that ensures consistency, that is

$$\Pi g(t, x(t)) \in \mathcal{W}_{K^*}, \text{ for all } t \geq t_0 \geq 0.$$

3.1 Existence and Uniqueness Problem

Let us start by recalling a technical lemma, which will be useful in our analysis (see Dai [13] for the detailed proofs).

Lemma 3.2 *Let $E, A \in \mathbb{R}^{n\times n}$ and (E, A) is regular if and only if two nonsingular matrices $S \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times n}$ may be chosen such that,*

$$SET = \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}, \quad SAT = \begin{pmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{r\times r}$.

For a well-posed problem to system (3.2), we impose the following assumptions.

Assumptions

1. The pair (E, A) is regular and impulse-free.
2. The function $g(t, x)$ satisfies the following relations $\forall t \geq 0,\ \forall x \in \mathbb{R}^n$ and $\forall \bar{x} \in \mathbb{R}^n$,

$$||g(t, x)||^2 \leq C_1(1 + ||x||^2), \tag{3.3}$$
$$||g(t, x) - g(t, \bar{x})|| \leq C_2||x - \bar{x}||, \tag{3.4}$$

where C_1 and C_2 are given strictly positive reals.

We have the following result for the existence and uniqueness of the solution to the system (3.2).

Lemma 3.3 *The stochastic perturbed singular system (3.2) has a unique solution if the above assumptions hold.*

Proof The pair (E, A) is regular and impulse-free, then from Lemma 3.2 there exist two nonsingular matrices $S \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ such that:

$$SET = \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}, \quad SAT = \begin{pmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{pmatrix}, \quad A_1 \in \mathbb{R}^{r\times r}.$$

That is, we obtain

$$SEdx(t) = SAx(t)dt + S\Pi g(t, x(t))dB_t$$

$$SETT^{-1}dx(t) = SATT^{-1}x(t)dt + S\Pi g(t, x(t))dB_t.$$

Let $y(t) = T^{-1}x(t) = (y_1^T(t),\ y_2^T(t))^T$, where $y_1(t) \in \mathbb{R}^r$ and $y_2(t) \in \mathbb{R}^{n-r}$.

Based upon the form of SET, then we may choose Π in the following form:

$$\Pi = \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}.$$

As a result for $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, we obtain

$$\begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} \begin{pmatrix} dy_1(t) \\ dy_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{pmatrix} dt + S \begin{pmatrix} g_1(t, Ty(t)) \\ 0 \end{pmatrix} dB_t.$$

Let $\hat{g}_1(t, y(t)) = S \begin{pmatrix} g_1(t, Ty(t)) \\ 0 \end{pmatrix}$, which in turn gives

$$\begin{cases} dy_1(t) = A_1 y_1 dt + \hat{g}_1(t, y(t))dB_t \\ 0 = y_2(t)dt, \end{cases} \tag{3.5}$$

with initial condition y_0 satisfying $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} y_0 = \begin{pmatrix} y_{10} \\ 0 \end{pmatrix}$.

Since the function $g(t, x)$ satisfies the conditions (3.3) and (3.4). Then, we obtain the conditions of the usual existence and uniqueness theorem for SDEs (see e.g. [17,18], for the regular SDE (3.5)):

- Lipschitz condition: The coefficient $\hat{g}_1$ with $\hat{g}_1(t, y(t)){:=}S \begin{pmatrix} g_1(t, Ty(t)) \\ 0 \end{pmatrix}$ is Lipshitz-continuous with respect to y_1 with a constant $L_{\hat{g}_1} \leq C_2||S||||T||$, where C_2 is the Lipshitz constant for the function g.
- Growth condition: Since g depends continuously on t, also $\hat{g}_1$ depends continuously on t. Hence, the growth condition follows from the global lipshitz condition.

Employing the usual existence and uniqueness theorem for SDEs to the regular stochastic system (3.5), we obtain the regular SDE (3.5) has a unique solution on the interval $[0, \infty)$.

Then, $x(t) = Ty(t)$ is the unique solution to (3.2) on $t \geq 0$. □

3.2 Vanishing Perturbation

In this paragraph, we suppose that the perturbation g vanishes at zero, that is, $g(t, 0) = 0,\ \forall t \geq 0$. Then, we analyze the stability behavior of the origin as an equilibrium point of the stochastic perturbed singular system (3.2) based on Lyapunov techniques.

Definition 3.1 The stochastic perturbed singular system (3.2) is said to be uniformly exponentially stable in a mean square, if there exist positive constants β_1 and β_2, such that for all $t_0 \in \mathbb{R}_+$, and all consistent initial conditions x_0,

$$\mathbb{E}(||x(t)||^2) \leq \beta_1 ||x_0||^2 e^{-\beta_2(t-t_0)}, \quad \text{for all } t \geq t_0 \geq 0. \tag{3.6}$$

Theorem 3.4 [18] *Let $x(t)$ be a n-dimensional Itô process on $t \geq 0$ satisfies the following stochastic differential equation*

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB_t,$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$.

Let $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$: the family of all non-negative functions $V(t, x(t))$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ which are once continuously differentiable in t and twice in x.

Then, $V(t, x(t))$ is an ltô process and

$$dV(t, x(t)) = LV(t, x(t))dt + V_x(t, x(t))g(t, x(t))dB_t,$$

where

$$LV(t, x) := V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2} trace\left[g^T(t, x)V_{xx}(t, x)g(t, x)\right],$$

$$V_t(t, x) = \frac{\partial V}{\partial t}(t, x)\ ;\ V_x(t, x) = \left(\frac{\partial V}{\partial x_1}(t, x), \frac{\partial V}{\partial x_2}(t, x), \ldots, \frac{\partial V}{\partial x_n}(t, x)\right)\ ;\ V_{xx}(t, x) = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}(t, x)\right)_{n\times n}.$$

Remark 3.5 Wealthy historic background recorded in the investigation of the stability of linear time-invariant singular systems (2.1) via a Lyapunov-like approach, we would like to mention here the references [6,13,15,20], among others.

It is well known that one seeks for solutions $P \in \mathbb{R}^{n\times n}$ as well as $Q \in \mathbb{R}^{n\times n}$ of the Lyapunov equation

$$A^T PE + E^T PA = -Q, \tag{3.7}$$

and the corresponding Lyapunov function candidate is

$$V : \mathcal{W}_{K^*} \to \mathbb{R}, \quad x \longmapsto (Ex)^T P(Ex). \tag{3.8}$$

Our approach structured in this paper is to use the Lyapunov function (3.8) for the deterministic linear singular system (2.1) as a Lyapunov function candidate for the stochastic perturbed singular system (3.2) under some assumptions in the perturbation term.

Now, we are in a position to state our main result in this paragraph.

Theorem 3.6 *The stochastic perturbed singular system* (3.2) *is uniformly exponentially stable in mean square, if there exists a positive definite symmetric matrix P, being the solution of Lyapunov matrix equation* (3.7)*, with matrix $Q = Q^T > 0$, such that*

$$x^T Qx > 0, \quad \forall x \in \mathcal{W}_{K^*}\backslash\{0\}, \tag{3.9}$$

where $\mathcal{W}_{K^*}$ is the subspace of consistent initial conditions.

Furthermore, the perturbation term satisfies the following condition:

$$||g(t,x)|| \leq \gamma ||x||, \ a.s. \quad \forall (t,x) \in \mathbb{R}_+ \times \mathcal{W}_{K^*}, \tag{3.10}$$

where γ is a positive constant.

To prove this Theorem, we need an essential Gronwall lemma revealed in [21].

Lemma 3.7 *Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous function, ε is a positive real number and η is a strictly positive real number. Assume that for all $t \in \mathbb{R}_+$ and $0 \leq v \leq t$, we have*

$$\phi(t) - \phi(v) \leq \int_v^t (-\eta\phi(s) + \varepsilon) ds.$$

Then,

$$\phi(t) \leq \frac{\varepsilon}{\eta} + \phi(0)\exp(-\eta t).$$

Note that, the previous result remains true if we replace $\phi(0)$ by $\phi(t_0)$ with $0 \leq t_0 \leq v \leq t$.

Proof of Theorem (3.6) To prove sufficiency, note that (2.2) reveals that:

$$V(x) = x^T E^T P E x,$$

is a positive quadratic form on $\mathcal{W}_{K^*}$. Furthermore, all smooth solutions $x(t)$ evolve in $\mathcal{W}_{K^*}$. Hence, $V(x)$ can be used as a Lyapunov function for the system under consideration.

That is, there exist $\lambda_1, \lambda_2 > 0$ such that:

$$\lambda_1 x^T x \leq x^T E^T P E x \leq \lambda_2 x^T x, \quad \forall t \geq 0, \ \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \tag{3.11}$$

Applying Itô's formula to $V(x(\cdot))$ where $x(\cdot)$ is a trajectory (solution) of the stochastic singular perturbed system (3.2), we obtain

$$\begin{aligned} LV(x(t)) &= x^T(t)\Big(A^T P E + E^T P A\Big)x(t) + (\Pi g(t,x(t)))^T P(\Pi g(t,x(t))) \\ &= -x^T(t) Q x(t) + (\Pi g(t,x(t)))^T P(\Pi g(t,x(t))). \end{aligned}$$

It follows from (3.10) that,

$$\begin{aligned} (\Pi g(t,x(t)))^T P(\Pi g(t,x(t))) &\leq ||\Pi|| \ ||g(t,x(t))|| \ ||P|| \ ||\Pi|| \ ||g(t,x(t))|| \\ &= ||\Pi||^2 ||P|| \ ||g(t,x(t))||^2 \\ &\leq \gamma^2 ||\Pi||^2 \lambda_{\max}(P) ||x||^2. \end{aligned}$$

Using the Euclidean matrix norm, we obtain

$$(\Pi g(t,x(t)))^T P(\Pi g(t,x(t))) \leq \gamma^2 \lambda_{\max}(\Pi^T \Pi) \lambda_{\max}(P) ||x||^2. \tag{3.12}$$

It follows from condition (3.9), that there exist positive constants q_1 and q_2, such that

$$q_1 x^T x \leq x^T Q x \leq q_2 x^T x, \quad \forall t \geq 0, \ \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \tag{3.13}$$

Using (3.12) and (3.13), we obtain

$$\begin{aligned} LV(x(t)) &\leq -q_1 x^T(t)x(t) + \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P) x^T(t)x(t) \\ &= -(q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)) x^T(t)x(t). \end{aligned}$$

Without loss of generality, we may assume that $q_1 > \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)$.
Then, we deduce from (3.11) that,

$$LV(x(t)) \leq -\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2} V(x(t)).$$

By Dynkin's formula [17], we have

$$\mathbb{E}(V(x(t))) - V(x(t_0)) = \int_{t_0}^{t} \mathbb{E}(LV(x(s)))ds.$$

Thus, for all v, t with $0 \leq t_0 \leq v \leq t \leq \infty$, we obtain

$$\begin{aligned} 0 \leq \mathbb{E}(V(x(t))) - \mathbb{E}(V(x(v))) &\leq \int_{v}^{t} \mathbb{E}(LV(x(s)))ds \\ &\leq \int_{v}^{t} -\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2} \mathbb{E}(V(x(s)))ds. \end{aligned}$$

Thanks to Lemma 3.7, the above inequality implies that

$$\mathbb{E}(V(x(t))) \leq \mathbb{E}(V(x(t_0))) \exp\left(-\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right).$$

Now, we are in a position to derive an estimate for the expectation of the norm of $x(\cdot)$.

$$\begin{aligned} \mathbb{E}(||x(t)||^2) &\leq \frac{1}{\lambda_1} \mathbb{E}((Ex(t))^T P(Ex(t))) \\ &\leq \frac{1}{\lambda_1} \mathbb{E}(V(x(t_0))) \exp\left(-\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right) \\ &= \frac{1}{\lambda_1} x^T(t_0) E^T P E x(t_0) \exp\left(-\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right) \\ &\leq \frac{\lambda_2}{\lambda_1} ||x_0||^2 \exp\left(-\frac{q_1 - \gamma^2 \lambda_{\max}(\Pi^T \Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right). \end{aligned}$$

Finally, for all $t \geq t_0 \geq 0$, and all consistent initial conditions x_0, we have

$$\mathbb{E}(||x(t)||^2) \leq \frac{\lambda_2}{\lambda_1}||x_0||^2 \exp\left(-\frac{q_1 - \gamma^2\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right).$$

Setting, $\beta_1 = \frac{\lambda_2}{\lambda_1}$, and $\beta_2 = \frac{q_1 - \gamma^2\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}$, we close that the stochastic perturbed singular system (3.2) is uniformly exponentially stable in mean square. □

3.3 Non-vanishing Perturbation

Now, we suppose that $g(t, 0)$ is not necessarily zero. That is, the origin is no longer an equilibrium point for the stochastic perturbed singular system (3.2). To deal with this situation, we need to introduce a new notion of stability in the sense that the trajectories converge in probability, to a small neighborhood of the origin.

The study of the asymptotic behavior of solutions leads to analyze the stability behavior of a ball centered at the origin: $B_r := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}, \; r > 0$.

Definition 3.2 (i) B_r is uniformly exponentially stable in mean square, if there exist positive constants α_1 and α_2, such that for all $t_0 \in \mathbb{R}_+$, and all consistent initial conditions x_0,

$$\mathbb{E}(||x(t)||^2) \leq \alpha_1||x_0||^2 e^{-\alpha_2(t-t_0)} + r, \quad \forall t \geq t_0 \geq 0. \tag{3.14}$$

(ii) The stochastic perturbed singular system (3.2) is said to be practically uniformly exponentially stable in a mean square, if there exists $r > 0$, such that B_r is uniformly exponentially stable in mean square.

Sufficient conditions on practical uniform exponential stability in the mean square of the stochastic perturbed singular system based on the Lyapunov approach will be proved next.

Now, we consider the following assumption:

(H) The perturbation term $g(t, x)$ satisfies for all $t \geq 0$, and all $x \in \mathcal{W}_{K^*}$, the following condition:

$$||g(t, x)||^2 \leq c||x||^2 + \varphi(t), \quad \text{a.s.} \tag{3.15}$$

where c is a positive constant and φ is a nonnegative bounded continuous function.

Theorem 3.8 *Consider the stochastic perturbed singular system* (3.2)*, if there exists a positive definite symmetric matrix P, such that condition* (3.7) *holds with matrix $Q = Q^T > 0$ and Q satisfies condition* (3.9)*. Furthermore, if (H) holds, then the stochastic perturbed singular system* (3.2) *is practically uniformly exponentially stable in mean square.*

Proof We consider the following Lyapunov-like function:

$$V(x) = x^T E^T P E x,$$

which is a positive quadratic form on $\mathcal{W}_{K*}$.

Applying the Itô's formula to $V(x(.))$, where $x(.)$ is a trajectory of the stochastic perturbed singular system (3.2), we have

$$\begin{aligned} LV(x(t)) &= x^T(t)\Big(A^T PE + E^T PA\Big)x(t) + (\Pi g(t,x(t)))^T P(\Pi g(t,x(t))) \\ &\leq -x^T(t)Qx(t) + ||\Pi||^2||P||||g(t,x(t))||^2 \\ &\leq -x^T(t)Qx(t) + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)||g(t,x(t))||^2. \end{aligned}$$

Recall that, Q is a positive quadratic form on $\mathcal{W}_{K^*}\backslash\{0\}$. Then,

$$\exists q_1,\ q_2>0: \quad q_1x^Tx \leq x^TQx \leq q_2x^Tx, \quad \forall t\geq 0,\ \forall x\in \mathcal{W}_{K^*}\backslash\{0\}. \tag{3.16}$$

Using (3.15) and (4.4), we obtain

$$\begin{aligned} LV(x(t)) &\leq -q_1x^T(t)x(t) + c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)x^T(t)x(t) \\ &\quad + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\varphi(t) \\ &= -(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))x^T(t)x(t) + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\varphi(t). \end{aligned}$$

Furthermore, V is a positive quadratic form on $\mathcal{W}_{K^*}\backslash\{0\}$. Subsequently,

$$\exists\lambda_1,\ \lambda_2>0: \quad \lambda_1x^Tx \leq x^TE^TPEx \leq \lambda_2x^Tx, \quad \forall t\geq 0,\ \forall x\in \mathcal{W}_{K^*}\backslash\{0\}.$$

Without loss of generality, we may assume that

$$q_1 > c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P). \tag{3.17}$$

Then, we deduce that

$$LV(x(t)) \leq -\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}\quad V(x(t)) + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\varphi(t).$$

On the other hand, since the function $t\mapsto\varphi(t)$ is a nonnegative bounded on $[0,+\infty)$, then there exists $M>0$, such that $\varphi(t)\leq M$.

As a result, we get

$$LV(x(t)) \leq -\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}V(x(t)) + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M.$$

By using Dynkin's formula [17], it follows that

$$\mathbb{E}(V(x(t))) - V(x(t_0)) = \int_{t_0}^{t}\mathbb{E}(LV(x(s)))ds.$$

That is, for all v, t such that $0 \leq t_0 \leq v \leq t \leq \infty$, we acquire

$$
\begin{aligned}
0 \leq \mathbb{E}(V(x(t))) - \mathbb{E}(V(x(v))) &\leq \int_v^t \mathbb{E}(LV(x(s)))ds \\
&\leq \int_v^t -\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}\mathbb{E}(V(x(s))) \\
&\quad + \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)Mds.
\end{aligned}
$$

By applying Lemma 3.7, the previous inequality involve that

$$
\begin{aligned}
&\mathbb{E}(V(x(t))) \leq \mathbb{E}(V(x(t_0)))\exp(-\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)) \\
&\quad + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}.
\end{aligned}
$$

Now, we are at a point to acquire an estimate for the expectation of the norm of $x(\cdot)$.

$$
\begin{aligned}
\mathbb{E}(||x(t)||^2) &\leq \frac{1}{\lambda_1}\mathbb{E}((Ex(t))^T P(Ex(t))) \\
&\leq \frac{1}{\lambda_1}\mathbb{E}(V(x(t_0)))\exp(-\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)) \\
&\quad + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{\lambda_1(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))} \\
&= \frac{1}{\lambda_1}x^T(t_0)E^T PEx(t_0)\exp(-\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)) \\
&\quad + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{\lambda_1(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))} \\
&\leq \frac{\lambda_2}{\lambda_1}||x_0||^2\exp(-\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)) \\
&\quad + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{\lambda_1(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))}.
\end{aligned}
$$

Eventually, for all $t \geq t_0 \geq 0$, and all consistent initial conditions x_0, we procure

$$
\begin{aligned}
&\mathbb{E}(||x(t)||^2) \leq \frac{\lambda_2}{\lambda_1}||x_0||^2\exp\left(-\frac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}(t - t_0)\right) \\
&\quad + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{\lambda_1(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))}.
\end{aligned}
$$

Then, we conclude that the stochastic perturbed singular system (3.2) is practically uniformly exponentially stable in mean square, with $\alpha_1 = \frac{\lambda_2}{\lambda_1}$, $\alpha_2 = \dfrac{q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)}{\lambda_2}$, and $r = \dfrac{\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)M\lambda_2}{\lambda_1(q_1 - c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P))}$, □

4 Stabilization

In this section, we discuss the stabilization problem for a class of stochastic singular systems.

Consider the following problem:

$$\begin{cases} \bar{E}dx(t) = [\bar{A}x(t) + Hu]dt + \Pi g(t, x(t))dB_t \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $B_t \in \mathbb{R}$ is a standard Brownian motion defined on a complete probability space (Ω, F, P) with $B_0 = 0$. $\bar{E}, \bar{A} \in \mathbb{R}^{n\times n}$ are constant matrices, with $\bar{E}$ singular and $\text{rank}(\bar{E}) = r < n$, and $H \in \mathbb{R}^{n\times m}$ is a constant matrix.

We assume that $(\bar{E}, \bar{A})$ is regular and impulse free, and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies both of conditions (3.3) and (3.4).

Definition 4.1 System (4.1) is said to be uniformly exponentially stabilizable in mean square, if there exists a state feedback control law: $u = u(x)$, such that the closed-loop stochastic singular system

$$\bar{E}dx(t) = [\bar{A}x(t) + Hu(x)]dt + \Pi g(t, x(t))dB_t, \tag{4.2}$$

is uniformly exponentially stable in mean square.

Definition 4.2 System (4.1) is said to be practically uniformly exponentially stabilizable in mean square, if there exists a state feedback control law: $u = u(x)$, such that the closed-loop stochastic singular system (4.2) is uniformly exponentially practically stable in mean square.

In the sequel, we will define the feedback law as follows: $u(x) = Fx$, which stabilizes the linear part, where $F \in \mathbb{R}^{m\times n}$ is a constant matrix.

Let us state now some assumptions, which we will impose it later on:

(H_1) There exists a positive definite symmetric matrix P, solution of the following Lyapunov matrix equation:

$$(\bar{A} + HF)^T P\bar{E} + \bar{E}^T P(\bar{A} + HF) = -Q,$$

with matrix $Q = Q^T > 0$, satisfying condition (3.9).

(H_2) There exists a positive constant $\bar{\mu}_1$, such that

$$||g(t, x)|| \le \bar{\mu}_1||x||, \text{ a.s. } \forall(t, x) \in \mathbb{R}_+ \times \mathcal{W}_{K^*}.$$

Moreover, $g(t, 0) = 0$ for all $t \geq 0$.

(H_3) There exists a continuous nonnegative bounded function μ_2, such that

$$||g(t, x)|| \leq \mu_2(t), \text{ a.s. } \quad \forall (t, x) \in \mathbb{R}_+ \times \mathcal{W}_{K^\star}.$$

In addition, there exists t such that $g(t, 0) \neq 0$.

Remark 4.1 $\mathcal{W}_{K^*}$ is the subspace of consistent initial conditions corresponding to $(\bar{E}, \bar{A} + HF)$, and im $\Pi = \mathcal{W}_{K^*}$.

Now, we are in the position to state the following stabilization result.

Theorem 4.2 *Under assumptions* (H_1) *and* (H_2), *the closed-loop stochastic singular system* (4.2) *is uniformly exponentially stable in mean square.*

Proof Define the Lyapunov function in the following form,

$$V(x) = x^T \bar{E}^T P \bar{E} x, \tag{4.3}$$

which is a positive quadratic form on $\mathcal{W}_{K*}$.

Invoking Itô's formula for $V(\cdot)$ along the trajectory $x(\cdot)$ of the stochastic singular system (4.2), we get

$$LV(x(t)) = x^T(t)\Big((\bar{A} + HF)^T P \bar{E} + \bar{E}^T P(\bar{A} + HF)\Big)x(t) + (\Pi g(t, x(t)))^T P(\Pi g(t, x(t))).$$

Taking into account assumption (H_1) yields

$$LV(x(t)) \leq -x^T(t) Q x(t) + ||\Pi||^2 ||P|| ||g(t, x(t))||^2.$$

Moreover, assumption (H_2) implies

$$LV(x(t)) \leq -x^T(t) Q x(t) + \lambda_{\max}(\Pi^T \Pi) \lambda_{max}(P) \bar{\mu}_1^2 x^T(t) x(t).$$

Since, Q is a positive quadratic form on $\mathcal{W}_{K^*} \backslash \{0\}$. Then,

$$\exists q_1', \; q_2' > 0 : \quad q_1' x^T x \leq x^T Q x \leq q_2' x^T x, \quad \forall t \geq 0, \; \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \tag{4.4}$$

Consequently, we obtain

$$LV(x(t)) \leq -q_1' x^T(t) x(t) + \lambda_{\max}(\Pi^T \Pi) \lambda_{max}(P) \bar{\mu}_1^2 x^T(t) x(t).$$

On the other hand, V is a positive quadratic form on $\mathcal{W}_{K^*} \backslash \{0\}$. Subsequently,

$$\exists \lambda_1', \; \lambda_2' > 0 : \quad \lambda_1' x^T x \leq x^T \bar{E}^T P \bar{E} x \leq \lambda_2' x^T x, \quad \forall t \geq 0, \; \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \tag{4.5}$$

Hence, we see that

$$\begin{aligned} LV(x(t)) &\leq -\frac{q_1'}{\lambda_2'}(\bar{E}x(t))^T P(\bar{E}x(t)) + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_1^2}{\lambda_2'}(\bar{E}x(t))^T P(\bar{E}x(t)) \\ &= -\frac{\left(q_1' - \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\bar{\mu}_1^2\right)}{\lambda_2'} V(x(t)) = -\varrho V(x(t)), \end{aligned}$$

where $\varrho = \dfrac{(q_1' - \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\bar{\mu}_1^2)}{\lambda_2'}$.

Without loss of generality, we may assume that,

$$q_1' > \lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P)\bar{\mu}_1^2.$$

Moreover, because of Dynkin's formula [17], we have

$$\mathbb{E}(V(x(t))) - V(x(0)) = \int_0^t \mathbb{E}(LV(x(s)))ds.$$

Thus one has $\forall v, t,\ 0 \leq t_0 \leq v \leq t \leq \infty$,

$$\begin{aligned} 0 \leq \mathbb{E}(V(x(t))) - \mathbb{E}(V(x(v))) &\leq \int_v^t \mathbb{E}(LV(x(s)))ds \\ &\leq \int_v^t -\varrho\mathbb{E}(V(x(s)))ds. \end{aligned}$$

By using Lemma 3.7, it follows that

$$\mathbb{E}(V(x(t))) \leq \mathbb{E}(V(x(t_0))) \exp(-\varrho(t - t_0)).$$

Thus, taking into consideration (4.5), it follows that

$$\begin{aligned} \mathbb{E}(||x(t)||^2) &\leq \frac{1}{\lambda_1'}\mathbb{E}((Ex(t))^T P(Ex(t))) \\ &\leq \frac{1}{\lambda_1'}\mathbb{E}(V(x(t_0))) \exp(-\varrho(t - t_0)) \\ &= \frac{1}{\lambda_1'}x^T(t_0)E^T PEx(t_0) \exp(-\varrho(t - t_0)) \\ &\leq \frac{\lambda_2'}{\lambda_1'}||x_0||^2 \exp(-\varrho(t - t_0)). \end{aligned}$$

Hence, we deduce that for all $t \geq t_0 \geq 0$, and all consistent initial conditions x_0,

$$\mathbb{E}(||x(t)||^2) \leq \frac{\lambda_2'}{\lambda_1'}||x_0||^2 \exp(-\varrho(t - t_0)).$$

That is, the closed-loop stochastic singular system (4.2) is uniformly exponentially stable in mean square. □

Theorem 4.3 *Under assumptions* (H_1) *and* (H_3) *the closed-loop stochastic singular system* (4.2) *is uniformly practically exponentially stable in mean square.*

Proof We consider the Lyapunov function (4.3), and we apply the Itô's formula for $V(\cdot)$ along the trajectory $x(.)$ of the stochastic singular system (4.2).

Then, we obtain

$$LV(x(t)) = x^T(t)\Big((\bar{A}+HF)^T P\bar{E} + \bar{E}^T P(\bar{A}+HF)\Big)x(t) \\ +(\Pi g(t,x(t)))^T P(\Pi g(t,x(t))).$$

From (H_1), and (H_3), we have

$$LV(x(t)) \le -x^T(t)Qx(t) + \lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\mu_2^2(t).$$

Furthermore, since $\mu_2(t)$ is a continuous nonnegative bounded function, then there exists $\bar{\mu}_2 > 0$, such that

$$||\mu_2(t)|| \le \bar{\mu}_2, \quad \forall t \ge t_0 \ge 0.$$

Hence, we get

$$LV(x(t)) \le -x^T(t)Qx(t) + \lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2.$$

Due to (4.4) and (4.5), we obtain

$$LV(x(t)) \le -\frac{q_1'}{\lambda_2'}(\bar{E}x(t))^T P(\bar{E}x(t)) + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{\lambda_2'}.$$

From Dynkin's formula [17], we get

$$\mathbb{E}(V(x(t))) - V(x(0)) = \int_0^t \mathbb{E}(LV(x(s)))ds.$$

That is, $\forall v, t,\ 0 \le t_0 \le v \le t \le \infty$, we obtain

$$0 \le \mathbb{E}(V(x(t))) - \mathbb{E}(V(x(v))) \le \int_v^t \mathbb{E}(LV(x(s)))ds \\ \le -\frac{q_1'}{\lambda_2'}\mathbb{E}(V(x(s))) + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{\lambda_2'}ds.$$

Therefore, from the Gronwall lemma (3.7), we obtain

$$\mathbb{E}(V(x(t))) \le \mathbb{E}(V(x(t_0)))\exp\left(-\frac{q_1'}{\lambda_2'}(t-t_0)\right) + \frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{q_1'}.$$

Taking into consideration (4.5), it follows that

$$\begin{aligned}\mathbb{E}(||x(t)||^2) &\leq \frac{1}{\lambda_1'}\mathbb{E}((\bar{E}x(t))^T P(\bar{E}x(t)))\\ &\leq \frac{1}{\lambda_1'}x^T(t_0)\bar{E}^T P\bar{E}x(t_0)\exp\left(-\frac{q_1'}{\lambda_2'}(t-t_0)\right)+\frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{q_1'\lambda_1'}\\ &\leq \frac{\lambda_2'}{\lambda_1'}||x_0||^2\exp\left(-\frac{q_1'}{\lambda_2'}(t-t_0)\right)+\frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{q_1'\lambda_1'}.\end{aligned}$$

Finally, for all $t \geq t_0 \geq 0$, and all consistent initial conditions x_0, we obtain

$$\mathbb{E}(||x(t)||^2) \leq \frac{\lambda_2'}{\lambda_1'}||x_0||^2\exp\left(-\frac{q_1'}{\lambda_2'}(t-t_0)\right)+\frac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{q_1'\lambda_1'}.$$

Setting, $\alpha_1 = \frac{\lambda_2'}{\lambda_1'}$, $\alpha_2 = q_1'\lambda_2'$, and $r = \dfrac{\lambda_{\max}(\Pi^T\Pi)\lambda_{max}(P)\bar{\mu}_2^2}{q_1'\lambda_1'}$, we close that the closed-loop stochastic singular system (4.2) is practically uniformly exponentially stable in mean square.

Thus, proving the theorem. □

Remark 4.4 Based on the discussion of Theorem 3.8, the process of analyzing the practical stability of the perturbed singular system rests on the next several steps:

Step 1:If the pair (E, A) is regular and impulse–free, we should find the set of consistent initial conditions of the linear singular system (2.1).
Step 2:Solve the generalized Lyapunov equation (3.7) and obtain q_1 and $\lambda_{\max}(P)$ by a simple calculation.
Step 3: For a given scalar c, q_1, $\lambda_{\max}(\Pi^T\Pi)$ and $\lambda_{\max}(P)$, consider assumption (3.15) and inequality (3.17).
Step 4: End.

5 Example

We provide the following illustrative example to show the applicability of our main result.

Example 5.1 Consider the following stochastic singular system:

$$Edx(t) = Ax(t)dt + \Pi g(t, x(t))dB_t, \tag{5.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

$$E = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 & 0\\ 0 & -1 & 0\\ -1 & 0 & -1\end{pmatrix}, \quad g(t,x) = \begin{pmatrix} g_1(t,x)\\ g_2(t,x)\\ g_3(t,x)\end{pmatrix},$$

with,

$$\begin{cases} g_1(t,x) = \frac{1}{16}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2+x_3^2}} + \frac{1}{4}\cos(2\pi t) \\ g_2(t,x) = \frac{1}{16}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2+x_3^2}} \\ g_3(t,x) = \frac{1}{8}\frac{x_3^2}{1+\sqrt{x_1^2+x_2^2+x_3^2}} + \frac{1}{\sqrt{2}^3}e^{-t}. \end{cases}$$

System (5.1) can be regarded as a singular perturbed system of

$$Edx(t) = Ax(t)dt. \tag{5.2}$$

One has, $det(zE - A) = (z+1)^2 \neq 0$ for some $z \in \mathbb{C}$. Moreover, $degree(det(zE - A)) = rank(E) = 2$. That is, the singular nominal system (5.1) is regular and impulse-free.

(1) Consistent initial conditions with the method of Campbell:

$$\widehat{E} = (\lambda E + A)^{-1}_{|\lambda=0}.\, E.$$

$$\widehat{E} = A^{-1}E = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of matrix $\widehat{E}$ are:

$$\sigma(\widehat{E}) = \{0,\ -1,\ -1\}.$$

Then with the method of Campbell [7], $\widehat{E}^D$ is given by

$$\widehat{E}^D = \widehat{E}^2(3\mathbb{I} + 2\widehat{E}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

$$\widehat{E}\widehat{E}^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} \aleph(\mathbb{I} - \widehat{E}\widehat{E}^D) &= (\mathbb{I} - \widehat{E}\widehat{E}^D)x_0 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_0 = 0. \end{aligned}$$

Then, we obtain

$$\aleph(\mathbb{I} - \widehat{E}\widehat{E}^D) = \mathcal{W}_{K^*} = \{x \in \mathbb{R}^3 : x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 \in \mathbb{R}\}. \tag{5.3}$$

(2) Stability analysis:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} = P^T, \quad Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} = Q^T.$$

$$\begin{aligned} A^T PE + E^T PA &= \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2p_{11} - 2p_{13} & -p_{11} - p_{12} & -2p_{13} - p_{33} \\ -p_{11} - p_{12} & 0 & -p_{13} - p_{23} \\ -2p_{13} - p_{33} & -p_{13} - p_{23} & -2p_{33} \end{pmatrix} = -Q. \end{aligned}$$

It is apparent that $q_{22} = 0$.

Let $p_{12} = p_{23} = p_{13} = 0$, then the matrix Q is in the following form:

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & 0 & 0 \\ q_{13} & 0 & q_{33} \end{pmatrix}$$

where, $q_{11} = 2p_{11} \neq 0, q_{12} = p_{11} \neq 0, q_{13} = p_{33} \neq 0,\ q_{33} = 2p_{33} \neq 0$.

We can adopt,

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = P^T, \quad Q = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} = Q^T,$$

and we compute

$$\begin{aligned} x^T Qx &= (x_1\ x_2\ x_3) \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= (x_1^2 + x_1x_2 + x_1x_3 + x_3^2)_{x_2=0}, \\ &= x_1^2 + x_1x_3 + x_3^2 > 0, \quad \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} V(x) = x^T E^T PEx &= (x_1\ x_2\ x_3) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}x_3^2 > 0, \quad \forall x \in \mathcal{W}_{K^*} \backslash \{0\}. \end{aligned}$$

Therefore, $V(x)$ can be used as a Lyapunov function for the system (5.1).

Based on the set of consistent initial condition (5.3) found out, we may choose Π in the following form:

$$\Pi = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{25} \\ 0 & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{40} \end{pmatrix}.$$

By using MATLAB, we can find $q_1 = \frac{1}{2},\ q_2 = \frac{3}{2}$, and $\lambda_{\max}(\Pi^T \Pi) = 0.12$.

On the other hand, we have

$$\Pi g(t,x) = \begin{pmatrix} \frac{1}{4}g_1(t,x) + \frac{1}{8}g_2(t,x) + \frac{1}{25}g_2(t,x) \\ 0 \\ \frac{1}{16}g_1(t,x) + \frac{1}{4}g_2(t,x) + \frac{1}{40}g_3(t,x) \end{pmatrix}$$

and,

$$||g(t,x)||^2 = g_1^2(t,x) + g_2^2(t,x) + g_3^2(t,x)$$

where, $||.||$ represents the Euclidean norm.

Based on the fact that, $(a+b)^n \le 2^{n-1}(a^n + b^n)$, for all $a, b \ge 0$, $\mathrm{n} \ge 1$, one can get:

$$||g(t,x)||^2 \le \frac{1}{64}(x_1^2 + x_2^2 + x_3^2) + \frac{1}{2}(e^{-2t} + \cos^2(2\pi t)).$$

Consequently, for all $t \ge 0$, and all $x \in \mathcal{W}_{K^*}$, we get

$$||g(t,x)||^2 \le \frac{1}{64}\left(x_1^2 + x_3^2\right) + \frac{1}{2}(e^{-2t} + \cos^2(2\pi t)).$$

That is, the constants in Theorem 3.8 become $q_1 = \lambda_{\max}(P) = \frac{1}{2}$, $c = \frac{1}{64}$, $\lambda_{\max}(\Pi^T\Pi) = 0.12$, and $\varphi(t) = \frac{1}{2}(e^{-2t} + \cos^2(2\pi t))$. It is easy to check that, $c\lambda_{\max}(\Pi^T\Pi)\lambda_{\max}(P) < q_1$ and $\varphi(t)$ is a nonnegative bounded function.

Thus, all conditions of Theorem 3.8 are satisfied. Hence, we deduce that the perturbed singular system (5.1) is practically uniformly exponentially stable in mean square (Fig. 1).

6 Conclusion

In this paper, stability for certain classes of stochastic perturbed singular systems has been developed. Some stability criteria have been established. Using the generalized Gronwall inequality and Lyapunov techniques, the notion of exponential stability as well as practical exponential stability in mean square have been studied for linear time-invariant singular systems under Brownian motion perturbations. In this context, some sufficient conditions are presented to ensure the desired stability property. An example has been introduced to validate the developed methods.

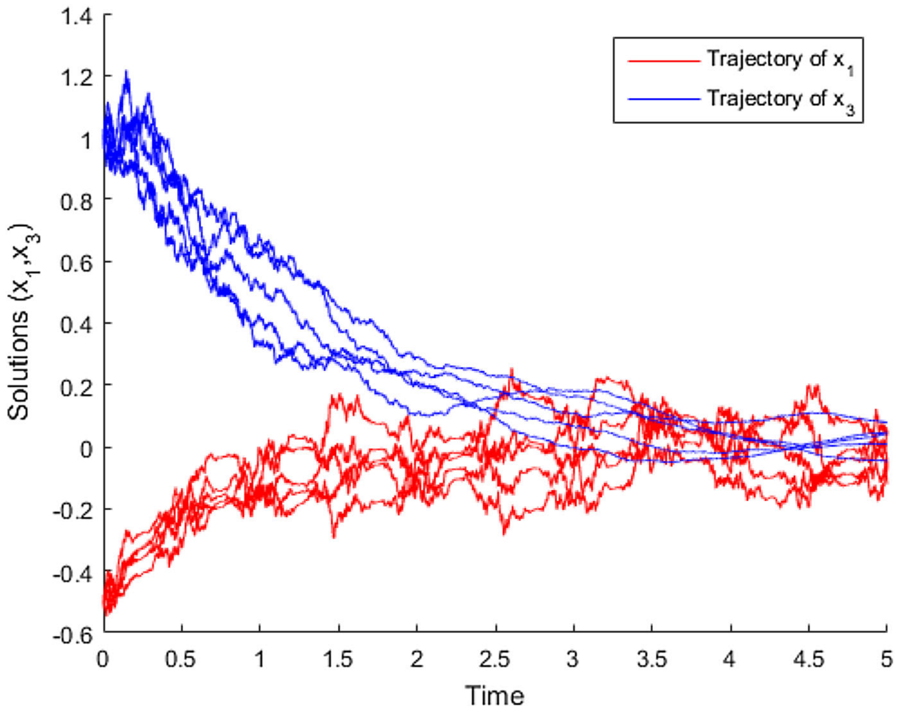

Fig. 1 The initial response of the system (5.1), with 3 different Brownian motions, and $x_0 = [-\frac{1}{2},\ 0,\ 1]^T$

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