# On the Categorical Theory of Persistence Modules

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*This thesis symbolises an important step in my journey through mathematics. I want to thank my advisor, Rocío González Díaz, for introducing me to the field of topological data analysis and for her invaluable guidance and encouragement through this last year. I cannot overstate the help of Manuel Soriano Trigueros, my unofficial coadvisor, whose insight and helpful comments have helped me shape this thesis. Both Rocío and Manuel have provided an important and constant support that has made this an exciting and delightful endeavour.*

*I am most thankful to my parents, whose unceasing support and time has made everything possible for me. Finally, I thank Andrea, for her encouragement and companionship.*

# Resumen

¿Cómo se infieren formas a partir de datos? Se pueden emplear técnicas del álgebra y la topología computacional para responder a esta pregunta, dando lugar al análisis topológico de datos (TDA), un campo que experimenta actualmente un gran crecimiento. La homología persistente es una herramienta clave en TDA que computa características topológicas de un espacio en diferentes resoluciones espaciales a partir de la construcción de complejos simpliciales y su caracterización. Esta herramienta tiene aplicaciones en varias áreas de la matemáticas aplicada y una base teórica sólida formalizada en el marco de la teoría de categorías.

En esta tesis hacemos una revisión de la teoría en torno a uno de los pilares centrales de la homología persistente, los módulos de persistencia. Examinamos la descomposición y comparación de estos y teoremas de estabilidad relacionados. En este proceso, nos centramos en buscar el enfoque matemático más útil para expresar estas ideas, que quedan cohesionados por la teoría de categorías. Además de la literatura revisada, también aportamos una serie de contribuciones propias en materia de una construcción particular de módulos de persistencia denominados módulos escalera (*ladder modules*). Asimismo, presentamos este trabajo de forma lo más autocontenida posible, para que sirva de introducción fluida tanto a esta vertiente de la teoría de categorías como a los conceptos de homología persistente aquí empleados.

**Palabras clave:** Homología persistente, teoría de categorías, análisis topológico de datos, módulos de persistencia, módulos escalera

# Abstract

How is shape infered from data? Algebraic and topological techniques can be employed computationally to answer this question, giving birth to the fast growing field of topological data analysis (TDA). Persistent homology is a key tool in TDA that computes topological features of a space at different spatial resolutions from the construction of simplicial complexes and their characterization. It has both a broad applicability in various areas of applied mathematics and a strong theoretical base that has been formalized in the framework of category theory.

In this thesis we review the theory around one of persistent homology's key elements: persistence modules. We examine their decomposition and comparison and related stability theorems. Along the way, we focus on finding the most useful mathematical language to convey these ideas, which end up being glued by category theory. In addition to the reviewed literature on persistent homology, we also add some contributions of our own regarding ladder modules, a particular construction of persistence modules. Furthermore, we present this work in a self-contained way, to serve as a painless introduction both to this side of category theory and to persistent homology.

**Keywords:** Persistent homology, category theory, topological data analysis, persistence modules, ladder modules

# Contents

# 1 | Introduction

The purpuse of this work is twofold. Firstly, we review existing literature around persistence modules, an algebraic generalization of persistent homology which is a fundamental tool in topological data analysis. We focus our attention especially on ladder modules. Secondly, we approach this review from the perspective of category theory, formalizing many concepts under a common framework. The reason to do so is similar to the origins of category theory itself, grounded in the pursuit to formalize the intuitions behind geometric homology into what is now known as the field of homological algebra. Moreover, we tried to make the chapter devoted to category theory as self contained as possible, to allow a fast introduction for the reader unfamiliar with this theory.

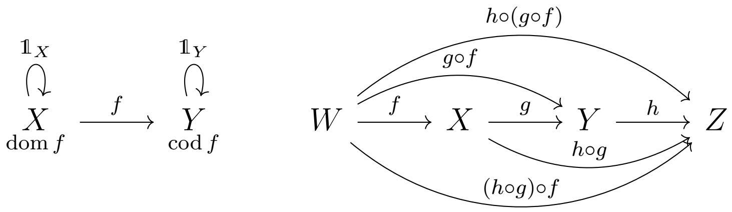
The thesis is structured as follows. Chapter 2 introduces some concepts and terminology of category theory, used throughout the rest of the work. Chapter 3 introduces the field of persistent homology from the categorical perspective. Here different definitions of ladder modules are recalled, and we propose a new definition and prove the equivalence to the former ones. We finish the chapter focusing on filtrations and proving an existence theorem of ladder modules arising from Vietoris-Rips filtrations. Chapter 4 discusses the building blocks of persistence modules, the intervals, and when and how certain classes of persistence modules can be decomposed into them according to the literature. Chapter 5 reviews some distances to compare persistence modules with each other and stability theorems that relate those distances. Lastly, we give some concluding remarks and ideas to develop in future work in Chapter 6.

Even though this work is mainly a bibliographical review, we add some of our own contributions scattered throughout the work. We will therefore denote all cited definitions and results from the original source, to make as clear as possible the distinction between cited and original content.

# 2 | Category Theory

Here we will present the notions of category theory that appear throughout this work. There are many alternative definitions of these concepts, so we have adapted the cited definitions (mainly from [1, 34]) to a cohesive notation, and the citations are thus not verbatim. Non-referenced proofs and examples are understood to be our own.

**Definition 2.0.1 (Category [1, Def 3.1]).** *A* category $\mathbf{C}$ *consists of a class of objects* $\mathbf{C}_0$*, and for each pair of objects* $X, Y \in \mathbf{C}_0$*, a set of morphisms or arrows* $\mathbf{C}(X,Y) \in \mathbf{C}_1$ *called the* homset*, which may be empty. Given a morphism* $f \in \mathbf{C}(X,Y)$*,* $X$ *and* $Y$ *are referred to as the domain and codomain of* $f$*, respectively. For each pair of morphisms* $f \in \mathbf{C}(X,Y)$ *and* $g \in \mathbf{C}(Y,Z)$ *there is a composition morphism* $g \circ f \in \mathbf{C}(X,Z)$*, which is associative: Given* $W, X, Y, Z \in \mathbf{C}_0$*,* $f \in \mathbf{C}(W,X)$*,* $g \in \mathbf{C}(X,Y)$ *and* $h \in \mathbf{C}(Y,Z)$*, we have* $(f \circ g) \circ h = f \circ (g \circ h)$*. For each object* $X \in \mathbf{C}_0$*, there is an unique identity morphism* $\mathbb{1}_X$ *(or* $\mathrm{id}_X$*) in* $\mathbf{C}(X,X)$*. The identity satisfies* $\mathbb{1}_X \circ f = f$ *and* $g \circ \mathbb{1}_X = g$ *for all* $f \in \mathbf{C}(W,X)$ *and* $g \in \mathbf{C}(X,Y)$*.*




(2.1)

Figure 2.1: *Diagrams* play a central role in category theory, shifting the focus from objects to morphisms.

Usually a morphism $f \in \mathbf{C}(X,Y)$ is expressed as $f \colon X \to Y$. We usually draw dotted arrows as in $X \dashrightarrow Y$ to emphasize their uniqueness. We will abuse notation and write objects as $X \in \mathbf{C}$ and arrows $f \in \mathbf{C}$, dropping the subscript and $\mathrm{Hom}(X,Y)$ instead of $\mathbf{C}(X,Y)$ when the category is understood to be $\mathbf{C}$. When the context is clear, composition will be written as $gf$, dropping the $\circ$ operator.

*Example* 2.0.2. Some examples of categories relevant to this work are:

- **Set**, of sets and functions between sets. Identity morphisms $\mathbb{1}_S \colon S \to S$ are the functions $\mathbb{1}_S(s) = s \; \forall s \in S$. Composition is defined by standard function

composition: given $f\colon A \to B$ and $g\colon B \to C$, then $g \circ f\colon A \to C$ is defined as $g \circ f(a) = g(f(a))\ \forall a \in A$.
- **Ab**, of abelian groups and group homomorphisms.
- **Vec**, of vector spaces and linear maps.
- **FinVec**, of finite vector spaces and linear maps.
- **SCpx**, of simplicial complexes and simplicial maps (cf. Definition 3.1.1).
- **Top**, of topological spaces and continuous maps. The identity map is continuous, and composition of continuous maps too.

*Remark* 2.0.3. The word *class* is used in category theory to talk about collections of sets. This is usually introduced in the foundations [34, Sec.I.6] [1, Sec.2.2] to overcome Russell's paradox in set theory. A *small class* is a set (that has a cardinality), and a *proper* or *large class* is a collection bigger than a set. A *small category* is a category with a set of objects and a set of morphisms.

**Definition 2.0.4 (Subcategory [34, Sec.I.3]).** *A* subcategory $\mathbf{D}$ *of a category* $\mathbf{C}$ *is a collection of some objects and some arrows of* $\mathbf{C}$*. For each arrow* $f \in \mathbf{D}_1$*,* $\operatorname{dom} f$ *and* $\operatorname{cod} f$ *are in* $\mathbf{D}_0$*. For each object* $X \in \mathbf{D}_0$*,* $\mathbb{1}_X$ *is in* $\mathbf{D}_1$*, and for each pair of composable arrows* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *in* $\mathbf{D}$*,* $g \circ f$ *is in* $\mathbf{D}_1$ *too. These conditions ensure that* $\mathbf{D}$ *is itself a category. For any two objects* $X, Y \in \mathbf{D}_0$*, in general we have* $\mathbf{D}(X,Y) \subsetneq \mathbf{C}(X,Y)$*. We say that* $\mathbf{D}$ *is a* full subcategory *of* $\mathbf{C}$ *if* $\mathbf{D}(X,Y) = \mathbf{C}(X,Y)$*, i.e. all morphisms between* $X$ *and* $Y$ *in* $\mathbf{C}_1$ *are in* $\mathbf{D}_1$ *too.*

*Example* 2.0.5. **FinVec** is a full subcategory of **Vec** because it contains only some objects of **Vec** (only the finite-dimensional vector spaces), but all linear maps between finite-dimensional vector spaces are contained in **FinVec**.

**Definition 2.0.6 (Initial, terminal and zero objects, zero arrow [34, Sec.I.5]).** *An object* $i$ *is* initial *in* $\mathbf{C}$ *of to each object* $a$ *there is exactly one arrow* $i \to a$*. An object* $t$ *is* terminal *in* $\mathbf{C}$ *of to each object* $a$ *there is exactly one arrow* $a \to t$*. A* zero object $0$ *in* $\mathbf{C}$ *is an object which is both initial and terminal. The* zero arrow *between two objects* $a, b$ *of* $\mathbf{C}$ *is the unique composite through z:* $a \to 0 \to b$*.*

*Example* 2.0.7. In **Vec**, the initial and terminal objects are both the zero-dimensional vector space 0, as there is only one linear map sending 0 to the zero vector in any vector space, and exactly one linear map sending all elements of a vector space to 0. Therefore 0 is the zero object of **Vec**.

**Definition 2.0.8 (Epimorphism [34, Sec.I.5]).** *A morphism* $f\colon X \to Y$ *in some category* $\mathbf{C}$ *is called* epimorphism *(or just* epi*) if for every other object* $Z \in \mathbf{C}$ *and every pair of morphisms* $g_1, g_2\colon Y \to Z$ *then* $g_1 \circ f = g_2 \circ f$ *implies* $g_1 = g_2$*. It is a generalization of surjective functions between sets.*

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

**Definition 2.0.9 (Monomorphism [34, Sec.I.5]).** *A morphism* $f\colon X \to Y$ *in some category* $\mathbf{C}$ *is called* monomorphism *(or just* mono*) if for every other object* $Z \in \mathbf{C}$

*and every pair of morphisms $g_1, g_2 \colon Z \to X$ then $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. It is a generalization of injective maps between sets. We sometimes emphasize that a morphism is mono drawing it with a hook as in*

$$Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \overset{f}{\hookrightarrow} Y$$

The above definition can be restated to emphasize the *universal property* of a monomorphism: There is at most one morphism $g \colon Z \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & Y \\ {\scriptstyle g}\uparrow & \nearrow & \\ Z & & \end{array} \tag{2.2}$$

**Definition 2.0.10 (Functor [34, Sec.I.3]).** *Let $\mathbf{C}, \mathbf{D}$ be categories. A* functor $F \colon \mathbf{C} \to \mathbf{D}$ *consists of a a map $F \colon \mathbf{C}_0 \to \mathbf{D}_0$, and for each pair $X, Y \in \mathbf{C}_0$, a map $F \colon \mathbf{C}(X,Y) \to \mathbf{D}(F(X), F(Y))$. These maps are compatible with composition and identity: for $f \colon X \to Y$ and $g \colon Y \to Z$ we have $F(gf) = F(g)F(f)$, and $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ for $X \in \mathbf{C}_0$.*

**Definition 2.0.11 (Natural transformation [34, Sec.I.4]).** *Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be functors. A* natural transformation $\eta \colon F \Rightarrow G$ *consists of, for all $X \in \mathbf{C}_0$, a morphism $\eta_X \colon F(X) \to G(X)$ in $\mathbf{D}$, called the* component *of $\eta$ for $X$, such that for all morphisms $f \colon X \to Y$ the diagram*

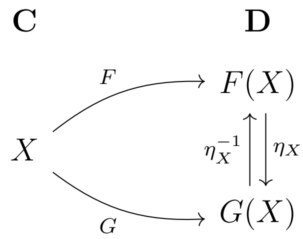
$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

*commutes, which is to say that $\eta_Y \circ F(f) = G(f) \circ \eta_X$. The above diagram is also drawn as follows:*

$$\begin{array}{ccc} F(X) & & G(X) \\ {\scriptstyle F(f)}\downarrow & \overset{\eta}{\Rrightarrow} & \downarrow{\scriptstyle G(f)} \\ F(Y) & & G(Y), \end{array} \qquad \text{or} \qquad \mathbf{C} \underset{G}{\overset{F}{\rightrightarrows}} \!\!\Downarrow\!\eta\ \mathbf{D}.$$

*The set of natural transformations from $F$ to $G$ is denoted $\mathbf{D}^{\mathbf{C}}(F, G)$.*

**Definition 2.0.12 (Natural isomorphism [34, Sec.I.4]).** *Given two functors $F, G \colon \mathbf{C} \to \mathbf{D}$, a* natural isomorphism *between $F$ and $G$ is a natural transformation $\eta \colon F \Rightarrow G$ with every component $\eta_X$ invertible in $\mathbf{D}$. That is, $\eta_X \colon F(X) \to G(X)$ has an inverse arrow $\eta_X^{-1} \colon G(X) \to F(X)$. We write $\eta \colon F \cong G$, and the inverses $(\eta_X)^{-1}$ in $\mathbf{D}$ are the* *components of a natural isomorphism* $\eta^{-1}\colon G \Rightarrow F$.




**Definition 2.0.13 (Equivalence [34, Sec.I.4]).** *An* equivalence *between two categories* $\mathbf{C}$, $\mathbf{D}$ *is defined as a pair of functors* $F\colon \mathbf{C} \to \mathbf{D}$, $G\colon \mathbf{D} \to \mathbf{C}$ *together with natural isomorphisms* $\mathbb{1}_{\mathbf{C}} \cong G \circ F$, $\mathbb{1}_{\mathbf{D}} \cong F \circ G$.

**Definition 2.0.14 (Preordered set as category [34, Sec.I.2]).** *A preorder is a category* $\mathbf{P}$ *where, for any two objects* $a, b \in \mathbf{P}$ *there is at most one arrow* $a \to b$. *This arrow is the binary relation* $a \leq b$ *which is reflexive (there is always the identity arrow* $a \leq a$*) and transitive (composition:* $a \leq b \leq c$ *implies* $a \leq c$*).*

*Example* 2.0.15. The reals $(\mathbb{R}, \leq)$ are in particular a preorder (ignoring the antisymmetry condition that $a \leq b \leq a$ implies $a = b$), so it is a category, where the objects are real numbers, and morphisms are inequalities: for any two reals $a, b$ the homset $\mathbb{R}(a, b)$ contains one map $a \leq b$. Composition is the transitivity of inequalities:

$$\begin{aligned} \mathbb{R}(b, c) \times \mathbb{R}(a, b) &\to \mathbb{R}(a, c) \\ b \leq c, \quad a \leq b \quad &\mapsto a \leq c \end{aligned}$$

For every $a \in \mathbb{R}$ there is an identity morphism $\mathbb{1}_a \in \mathbb{R}(a, a)$, namely $a \leq a$, such that for every $f \in \mathbb{R}(a, b)$, it holds that $f \circ \mathbb{1}_a = f$, and similarly $\mathbb{1}_a \circ g = g$ for every $g \in \mathbb{R}(c, a)$. Associativity is also immediate: for every $a, b, c, d \in \mathbb{R}$ where $f\colon a \leq b$, $g\colon b \leq c$, $h\colon c \leq d$:

$$\begin{aligned} h \circ (g \circ f) &= (h \circ g) \circ f \\ a \leq c \leq d \quad &\quad a \leq b \leq d \end{aligned}$$

*Example* 2.0.16. A finite set $\{k \in \mathbb{Z}_+ \mid k < n\}$ for some positive integer $n$ can be regarded as a category $\mathbf{n}$, where the generating morphisms connect consecutive elements:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

with the usual composition and identity. We choose not to include 0 so we can say that $\mathbf{n}$ has exactly $n$ objects. It should not be confused with the discrete category of $n$ elements, where the only morphisms are the identity morphisms:

$$1 \qquad 2 \qquad \cdots \qquad n$$

**Definition 2.0.17 (Functor category, diagram [34, Sec.II.4]).** *Let* $\mathbf{C}, \mathbf{J}$ *be categories. The* functor category $\mathbf{C}^{\mathbf{J}}$ *is the category whose objects are functors* $F\colon \mathbf{J} \to \mathbf{C}$*, and morphisms are natural transformations between these functors. A functor* $F\colon \mathbf{J} \to \mathbf{C}$ *with* $\mathbf{J}$ *(usually very) small*[(1)] *is called a* $\mathbf{J}$-indexed diagram in $\mathbf{C}$*. We also talk about* the diagram $F$ in $\mathbf{C}^{\mathbf{J}}$*. Here* $\mathbf{C}$ *is denoted the target category, and* $\mathbf{J}$ *the indexing category.*

*Example* 2.0.18. A $(\mathbb{Z}_+, \leq)$-indexed diagram $F$ in a category $\mathbf{C}$ is a sequence of objects of $\mathbf{C}$ connected by morphisms:

$$F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots$$

Likewise, a diagram $F \in \mathbf{C}^{(\mathbb{R},\leq)}$ has objects $F_a$ for all $a \in \mathbb{R}$ and for each $a \leq b$, a morphism $F_a \to F_b$.

The term *diagram* is also used to talk about the image of a certain diagram in the above sense, that is, some objects of a category $\mathbf{C}$ connected by some morphisms of $\mathbf{C}$, like (2.1) or (2.2).

**Definition 2.0.19 (Arrow category [4, 1.6.3]).** *For* $\mathbf{C}$ *any category, its* arrow category*, denoted* $\mathrm{Arr}(\mathbf{C})$ *or* $\mathbf{C}^{\mathbf{2}}$*, has arrows (morphisms) of* $\mathbf{C}$ *as objects, and an arrow* $g$ *from* $f\colon A \to B$ *to* $f'\colon A' \to B'$ *in* $\mathrm{Arr}(\mathbf{C})$ *is a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ B & \xrightarrow{g_2} & B' \end{array}$$

*where* $g_1$ *and* $g_2$ *are arrows in* $\mathbf{C}$ *such that* $g_2 \circ f = f' \circ g_1$*. The identity arrow* $\mathbb{1}_f$ *on an object* $f\colon A \to B$ *is the pair* $(\mathbb{1}_A, \mathbb{1}_B)$*. Composition of arrows is done componentwise:*

$$(h_1, h_2) \circ (g_1, g_2) = (h_1 \circ g_1, h_2 \circ g_2)$$

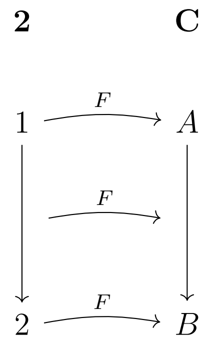
*Remark* 2.0.20. One might wonder why the arrow category of $\mathbf{C}$ is denoted $\mathbf{C}^{\mathbf{2}}$. This is because it is a special case of a functor category. Indeed, a functor $F$ from $\mathbf{2}$ (the category containing two objects and a morphism $1 \to 2$) to $\mathbf{C}$ sends

- the object 1 in $\mathbf{2}$ to an object $A$ in $\mathbf{C}$.
- the object 2 in $\mathbf{2}$ to an object $B$ in $\mathbf{C}$.
- the morphism $1 \to 2$ in $\mathbf{2}$ to a morphism $A \to B$ in $\mathbf{C}$.

so it effectively describes two objects of $\mathbf{C}$ and a morphism between them. The category $\mathbf{C}^{\mathbf{2}}$ has *as objects* these functors $F$, or more pedantically, the image of these functors,

[(1)] Recall Remark 2.0.3. We say informally that $\mathbf{J}$ is *very* small to emphasize that most of the time $\mathbf{J}$ contains no more than a handfull of objects.

i.e. diagrams $A \to B$ in $\mathbf{C}$.




**Definition 2.0.21 (Adjunction [43, Def.4.1.1]).** *An* adjunction *between categories* $\mathbf{C}$ *and* $\mathbf{D}$*, denoted* $\mathbf{C} \rightleftharpoons \mathbf{D}$*, consists of a pair of functors* $F\colon \mathbf{C} \to \mathbf{D}$ *and* $G\colon \mathbf{D} \to \mathbf{C}$ *together with an isomorphism*

$$\mathbf{D}(Fc, d) \cong \mathbf{C}(c, Gd)$$

*for each* $c \in \mathbf{C}$ *and* $d \in \mathbf{D}$ *that is* natural *in both variables. Here* $F$ *is the* left adjoint to $G$ *and* $G$ *is the* right adjoint to $F$.

**Definition 2.0.22 (Product category, bifunctor [34, Sec.II.3]).** *For two categories* $\mathbf{C}$ *and* $\mathbf{D}$*, the* product category $\mathbf{C} \times \mathbf{D}$ *is the category whose objects are ordered pairs* $(c, d)$ *with* $c$ *an object of* $\mathbf{C}$ *and* $d$ *an object of* $\mathbf{D}$*. Morphisms are ordered pairs* $(f, g) : (c, d) \to (c', d')$*, and composition is defined componentwise:*

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$$

*A functor* $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ *from a product category* $\mathbf{A} \times \mathbf{B}$ *is called a* bifunctor.

*Proposition* 2.0.23. For categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, there is a natural isomorphism $\mathbf{C}^{\mathbf{A}\times\mathbf{B}} \cong (\mathbf{C}^{\mathbf{B}})^{\mathbf{A}}$.

*Proof.* This is a standard exercise in Category Theory. See for example [34, II.5 Ex.2]. For a natural isomorphism to exist, we define two functors

$$\varphi\colon \mathbf{C}^{\mathbf{A}\times\mathbf{B}} \to (\mathbf{C}^{\mathbf{B}})^{\mathbf{A}}$$
$$\psi\colon (\mathbf{C}^{\mathbf{B}})^{\mathbf{A}} \to \mathbf{C}^{\mathbf{A}\times\mathbf{B}}$$

- $\varphi$ acting on objects of $\mathbf{C}^{\mathbf{A}\times\mathbf{B}}$: Given $T\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$, $\varphi(T)\colon \mathbf{A} \to \mathbf{C}^{\mathbf{B}}$ maps
  - objects $a \in \mathbf{A}$ to functors $T(a, -)\colon \mathbf{B} \to \mathbf{C}$.
  - morphisms $f\colon a \to a'$ to natural transformations $\varphi(T)(f)\colon \mathbf{C}^{\mathbf{B}} \to \mathbf{C}^{\mathbf{B}}$ with components $\varphi(T)(f)_b\colon T(a, b) \to T(a', b)$ such that for every functor $g\colon b \to$

$b'$ in $\mathbf{B}$, the following diagram commutes:

$$\begin{array}{c} b \\ \Big\downarrow g \\ b' \end{array} \qquad \begin{array}{ccc} \varphi(T)(a)(b) & \xrightarrow{\varphi(T)(f)_b} & \varphi(T)(a')(b) \\ {\scriptstyle \varphi(T)(a)(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi(T)(a')(g)} \\ \varphi(T)(a)(b') & \xrightarrow{\varphi(T)(f)_{b'}} & \varphi(T)(a')(b') \end{array} \qquad \begin{array}{ccc} T(a,b) & \xrightarrow{T(f,b)} & T(a',b) \\ {\scriptstyle T(a,g)}\Big\downarrow & & \Big\downarrow{\scriptstyle T(a',g)} \\ T(a,b') & \xrightarrow{T(f,b')} & T(a',b') \end{array}$$

- $\varphi$ acting on morphisms of $\mathbf{C}^{\mathbf{A}\times\mathbf{B}}$: Given $S, T\colon \mathbf{A}\times\mathbf{B} \to \mathbf{C}$, $\varphi$ maps the morphism $\eta\colon S \to T$ to a natural transformation $\varphi(\eta)$ with components

$$\varphi(\eta)_a = \eta(a, -)\colon \varphi(S)(a) \to \varphi(T)(a)$$

such that the following diagram commutes:

$$\begin{array}{c} a \\ \Big\downarrow f \\ a' \end{array} \qquad \begin{array}{ccc} \varphi(S)(a) & \xrightarrow{\varphi(\eta)_a} & \varphi(T)(a) \\ {\scriptstyle \varphi(S)(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi(T)(f)} \\ \varphi(S)(a') & \xrightarrow{\varphi(\eta)_{a'}} & \varphi(T)(a') \end{array} \qquad \begin{array}{ccc} S(a,-) & \xrightarrow{\eta(a,-)} & T(a,-) \\ {\scriptstyle S(f,-)}\Big\downarrow & & \Big\downarrow{\scriptstyle T(f,-)} \\ S(a',-) & \xrightarrow{\eta(a',-)} & T(a',-) \end{array}$$

- $\psi$ acting on objects of $(\mathbf{C}^{\mathbf{B}})^{\mathbf{A}}$: Given $T\colon \mathbf{A} \to \mathbf{C}^{\mathbf{B}}$, $\psi(T)\colon \mathbf{A}\times\mathbf{B} \to \mathbf{C}$ maps

  - objects $(a, b) \in \mathbf{A}\times\mathbf{B}$ to objects $\psi(T)((a,b)) = T(a)(b)$.
  - pairs of morphisms $(f, g)\colon (a,b) \to (a',b')$ in $\mathbf{A}\times\mathbf{B}$ to a morphism

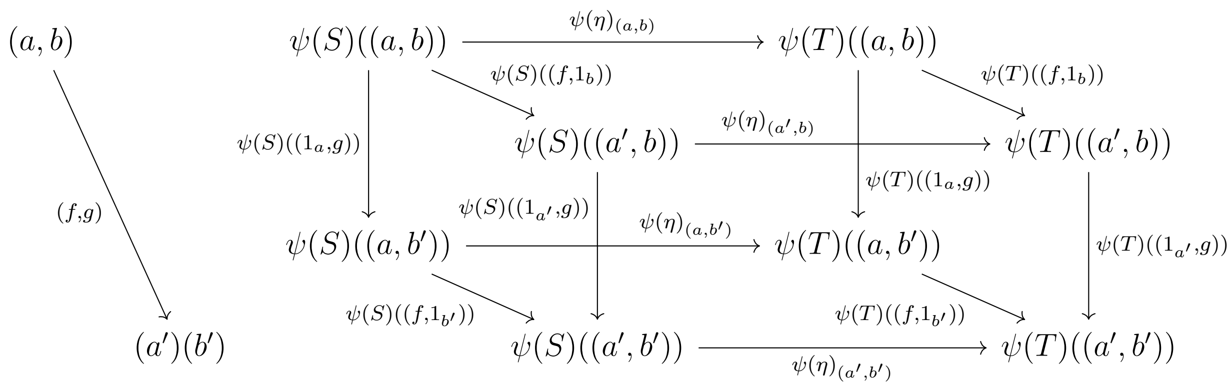
$$\psi(T)((f,g)) = T(f)_{b'} \circ T(a)(g)\colon \psi(T)((a,b)) \to \psi(T)((a',b')) \tag{2.3}$$

  which we will name $T(f)(g)$ for brevity.

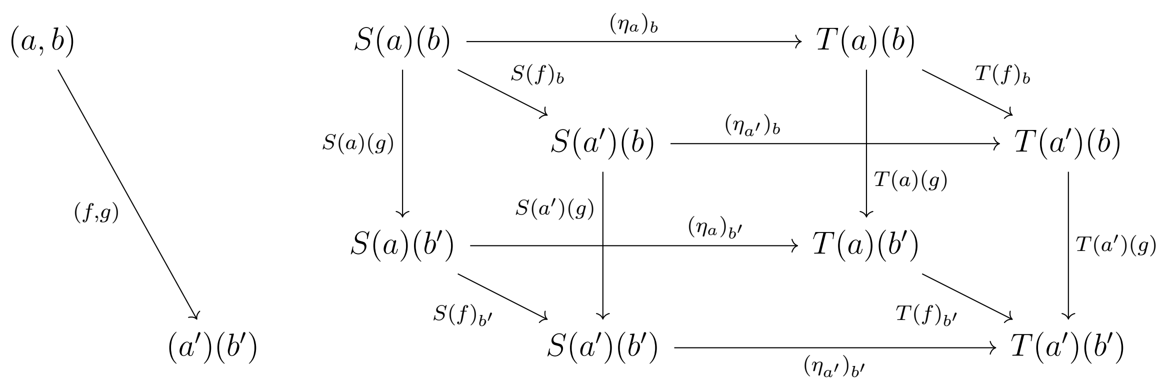
- $\psi$ acting on morphisms of $(\mathbf{C}^{\mathbf{B}})^{\mathbf{A}}$: Given $S, T\colon \mathbf{A} \to \mathbf{C}^{\mathbf{B}}$, $\psi$ maps the morphism $\eta\colon S \to T$ to a natural transformation $\psi(\eta)$ with components $\psi(\eta)_{(a,b)} = (\eta_a)_b$, such that the following diagram commutes:

$$\begin{array}{c} (a,b) \\ \Big\downarrow (f,g) \\ (a',b') \end{array} \qquad \begin{array}{ccc} \psi(S)((a,b)) & \xrightarrow{\psi(\eta)_{(a,b)}} & \psi(T)((a,b)) \\ {\scriptstyle \psi(S)((f,g))}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi(T)((f,g))} \\ \psi(S)((a',b')) & \xrightarrow{\psi(\eta)_{(a',b')}} & \psi(T)((a',b')) \end{array} \qquad \begin{array}{ccc} S(a)(b) & \xrightarrow{(\eta_a)_b} & T(a)(b) \\ {\scriptstyle S(f)(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle T(f)(g)} \\ S(a')(b') & \xrightarrow{(\eta_{a'})_{b'}} & T(a')(b') \end{array}$$

Recall that $\psi(T)$ maps the pair $(f, g)$ to a composition of the $b'$ component of the natural transformation $T(f)$ after the functor $T(a)(g)$ (2.3). The above commuting square is thus the diagonal square of the following commuting cube:




which evaluates to




∎

*Corollary* 2.0.24. For categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, there is a natural isomorphism $(\mathbf{C}^{\mathbf{B}})^{\mathbf{A}} \cong (\mathbf{C}^{\mathbf{A}})^{\mathbf{B}}$.

*Proof.* There is a natural isomorphism $\mathbf{A} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A}$. For objects $a \in \mathbf{A}$, $b \in \mathbf{B}$ and morphisms $f\colon a \to a'$ in $\mathbf{A}$ and $g\colon b \to b'$ in $\mathbf{B}$:

$$(a,b) \cong (b,a)$$
$$(f,g) \cong (g,f)$$

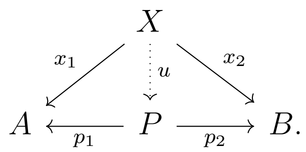
Then by Proposition 2.0.23 we have $(\mathbf{C}^{\mathbf{B}})^{\mathbf{A}} \cong \mathbf{C}^{\mathbf{A}\times\mathbf{B}} \cong \mathbf{C}^{\mathbf{B}\times\mathbf{A}} \cong (\mathbf{C}^{\mathbf{A}})^{\mathbf{B}}$. Note that this agrees with general rules of exponentiation in arithmetic. ∎

**Definition 2.0.25 (Product [4, Def.2.15]).** *In a category* $\mathbf{C}$*, a* product diagram *for objects* $A$ *and* $B$ *consists of an object* $P$ *(the* product*) and arrows*

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

*such that for any diagram of the form* $A \xleftarrow{x_1} X \xrightarrow{x_2} B$ *there exists a unique* $u\colon X \to P$

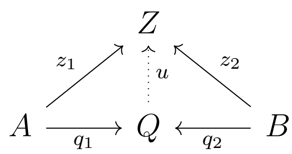
*such that $x_1 = p_1 u$ and $x_2 = p_2 u$, making the following diagram commute:*




**Definition 2.0.26 (Coproduct [4, Def.3.3]).** *Dually, a* coproduct *of objects $A$ and $B$ in a category* **C** *consists of an object $Q$ (the* coproduct*) and arrows*

$$A \xrightarrow{q_1} Q \xleftarrow{q_2} B$$

*such that for any diagram of the form $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there exists a unique $u\colon Q \to Z$ such that $uq_1 = z_1$ and $uq_2 = z_2$ in*



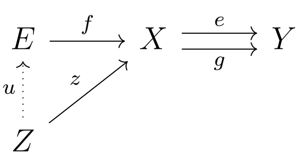

*Example* 2.0.27 ([4, 2.4,3.2]). In **Set**, a product of two sets $A$ and $B$ is the *cartesian product* $A \times B = \{(a, b) : a \in A, b \in B\}$. Their coproduct is their *disjoint union* $A + B = \{(a, 1) : a \in A\} \cup \{(b, 2) : b \in B\}$.

*Example* 2.0.28. In **Vec**, the product of two vector spaces $V$ and $W$ is the usual product $V \times W = \{(v, w) : v \in V, w \in W\}$, and the coproduct is the direct sum $V \oplus W$.

**Definition 2.0.29 (Equalizer [4, Def.3.13]).** *In a category* **C**, *given two parallel arrows (parallel as having same domain and codomain)*

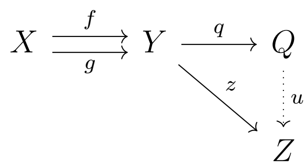
$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

*an* equalizer *of $f$ and $g$ consists of an object $E$ and an arrow $e\colon E \to X$ such that $fe = ge$ and for any $z\colon Z \to X$ with $fz = gz$, there is a unique $u\colon Z \to E$ with $eu = z$, all as in the diagram*




**Definition 2.0.30 (Coequalizer [4, 3.18]).** *Given two parallel arrows $f, g\colon X \to Y$ in a category* **C**, *a* coequalizer *consists of $Q$ and $q\colon Y \to Q$ such that $qf = qg$ and for*

*any $z\colon Y \to Z$ with $zf = zg$, then there exists a unique $u\colon Q \to Z$ such that $uq = z$.*




**Definition 2.0.31 (Kernel, cokernel [43, Def.E.5.1]).** *The kernel* $\ker(f)$ *of a morphism $f\colon X \to Y$ is the equalizer of $f$ with the zero arrow, and the cokernel* $\operatorname{coker}(f)$ *is the coequalizer of $f$ with the zero arrow.*

$$K \xrightarrow{\ker f} X \underset{0}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\operatorname{coker} f} C$$

*Example* 2.0.32. In **Vec**, the kernel and cokernel of $f\colon V \to W$ coincide with the linear algebra notions of the terms, respectively the nullspace of $f$ and the quotient space $W/\operatorname{im}(f)$.

**Definition 2.0.33 (Abelian category [34, Sec.VIII.3]).** *A category* **C** *is abelian if*

- *it has a zero object* $0$,
- *for every two objects, their product and coproduct always exist,*
- *for every morphism, its kernel and cokernel always exist, and*
- *all monomorphisms and epimorphisms are kernels and cokernels, respectively. This means that if $f\colon X \to Y$ is monic, then $f = \ker g$ for some $g\colon Y \to Z$. If $f\colon X \to Y$ is epi, then $f = \operatorname{coker} g$ for some $g\colon Z \to X$.*

*Example* 2.0.34. **Vec** is abelian. The first three conditions are met by Example 2.0.7, Example 2.0.28 and Example 2.0.32. The last one is proved next.

Let $f\colon X \to Y$ be a monomorphism. The quotient space $Y/X$ exists by Example 2.0.32. Let $g\colon Y \to Y/\operatorname{im}(f)$ be the projection sending $y \in Y$ to the equivalence class $y + f(X) \in Y/\operatorname{im}(f)$. The kernel of $g$ is some morphism $h\colon E \to Y$ such that $gh = 0h$.

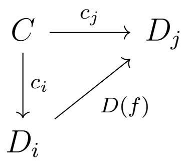
$$gh\colon E \to Y/\operatorname{im}(f) \qquad\qquad 0h\colon E \to Y/\operatorname{im}(f)$$
$$e \mapsto g(e) + f(X) \qquad\qquad e \mapsto 0 + f(X)$$

A perfect candidate for $h\colon E \to Y$ where $g(e) + f(X) = 0 + f(X)$ is of course $h = f$, $E = X$. To really prove that $f = \ker g$ though, we need to show that given any other $h\colon E \to Y$, there exists a unique $u\colon E \to X$. But this is just fulfilled by the universal property of $f$ being a monomorphism (2.2).

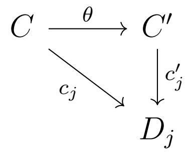
Any epimorphism $f\colon X \to Y$ is the cokernel of $\ker f$ by a dual argument.

*Proposition* 2.0.35 ([34, Sec.VIII.3]). If **C** is an abelian category then so is $\mathbf{C}^{\mathbf{D}}$ for any small category **D**.

**Definition 2.0.36 (Cone [4, Def.5.15]).** *A* cone *to a diagram $D \in \mathbf{C}^{\mathbf{J}}$ consists of an object $C$ in $\mathbf{C}$ and a family $c_j \colon C \to D_j$ of arrows in $\mathbf{C}$, one for each object $j \in \mathbf{J}$, such that for each arrow $f : i \to j$ in $\mathbf{J}$ the following triangle commutes:*




*A* morphism *of cones $\theta \colon (C, c_j) \to (C', c'_j)$ is an arrow $\theta$ in $\mathbf{C}$ making each triangle*




*commute. This forms a category $\mathbf{Cone}(D)$ of cones to $D$.*

**Definition 2.0.37 (Limit [4, Def.5.16]).** *A* limit *for a diagram $D \colon \mathbf{J} \to \mathbf{C}$ is a terminal object in $\mathbf{Cone}(D)$, and it is denoted $\varprojlim_j D_j$. The limit has projections*

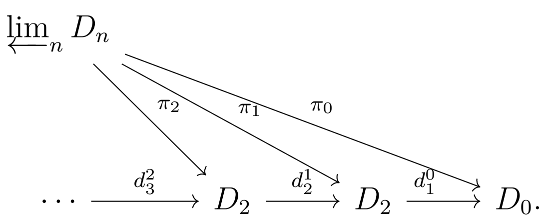
$$\pi_i : \varprojlim_j D_j \to D_i$$

**Definition 2.0.38 (Inverse limit [1, Ch.III Ex.11.4.3]).** *Let $\mathbf{I}$ be a poset considered as a category (Example 2.0.15 treats $\mathbb{R}$ as a preordered set, but the same construction can be adapted to $\mathbb{R}$ as a poset). The limits of $\mathbf{I}$-indexed diagrams are called* inverse *(or* projective*) limits.*

*Example* 2.0.39 (Inverse limit in $\mathbf{Set}^{(\mathbb{N},\geq)}$ [1, Ch.III Ex.11.4.3]). Let $\mathbf{I} = (\mathbb{N}, \geq)$ be the category of natural numbers where there is an arrow $i \to j$ whenever $i \geq j$ (opposite of the usual ordering). A diagram $D$ in $\mathbf{Set}^{(\mathbb{N},\geq)}$ is a sequence

$$\cdots \xrightarrow{d_3^2} D_2 \xrightarrow{d_2^1} D_1 \xrightarrow{d_1^0} D_0.$$

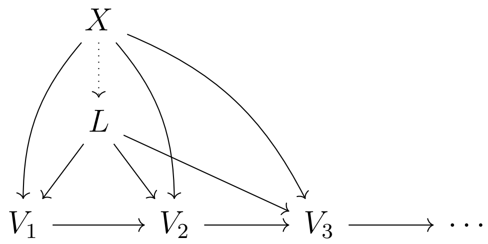
A projective limit of $D$ is the set $\varprojlim_n D_n$ of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D_n$ and $d_{n+1}^n(x_{n+1}) = x_n$ for each $n \in \mathbb{N}$. The map $\pi_m$ is a restriction of the $m$-th projection $p_m \colon \Pi_{n\in\mathbb{N}} D_n \to D_m$.




*Example* 2.0.40 (Cones and inverse limit in **Vec**). Let $\mathbf{I} = (\mathbb{Z}_+, \leq)$ be the category of positive integers where there is an arrow $i \to j$ whenever $i \leq j$. A diagram $V$ in $\mathbf{Vec}^{(\mathbb{Z},\geq)}$ is a sequence

$$V_1 \xrightarrow{v_1^2} V_2 \xrightarrow{v_2^3} V_3 \xrightarrow{v_3^4} \cdots$$

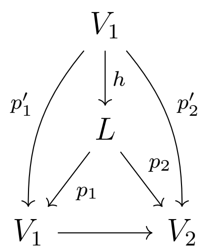
A cone to the diagram $V$ is a vector space (object of $\mathbf{Vec}$) and a family of linear maps to $(V_i)_{i\in\mathbb{Z}_+}$ that commute with the arrows $v_i^j : V_i \to V_j$ of the above diagram. There are many such cones $(X,Y,...)$ but only one of them, which we call $L = \varprojlim_n V_n$, has the *universal property* that all other cones have a single arrow to it which commutes with the rest of the diagram. It is thus a terminal object in $\mathbf{Cone}(V)$, i.e. the inverse limit.



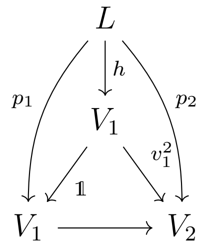
*Proposition* 2.0.41. Let $\mathbf{I}$ be a poset with an initial object 1. The inverse limit $L$ of a diagram $V$ in $\mathbf{Vec}^{\mathbf{I}}$ is $V_1$.

*Proof.* We prove by contradiction that $L = V_1$.

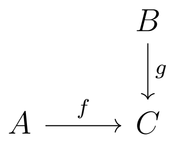
$L \subsetneq V_1$ Suppose $L$ were a strict subspace of $V_1$. Then there exists a cone whose projections onto $V$ would not commute, namely $V_1$: $p_1 \circ h \neq p_1' = \mathbb{1}$.



$L \supsetneq V_1$ Suppose $V_1$ were a strict subspace of $L$. One could always choose $V_1$ as a cone to which the map $h = p_1 : L \to V_1$ commutes with the rest of the diagram, and $L$ is thus no longer a terminal object in $\mathbf{Cone}(V)$.

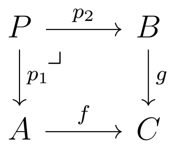


**Definition 2.0.42 (Pullback [4, Def.5.4]).** *Let $f, g$ be two arrows in a category* $\mathbf{C}$ *such that* $\mathrm{cod}(f) = \mathrm{cod}(g)$.

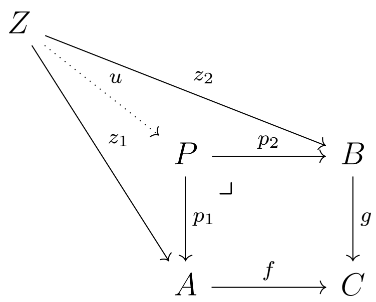



(2.4)

*The* pullback *of $f$ and $g$ consists of an object $P$ and arrows $p_1, p_2$ such that the diagram*




*commutes, i.e.* $fp_1 = gp_2$*, and for any other* $z_1 \colon Z \to A$*,* $z_2 \colon Z \to B$ *with* $fz_1 = gz_2$ *there exists a unique* $u \colon Z \to P$ *with* $z_1 \colon p_1 u$ *and* $z_2 \colon p_2 u$*, i.e. such that the following diagram commutes:*




*We usually add the* ⌟ *symbol to indicate the pullback operation.*

*Example* 2.0.43 (Pulbacks in **Set** [41]). In **Set**, the pullback of (2.4) is given by $\{(a, b) \in A \times B : f(a) = g(b)\}$. If $f \colon A \to C$ is the inclusion of a subset (i.e. a monomorphism), then the pullback is given by $\{b \in B : g(b) \in A\}$. So this is given by restricting $g$ to the elements that are mapped into $A$.

This was an exhaustive compilation of the category theoretic concepts that come up throughout the rest of the work. A reader introducing himself for the first time in it may find some parts a bit too terse. If we had to prioritize something, we highlight two concepts:

- A functor $F$ from a category $\mathbf{C}$ to a category $\mathbf{D}$, $F \colon \mathbf{C} \to \mathbf{D}$, is the same as an object of the category $\mathbf{D}^{\mathbf{C}}$ (Definition 2.0.17).
- A functor $F \colon (\mathbb{Z}_+, \leq) \to \mathbf{C}$ is called a $(\mathbb{Z}_+, \leq)$-indexed diagram in $\mathbf{C}$ because it represents a sequence of objects of $\mathbf{C}$ indexed by $\mathbb{Z}_+$ and connected by morphisms (Example 2.0.18).

# 3 | Persistent homology

Persistent homology [30] is a tool in Topological Data Analysis (TDA) that is developed to address the non-robustness of traditional topological invariants like Betti numbers or the fundamental group to noise and other discontinuous changes in the space under consideration. Consider for example the Vietoris-Rips complex constructed from a point cloud in $\mathbb{R}^n$ and a fixed diameter $\delta$. The 0-th Betti number is the number of connected components, which is highly sensitive to the picked $\delta$. A slightly larger $\delta$ might connect two previously disconnected components. Persistent homology is on the other hand scale-invariant because it uses tools like *barcodes* which can be thought of as parametrized versions of the Betti numbers: $\delta$ is not fixed anymore, and the 0-th barcode represents the evolution of the number of connected components varying the $\delta$ parameter.

We will first give some definitions of algebraic topology and a classical introduction to simplicial homology. We cover its relation to singular homology and the functor definition. Next we introduce persistence modules, its traditional definition and several generalizations of the concept. Lastly, a particular construction of persistence modules from data is discussed.

## 3.1 Complexes

Homology theory builds on concepts in algebraic topology. Here we present the basic definitions, taken mainly from [38] and [31].

**Definition 3.1.1 (Abstract simplicial complex [38, Sec.3]).** *An* abstract simplicial complex *is a collection $K$ of finite nonempty sets, such that if $A$ is an element of $K$, so is every nonempty subset of $A$.*

We will refer to abstract simplicial complexes just as simplicial complexes. The element $A$ of $K$ is called a *simplex* of $K$; its *dimension* is one less than the number of its elements. Each nonempty subset of $A$ is called a *face* of $A$. The *dimension* of $K$ is the largest dimension of one of its simplices, or infinite if there is no such largest dimension. The *vertex set* of $K$, denoted $K^{(0)}$, is the union of the *vertices* or 0-simplices of $K$. Given two simplicial complexes $K$, $L$, a *simplicial map* $f\colon K \to L$ is a continuous map that maps each simplex of $K$ linearly onto a simplex of $L$. Simplicial complexes and simplicial maps form a category **SCpx**.

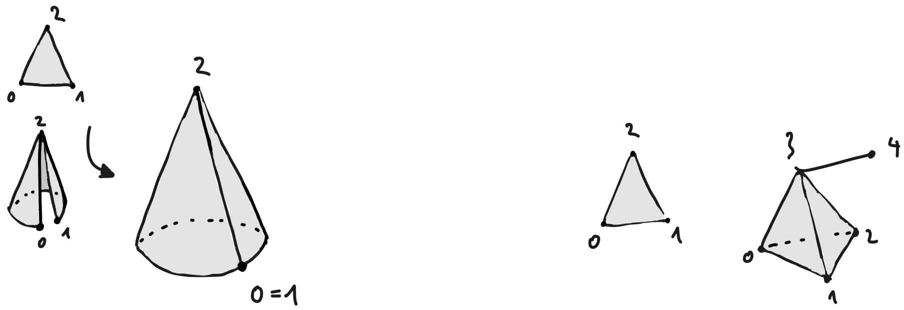
**Definition 3.1.2 (Standard $n$-simplex [31, Sec.2.1]).** *A* standard $n$-simplex *is*

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i \right\}.$$

**Definition 3.1.3 ($\Delta$-complex [31, Sec.2.1]).** *A* $\Delta$-complex *is constructed in two steps:*

1. *Start with a collection of disjoint simplices $\Delta^n_\alpha$ of various dimensions (where $\alpha$ is an enumeration of the simplices) and some sets of faces $\mathcal{F}_i$ of the simplices, where all the faces in each $\mathcal{F}_i$ has the same dimension.*
2. *Form a quotient space of the disjoint union $\sqcup_\alpha \Delta^n_\alpha$ by identifying all the faces in each $\mathcal{F}_i$ to a single simplex via the canonical linear homeomorphisms between them.*

A simplicial complex can be defined alternatively to Definition 3.1.1 as a $\Delta$-complex whose simplices are uniquely determined by their vertices [31, Sec.2.1]. See Figure 3.1 for an illustration.




(a) This $\Delta$-complex is not a simplicial complex because the vertex $0 = 1$ is degenerate.

(b) These two $\Delta$-complexes are simplicial complexes too because every simplex is uniquely determined by their vertices.

Figure 3.1: Examples of $\Delta$-complex and simplicial complex.

**Definition 3.1.4 (Isomorphic complexes [38]).** *Let $K$, $L$ be two (abstract) simplicial complexes. Suppose $f\colon K^{(0)} \to L^{(0)}$ is a bijective correspondence such that the vertices $x_0, \dots, x_n$ of $K$ span a simplex of $K$ if and only if $f(x_0), \dots, f(x_n)$ span a simplex of $L$. The induced simplicial map is an* isomorphism *of $K$ with $L$.*

## 3.2 Homology

Given a simplicial complex $K$, the $k$-th homology group $H_k(K)$ over the ring $\mathbb{Z}$ is defined by the quotient $\ker(\partial_k)/\operatorname{im}(\partial_{k+1}) = Z_k(K)/B_k(K)$ of cycles by boundaries, where the homomorphisms $\partial_k$ are the boundary operators connecting the sequence of abelian groups that forms the chain complex $C(K)$:

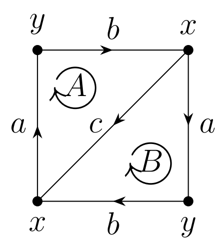
$$\cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Each $C_k$ has an abelian group structure because its simplices admit a symbolic sum with coefficients in $\mathbb{Z}$. The rank of infinite cyclic groups is what we call the Betti numbers: $b_k = \operatorname{rank} H_k(K)$. The finite cyclic groups are the torsions coefficients. Recall that a cyclic group is a group that is generated by a single element, and a cyclic group is said to be infinite if it is isomorphic to $\mathbb{Z}$. By default, the coefficients are in the ring $\mathbb{Z}$, and $H_k(K)$ is a finitely generated abelian group that has in general the form:

$$H_k(K) = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\text{infinite cyclic groups}} \oplus \underbrace{\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_n}}_{\text{finite cyclic groups}}$$

The chain complex can also be calculated over a field $\mathbb{F}$ (like $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{R}$), which results in the computational advantage of obtaining a chain complex of vector spaces over $\mathbb{F}$. Throughout this work, we will consider the homology operation to have coefficients in $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ and omit it from the notation, denoting it simply as $H_k(K)$.

*Example* 3.2.1. To show the difference of taking coefficients over $\mathbb{Z}$ and over $\mathbb{Z}_2$, we will calculate the homology of the (real) projective plane $P$ described as the following $\Delta$-complex:




The chain complex over $\mathbb{Z}$ is

$$0 \xrightarrow{\partial_3} C_2 = \langle A, B\rangle \xrightarrow{\partial_2} C_1 = \langle a, b, c\rangle \xrightarrow{\partial_1} C_0 = \langle x, y\rangle \xrightarrow{\partial_0 = 0} 0$$

where $C_2 = \langle A, B\rangle$ means that $C_2$ is an abelian group generated by $A$ and $B$.

- 0-homology: The 0-cycles and 0-boundaries are:

$$Z_0 = \ker \partial_0 = \langle x, y \rangle$$
$$B_0 = \operatorname{im} \partial_1 = \langle x - y \rangle$$

  where the boundaries are generated by $x - y$ because $\partial a = y - x$, $\partial b = x - y$, $\partial c = x - x = 0$.

$$H_0(P, \mathbb{Z}) = Z_0/B_0 \simeq \mathbb{Z}$$

- 1-homology: Similarly:

$$H_1(P, \mathbb{Z}) = Z_1/B_1 = \langle a+b, c\rangle/\langle a+b+c, a+b-c\rangle \simeq \langle a+b, c\rangle/\langle a+b, 2c\rangle \simeq \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$$

- 2-homology:

$$H_2(P, \mathbb{Z}) = Z_2/B_2 = 0/0 = 0$$

The chain complex over $\mathbb{Z}_2$ produces the same 0- and 2-homology. However for the 1-homology:

$$B_1(P, \mathbb{Z}_2) = \langle a + b + c, a + b - c\rangle = \langle a + b + c\rangle$$

and thus

$$H_1(P, \mathbb{Z}_2) = \langle a + b, c\rangle/\langle a + b\rangle \simeq \mathbb{Z}_2.$$

Even though $H_1(P, \mathbb{Z}) = \mathbb{Z}_2$ and $H_1(P, \mathbb{Z}_2) = \mathbb{Z}_2$, the first case is a finite cyclic group with a torsion coefficient of 2 respect to the base field $\mathbb{Z}$ and its Betti number is $b_1 = 0$. In the latter case, the 1-homology is an infinite cyclic group and $b_1 = 1$. This reflects the fact that the projective plane is even-dimensional, and thus non-orientable. Calculating the Betti number over $\mathbb{Z}$ produces a torsion coefficient, and working over $\mathbb{Z}_2$ forgets orientation, generating a cycle in its place.

### 3.2.1 Homology as functor

The homology explained and exemplified above is called *simplicial homology*. It defines a functor

$$H_k \colon \mathbf{SCpx} \to \mathbf{Vec} \tag{3.1}$$

from the category of simplicial complexes and simplicial maps to the category of vector spaces and linear maps. [(1)] Given two simplicial complexes connected by a simplicial map $K \xrightarrow{\varphi} L$, $H_k$ outputs two vector spaces connected by a linear map $H_k(K) \to H_k(L)$. It is a functor because it respects identity maps and composition.

*Example* 3.2.2. The 0-homology functor applied to $K \xrightarrow{\varphi} L$ in Figure 3.2 outputs

[(1)] Recall that we are working over $\mathbb{Z}_2$. In case of homology over $\mathbb{Z}$, it would be a functor $H_n \colon \mathbf{SCpx} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups and group homomorphisms.

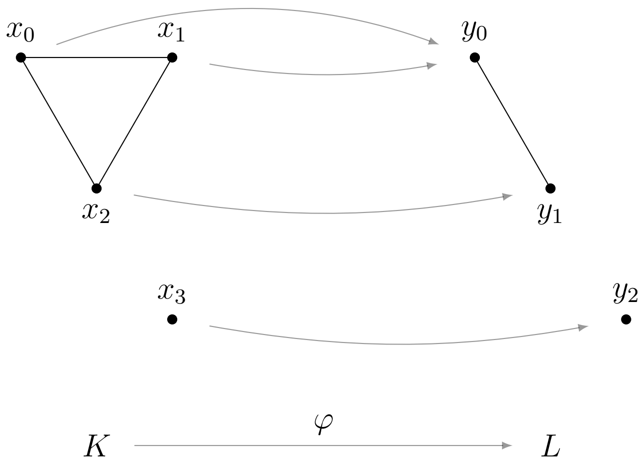



Figure 3.2: Two simplicial complexes connected by a simplicial map.

$\mathbb{F}^2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \mathbb{F}^2$, and the 1-homology functor outputs $\mathbb{F} \xrightarrow{0} 0$. Both are vector spaces connected by linear maps.

*Singular homology* is another notion of homology that is defined for topological spaces. Simplices are replaced with *singular simplices*, which are continuous maps $\sigma\colon \Delta^n \to X$ from the standard $n$-simplex to the topological space, and $k$-chains are formal sums of $k$-dimensional singular simplices. This defines a functor [11, Ex.2.4]

$$H_k\colon \mathbf{Top} \to \mathbf{Vec}$$

from the category of topological spaces and continuous maps to **Vec**. As remarked in [18], singular and simplicial homologies coincide whenever $X$ is homeomorphic to the geometric realization of a simplicial complex, and thus we are able to indifferently talk about simplicial or singular homology for simplicial complexes and topological spaces. We could also work with homology of CW complexes, $\Delta$-complexes (as in Example 3.2.1), etc.

In general, the definition of a $k$-homology functor can be given for any abelian category.

## 3.3 Persistence modules and generalizations

The classical definitions of persistence [25] and persistence modules introduced in computational topology have been formalized and expanded in the framework of category theory in several ways. We give next various definitions of persistence modules and related constructions.

**Definition 3.3.1 (Persistence module [11]).** *Diagrams in* $\mathbf{Vec}^{(\mathbb{R},\leq)}$ *are often called* persistence modules*. A persistence module is a functor* $\mathbb{V}$ *from* $(\mathbb{R},\leq)$ *to* **Vec** *that sends real numbers to vector spaces and inequalities between two reals to linear maps:*[(2)]

$$\begin{aligned} \mathbb{V}\colon (\mathbb{R},\leq) &\to \mathbf{Vec} \\ t &\mapsto V_t \\ s \leq t &\mapsto v_s^t\colon V_s \to V_t \end{aligned}$$

*where composition and identity are expressed as*

$$\begin{aligned} v_s^t v_r^s &= v_r^t \qquad \forall r \leq s \leq t \text{ in } \mathbb{R}, \\ v_t^t &= \mathbb{1}_{V_t} \qquad \forall t \in \mathbb{R}. \end{aligned}$$

$\mathbb{1}_{V_t}\colon V_t \to V_t$ *is the identity morphism sending* $V_t$ *to itself. A morphism* $\varphi$ *between two persistence modules* $\mathbb{V},\mathbb{W}$ *is a* $(\mathbb{R},\leq)$*-indexed family of linear maps* $\{\varphi_t\}_{t\in(\mathbb{R},\leq)}$ *such that for any* $s,t\in\mathbb{R}$ *where* $s\leq t$*, the following diagram commutes:*

$$\begin{array}{ccc} V_s & \xrightarrow{v_s^t} & V_t \\ \downarrow{\scriptstyle \varphi_s} & & \downarrow{\scriptstyle \varphi_t} \\ W_s & \xrightarrow{w_s^t} & W_t \end{array}$$

*The set of all morphisms from* $\mathbb{V}$ *to* $\mathbb{W}$ *is denoted* $\mathrm{Hom}(\mathbb{V},\mathbb{W})$*. An identity morphism from* $\mathbb{V}$ *to itself is denoted* $\mathbb{1}_{\mathbb{V}}$*, and it consists of a family of identity maps* $\mathbb{1}_{V_t} : V_t \to V_t$*. The set of all* endomorphisms *of* $\mathbb{V}$*, that is, morphisms from* $\mathbb{V}$ *to itself, is denoted as* $\mathrm{End}(\mathbb{V})$*.*

Some variations of Definition 3.3.1 are made where the indexing category are the reals $(\mathbb{R},\leq)$ (Example 2.0.15), the integers $(\mathbb{Z},\leq)$, or a finite set $\mathbf{n}$ (Example 2.0.16). Definition 3.3.1 is generalized in [10] by letting the indexing category to be any preordered set $\mathbf{P}$, and the target category (usually **Vec**) to be any category $\mathbf{C}$. Further generalizations as [21] are out of the scope of this work. The level of abstraction that we are going to work with here is to consider persistence modules as $\mathbf{P}$-indexed diagrams in **Vec** where $\mathbf{P}$ is usually $(\mathbb{R},\leq)$ or $\mathbf{n}$.

*Remark* 3.3.2. The indexing category $(\mathbb{R},\leq)$ includes the cases $(\mathbb{Z},\leq)$, $(\mathbb{Z}_+,\leq)$ and $\mathbf{n}$ by suitable inclusion and retraction functors [11, 2.2.3]. For example the inclusion functor $\mathbf{i}\colon \mathbb{Z}\to\mathbb{R}$ given by $\mathbf{i}(j)=j$ has a (left adjoint) ceiling functor $\mathbf{c}\colon \mathbb{R}\to\mathbb{Z}$ given by $\mathbf{c}(t)=\lceil t\rceil$.

*Proposition* 3.3.3. These categories of persistence modules, that is, of $\mathbf{P}$-indexed diagrams in **Vec**, are abelian categories.

(2) In this work, a map from $V_s$ to $V_t$ is denoted $v_s^t$, while in [19] they are denoted $\rho_{ts}$, and in [16] the super- and subindex are flipped as in $v_t^s$.

*Proof.* Since **Vec** is an abelian category (Example 2.0.34), these categories are too by Proposition 2.0.35. ∎

*Remark* 3.3.4 ([10, Sec.1.1]). A persistence module in $\mathbf{Vec}^{\mathbf{n}}$ can be regarded as the composition of a functor $F\colon \mathbf{n} \to \mathbf{SCpx}$ and the simplicial homology functor (3.1) which takes the *diagram* of simplicial complexes connected by simplicial maps $S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n$ and outputs a *diagram* of vector spaces connected by linear maps $V_1 \to V_2 \to \cdots \to V_n$ as in Example 3.2.2. A complete picture of a persistence module is then

$$\mathbf{n} \xrightarrow{F} \mathbf{SCpx} \xrightarrow{H_k} \mathbf{Vec}.$$

We will discuss the nature of the functor $F$ later in Section 3.4.

### 3.3.1 Zigzag modules

From the applied viewpoint, there are many techniques to construct a simplicial complex from a given point cloud in $\mathbb{R}^n$: the Čech complex, the Vietoris-Rips complex, the alpha complex, etc. A limitation of persistent homology is that the resulting sequence of spaces must be a nested family for the homology functor to produce a persistence module, and several situations in topological data analysis arise where this sequential nesting is too limited [44]. Zigzag persistence are introduced in [13] to address this limitation borrowing results from quiver theory (graph representation theory) and extends diagrams in $\mathbf{Vec}^{\mathbf{n}}$ by not requiring the morphisms to go all in the same direction. Some background terminology of quiver theory is given next.

**Definition 3.3.5 (Quiver).** *A* quiver $Q = (Q_0, Q_1)$ *is a directed graph, also called multigraph, with a set of vertices* $Q_0$ *and a set of arrows* $Q_1$*. Each arrow in* $Q_1$ *has a source* $s \in Q_0$ *and a target* $t \in Q_0$*.*

*Example* 3.3.6 ($\mathbb{A}_n(\boldsymbol{\tau})$). Given an orientation $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{n-1})$, the $\mathbb{A}_n(\tau)$ quiver is defined as

$$\mathbb{A}_n(\boldsymbol{\tau}):\ 1 \xleftrightarrow{\tau_1} 2 \xleftrightarrow{\tau_2} \cdots \xleftrightarrow{\tau_{n-1}} n$$

where the $i$-th arrow is $i \longrightarrow (i+1)$ if $\tau_i = \vec{\tau}$ and $i \longleftarrow (i+1)$ if $\tau_i = \overleftarrow{\tau}$.

**Adjunction Cat $\rightleftharpoons$ Quiv [42, 4.].** We can think of quivers as small categories with identity morphisms and composition forgotten. The functor Forget: **Cat** $\to$ **Quiv** from **Cat** (the category of small categories) to **Quiv** (the category of directed graphs) sends each category (an object of **Cat**) to its underlying directed graph (an object of **Quiv**). The left adjoint of this functor gives the *free category* from a directed graph: Free: **Quiv** $\to$ **Cat**. The *free category* or *path category* Free$(Q)$ of a graph $Q$ [9, 5.1] has the vertices of the graph as objects and the set of paths starting in $x$ and ending at $y$ (i.e. all compatible compositions of arrows in the graph) as the set of morphisms between $x$ and $y$.

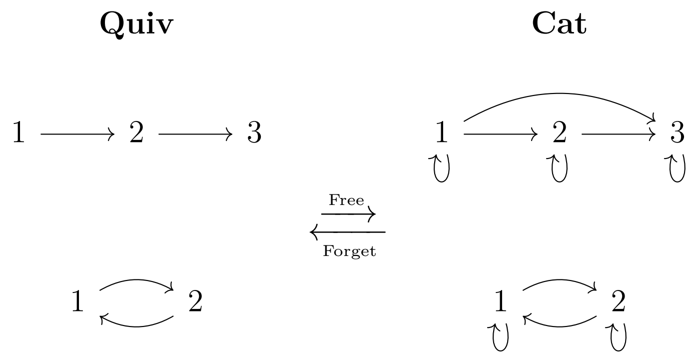



Figure 3.3: Example of two objects of **Quiv** and their correponding free categories, objects of **Cat**.

*Example* 3.3.7. Free $(\mathbb{A}_n(\boldsymbol{\tau}))$ with $\boldsymbol{\tau} = (\vec{\tau}, \ldots, \vec{\tau})$ (i.e. all forward arrows) is isomorphic to **n**. Free $(-)$ assigns an object to each vertex in $\mathbb{A}_n(\boldsymbol{\tau})_0$ and a morphism to each arrow in $\mathbb{A}_n(\boldsymbol{\tau})_1$.

The functor Free $(-)$ together with the $\mathbb{A}_n(\boldsymbol{\tau})$ quiver generalize **n** as an indexing category by not requiring the orientation to be all forward maps, and allows to describe zigzag modules as diagrams in $\mathbf{Vec}^{\mathrm{Free}(\mathbb{A}_n(\boldsymbol{\tau}))}$.

**Definition 3.3.8 (Representation of a quiver [42, 4.]).** *The* representation of a quiver *$Q$ is an object of* $\mathbf{Vec}^{\mathrm{Free}(Q)}$*, i.e. a functor from its free category* Free $(Q)$ *to* **Vec**

$$R\colon \mathrm{Free}\,(Q) \to \mathbf{Vec}$$

*where $R$ sends each vertex to a vector space and each edge to a linear map. Here* Free $(Q)$ *is not an object of* **Cat** *but itself a category.*

*Example* 3.3.9. Let $\mathbb{A}_3(\boldsymbol{\tau})$ be the $\mathbb{A}_n(\boldsymbol{\tau})$ quiver with three vertices and $\boldsymbol{\tau} = (\vec{\tau}_1, \vec{\tau}_2)$. Its free category Free $(\mathbb{A}_3(\boldsymbol{\tau}))$ has three objects, three identity morphisms, two morphisms $\vec{\tau}_1, \vec{\tau}_2$ and all compositions between these morphisms. For example, $\vec{\tau}_2 \circ \mathbb{1}_2 \circ \vec{\tau}_1$ is a valid morphism in Free $(\mathbb{A}_3(\boldsymbol{\tau}))$.

The representation of $\mathbb{A}_3(\boldsymbol{\tau})$ has three vector spaces, three identity linear maps, two linear maps $v_1^2$ and $v_2^3$ and all compositions of morphisms, e.g. $v_1^3 = v_2^3 \circ v_1^2 \in \mathbf{Vec}(V_1, V_3)$. We will not write $R \circ \mathrm{Free}\,(Q)$ but rather $R \in \mathbf{Vec}^{\mathrm{Free}(\mathbb{A}_3(\boldsymbol{\tau}))}$, because Free $(-)$ is a functor that acts on quivers as objects of **Quiv** and outputs a category as an object of **Cat**, while $R$ takes Free $(G)$ not as an object of **Cat** but as a category by itself. $R$ is a particular representation of Free $(\mathbb{A}_3(\boldsymbol{\tau}))$, fixing specific vector spaces and linear maps. Another $R'$ may represent Free $(\mathbb{A}_3(\boldsymbol{\tau}))$ with another set of vector spaces and linear maps.

$$\mathrm{Free}\left(\mathbb{A}_3(\boldsymbol{\tau})\right) \quad \xrightarrow{R} \quad \mathbf{Vec}$$

$$1 \xrightarrow{\vec{\tau}} 2 \xrightarrow{\vec{\tau}} 3 \qquad\qquad V_1 \xrightarrow{v_1^2} V_2 \xrightarrow{v_2^3} V_3$$

A zigzag module is a sequence of vector spaces and linear maps of length $n$, where the directions of the maps are encoded in the orientation $\boldsymbol{\tau}$. It is a representation of a $\mathbb{A}_n(\boldsymbol{\tau})$ quiver. The orientation of a zigzag module is a sequence of $n-1$ symbols in $\{\vec{\tau}, \overleftarrow{\tau}\}$ where $\vec{\tau}$ stands for forward maps and $\overleftarrow{\tau}$ for backward maps. For example, the zigzag module $V_1 \to V_2 \leftarrow V_3 \leftarrow V_4$ is of length 4 and has orientation $\boldsymbol{\tau} = (\vec{\tau}, \overleftarrow{\tau}, \overleftarrow{\tau})$. Zigzag modules of a given length $n$ and an orientation $\boldsymbol{\tau}$ are called $\boldsymbol{\tau}$-modules, and the class[(3)] of $\boldsymbol{\tau}$-modules is denoted by $\boldsymbol{\tau}\mathbf{Mod}$. Persistence modules fit naturally in this definition, where their orientation is composed of only forward maps $\vec{\tau}$.

*Proposition* 3.3.10. $\boldsymbol{\tau}\mathbf{Mod}$ is an abelian category.

*Proof.* This is remarked in [13, p.5], defining the relevant kernels, images and cokernels. A shorter proof can be made by noting that $\boldsymbol{\tau}\mathbf{Mod} \simeq \mathbf{Vec}^{\mathrm{Free}(\mathbb{A}_n(\boldsymbol{\tau}))}$ and applying Proposition 3.3.3. ∎

### 3.3.2 Ladder modules

Motivated by these connections between persistent homology and quiver theory and also by TDA applications in material science and protein structural analysis, ladder modules are introduced in [3] as persistence modules on the commutative ladder. A commutative ladder is another example of a quiver, which we define next.

*Example* 3.3.11 (Ladder quiver $L_n(\boldsymbol{\tau})$). Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{n-1})$ be an orientation. The *ladder quiver* $L_n(\boldsymbol{\tau})$ is the following quiver:

$$L_n(\boldsymbol{\tau})\colon \quad \begin{array}{ccccccc} 1 & \overset{\tau_1}{\longleftrightarrow} & 2 & \overset{\tau_2}{\longleftrightarrow} & \cdots & \overset{\tau_{n-1}}{\longleftrightarrow} & n \\ \downarrow & & \downarrow & & & & \downarrow \\ 1' & \overset{\tau_1}{\longleftrightarrow} & 2' & \overset{\tau_2}{\longleftrightarrow} & \cdots & \overset{\tau_{n-1}}{\longleftrightarrow} & n' \end{array}$$

where the directions of the arrows on both the top and bottom rows are determined by the orientation $\boldsymbol{\tau}$. The *commutative ladder* $CL_n(\boldsymbol{\tau})$ is the ladder quiver $L_n(\boldsymbol{\tau})$ bound by commutative relations. Intuitively this means that any two paths with common source and targets are equivalent (we take the quotient by this equivalence relation).

Ladder modules can be defined in multiple ways:

(3) see Remark 2.0.3.

- A persistence module on the commutative ladder $CL_n(\boldsymbol{\tau})$ [3, Def.2].
- A representation of the $CL_n(\boldsymbol{\tau})$ quiver [3, Def.2].
- A diagram in $\mathbf{Vec}^{\mathrm{Free}(CL_n(\boldsymbol{\tau}))}$.
- A morphism between $\mathbb{A}_n(\boldsymbol{\tau})$ quivers. This is proved in [3, Th.2], where $\operatorname{Rep} Q$ is a notation for an object of $\mathbf{Vec}^{\mathrm{Free}(Q)}$ for some quiver $Q$ and $\operatorname{Arr}(\mathbf{C})$ is the arrow category of a category $\mathbf{C}$:

$$\operatorname{Rep} CL_n(\boldsymbol{\tau}) \cong \operatorname{Arr}(\operatorname{Rep} \mathbb{A}_n(\boldsymbol{\tau}))$$

*Example* 3.3.12 ([10, Ex.1.1.2]). To show how such a construction can arise from a practical example, consider two nested families of abstract simplicial complexes $(K_i)$ and $(L_i)$ that we want to relate with each other,

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4, \qquad L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4.$$

These can be thought of as functors $K, L\colon \mathbf{4} \to \mathbf{SCpx}$ where $\mathbf{4}$ is the **n** category with four objects. A natural question would be to ask if there is a morphism between $(K_i)$ and $(L_i)$. Suppose $\varphi_4\colon K_4 \to L_4$ is a simplicial map which restricts to simplicial maps $\varphi_i\colon K_i \to L_i$ for all $i$. Then $\varphi = (\varphi_i)$ is a natural transformation $K \Rightarrow L$. The commutative diagram

$$\begin{array}{ccccccc} K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & K_4 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} \\ L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & L_4 \end{array}$$

shows the maps and contains all required commutative squares.

#### 3.3.2.1 Alternative definition of ladder modules

We propose an alternative definition of ladder module taking inspiration from Definition 3.3.1. First we fix the target category to be the arrow category of $\mathbf{Vec}$:

**Definition 3.3.13.** *The* arrow category *of* $\mathbf{Vec}$*, notated as* $\operatorname{Arr}(\mathbf{Vec})$ *or* $\mathbf{Vec}^{\mathbf{2}}$*, is a functor category from* $\mathbf{2}$ *to* $\mathbf{Vec}$*.*

- *Objects: Linear maps* $\varphi\colon V \to W$ *of* $\mathbf{Vec}$*, where* $V$ *and* $W$ *are objects of* $\mathbf{Vec}$*.*
- *Morphisms: A morphism* $\rho_s^t = (v_s^t, w_s^t)\colon \varphi_s \to \varphi_t$ *from an object* $\varphi_s\colon V_s \to W_s$ *to* $\varphi_t\colon V_t \to W_t$ *is a pair of linear maps* $(v_s^t\colon V_s \to V_t, w_s^t\colon W_s \to W_t)$ *of* $\mathbf{Vec}$*, such that the following diagram commutes:*

$$\begin{array}{ccc} V_s & \xrightarrow{v_s^t} & V_t \\ \downarrow{\scriptstyle \varphi_s} & & \downarrow{\scriptstyle \varphi_t} \\ W_s & \xrightarrow{w_s^t} & W_t \end{array}$$

**Definition 3.3.14.** *A* ladder module *is a functor from* $(\mathbb{R}, \leq)$ *to* $\mathrm{Arr}(\mathbf{Vec})$.

$$\begin{aligned} M\colon (\mathbb{R}, \leq) &\to \mathrm{Arr}(\mathbf{Vec}) \\ t &\mapsto \varphi_t \\ s \leq t &\mapsto \rho_s^t\colon \varphi_s \to \varphi_t \end{aligned}$$

A ladder module can be equivalently defined as an object of $(\mathrm{Arr}(\mathbf{Vec}))^{(\mathbb{R},\leq)}$, i.e. a $(\mathbb{R}, \leq)$-indexed diagram in $\mathrm{Arr}(\mathbf{Vec})$. In the case of a finite ladder module, it is an object of $(\mathrm{Arr}(\mathbf{Vec}))^{\mathbf{n}}$.

*Proposition* 3.3.15. Consider $\mathbb{A}_n(\boldsymbol{\tau})$ and $\mathbf{n}$ the indexing category with $n$ objects. There exists an equivalence between $(\mathrm{Arr}(\mathbf{Vec}))^{\mathbf{n}}$ and $\mathrm{Arr}(\mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau}))$ with $\boldsymbol{\tau} = (\vec{\tau}, \dots, \vec{\tau})$.

*Proof.* The chain of equivalences follow:

$$(\mathrm{Arr}(\mathbf{Vec}))^{\mathbf{n}} \simeq \left(\mathbf{Vec}^{\mathbf{2}}\right)^{\mathbf{n}} \simeq (\mathbf{Vec}^{\mathbf{n}})^{\mathbf{2}} \simeq (\mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau}))^{\mathbf{2}} \simeq \mathrm{Arr}(\mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau}))$$

The first and last equivalences are clear by the definition of arrow category.

We can prove the equivalence of the categories $(\mathbf{Vec}^{\mathbf{n}})^{\mathbf{2}} \simeq (\mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau}))^{\mathbf{2}}$ by proving the equivalence of the underlying categories $\mathbf{Vec}^{\mathbf{n}} \simeq \mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau})$. This follows from $n \simeq F(\mathbb{A}_n(\boldsymbol{\tau}))$ with $\boldsymbol{\tau} = (\vec{\tau}, \dots, \vec{\tau})$ (Example 3.3.7) and $\mathrm{Rep}\,\mathbb{A}_n(\boldsymbol{\tau})$ being notation for an object of $\mathbf{Vec}^{F(\mathbb{A}_n(\boldsymbol{\tau}))}$.

Letting $\mathbf{A} = \mathbf{2}$, $\mathbf{B} = \mathbf{n}$ and $\mathbf{C} = \mathbf{Vec}$, we have $(\mathbf{Vec}^{\mathbf{n}})^{\mathbf{2}} \cong \left(\mathbf{Vec}^{\mathbf{2}}\right)^{\mathbf{n}}$ by Corollary 2.0.24. This is clear if we note that objects of both categories are partial applications of the bifunctor from the product category $\mathbf{n} \times \mathbf{2}$ to $\mathbf{Vec}$, which is what Corollary 2.0.24 tells us. As the cartesian product of two posets is also a poset (with a chosen order, like lexicographic order), the product category $\mathbf{n} \times \mathbf{2}$ is also an indexing category that defines a diagram. In the case of $(\mathbf{Vec}^{\mathbf{n}})^{\mathbf{2}}$, the bifunctor is first applied in $\mathbf{2}$ and then in $\mathbf{n}$, and viceversa for $\left(\mathbf{Vec}^{\mathbf{2}}\right)^{\mathbf{n}}$.

∎

*Example* 3.3.16. Let $\mathbf{n} = \mathbf{3}$ be the indexing category with three elements and morphisms between them:

$$1 \longrightarrow 2 \longrightarrow 3$$

We omit the identity morphisms and the compositions from the diagram. The isomorphism $(\mathbf{Vec}^{\mathbf{3}})^{\mathbf{2}} \cong (\mathbf{Vec}^{\mathbf{2}})^{\mathbf{3}}$ amounts to saying that arrows $V_i \to W_i$ indexed by $\mathbf{3}$ hold the same information than an arrow between persistence modules $\mathbb{V}$ and $\mathbb{W}$: 6 vector spaces connected by some arrows. The difference in hierarchy of single and double arrows can be illustrated as follows (recall Definition 2.0.11 for the diagramatic notation).

$$\begin{array}{ccccc} V_1 & & V_2 & & V_3 \\ \downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow \\ W_1 & & W_2 & & W_3 \end{array} \qquad \begin{array}{ccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 \\ & & \Downarrow & & \\ W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 \end{array}$$

## 3.4 Filtrations

Topological persistence emerges as a tool to infer the underlying geometric and topological structure of a finite point cloud in a metric space using what is called a filtering function [15,45]. In keeping with the theme of this work, we will introduce it's classical definition and it's categorical reformulation. In Section 3.4.1 we will look at conditions under which morphisms between filtrations exist, and finish with a theorem of existence of ladder modules.

**Definition 3.4.1 (Filtering function in Set [36, Sec.3.]).** *Let $S$ be a finite set, and $\mathcal{P}(S)$ its power set. A* filtering function *on $S$ is any function $f_S \colon \mathcal{P}(S) \to \mathbb{R}$ which satisfies the monotonicity condition: $f_S(\varsigma) \leq f_S(\sigma)$, for all subsets $\varsigma \subseteq \sigma \subseteq S$.*

We will omit the subscript and write $f_S$ just as $f$ if it is clear from the context. The definition of a filtering function is not particular to **Set**. In fact, it is common to define it for simplicial complexes too:

**Definition 3.4.2 (Filtering function in SCpx).** *Let $K$ be a simplicial complex. A filtering function on $K$ is a function $f_K \colon K \to \mathbb{R}$ satisfies the monotonicity condition: $f_K(\varsigma) \leq f_K(\sigma)$ for any simplex $\varsigma$ contained in another $\sigma$ (as a face) in $K$.*

This leads us to define a general *filtered object* in an arbitrary category.

**Definition 3.4.3 (Filtered object, filtration [40]).** *Given a category $\mathbf{C}$, a* filtered object *is an object $X$ of $\mathbf{C}$ equipped with a filtration.*[4] *A* filtration *of $X$ is a sequence of morphisms (often required to be monomorphisms) of the form*

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots \to X$$

*where $X_n$ are objects of $\mathbf{C}$ and $X$ is the last element of the sequence. We will refer to a filtration of $X$ with the symbol $\mathbb{X}$.*

[4] We have kept only the definition of *descending* filtration. A *descending filtration* of $X$ is typically denoted in category theory as $\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X$, but we have flipped the indexing order here to accomodate the usual order of persistence modules.

Two ways of constructing a filtration of an object $X$ are:

- Using the previously defined filtering functions $f_X$. This is known as the *inverse-image construction* [10, 3.2], or *sublevelset construction* [16, Sec.2.1]. See Example 3.4.4.
- Defining an indexing functor directly and looking at its image. See Example 3.4.6.

*Example* 3.4.4 (Sublevelset). Given a filtering function in **Set** $f_S\colon \mathcal{P}(S) \to \mathbb{R}$, the *sublevelset construction* or *inverse-image construction* $F_S$ is a functor

$$\begin{aligned} F_S\colon (\mathbb{R}, \leq) &\to \mathbf{Set} \\ s &\mapsto f^{-1}(-\infty, s] \\ s \leq t &\mapsto f^{-1}(-\infty, s] \subseteq f^{-1}(-\infty, t] \end{aligned}$$

This is defined similarly as $F_K\colon (\mathbb{R}, \leq) \to \mathbf{SCpx}$ for $f_K\colon K \to \mathbb{R}$ with $K$ a simplicial complex, $f_{\mathcal{X}}\colon \mathcal{X} \to \mathbb{R}$ for $\mathcal{X}$ a topological space, and filtering functions in other categories. We say that these filtering functions $f$ encode the *times of appearance* of the sets/simplices/spaces.

The next construction starts from a point cloud in a finite metric space, which we define first for good measure.

**Definition 3.4.5 (Metric, metric space [12, Def.1.1.1]).** *Let $X$ be an arbitrary set. A function $d\colon X \times X \to \mathbb{R} \cup \{\infty\}$ is a* metric *on $X$ if the following conditions are satisfied for all $x, y, z \in X$.*

- *Positiveness: $d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$.*
- *Symmetry: $d(x, y) = d(y, x)$.*
- *Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.*

*A* metric space *$(X, d_X)$ is a set $X$ with a metric $d_X$ on it. A metric space $(X, d_X)$ is* finite *if the set $X$ is finite. A metric space $(X, d_X)$ is* bounded *if there exists some number $r$ such that $d(x, y) \leq r$ for all $x, y \in X$.*

*Example* 3.4.6 (Vietoris-Rips [26, Sec.III.2]). Let $(X, d_X)$ be a finite metric space, for example a set of points in $\mathbb{R}^n$ with the usual Euclidean distance. The *Vietoris-Rips complex* of radius $s$, denoted $\mathcal{VR}_s(X)$, is the maximal simplicial complex built from $\{\sigma \subseteq X : d_X(x_i, x_j) < 2s,\ \forall x_i, x_j \in \sigma\}$.

$$\begin{aligned} \mathcal{VR}\colon (\mathbb{R}, \leq) &\to \mathbf{SCpx} \\ s &\mapsto \mathcal{VR}_s(X) \\ s \leq t &\mapsto \mathcal{VR}_s(X) \subseteq \mathcal{VR}_t(X) \end{aligned}$$

$\mathcal{VR}$ is an indexing functor from the proset of reals $(\mathbb{R}, \leq)$ to the category **SCpx**. $\mathcal{VR}$ is an object of $\mathbf{SCpx}^{(\mathbb{R},\leq)}$.

*Remark* 3.4.7. As any finite filtration, the Vietoris-Rips construction can be written as some sublevelset filtration, concretely for

$$f(\sigma) = \max_{x_i, x_j \in \sigma} d_X(x_i, x_j) \tag{3.2}$$

given any $n$-simplex $\sigma$ [36, Sec.5.1]. This includes the case of 0-simplices (vertices) $f(\{x_i\}) = d_X(x_i, x_i) = 0$. So $\mathcal{VR}_s(X) = f^{-1}(-\infty, s]$, and the filtration extends trivially to $s < 0$ as $\mathcal{VR}_s(X) = \emptyset$.

### 3.4.1 Morphisms between filtrations

We focus next our attention to the conditions under which morphisms $(\varphi_i)$ between filtrations in Example 3.3.12 exist. We will consider filtrations obtained from sublevelsets as they encompass the case of Vietoris-Rips filtrations. In this section, $K$, $L$ are two simplicial complexes, and $\mathbb{K}$, $\mathbb{L}$ two filtrations obtained from the sublevelsets of filtering functions $f : K \to \mathbb{R}$ and $g : L \to \mathbb{R}$ respectively. $\mathbb{K}$ is a sequence of subcomplexes of $K$, where the $s$-th subcomplex is denoted $K_s$, and two subcomplexes are connected by maps $k_s^t \colon K_s \to K_t$ for $s, t \in \mathbb{R}$ such that $s \leq t$. If we omit the superindex, the map $k_s$ is understood to go to $K$.

**Definition 3.4.8 (Morphism between filtrations).** *Given a category* $\mathbf{C}$, *a* morphism $\Phi$ *between* filtrations $\mathbb{X}$ *and* $\mathbb{Y}$ *is a family of morphisms* $\varphi_s \colon X_s \to Y_s$ *that commute with the monomorphisms* $X_s \hookrightarrow X_t$ *with* $s \leq t$. *These morphisms* $\Phi$ *form a set* $\mathrm{Hom}(\mathbb{X}, \mathbb{Y})$, *and we will write* $\Phi = (\varphi_s)$ *to denote the same morphism.*

An example of a commutative diagram of a morphism between filtrations $(\varphi_s) \colon \mathbb{K} \to \mathbb{L}$ is the one shown in Example 3.3.12, repeated here for clarity:

$$\begin{array}{ccccccc} K_1 & \xrightarrow{k_1^2} & K_2 & \xrightarrow{k_2^3} & K_3 & \xrightarrow{k_3^4} & K_4 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} \\ L_1 & \xrightarrow{l_1^2} & L_2 & \xrightarrow{l_2^3} & L_3 & \xrightarrow{l_3^4} & L_4 \end{array} \tag{3.3}$$

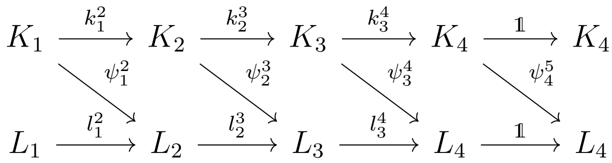
We can generalize this definition to allow for an offset between the indices.

**Definition 3.4.9 (Shifted morphism between filtrations).** *Given a category* $\mathbf{C}$, *a* $\delta$-shifted morphism $\Phi$ *between filtrations* $\mathbb{X}$ *and* $\mathbb{Y}$ *is a family of morphisms* $\varphi_s^{s+\delta} \colon X_s \to Y_{s+\delta}$ *that commute with the monomorphisms* $X_s \hookrightarrow X_t$ *with* $s \leq t$. *These morphisms* $\Phi$ *form a set* $\mathrm{Hom}^\delta(\mathbb{X}, \mathbb{Y})$, *and we will write* $\Phi = (\varphi_s^{s+\delta})$ *to denote the same morphism.*

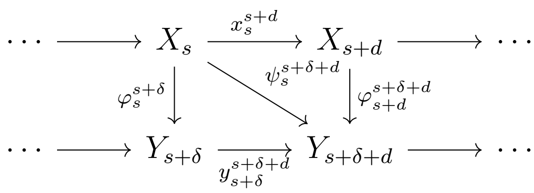
*Remark* 3.4.10. This definition is similar to $\delta$-simplicial maps defined in [17, Def.3.1], although the focus there is on $\delta$-shifted morphisms induced from a vertex map $X^{(0)} \to$

$Y^{(0)}$.

Given the same filtrations of (3.3), a 1-shifted morphism $\Psi = (\psi_s^{s+1}) \in \mathrm{Hom}^1(\mathbb{K}, \mathbb{L})$ would commute the following diagram:



where we extend $\mathbb{L}$ with $L_5 = L_4$ and the identity map $\mathbb{1}\colon L_4 \to L_4$. A morphism between filtrations is thus the same as a 0-shifted morphism. Note that given a $\delta$-shifted morphism $\Phi \in \mathrm{Hom}^\delta(\mathbb{X}, \mathbb{Y})$, one can always construct a $(\delta+d)$-shifted morphism $\Psi \in \mathrm{Hom}^{\delta+d}(\mathbb{X}, \mathbb{Y})$ by precomposing $\Phi$ with $(x_s^{s+d})$ or postcomposing $\Phi$ with $(y_s^{s+d})$. These two ways produce the same $(\delta+d)$-shifted morphism because of the commuting condition of $\Phi$: $\varphi_s^{s+\delta} \circ y_{s+\delta}^{s+\delta+d} = x_s^{s+d} \circ \varphi_{s+d}^{s+d+\delta}$



We say that a function $f$ *dominates* another function $g$ over a map $\varphi$ if $f(\sigma) \geq g \circ \varphi(\sigma)$ for any simplex $\sigma \in K$.

*Proposition* 3.4.11. If $f$ dominates $g$ over a simplicial map $\varphi : K \to L$, then $\varphi$ defines a morphism $\Phi \in \mathrm{Hom}(\mathbb{K}, \mathbb{L})$.

*Proof.* $\varphi$ is a simplicial map from $K$ to $L$, i.e. it maps every simplex $\sigma \in K$ to a simplex $\varphi(\sigma) \in L$. Let $a = f(\sigma)$ be the value that $f$ takes at the simplex $\sigma$. $f(\sigma) \geq g \circ \varphi(\sigma)$, thus $\varphi(\sigma) \in g^{-1}(-\infty, a]$. ∎

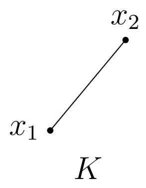
*Corollary* 3.4.12. If $f$ dominates $g$ over a simplicial map $\varphi : K \to L$ with offset $\delta$, that is, $f(\sigma) \geq g \circ \varphi(\sigma) + \delta$ for every $\sigma \in K$, then $\varphi$ defines a shifted morphism $\Phi \in \mathrm{Hom}^\delta(\mathbb{K}, \mathbb{L})$.

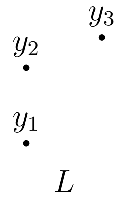
The next two examples illustrate why we require $\varphi$ to be a simplicial map and why $f$ has to dominate $g$ over $\varphi$.

*Example* 3.4.13. Given two simplicial complexes $K$ and $L$ with vertex sets $K^{(0)} = \{x_0, x_1\}$ and $L^{(0)} = \{y_0, y_1, y_2\}$, and filtrations $\mathbb{K}$, $\mathbb{L}$ constructed from $f$, $g$.

| $K$ | $f$ |
|---|---|
| $\{x_0\}$ | 0 |
| $\{x_1\}$ | 1 |
| $\{x_0, x_1\}$ | 2 |

| $L$ | $g$ |
|---|---|
| $\{y_0\}$ | 0 |
| $\{y_1\}$ | 1 |
| $\{y_2\}$ | 2 |




$$K_0 = \{\{x_0\}\} \xhookrightarrow{k_0^1} K_1 = \{\{x_0\}, \{x_1\}\} \xhookrightarrow{k_1^2} K_2 = \{\{x_0\}, \{x_1\}, \{x_0, x_1\}\},$$

$$L_0 = \{\{y_0\}\} \xhookrightarrow{l_0^1} L_1 = \{\{y_0\}, \{y_1\}\} \xhookrightarrow{l_1^2} L_2 = \{\{y_0\}, \{y_1\}, \{y_2\}\}.$$

A map sending $x_1 \mapsto y_1$, $x_2 \mapsto y_2$ is not a simplicial map, but any map sending $x_1$ and $x_2$ to the same vertex in $L$ is a simplicial map and induces a morphism between the filtrations.

*Example* 3.4.14. Suppose $\mathbb{K}$ and $\mathbb{L}$ are two filtrations constructed from $f$ and $g$ as shown next: Let $\varphi \colon K \to L$ map $\{x_0\}$ to $\{y_1\}$, $\{x_1\}$ to $\{y_0\}$ and $\{x_0, x_1\}$ to $\{y_0, y_1\}$.

| $K$ | $f$ | $L$ | $g$ |
|---|---|---|---|
| $\{x_0\}$ | 0 | $\{y_0\}$ | 0 |
| $\{x_1\}$ | 1 | $\{y_1\}$ | 1 |
| $\{x_0, x_1\}$ | 2 | $\{y_0, y_1\}$ | 2 |

$$\{\{x_0\}\} \xhookrightarrow{k_0^1} \{\{x_0\}, \{x_1\}\} \xhookrightarrow{k_1^2} \{\{x_0\}, \{x_1\}, \{x_0, x_1\}\},$$

$$\{\{y_0\}\} \xhookrightarrow{l_0^1} \{\{y_0\}, \{y_1\}\} \xhookrightarrow{l_1^2} \{\{y_0\}, \{y_1\}, \{y_0, y_1\}\}.$$

This defines $\varphi_2$ and is a valid simplicial map. However, $0 = f(\{x_0\}) \not\geq g \circ \varphi(\{x_0\}) = g(\{x_1\}) = 1$ so there is no $\varphi_0$ such that $l_0^2 \circ \varphi_0 = \varphi_2 \circ k_0^2$ and therefore we cannot define a morphism $\Phi : \mathbb{K} \to \mathbb{L}$.

Up until now we have given some conditions on $f$, $g$ and $\varphi$ in order to assert that a morphism between $\mathbb{K}$ and $\mathbb{L}$ exists. But what if we start directly with the filtrations $\mathbb{K}$ and $\mathbb{L}$? Can we still prove when a morphism $\Phi \colon \mathbb{K} \to \mathbb{L}$ exists? Yes! We answer this in our next proposition, employing pullbacks, defined at the end of Chapter 2 (Definition 2.0.42). Even though the core argument is about sets, we use this category-theoretic construction in line with the main objective of this work.

*Proposition* 3.4.15. Let $K$, $L$ be two objects of **SCpx**. Let $\mathbb{K}, \mathbb{L}$ be filtrations of $K$ and $L$, i.e. diagrams in **SCpx**. A morphism $\varphi : K \to L$ builds a morphism $(\varphi_s) : \mathbb{K} \to \mathbb{L}$ if and only if succesive pullbacks $P_s$ with projections $p_s$ and $q_s$ of diagrams

$$\begin{array}{ccc} P_s & \xrightarrow{q_s} & K \\ \downarrow{\scriptstyle p_s} & \lrcorner & \downarrow{\scriptstyle \varphi} \\ L_s & \xhookrightarrow{l_s} & L \end{array}$$

meet the condition $K_s \subseteq P_s$ for all filtration indices $s$. The family of simplicial maps $(\varphi_s)$ is then the family $(p_s|_{K_s}) : \mathbb{K} \to \mathbb{L}$.

*Proof.* This can be understood in terms of maps between sets. $l_s \colon L_s \to L$ is an inclusion, therefore the pullback is $P_s = \{\sigma \in K : \varphi(\sigma) \in L_s\}$ (in a way similar to Example 2.0.43). If $K_s \subseteq P_s$, then $\varphi(\sigma) \in L_s$ for all $\sigma \in K_s$ and this defines the simplicial map $\varphi_s = p_s|_{K_s}$. ∎

*Example* 3.4.16. Consider Example 3.4.14 again and let us try to construct a morphism $(\varphi_s) : \mathbb{K} \to \mathbb{L}$ from the simplicial map $\varphi$ defined there. At the index $s = 1$, the pullback is $P_1 = \{\{x_0\}, \{x_1\}\}$. $K_1 \subseteq P_1$ (they are actually equal) so $\varphi_1$ is defined. It maps $\{x_0\}$ to $\{y_1\}$ and $\{x_1\}$ to $\{y_0\}$. At the index $s = 0$, the pullback is $P_0 = \{\{x_1\}\}$. $K_0 \nsubseteq P_0$, so $\varphi_0$ cannot be defined and the morphism $(\varphi_s) : \mathbb{K} \to \mathbb{L}$ does not exist.

This next example illustrates the pullback construction better and shows a case where $K_s \subsetneq P_s$.

*Example* 3.4.17. Consider $\mathbb{K}$, $\mathbb{L}$ given by

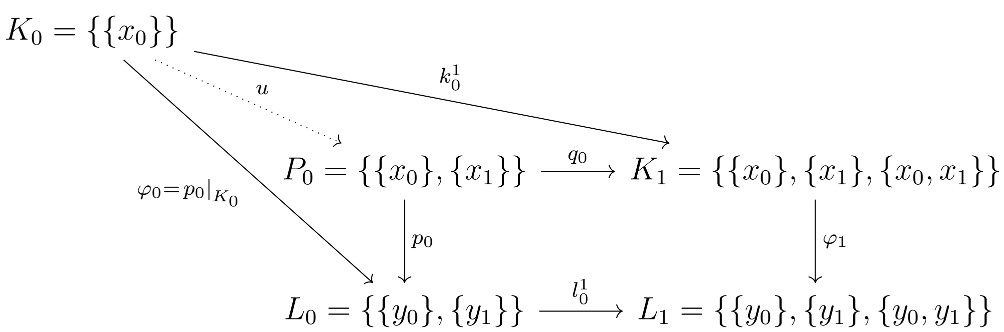
$$K_0 = \{\{x_0\}\} \xhookrightarrow{k_0^1} K_1 = \{\{x_0\}, \{x_1\}, \{x_0, x_1\}\},$$

$$L_0 = \{\{y_0\}, \{y_1\}\} \xhookrightarrow{l_0^1} L_1 = \{\{y_0\}, \{y_1\}, \{y_0, y_1\}\}.$$

and the 'identity' simplicial map:

$$\begin{aligned} \varphi \colon K &\to L \\ \{x_0\} &\mapsto \{y_0\} \\ \{x_1\} &\mapsto \{y_1\} \\ \{x_0, x_1\} &\mapsto \{y_0, y_1\} \end{aligned}$$

The pullback at $s = 0$ is $P_0 = \{\{x_0\}, \{x_1\}\}$




Here the map that is needed to define a morphism $(\varphi_s) \colon \mathbb{K} \to \mathbb{L}$ is not $p_0$ directly because the original $\mathbb{K}$ filtration has a complex $K_0 \subsetneq P_0$ strictly smaller that $P_0$. The map $p_0 : P_0 \to L_0$ that commutes with the diagram has to be restricted to the domain

$K_0$.

*Remark* 3.4.18. If we discard $\mathbb{K}$ and start only from a simplicial map $\varphi\colon K \to L$ and a filtration $\mathbb{L}$, we can construct with the above method a filtration on $K$. This introduced in [36, 4.1] as a *pullback filtration*.

We now focus our attention to morphisms between Vietoris-Rips filtrations and try to give a sufficient condition on distance matrices in the metric space that guarantees that such a morphism exists. Our initial data are finite metric spaces $(X, d_X)$ of cardinality $N$. For the reader unfamiliar with metric spaces, this is equivalent to a point cloud $X$ of $N$ points where the *distance matrix* is a $N \times N$ matrix $M_X$ that encodes all pairwise distances with entries $(M_X)_{ij} = d_X(i,j)$. We will first propose the condition using point clouds and distance matrices in Proposition 3.4.19, and later give a more concise definition using metric spaces in Corollary 3.4.21

*Proposition* 3.4.19. Consider two point clouds $X$, $Y$ with distance matrices $M_X$ and $M_Y$ and Vietoris-Rips filtrations $\mathcal{VR}(X)$ and $\mathcal{VR}(Y)$. Let $\varphi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$ be a simplicial map, and $\varphi^{(0)}\colon X \to Y$ its corresponding vertex map that maps a vertex $x_i \in X$ to $y_{i'} \in Y$. Then, given vertices $x_i$, $x_j$ in $X$, we denote by $y_{i'}$, $y_{j'}$ the corresponding vertices in $Y$, that is, $y_{i'} = \varphi^{(0)}(x_i)$ and $y_{j'} = \varphi^{(0)}(x_j)$. If $(M_X)_{ij} \geq (M_Y)_{i'j'}$ for every pair of vertices $x_i, x_j$ in $X$, then there exists a morphism $\Phi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$.

*Proof.* This is a corollary to Proposition 3.4.11. By Equation (3.2), the condition $f(\sigma) \geq g \circ \varphi(\sigma)$ translates to

$$\max_{x_i, x_j \in \sigma} \|x_i - x_j\| \geq \max_{y_i, y_j \in \varphi(\sigma)} \|y_i - y_j\|.$$

This condition is met if we require every distance to be reduced, that is, for every $x_i, x_j \in X$, $\|x_i - x_j\| \geq \|y_{i'} - y_{j'}\|$, which expressed in terms of distance matrices yields

$$(M_X)_{ij} \geq (M_Y)_{i'j'}. \tag{3.4}$$

∎

To express the above proposition in terms of metric spaces instead of point clouds and distance matrices, we first need to define the notion of continuous functions between metric spaces.

**Definition 3.4.20 (Short map [12, Def.1.4.1]).** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map $m\colon X \to Y$ sending $x_i \mapsto m(x_i) = y_i$ is called* Lipschitz *if there exists a $C \geq 0$ such that*

$$d_Y(y_i, y_j) \leq C \cdot d_X(x_i, x_j).$$

*A map with Lipschitz constant $C = 1$ is called* non-expanding *or* short map.

*Corollary* 3.4.21. Consider two metric spaces $(X, d_X)$, $(Y, d_Y)$ and Vietoris-Rips filtrations $\mathcal{VR}(X)$ and $\mathcal{VR}(Y)$. Let $\varphi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$ be a simplicial map, and

$\varphi^{(0)}\colon X \to Y$ its corresponding vertex map sending $x_i \mapsto \varphi^{(0)}(x_i) = y_i$. If $\varphi^{(0)}$ is a short map, then there exists a morphism $\Phi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$.

*Proof.* Write (3.4) as $d_X(x_i, x_j) \geq d_Y(y_i, y_j)$, which is the definition of short map. ∎

*Corollary 3.4.22.* Let $(X, d_X)$, $(Y, d_Y)$ and $\varphi$ be as above. If

$$d_X(x_i, x_j) \geq d_Y(y_i, y_j) + \delta \quad \forall x_i, x_j \in X$$

then there exists a $\delta$-shifted morphism $\Phi \in \mathrm{Hom}^\delta(\mathcal{VR}(X), \mathcal{VR}(Y))$.

*Proof.* Proceeding in the same way, for all $x_i, x_j$ in $X$:

$$d_X(x_i, x_j) \geq d_Y(y_i, y_j) + \delta$$

This inequality holds for every pair of vertices $x_i$,$x_j$ in $X$. It holds then in particular for the most distant pair of vertices in each simplex $\sigma \in \mathcal{VR}(X)$:

$$\max_{x_i, x_j \in \sigma} d_X(x_i, x_j) \geq \max_{y_i, y_j \in \varphi(\sigma)} d_Y(y_i, y_j) + \delta$$

which by (3.2) is

$$f(\sigma) \geq g \circ \varphi(\sigma) + \delta$$

and we have $\Phi$ by Corollary 3.4.12. ∎

We have thus given sufficient conditions on the distance matrices of two point clouds under which there exist morphisms between their respective Vietoris-Rips filtrations. But recall why we even started talking about morphisms between filtrations. It was to find out when a construction like Example 3.3.12 existed.

Let us examine what the morphism $\Phi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$ really is. $\mathcal{VR}(X)$ and $\mathcal{VR}(Y)$ are objects of $\mathbf{SCpx}^{(\mathbb{R},\leq)}$. We have previously seen in Remark 2.0.20 a construction of two objects of a category and a morphism between them. $\Phi$ is thus a functor from $\mathbf{2}$ to $\mathbf{SCpx}^{(\mathbb{R},\leq)}$. Let us break down $\Phi$ even further as:[(5)]

$$\underbrace{\mathbf{2} \longrightarrow (\mathbb{R}, \leq) \longrightarrow}_{\Phi} \mathbf{SCpx}$$

By Corollary 2.0.24 we can exchange $\mathbf{2}$ and $(\mathbb{R}, \leq)$:

$$(\mathbb{R}, \leq) \longrightarrow \mathbf{2} \longrightarrow \mathbf{SCpx}$$

[(5)] We might think of this as *defunctionalizing a high-order function*, if one is familiar with this programming terminology.

If we post-compose with the homology functor $H_k$:

$$(\mathbb{R}, \leq) \longrightarrow \mathbf{2} \longrightarrow \mathbf{SCpx} \xrightarrow{H_k} \mathbf{Vec} \tag{3.5}$$

and interpret $(\mathbb{R}, \leq)$ and $\mathbf{2}$ as indexing categories, we get back an object of $((\mathbf{Vec})^{\mathbf{2}})^{(\mathbb{R},\leq)}$. This is a ladder module![6]

To close this section, we can summarize the above in the following theorem.

**Theorem 3.4.23.** *Let $(X, d_X)$, $(Y, d_Y)$ be two metric spaces, and $\mathcal{VR}(X)$ and $\mathcal{VR}(Y)$ their Vietoris-Rips filtrations. Let $\varphi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$ be a simplicial map, and $\varphi^{(0)} \colon X \to Y$ its corresponding vertex map sending $x_i \mapsto \varphi^{(0)}(x_i) = y_i$. If $\varphi^{(0)}$ is a short map, then there exists a ladder module between the persistence modules $H_k(\mathcal{VR}(X))$ and $H_k(\mathcal{VR}(Y))$.*

*Proof.* By Corollary 3.4.21 we have a morphism $\Phi : \mathcal{VR}(X) \to \mathcal{VR}(Y)$, and by the reasoning above (3.5), composing $\Phi$ with $H_k$ we obtain an object of $(\mathrm{Arr}(\mathbf{Vec})^{(\mathbb{R},\leq)}$, i.e. a ladder module. ∎

This result ties up our definition of ladder modules (Definition 3.3.14) with the introduction of filtrations and their categorical conditions of existence (Proposition 3.4.15) which translate to a condition on metric spaces (Corollary 3.4.21) in the special case of Vietoris-Rips filtrations.

### 3.4.2 Vietoris-Rips as bifunctor

We have defined the Vietoris-Rips filtration as an indexed category from $(\mathbb{R}, \leq)$ to $\mathbf{SCpx}$. The point cloud data is implicit in the functor definition. If we want to make this dependence explicit, we would like to define a 'category of point clouds'. We first give the necessary definitions to formalize a metric space as a category.

**Definition 3.4.24 (Symmetric monoidal preorder [28, Def.2.2]).** *A* symmetric monoidal preorder $\mathbf{V}$ *is a preorder $(X, \leq)$ with*

- *a monoidal unit $I \in \mathbf{V}_0$, and*
- *a monoidal product $\otimes \colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$*

*such that for all $x_1, x_2, y_1, y_2, x, y, z \in \mathbf{V}_0$ the following hold:*

1. *Monotonicity: If $x_1 \leq y_1$ and $x_2 \leq y_2$, then $x_1 \otimes y_1 \leq x_2 \otimes y_2$.*
2. *Unitality: $I \otimes x = x$ and $x \otimes I = x$.*

[6] We could even define ladder modules in the $\mathbf{SCpx}$ category as objects of $(\mathrm{Arr}(\mathbf{SCpx}))^{(\mathbb{R},\leq)}$ without applying $H_k$, but to avoid confusion, we will apply $H_k$ to get exactly the same objects as the original definition.

3. *Associativity:* $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.
4. *Symmetry:* $x \otimes y = y \otimes x$.

*Example* 3.4.25. $([0, \infty], \geq)$ [28, Ex.2.37] and other categories that we have seen throughout this work like $(\mathbb{R}, \leq)$, $\mathbf{n}$ and $(\mathbb{Z}, \leq)$ are symmetric monoidal preorders.

An enriched category is a category where homsets (the set of morphisms between two objects) have additional structure: they form a vector space, or a topological space, etc. These homsets themselves become objects of a suitable monoidal category $K$, which has to be monoidal because composition in the enriched category is defined in terms of the tensor product:

$$\circ\colon \operatorname{Hom}(y, z) \otimes \operatorname{Hom}(x, y) \to \operatorname{Hom}(x, z)$$

Such a category is called a *category enriched over* $K$, or a $K$*-category.*

**Definition 3.4.26 (V-category [28, Def.2.46]).** *A category* $\mathbf{C}$ *enriched over a symmetric monoidal category* $\mathbf{V}$ *has*

- *a set of objects* $\mathbf{C}_0$.
- *for each pair of objects* $a, b \in \mathbf{C}_0$*, a hom-object* $\mathbf{C}(a, b) \in \mathbf{V}_0$.

*such that for all* $a, b, c \in \mathbf{C}_0$

1. *there exists an identity element* $j_a\colon I \to \mathbf{C}(a, a)$*, and*
2. *there exists a composition morphism* $\circ_{a,b,c}\colon \mathbf{C}(b, c) \otimes \mathbf{C}(a, b) \to \mathbf{C}(a, c)$ *in* $\mathbf{V}$ *which is associative and unital.*

**Definition 3.4.27 (Lawvere metric space [28, Sec.2.3.3]).** *A* Lawvere metric space *is a metric space including* $\infty$ *but with less constraints: It has a metric* $d$*, which for elements* $x, y, z$ *satisfies two conditions:*

1. $d(x, x) = 0$
2. *Triangle inequality:* $d(x, y) + d(y, z) \geq d(x, z)$.

*The symmetry axiom is not imposed: In general,* $d(x, y) \neq d(y, x)$.

*Proposition* 3.4.28 ([32]). Lawvere metric spaces are categories enriched in the symmetric monoidal preorder $([0, \infty], \geq)$, where tensor product is addition and composition is the triangle inequality. We will denote a Lawvere metric space as a category $\mathbf{L}$.

Objects of $\mathbf{L}$ are points. Building on $\mathbf{L}$, we are interested in defining ordered sets of points in this Lawvere metric space. However we are interested in defining only morphisms that translate to inclusions of Vietoris-Rips complexes later so that Corollary 3.4.21 becomes a consequence of functoriality, thus we define a subcategory of $[\mathbf{n}, \mathbf{L}]$ as follows:

**Definition 3.4.29.** *The category* **PCloud** *is a subcategory of* $[\mathbf{n}, \mathbf{L}]$ *consisting of*

- *Objects: Point clouds, that is, ordered sets of* $n$ *objects in* $\mathbf{L}$*:* $X = \{x_1, x_2, \ldots, x_n \in \mathbf{L}\}$*. These form metric spaces* $(X, d_X)$ *restricting the domain of* $d$ *to* $X \times X$*.*
- *Morphisms: Short maps between point clouds, that is, a map* $m \colon X \to Y$ *such that* $d_X(i,j) \geq d_Y(i,j)$*. They are composable, as two maps* $X \xrightarrow{m} Y \xrightarrow{\tilde{m}} Z$ *compose to* $\tilde{m} \circ m \colon X \to Y$ *with* $d_X(i,j) \geq d_Z(i,j)$*.*

A Vietoris-Rips complex is then a bifunctor

$$\begin{aligned} \mathcal{VR} \colon ([0,\infty], \geq) \times \mathbf{PCloud} &\to \mathbf{SCpx} \\ (r, X) &\mapsto \mathcal{VR}_r(X) = \Big\{ \{u_1, u_2, \ldots\} \in X \;\Big|\; \|u_i - u_j\| \leq r \Big\} \\ (r \leq s, X) &\mapsto \mathcal{VR}_r(X) \hookrightarrow \mathcal{VR}_s(X) \\ (r, X \to Y) &\mapsto \mathcal{VR}_r(X) \hookrightarrow \mathcal{VR}_r(Y) \end{aligned}$$

where there are suitable inclusions of $\mathcal{VR}$ complexes when changing the radius $r \leq s$ and also when changing between point clouds where the distance matrix has contracted. This way we do not lose information about the points positions.

Theorem 3.4.23 becomes then a consequence of functoriality: short maps between metric spaces get mapped to inclusions of Vietoris-Rips complexes.

# 4 Decomposition of persistence modules

To analyze persistence modules, it is useful to have a decomposition into indecomposable building blocks. This section is devoted to the decomposition of certain persistence modules defined in Section 3.3.

Section 4.1 presents the tools used in classical persistent homology to tackle decompositions of persistence modules in $\mathbf{FinVec}^{(\mathbb{Z},\leq)}$. We then focus on one hand in decompositions of zigzag modules by [13] in Section 4.2, because they offer a constructive decomposition algorithm which highlights the computational applications of persistent homology theory. On the other hand, we give an overview in Section 4.3 of the work by [19] describing decomposition of persisence modules over $(\mathbb{R}, \leq)$, to discuss some techniques from category theory (and originally from representation theory) used in persistent homology. Although not reviewed in this chapter, we cite [27, Theorems 3 and 4], where the decomposition of ladder modules is addressed.

## 4.1 Interval decomposition

The idea of combining two 'simple' modules and get a 'bigger' one is through direct sums.

**Definition 4.1.1 (Direct sum [16, Sec.2.5]).** *The* direct sum $\mathbb{W} = \mathbb{U} \oplus \mathbb{V}$ *of two persistence modules* $\mathbb{U}, \mathbb{V} \in \mathbf{Vec}^{\mathbf{P}}$ *is the pointwise direct sum in* $\mathbf{Vec}$*:*

$$W_t = U_t \oplus V_t \qquad\qquad w_s^t = u_s^t \oplus v_s^t$$

This generalizes to arbitrary (finite or infinite) direct sums.

*Remark* 4.1.2. $\mathbb{W}$ is still a module over $\mathbf{P}$, as $\mathbf{Vec}^{\mathbf{P}}$ is closed under direct sums. The notion of direct sum of vector spaces coincides with coproduct in a general category. Also, as $\mathbf{Vec}^{\mathbf{P}}$ is an abelian category, there is a canonical isomorphism between (finite) products and coproducts.

**Definition 4.1.3 (Indecomposable persistence module [16, Sec.2.5]).** *A persistence module* $\mathbb{W}$ *is* indecomposable *if the only decompositions* $\mathbb{W} = \mathbb{U} \oplus \mathbb{V}$ *are the trivial decompositions* $\mathbb{W} \oplus 0$ *and* $0 \oplus \mathbb{W}$*, where* $0$ *is the* zero persistence module *consisting of zero vector spaces* $0$ *and identity morphisms between them.*

Our basic building blocks are *intervals*.

**Definition 4.1.4 (Interval [13, Def.2.3]).** *An* interval $\boldsymbol{\tau}$-module *(with* $\boldsymbol{\tau}$ *an orientation) or simply an* interval $\mathbb{I}_{\boldsymbol{\tau}}(b, d)$ *where* $1 \leq b \leq d \leq n = \mathrm{length}(\boldsymbol{\tau})$ *represents a feature which 'persists' over an interval specified by its birth index* $b$ *and its death index* $d$*.*

For example, given that persistence modules are build over a field $\mathbb{F}$, the interval $\mathbb{I}_{\boldsymbol{\tau}}(2, 3)$ with $\boldsymbol{\tau} = (\vec{\tau}, \overleftarrow{\tau}, \overleftarrow{\tau})$ is

$$1 \xrightarrow{0} \mathbb{F} \xleftarrow{1} \mathbb{F} \xleftarrow{1} 0.$$

When the orientation is known, we usually drop the subscript and write $\mathbb{I}(b, d)$. When $I$ denotes an interval in a totally ordered set (e.g. in $(\mathbb{R}, \leq)$), we write $\mathbb{I}(I)$ to denote the interval module supported over $I$.

Interval modules are indecomposable [16, Prop.2.6]. In the case of $\tau$-modules, the converse is also true. The theory of zigzag persistence is based on a special case of Gabriel's theorem [29] (that says which quiver representations are finitely decomposable) that applies to representations of $\mathbb{A}_n(\boldsymbol{\tau})$ quivers:

**Theorem 4.1.5.** *[13, Th.2.5] The indecomposable* $\boldsymbol{\tau}$*-modules are the intervals* $\mathbb{I}(b, d)$*. Equivalently, every* $\boldsymbol{\tau}$*-module can be written as a direct sum of intervals.*

However, for other kinds of persistence modules like ones that are not pointwise finite dimensional, not all indecomposable persistence modules admit an interval decomposition. See [16, Th.2.8(3)] for an example. In case that a persistence module is decomposable into interval modules, the next result asserts that the decomposition is unique.

*Proposition* 4.1.6 (Krull-Remak-Schmidt-Azumaya). Let $\mathbb{V}$ be a persistence module with two different interval decompositions

$$\mathbb{V} = \bigoplus_{m \in M} \mathbb{I}(b_m, d_m) = \bigoplus_{n \in N} \mathbb{I}(b'_n, d'_n).$$

Then there is some permutation $\sigma : M \to N$ such that $\mathbb{I}(b_m, d_m) = \mathbb{I}(b'_{\sigma(m)}, d'_{\sigma(m)})$ for all $m \in M$.

This is proved for $\boldsymbol{\tau}$-modules in [13, Prop.2.2], and for finite type persistence diagrams (i.e. ones that admit such interval decompositions) in $\mathbf{Vec}^{(\mathbb{R}, \leq)}$ in [11, Cor.4.7] and [16, Th.2.7]. The uniqueness of the interval decomposition can be thought of as the uniqueness up to isomorphism of coproducts.

Once our goal of decomposing persistence modules is formalized as finding a direct sum of intervals, realise that the essence of a decomposition is just the birth and death indices. This data is encoded in *persistence diagrams* and *barcodes*.

**Definition 4.1.7 (Persistence [13, Def.2.6]).** *Let $\mathbb{V}$ be a persistence module of length $n$ decomposable as $\mathbb{I}(b_1, d_1) \oplus \cdots \oplus \mathbb{I}(b_N, d_N)$. The* persistence *of $\mathbb{V}$ is defined to be the multiset*

$$\mathrm{Pers}(\mathbb{V}) = \{[b_j, d_j] \subseteq \{1, \dots, n\} \mid j = 1, \dots, N\}$$

Proposition 4.1.6 asserts that this definition is unique up to reordering.

$\mathrm{Pers}(\mathbb{V})$ is represented graphically as a set of lines against a single axis (the *barcode*), or as a multiset of points in $\mathbb{R}^2$ lying on or above the diagonal in the positive quadrant (the *persistence diagram*). See Figure 4.2 for an example of both. These graphical representations sometimes replace the notation of $\mathrm{Pers}(\mathbb{V})$ to $\mathrm{dgm}(\mathbb{V})$ [16, Sec.2.6] or $\mathcal{B}(\mathbb{V})$ [6], but they refer ultimately to the same data.

## 4.2 Decomposition algorithm for zigzag modules

Theorem 4.1.5 is the relation between quiver representations and zigzag modules: Out of the many classes of quiver representations, $\mathbb{A}_n$ quivers (graphs) have the same structure as zigzag modules, which can be decomposed as a sum of intervals thanks to this corollary of Gabriel's theorem. Isomorphism classes of persistence diagrams of zigzag modules are thus a special case of the classification problem for quiver representations.

Theorem 4.1.5 is an existence theorem. Carlsson and De Silva give in [13] a constructive proof of it and build an algorithm to compute the interval summands of a given $\boldsymbol{\tau}$-module. We will give here a brief summary of the proof and the algorithm, applying it at the end in an example. The general strategy is to construct the decomposition by an iterative process.

**Definition 4.2.1 (Submodule, summand, direct sum [13, 2.2]).** *A submodule $\mathbb{W}$ of a $\boldsymbol{\tau}$-module $\mathbb{V}$ is defined by subspaces for all $i$:*

$$W_i \subseteq V_i \quad s.t. \quad \begin{cases} \vec{\tau}_i(W_i) \subseteq W_{i+1} & \text{case } \vec{\tau}_i \\ W_i \supseteq \overleftarrow{\tau}_i(W_{i+1}) & \text{case } \overleftarrow{\tau}_i \end{cases}$$

*A submodule $\mathbb{W}$ is called a summand of $\mathbb{V}$ if there exists a submodule $\mathbb{X} \subseteq \mathbb{V}$ which is complementary to $\mathbb{W}$, i.e. $V_i = W_i \oplus X_i$ for all $i$. In that case $\mathbb{V}$ is the direct sum of $\mathbb{W}, \mathbb{X}$, and we write $\mathbb{V} = \mathbb{W} \oplus \mathbb{X}$.*

*Remark* 4.2.2. A submodule $\mathbb{W}$ of $\mathbb{V}$ corresponds to an object $\mathbb{W}$ of $\boldsymbol{\tau}$**Mod** with a monomorphism $\mathbb{W} \hookrightarrow \mathbb{V}$.

Filtered vector spaces are filtered objects (introduced in Section 3.4) in **Vec**. We briefly characterize this category, based on remarks in [13, 3.2], in Proposition 4.2.4 and Proposition 4.2.5.

**Definition 4.2.3 (Filt$_n$).** *The category of filtered vector spaces of length $n$ is defined by:*

- *Objects: Sequences of $n$ vector spaces connected by injections. $\mathbb{V}$ can be for example*
$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n.$$
- *Morphisms: A morphism $f\colon \mathbb{V} \to \mathbb{W}$ in $\mathbf{Filt}_n$ is a set of $n$ linear maps between two filtered vector spaces of length $n$ such that the diagram below commutes.*

$$\begin{array}{ccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_n \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & & & \big\downarrow{\scriptstyle f_n} \\ W_1 & \longrightarrow & W_2 & \longrightarrow & \cdots & \longrightarrow & W_n. \end{array} \tag{4.1}$$

Objects of $\mathbf{Filt}_n$ are filtered objects $\mathbb{V}$ in **Vec** ordered by $\mathbf{n}$, and morphisms in $\mathbf{Filt}_n$ are morphisms between filtered objects. $\mathbf{Filt}_n$ contains intervals of the form $\mathbb{I}(i,n) = (0,\dots,0,\mathbb{F},\dots,\mathbb{F})$, which constitute the indecomposables, and direct sums of them.

*Proposition* 4.2.4. $\mathbf{Filt}_n$ is not an abelian category (Definition 2.0.33), since morphisms do not generally have cokernels [13, 3.2].

*Proof.* Consider $\mathbb{V} = (0 \xrightarrow{0} \mathbb{F})$ and $\mathbb{W} = (\mathbb{F} \xrightarrow{\mathbb{1}} \mathbb{F})$ two filtered vector spaces in $\mathbf{Filt}_n$ with $n = 2$, and a morphism $f\colon \mathbb{V} \to \mathbb{W}$.

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{F} \\ \big\downarrow{\scriptstyle f_1=0} & & \big\downarrow{\scriptstyle f_2=\mathbb{1}} \\ \mathbb{F} & \xrightarrow{\mathbb{1}} & \mathbb{F} \end{array}$$

$f_1$ and $f_2$ have cokernels $\operatorname{coker}(f_1) = W_1/\operatorname{im}(f_1) = \mathbb{F}$ and $\operatorname{coker}(f_2) = W_2/\operatorname{im}(f_2) = 0$. However $\operatorname{coker}(f_1) \to \operatorname{coker}(f_2)$ is not a monomorphism, which is the condition for a sequence of vector spaces to be a filtered vector space. ∎

*Proposition* 4.2.5. $\mathbf{Filt}_n$ is a full subcategory (Definition 2.0.4) of $\mathbf{Vec}^{\mathbf{n}}$.

*Proof.* It is a subcategory, as objects in $\mathbf{Filt}_n$ are indexed by $\mathbf{n}$. Not all objects in $\mathbf{Vec}^{\mathbf{n}}$ are in $\mathbf{Filt}_n$ though, as only those which are filtrations are in $\mathbf{Filt}_n$. Next, let $\mathbb{V}, \mathbb{W}$ be two filtrations. As objects of $\mathbf{Vec}^{\mathbf{n}}$, $\mathbb{V}$ and $\mathbb{W}$ are functors from $\mathbf{n}$ to **Vec**. The commutativity condition of a natural transformation $\alpha\colon \mathbb{V} \to \mathbb{W}$ is equivalent to (4.1), thus the $n$ components of $\alpha$ are the linear maps $\alpha_i \colon V_i \to W_i$ that define a morphism in $\mathbf{Filt}_n$. Therefore every morphism in $\mathbf{Vec^n}$ is also in $\mathbf{Filt}_n$, making the latter a full subcategory of the former.

A specific filtered vector space can be obtained from a $\boldsymbol{\tau}$-module using the following operator.

**Definition 4.2.6 (Right-filtration operator [13, Def.3.1]).** *The* right-filtration *of a $\boldsymbol{\tau}$-module $\mathbb{V}$ of length $n$ is a filtered vector space $R(\mathbb{V}) = (R_0, R_1, \ldots, R_n) \in$ $\mathbf{Filt}_{n+1}$[(1)] where the $R_i$ are subspaces of $V_n$, the right-most vector space of $\mathbb{V}$, satisfying the inclusion relations*

$$0 = R_0 \subseteq R_1 \subseteq \cdots R_n = V_n.$$

*$R(\mathbb{V})$ is defined recursively, starting with the left-most vector space $V_1$ and incrementing in each step the scope to $V_1 \leftrightarrow V_2$, $V_1 \leftrightarrow V_2 \leftrightarrow V_3$ and so on.*

- *Base case: $R(V_1) = (0, V_1) = (R_0, R_1)$.*
- *Recursive step: Suppose we have alredy defined $R(V_1 \leftrightarrow \cdots \leftrightarrow V_k) = (R_0, R_1, \cdots, R_k)$ up to a step $k < n$. Depending on $\tau_k$ being $\vec{\tau}_k$ or $\overleftarrow{\tau}_k$, we define the next step as*

$$R(V_1 \leftrightarrow \cdots \leftrightarrow V_{k+1}) = \begin{cases} (\vec{\tau}_k(R_0), \vec{\tau}_k(R_1), \cdots, \vec{\tau}_k(R_k), V_{k+1}) & \textit{case } \vec{\tau}_k \\ (0, \overleftarrow{\tau}_k^{-1}(R_0), \overleftarrow{\tau}_k^{-1}(R_1), \cdots, \overleftarrow{\tau}_k^{-1}(R_k)) & \textit{case } \overleftarrow{\tau}_k \end{cases} \tag{4.2}$$

*Example 4.2.7.* Let $\mathbb{V} = (V_1 \xrightarrow{\vec{\tau}_1} V_2)$, with $V_1 = \mathbb{F}^2$, $V_2 = \mathbb{F}^3$ and

$$\vec{\tau}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $R(V_1) = (0, V_1)$ where

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V_1 = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$$

and the next right-filtration would be $R(V_1 \xrightarrow{\vec{\tau}_1} V_2) = (\vec{\tau}_1(0), \vec{\tau}_1(V_1), V_2)$ with

$$\vec{\tau}_1(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{\tau}_1(V_1) = \langle \vec{\tau}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{\tau}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \quad V_2 = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle.$$

**Definition 4.2.8 (Birth-time index [13, Def.3.6]).** *The* birth-time index $b(\boldsymbol{\tau}) =$ *$(b_1, b_2, \ldots, b_n)$ is a vector of integers $b_i$ which indicate the birth times associated with the subquotients $R_i/R_{i-1}$ of the right-filtration of a $\boldsymbol{\tau}$-module. This is defined recursively:*

(1) We have defined an object of $\mathbf{Filt}_n$ as a sequence of $n$ vector spaces connected by injections, by which $R(\mathbb{V})$ is in $\mathbf{Filt}_{n+1}$ because it consists of $n+1$ vector spaces.

- *Base case: For* $\mathbb{V}$ *of length* $1$, $b(\boldsymbol{\tau}) = (1)$.
- *Given* $b(\tau_1, \dots, \tau_{k-1}) = (b_1, \dots, b_k)$,

$$b(\tau_1, \dots, \tau_{k-1}, \tau_k) = \begin{cases} (b_1, \dots, b_k, k+1) & \textit{case } \vec{\tau}_k \\ (k+1, b_1, \dots, b_k) & \textit{case } \overleftarrow{\tau}_k \end{cases}$$

**Definition 4.2.9 ([13, Def.3.15]).** *A* $\boldsymbol{\tau}$*-module* $\mathbb{V}$ *is* (right-)streamlined *if each* $\vec{\tau}$ *is injective and each* $\overleftarrow{\tau}$ *is surjective.*

The following lemmas and theorem lead up to Theorem 4.2.13, which together with Algorithm 1 provide a constructive way of obtaining a decomposition of an arbitrary $\tau$-module.

*Lemma* 4.2.10 (Decomposition Lemma [13, Lemma 3.18]). Le $\mathbb{V}$ be a streamlined $\boldsymbol{\tau}$-module and let $\mathcal{R} = R(\mathbb{V})$. For any decomposition $\mathcal{R} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_N$, there exists a unique decomposition $\mathbb{V} = \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_N$ such that $\mathcal{S}_i = R(\mathbb{W}_j)$ for all $j$.

**Theorem 4.2.11 (Interval decomposition for streamlined modules [13, Th.3.19]).** *Let* $\mathbb{V}$ *be a streamlined* $\boldsymbol{\tau}$*-module of length* $n$*, and write* $\dim(R(\mathbb{V})) = (c_1, \dots, c_n)$ *and* $b(\boldsymbol{\tau}) = (b_1, \dots, b_n)$*. Then there is an isomorphism of* $\boldsymbol{\tau}$*-modules*

$$\mathbb{V} \cong \bigoplus_{q \le i \le n} c_i \mathbb{I}(b_i, n)$$

*where* $\mathbb{I}(b_i, n)$ *is a streamlined interval born in* $b_i$ *and dying in* $n$*.*

Given a $\boldsymbol{\tau}$-module $\mathbb{V} = V_1 \overset{\tau_1}{\longleftrightarrow} \cdots \overset{\tau_{n-1}}{\longleftrightarrow} V_n$, the truncation of $\mathbb{V}$ to length $k$ (with $1 \le k \le n$) is denoted $\mathbb{V}[k] = (V_1 \overset{\tau_1}{\longleftrightarrow} \cdots \overset{\tau_{k-1}}{\longleftrightarrow} V_k)$, and its orientation $\boldsymbol{\tau}[k]$ is a truncation of $\boldsymbol{\tau}$.

*Lemma* 4.2.12 ([13, Lemma 4.3]). Let $\mathbb{V} = V_1 \overset{\tau_1}{\longleftrightarrow} \cdots \overset{\tau_{n-1}}{\longleftrightarrow} V_n$ be a $\boldsymbol{\tau}$-module of length $n$. Then there exists a direct-sum decomposition

$$\mathbb{V} = \mathbb{V}^1 \oplus \mathbb{V}^2 \oplus \cdots \oplus \mathbb{V}^n$$

where each $\mathbb{V}^k$ is a $\boldsymbol{\tau}[k]$-module supported over the indices $\{1, 2, \dots, k\}$ and is right-streamlined over that range.

**Theorem 4.2.13 (Interval decomposition [13, Th.4.1]).** *Let $\mathbb{V}$ be a $\boldsymbol{\tau}$-module. For $1 \leq k \leq n$, write the birth vector*

$$(b_1^k, b_2^k, \dots, b_k^k) = b(\boldsymbol{\tau}[k]).$$

*Writing $\mathcal{R} = R(\mathbb{V}[k])$, define the* dimension vector *as*

$$(c_1^k, c_2^k, \dots, c_k^k) = \begin{cases} \dim(\mathcal{R}_k \cap \ker(\vec{\tau}_k)) \ ^{(2)} & \textit{case } \vec{\tau} \\ \dim(\mathcal{R}_k) - \dim(\mathcal{R}_k \cap \operatorname{im}(\overleftarrow{\tau}_k)) & \textit{case } \overleftarrow{\tau} \end{cases} \tag{4.3}$$

*when $k \neq n$, and*

$$(c_1^n, c_2^n, \dots, c_n^n) = \dim(\mathcal{R}_n).$$

*Then*

$$\mathbb{V} = \bigoplus_{1 \leq k \leq n} \mathbb{V}^k \cong \bigoplus_{1 \leq k \leq n} \left\{ \bigoplus_{1 \leq i \leq k} c_i^k \mathbb{I}(b_i^k, k) \right\}. \tag{4.4}$$

Equation (4.3) has to be understood elementwise: $\mathcal{R}_k \cap \ker(\vec{\tau}_k)$ is the sequence of vector spaces $(\mathcal{R}_0^k \cap \ker(\vec{\tau}_k), \dots, R_k^k \cap \ker(\vec{\tau}_k))$ and $\dim(\mathcal{R}_k \cap \ker(\vec{\tau}_k))$ is the vector $(c_1^k, \dots, c_k^k)$. Similarly for $\overleftarrow{\tau}$.

A priori zigzag modules generalize persistence modules because they allow the functions between spaces to be forwards or backwards ($\vec{\tau}$ or $\overleftarrow{\tau}$), but they force the $\vec{\tau}$ functions to be injective and $\overleftarrow{\tau}$ functions to be surjective in order to obtain a right-streamlined module. All intervals into which a module is decomposed die at $n$. However, later this restriction is lifted, firstly decomposing an arbitrary $\boldsymbol{\tau}$-module in a direct sum of modules which are right-streamlined in their rank by Lemma 4.2.12, and then decomposing these into intervals with Theorem 4.2.11. Finally, one obtains a double direct sum (4.4) of intervals $\mathbb{I}(b_i^k, k)$ with their respective multiplicities $c_i^k$, and the intervals are not forced to be born/die at a specific time.

The algorithm to compute the right-filtration $\mathcal{R}_k = R(\mathbb{V}[k])$, the birth-time index $b(\boldsymbol{\tau}[k])$ and the dimensions $c_i^k$ from (4.3) iteratively through $k = 1, 2, \dots$ is given next:

**Algorithm 1** Interval decomposition algorithm [13]

$k = 1$ ▷ Initialization

$\mathcal{R}_1 = (0, V_1)$
$b(\boldsymbol{\tau}[1]) = (1)$

**for** $k = 1, 2, \ldots, n-1$ **do** ▷ Iterative step

Calculate $\mathcal{R}_{k+1}$ from $\mathcal{R}_k = (R_0^k, R_1^k, \ldots, R_k^k)$ using (4.2):

$$(R_0^{k+1}, R_1^{k+1}, \ldots, R_{k+1}^{k+1}) = \begin{cases} (\vec{\tau}_k(R_0^k), \vec{\tau}_k(R_1^k), \ldots, \vec{\tau}_k(R_k^k), V_{k+1}) & \text{case } \vec{\tau} \\ (0, \overleftarrow{\tau}_k^{-1}(R_0^k), \overleftarrow{\tau}_k^{-1}(R_1^k), \ldots, \overleftarrow{\tau}_k^{-1}(R_k^k)) & \text{case } \overleftarrow{\tau} \end{cases}$$

Calculate $b(\boldsymbol{\tau}[k+1])$ from $b(\boldsymbol{\tau}[k]) = (b_1^k, b_2^k, \ldots, b_k^k)$ using Definition 4.2.8:

$$(b_1^{k+1}, \ldots, b_{k+1}^{k+1}) = \begin{cases} (b_1^k, \ldots, b_k^k, k+1) & \text{case } \vec{\tau} \\ (k+1, b_1^k, \ldots, b_k^k) & \text{case } \overleftarrow{\tau} \end{cases}$$

Calculate $(c_1^k, \ldots, c_k^k)$ using formula (4.3) from Theorem 4.2.13:

$$(c_1^k, c_2^k, \ldots, c_k^k) = \begin{cases} \dim(\mathcal{R}_k \cap \ker(\vec{\tau}_k)) & \text{case } \vec{\tau} \\ \dim(\mathcal{R}_k) - \dim(\mathcal{R}_k \cap \operatorname{im}(\overleftarrow{\tau}_k)) & \text{case } \overleftarrow{\tau} \end{cases}$$

**end for**

$k = n$ ▷ Terminal step

$(c_1^n, \ldots, c_n^n) = \dim(\mathcal{R}(\mathbb{V}))$

**return** For $1 \le i \le k \le n$, the interval $\mathbb{I}(b_i^k, k)$ occurs with multiplicity $c_i^k$.

We will next illustrate how the algorithm works with a simple example.

*Example* 4.2.14. Let $K$ and $L$ be two simplicial complexes with intersection $K \cap L$ such that the interval $[1, 3]$ of $H_1(K) \leftarrow H_1(K \cap L) \rightarrow H_1(L)$ is non zero, given by Figure 4.1. Algorithm 1 computes the complete decomposition. The multiplicity of the interval $[1, 3]$ has a certain interpretation in terms of the structure of both simplicial complexes.


(a) $K$.                (b) $L$.

Figure 4.1: Two simplicial complexes defined over the set of vertices $\{x_0, x_1, x_2, x_3, x_4\}$. Note the grayed out areas: $x_2$ is not included in $K$, and $x_0$ is not in $L$.

We define two simplicial complexes over the five vertices above, and their corresponding 1-homology vector spaces generated by vectors expressed each one in their own basis.

$$K = \{(x_0,x_1),(x_0,x_4),(x_1,x_3),(x_1,x_4),(x_3,x_4)\} \quad H_1(K) = \left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\rangle \cong \mathbb{F}^2$$

$$L = \{(x_1,x_2),(x_1,x_3),(x_1,x_4),(x_2,x_3),(x_3,x_4)\} \quad H_1(L) = \left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\rangle \cong \mathbb{F}^2$$

$$K \cap L = \{(x_1,x_3),(x_1,x_4),(x_3,x_4)\} \quad H_1(K\cap L) = \langle 1 \rangle \cong \mathbb{F}$$

The corresponding $\boldsymbol{\tau}$-module $\mathbb{V}$ of length $n = 3$ is:

$$H_1(K) \xleftarrow{\overleftarrow{\tau}} H_1(K\cap L) \xrightarrow{\overrightarrow{\tau}} H_1(L)$$

where we can choose explicit expressions for $\overleftarrow{\tau} = \begin{bmatrix}0\\1\end{bmatrix}$ and $\overrightarrow{\tau} = \begin{bmatrix}1\\0\end{bmatrix}$. Both functions are injective, non-surjective. Following Algorithm 1:

$$\mathcal{R}_1 = (0, H_1(K))$$
$$\mathcal{R}_2 = (0, \overleftarrow{\tau}^{-1}(0), \overleftarrow{\tau}^{-1}(H_1(K))) = (0, \langle 0\rangle, \langle 1\rangle) = (0, 0, H_1(K\cap L))$$
$$\mathcal{R}_3 = (\overrightarrow{\tau}(0), \overrightarrow{\tau}\overleftarrow{\tau}^{-1}(0), \overrightarrow{\tau}\overleftarrow{\tau}^{-1}(H_1(K)), H_1(L)) = (0, 0, \langle \begin{bmatrix}1\\0\end{bmatrix} \rangle, H_1(L))$$

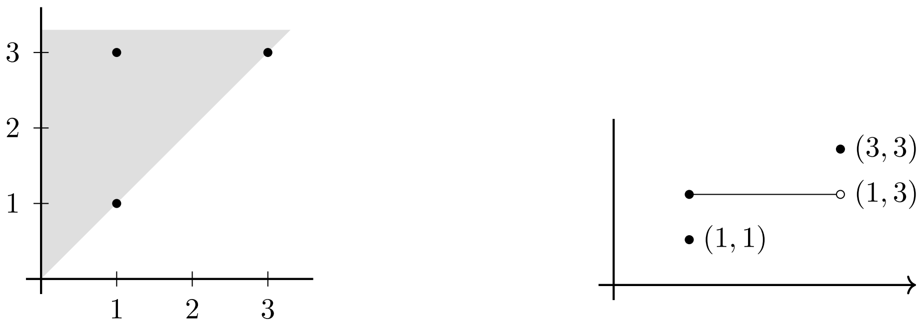
| $k$ | $\mathcal{R}_k$ | $b(\boldsymbol{\tau}[k])$ | $(c_1^k,\dots,c_k^k)$ | $(r_1^k,\dots,r_k^k)$ |
|---|---|---|---|---|
| 1 | $(0,\mathbb{F}^2)$ | $(1)$ | $(1)$ | $(2)$ |
| 2 | $(0,0,\mathbb{F})$ | $(2,1)$ | $(0,0)$ | $(0,1)$ |
| 3 | $(0,0,\mathbb{F},\mathbb{F}^2)$ | $(2,1,3)$ | $(0,1,1)$ | $(0,1,1)$ |

| $i$ | $k$ | $\mathbb{I}(b_i^k,k)$ | $c_i^k$ |
|---|---|---|---|
| 1 | 1 | $\mathbb{I}(1,1)$ | 1 |
| 1 | 2 | $\mathbb{I}(2,2)$ | 0 |
| 1 | 3 | $\mathbb{I}(2,3)$ | 0 |
| 2 | 2 | $\mathbb{I}(1,2)$ | 0 |
| 2 | 3 | $\mathbb{I}(1,3)$ | 1 |
| 3 | 3 | $\mathbb{I}(3,3)$ | 1 |

Table 4.1: Intervals and corresponding dimensions.

The dimension of a right-filtration $\mathcal{R}_k$ is defined as the dimensions of the succesive subquotients $r_i^k = \dim(R_i^k/R_{i-1}^k)$. The interval decomposition can be obtained from Table 4.1, or more compactly:

$$\mathbb{V} = \mathbb{I}(1,1) \oplus \mathbb{I}(1,3) \oplus \mathbb{I}(3,3)$$




(a) Persistence diagram. (b) Barcode.

Figure 4.2: Graphical representations of $\mathbb{V}$.

The multiplicity of $\mathbb{I}(1,3)$ ($c_2^3 = 1$) shows that there is a 1-homology feature (i.e. a hole) that persists through the entire $\boldsymbol{\tau}$-module. This is because the hole $\partial(x_1, x_3, x_4)$ is contained in both simplicial complexes $K$ and $L$.

## 4.3 Decomposition of persistence modules over $\mathbb{R}$

We have just seen a way to decompose a certain kind of persistence modules. This decomposition into *intervals* allow us to describe *features* that persist through a module. The idea of this section is to give an overview of a theorem that, even though it may seem more abstract, enables using barcodes not only to decompose persistence modules in $\mathbf{Vec}^{(\mathbb{Z},\leq)}$ but also to decompose pointwise finite dimensional (p.f.d.) persistence modules in $\mathbf{Vec}^{(\mathbb{R},\leq)}$. P.f.d. modules have $\dim(V_s) < \infty$ for any $s \in \mathbb{R}$, i.e. they are objects of $\mathbf{FinVec}^{(\mathbb{R},\leq)}$ [7, Def.2.1]. Although these results are proved in [19] for a totally ordered indexing set, we will present them here for the particular case of $(\mathbb{R}, \leq)$.

**Theorem 4.3.1 ([19, Th.1.1]).** *Any pointwise finite-dimensional persistence module is a direct sum of interval modules.*

We start by defining a property of persistence modules which is more general than pointwise finite-dimensionality.

**Definition 4.3.2 ([19, Sec. 1]).** *Let $\mathbb{V}$ be a persistence module in $\mathbf{Vec}^{(\mathbb{R},\leq)}$. $\mathbb{V}$ has the descending chain condition on images and kernels (DCCIK) if both conditions are met:*

- *For all $t, s_1, s_2, \dots \in \mathbb{R}$ such that $t \geq s_1 > s_2 > \dots$, the sequence*

$$V_t \supseteq \operatorname{im} v_{s_1}^t \supseteq \operatorname{im} v_{s_2}^t \supseteq \cdots$$

*stabilizes, i.e. there exists $n$ such that* $\operatorname{im} v^t_{s_n} = \operatorname{im} v^t_{s_{n+1}} = \dots$.

- *For all $t, r_1, r_2, \dots \in R$ such that $t \leq \dots < r_1 < r_2$, the sequence*

$$V_t \supseteq \ker v^{r_1}_t \supseteq \ker v^{r_2}_t \supseteq \cdots$$

*stabilizes, i.e. there exists $m$ such that* $\ker v^{r_m}_t = \ker v^{r_{m+1}}_t = \dots$.

P.f.d. persistence modules in particular satisfy DCCIK. The main theorem is the following:

**Theorem 4.3.3 ([19, Th.1.2]).** *Any persistence module with DCCIK is a direct sum of interval modules.*

The concept of *cuts* is presented next. Cuts can be used to define features that are born and die before and after a particular time in $\mathbb{R}$ without specifying closed or open ends yet. A related technique are decorated points [6, Sec.2.3].

**Definition 4.3.4 (Cut [19, Sec.2]).** *Let $\mathbb{V}$ be a persistence module with DCCIK. A* cut *$c$ for $\mathbb{R}$ is a pair of subspaces $(c^-, c^+)$ of $\mathbb{R}$*

$$c = (c^-, c^+) \qquad s.t. \qquad \begin{cases} \mathbb{R} = c^- \cup c^+ \\ s < t \qquad\qquad \forall s \in c^-,\ \forall t \in c^+ \end{cases}$$

- If $t \in c^+$, define the subspaces $\operatorname{im}^-_{ct} \subseteq \operatorname{im}^+_{ct} \subseteq V_t$ by

$$\operatorname{im}^-_{ct} = \bigcup_{s \in c^-} \operatorname{im} v^t_s \qquad ; \qquad \operatorname{im}^+_{ct} = \bigcap_{\substack{s \in c^+ \\ s \leq t}} \operatorname{im} v^t_s$$

  $\operatorname{im}^-_{ct}$ are features that persist at least from some time in $c^-$ until $t$, and $\operatorname{im}^+_{ct}$ are features that persist from the beginning of $c^+$ until $t$. The quotient $\operatorname{im}^+_{ct} / \operatorname{im}^-_{ct}$ is precisely the features that are born at the beginning of $l^+$ and are alive at least until $t$.

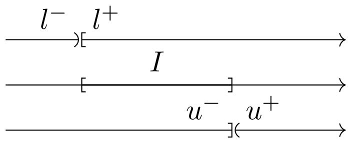
- If $t \in c^-$, define the subspaces $\ker^-_{ct} \subseteq \ker^+_{ct} \subseteq V_t$ by

$$\ker^-_{ct} = \bigcup_{\substack{r \in c^- \\ t \leq r}} \ker v^r_t \qquad ; \qquad \ker^+_{ct} = \bigcap_{r \in c^+} \ker v^r_t$$

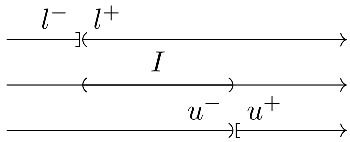
  $\ker^-_{ct}$ are features that die at some point in $c^-$ and $\ker^+_{ct}$ are features that are dead at some point in $c^+$. The inclusion $\ker^-_{ct} \subseteq \ker^+_{ct}$ is clear, as anything dead in $c^-$ is also dead in $c^+$.

For any interval $I$ in $\mathbb{R}$ there are unique cuts $l, u$ such that $I = l^+ \cap u^-$, where

$$l^- = \{t : t < s\ \forall s \in I\}, \quad l^+ = \{t : \exists s \in I : t \geq s\}$$
$$u^- = \{t : \exists s \in I : t \leq s\}, \quad u^+ = \{t : t > s\ \forall s \in I\}$$




(a) Cuts for $I$ a closed interval.




(b) Cuts for $I$ an open interval.

Given $t \in I$, we define $V_{It}^- \subseteq V_{It}^+ \subseteq V_t$:

$$\begin{aligned} V_{It}^- &= \left(\mathrm{im}_{lt}^- \cap \ker_{ut}^+\right) + \left(\mathrm{im}_{lt}^+ \cap \ker_{ut}^-\right) \\ V_{It}^+ &= \mathrm{im}_{lt}^+ \cap \ker_{ut}^+ \end{aligned}$$

Intuitively, $V_{It}^+$ represents features alive at least from the beginning of $l^+$ and at the beginning of $u^+$. $V_{It}^-$ includes modules born before $l$ and dead after $t$, and modules born before $t$ and dead before $u$. For $s \leq t$ in $I$, $v_s^t$ induces maps $V_{Is}^\pm \to V_{It}^\pm$ [19, Lemma 3.1][(3)]. One can then consider the inverse limit (Example 2.0.40)

$$V_I^\pm = \varprojlim_{t \in I} V_{It}^\pm$$

$V_I^\pm$ can be equivalently expressed as an equalizer of a product, i.e. the subset of the product of all $V_{It}^\pm$ in the interval, which are connected by $v_s^t$.

$$V_I^\pm = \left\{ x^\pm \in \prod_{t \in I} V_{It}^\pm \;\middle|\; x_b^\pm = v_a^b(x_a^\pm) \quad \forall a, b \in I \text{ with } a \leq b \right\}$$

Letting $\pi_t \colon V_I^+ \to V_{It}^+$ denote the natural map (projection of the $t$-th component), one can identify

$$V_I^- = \bigcap_{t \in I} \pi_t^{-1}(V_{It}^-) \subseteq V_I^+ \tag{4.5}$$

where the intersection has to be understood elementwise for each $t \in I$.

*Lemma* 4.3.5 ([19, Lemma 4.1]). For all $t \in I$, the induced map

$$\bar{\pi}_t \colon V_I^+ / V_I^- \to V_{It}^+ / V_{It}^-$$

is an isomorphism.

We know by (4.5) that $V_I^- \subseteq V_I^+$. We can choose a vector space complement (orthogonal complement) $W_I^0$ to $V_I^-$ in $V_I^+$:

$$V_I^+ = W_I^0 \oplus V_I^- \tag{4.6}$$

(3) We notate $V_{Is}^\pm \to V_{It}^\pm$ to describe two different maps $V_{Is}^+ \to V_{It}^+$ and $V_{Is}^- \to V_{It}^-$.

For $t \in I$, the restriction of $\pi_t$ to $W_I^0$ is injective by Lemma 4.3.5:

$$\begin{aligned} \pi_t\colon V_I^+ &\to V_{It}^+ \\ W_I^0 &\mapsto W_{It} \end{aligned}$$

*Lemma* 4.3.6 ([19, Lemma 5.1]). The assignment

$$W_{It} = \begin{cases} \pi_t(W_I^0) & (t \in I) \\ 0 & (t \notin I) \end{cases}$$

defines a submodule $\mathbb{W}_I$ of the persistence module.

*Lemma* 4.3.7 ([19, Lemma 5.2]). $V_{It}^+ = W_{It} \oplus V_{It}^-$ for all $t \in I$.

*Proof.* According to [19, Lemma 5.2], this follows from Lemma 4.3.5. We check this is the case starting from (4.6):

$$\begin{aligned} V_I^+ &= W_I^0 \oplus V_I^- \\ V_I^+/V_I^- &= W_I^0 \end{aligned}$$

Applying $\pi_t$ on each side:

$$V_{It}^+/V_{It}^- = W_{It}$$

and undoing the quotient of vector spaces:

$$V_{It}^+ = W_{It}^- \oplus V_{It}^-$$

Alternatively, applying $\pi_t$ to (4.6):

$$\pi_t V_I^+ = \pi_t(W_I^0 \oplus V_I^-) = \pi_t(W_I^0) \oplus \pi_t V_I^- = W_{It} \oplus V_{It}^-$$

∎

**Definition 4.3.8 (Section [19, Sec.6]).** *A section of a vector space $U$ is a pair $(F^-, F^+)$ of subspaces $F^- \subseteq F^+ \subseteq U$. We say that a set of sections $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$:*

- *is disjoint if:*
$$F_\lambda^+ \subseteq F_\mu^- \quad \text{or} \quad F_\mu^+ \subseteq F_\lambda^- \qquad \forall \lambda \neq \mu$$
- *covers $U$ if for all subspaces $X \subsetneq U$ there exists some $\lambda$ with*
$$X + F_\lambda^- \neq X + F_\lambda^+.$$

The proof of Theorem 4.3.3 considers the section $(F_{It}^-, F_{It}^+)$ of $V_t$ given by

$$F_{It}^\pm = \mathrm{im}_{lt}^- + \ker_{ut}^\pm \cap \mathrm{im}_{lt}^+$$

which turns out to be disjoint and covers $V_t$. By Lemma 4.3.7, $F^+_{It} = W_{It} \oplus F^-_{It}$ and by [19, Lemma 6.1] $\mathbb{V}$ is the direct sum of the submodules $\mathbb{W}_I$, which are direct sums of interval modules $\mathbb{I}(I)$ by [19, Lemma 5.3].

We have thus reviewed decomposition proofs of two kinds of persistence modules, one of them about zigzag modules alongside a constructive algorithm, and another focusing solely on the categorical techniques for persistence modules over $\mathbb{R}$. These two results support the soundness of taking the barcode of some persistence modules as a representation of the topological information within them. However, it is important to remember that there are other kinds of persistence modules that are not *representation-finite*, i.e. whose set of indecomposables is infinite. This is for example the case of ladder modules with $n \geq 5$, as discussed in [27, Theorems 3 and 4].

# 5 Comparison and stability of persistence modules

Once the concept of persistent modules is introduced and we know how to decompose certain classes of them, a reasonable question to ask is how we can compare them and obtain a 'distance' between their topological features.

The notion of 'closeness' or 'distance' of persistence modules was first quantified by the bottleneck distance. However, other kinds of distances are also used. For example, distance in input data is usually quantified by the Gromov-Hausdorff distance, calculated over a metric space. In this chapter we are going to deal mainly with the interleaving distance, applied both to ladder modules and Vietoris-Rips filtrations. Filtering functions are compared using the $L^\infty$ distance.

Furthermore, this motivates the study of stability, arguably one of the most important results in topological data analysis. Stability theorems relate closeness in different domains, e.g. how close two persistence modules are to how close the input data is. Figure 5.1 gives an overview of distances and stability theorems discussed in this chapter.

But first we start by motivating its use by defining the most fundamental metric structure to endow onto the space of persistence modules, the bottleneck distance.

## 5.1 Bottleneck distance

The bottleneck distance was first introduced in [25] to study the stability of persistence diagrams.

**Definition 5.1.1 (Bottleneck distance).** *Let $X, Y$ be multisets of points. The bottleneck distance between $X$ and $Y$ is*

$$\mathrm{d_B}(X, Y) = \inf_{\gamma} \sup_{x} \|y - \gamma(x)\|_\infty$$

*where $x \in X$ and $y \in Y$ and $\gamma$ are bijections from $X$ to $Y$.*

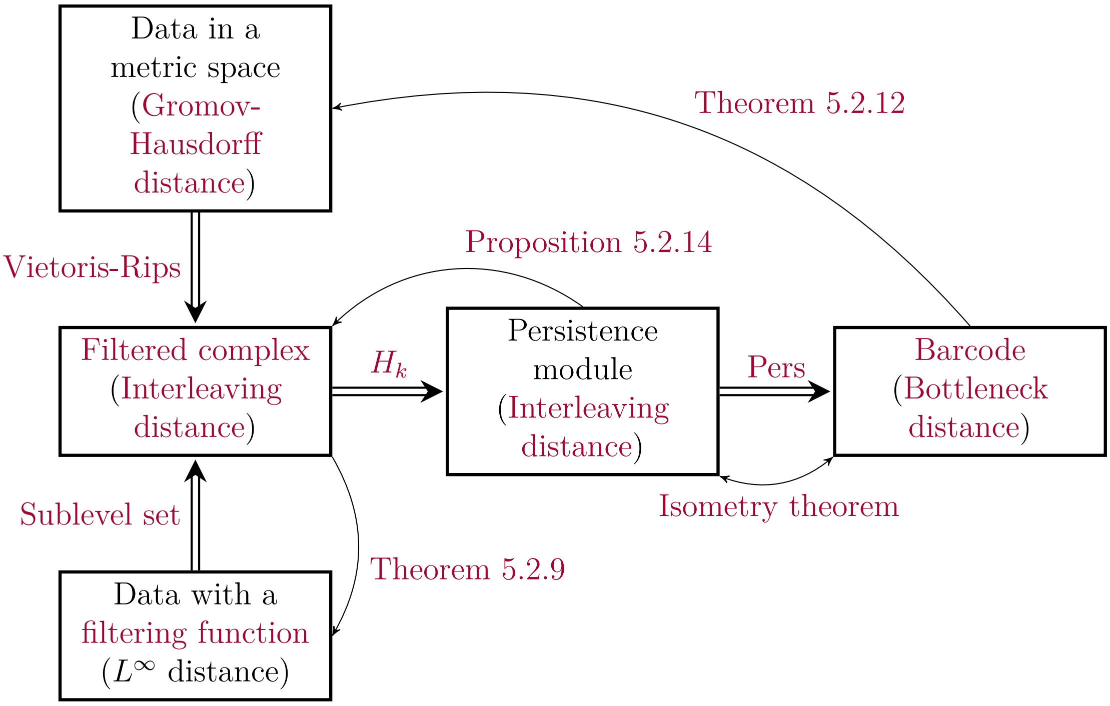



Figure 5.1: TDA workflow (double arrows) and stability theorems (single arrows). One can start either with data in a metric space or with a filtering function, from which one obtains a filtered simplicial complex. The homology functor produces the persistence module, and finally this is represented by a barcode. The 'comparison distance' used in each box is indicated between parenthesis. Single arrows indicate stability theorems between metrics in different boxes. Inspired by the diagram in [10, p.8].

The bottleneck distance between two persistence modules $\mathbb{V}$, $\mathbb{W}$ is understood to be between their persistence diagrams $\mathrm{Pers}(\mathbb{V})$ and $\mathrm{Pers}(\mathbb{W})$ extended with a set of diagonal points with infinite multiplicity [27, Eq.3.9].

This distance has some restrictive conditions like the continuity of the filtration function and triangulability of the space, and the next distance is introduced in [14] to address these limitations.

## 5.2 Interleaving distance

Bottleneck distances are difficult to generalize to the multidimensional case (that is, where the indexing set is multidimensional), and the interleaving distance provides a good alternative [33, p.4].

To present the interleaving distance, first we have to talk about isomorphic persistence modules. Two persistence modules $\mathbb{V}, \mathbb{W} \in \mathbf{Vec}^{(\mathbb{R},\leq)}$ are isomorphic if there are maps $\Phi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W})$, $\Psi \in \mathrm{Hom}(\mathbb{W}, \mathbb{V})$ such that $\Phi\Psi = \mathbb{1}_{\mathbb{V}}$ and $\Psi\Phi = \mathbb{1}_{\mathbb{W}}$ [16, Ch.4].

This condition is however too strong against uncertainty and noise in data. Therefore an 'approximate isomorphism' is defined in terms of shifted morphisms, which we defined for filtrations in Definition 3.4.9. The definition adapted for persistence modules in $\mathbf{Vec}^{(\mathbb{R},\leq)}$ is given next.

**Definition 5.2.1 (Shifted morphism [16, Sec.4.1]).** *A $\delta$-*shifted morphism *(or* shifted homomorphism*) $\Phi$ of degree $\delta$ between persistence modules in $\mathbf{Vec}^{(\mathbb{R},\leq)}$ shifts the value of the persistent index by $\delta$:*

$$\Phi = (\phi_t \colon V_t \to W_{t+\delta} \mid t, \delta \in \mathbb{R}) \in \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{W})$$

*such that the following diagram commutes:*

$$\begin{array}{ccccc} V_s & \xrightarrow{v_s^t} & V_t & & \\ & \searrow^{\phi_s} & & \searrow^{\phi_t} & \\ & & W_{s+\delta} & \xrightarrow{w_{s+\delta}^{t+\delta}} & W_{t+\delta} \end{array}$$

$\mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{W})$ is the set of all $\delta$-shifted morphisms from $\mathbb{V}$ to $\mathbb{W}$. $\mathrm{End}^{\delta}(\mathbb{V})$ are endomorphisms of degree $\delta$ in $V$, i.e. families of morphisms from $\mathbb{V}$ to itself shifting the persistent index by $\delta$. An important example is

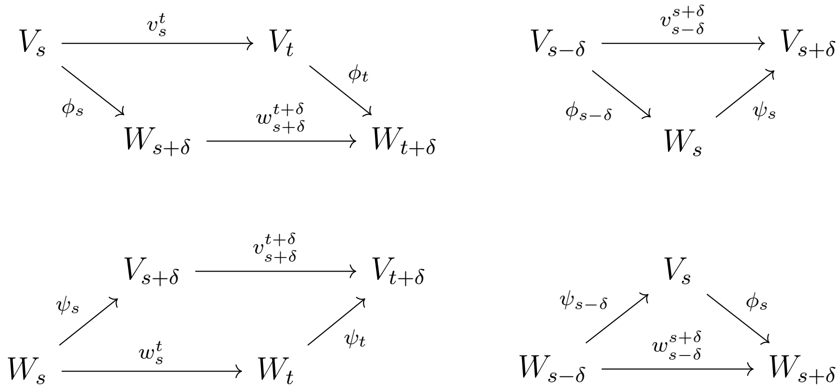
$$\mathbb{1}_{\mathbb{V}}^{\delta} = (v_t^{t+\delta} \colon V_t \to V_{t+\delta}) \in \mathrm{End}^{\delta}(\mathbb{V})$$

Note that by definition $\Phi \mathbb{1}_{\mathbb{U}}^{\delta} = \mathbb{1}_{\mathbb{V}}^{\delta} \Phi$ because composition is unique and associative. Both $\mathrm{Hom}^{\delta}$ and $\mathrm{End}^{\delta}$ are given categorical definitions in terms of translation functors in [11, Sec.3], [10, Sec.2.2]. Once a $\delta$-shifted morphism between persistence modules is defined, we can combine two $\delta$-shifted morphisms going back and forth between two persistence modules to define a distance between them.

**Definition 5.2.2 ($\delta$-interleaved persistence modules [16, Sec.4.2]).** *Two persistence modules $\mathbb{V}$, $\mathbb{W}$ are said to be $\delta$-*interleaved *if there are maps $\Phi \in \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{W})$, $\Psi \in \mathrm{Hom}^{\delta}(\mathbb{W}, \mathbb{V})$ such that*

$$\Phi\Psi = \mathbb{1}_{\mathbb{V}}^{2\delta} \qquad \Psi\Phi = \mathbb{1}_{\mathbb{W}}^{2\delta}$$

*and these four diagrams commute for all $s \leq t$:*




(5.1)

**Definition 5.2.3 (Interleaving distance [16, Sec.5.1]).** *The* interleaving distance *is defined as the infimum over such $\delta$-interleavings:*

$$d_I(\mathbb{V}, \mathbb{W}) = \inf\{\delta \mid \mathbb{V}, \mathbb{W} \text{ are } \delta\text{-interleaved}\}$$

Two important inequalities involving interleaving distances are the triangle inequality for any three persistence modules $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$ [16, Prop.5.3]:

$$d_I(\mathbb{U}, \mathbb{W}) \leq d_I(\mathbb{U}, \mathbb{V}) + d_I(\mathbb{V}, \mathbb{W})$$

and the next proposition:

*Proposition* 5.2.4 ([10, Th.2.5.5]). Let $F, G \in \mathbf{C}^{\mathbf{P}}$ be two persistence modules, where $\mathbf{P}$ is any preordered set as a category, and $\mathbf{C}$ is any category. Let $H: \mathbf{C} \to \mathbf{D}$ be any functor from $\mathbf{C}$ to any other category $\mathbf{D}$. Then $d_I(HF, HG) \leq d_I(F, G)$.

A theorem that relates the interleaving distance with the bottleneck distance is the isometry theorem.

**Theorem 5.2.5 (Isometry theorem [16, Th.5.14]).** *Let $\mathbb{V}$, $\mathbb{W}$ be two p.f.d.*[(1)] *persistence modules in $\mathbf{Vec}^{(\mathbb{R},\leq)}$. Then*

$$d_I(\mathbb{V}, \mathbb{W}) = d_B(\mathrm{Pers}(\mathbb{V}), \mathrm{Pers}(\mathbb{W}))$$

The interleaving distance $d_I$ is a generalization of the bottleneck distance $d_B$ in the sense that $d_B$ can be recovered as a specific interleaving distance [11, 4.3].

Additionally, Theorem 5.2.5 is also interesting because the interleaving distance can

(1) The cited theorem is stronger, as it applies to *q-tame* persistence modules (ones where $\mathrm{rank}(v_s^t) < \infty$ whenever $s < t$). A p.f.d. persistence module has $\dim(V_s) < \infty$, so it implies q-tameness (cf. [16, p.51]).

be defined in more situations than the bottleneck distance, for example on multidimensional persistence modules [33] and filtrations [14]. It has been generalized also to include other kinds of metrics as particular cases: the Hausdorff distance, $L^\infty$-distance on $\mathbb{R}^n$ and the $L^\infty$-distance between filtering functions among others [21, Sec.3]. This means that all boxes in Figure 5.1 have an associated interleaving distance, and that this is a useful concept to think about when comparing these objects.

In fact, in the next two subsection we are going to study interleaving distances applied to ladder modules and Vietoris-Rips filtrations, and give for each of them a boundness relation (Proposition 5.2.8 and Proposition 5.2.14).

### 5.2.1 Interleavings between ladder modules

We borrow the notation from Definition 5.2.2 and define $\delta$-interleavings and interleaving distance between ladder modules.

**Definition 5.2.6 ($\delta$-interleaving of ladder modules).** *Given $\delta \geq 0$, two ladder modules $\alpha, \beta \in (\mathbf{Vec^2})^{(\mathbb{R},\leq)}$ (interpreted here as $(\mathbb{R},\leq)$-indexed arrows $\alpha_i \colon V_i \to W_i$ and $\beta_i \colon V'_i \to W'_i$) are $\delta$-*interleaved *if there exist $\delta$-shifted morphisms*

$$\Phi \in \operatorname{Hom}^\delta(\alpha, \beta), \qquad \Psi \in \operatorname{Hom}^\delta(\beta, \alpha)$$

*such that*

$$\Psi\Phi = \mathbb{1}_\alpha^{2\delta}, \qquad \Phi\Psi = \mathbb{1}_\beta^{2\delta}.$$

*More expansively, this means that there are pairs of maps*

$$\phi_t \colon \alpha_t \to \beta_{t+\delta} \qquad \psi_t \colon \beta_t \to \alpha_{t+\delta}$$

*defined for all $t \in \mathbb{R}$, such that the following diagrams*

$$\begin{array}{ccc} \alpha_s & \longrightarrow & \alpha_t \\ {\scriptstyle \phi_s}\searrow & & \searrow{\scriptstyle \phi_t} \\ \beta_{s+\delta} & \longrightarrow & \beta_{t+\delta} \end{array} \qquad \begin{array}{ccc} \alpha_{s-\delta} & \longrightarrow & \alpha_{s+\delta} \\ {\scriptstyle \phi_{s-\delta}}\searrow & & \nearrow{\scriptstyle \psi_s} \\ & \beta_s & \end{array}$$

$$\begin{array}{ccc} \alpha_{s+\delta} & \longrightarrow & \alpha_{t+\delta} \\ {\scriptstyle \psi_s}\nearrow & & \nearrow{\scriptstyle \psi_t} \\ \beta_s & \longrightarrow & \beta_t \end{array} \qquad \begin{array}{ccc} & \alpha_s & \\ {\scriptstyle \psi_{s-\delta}}\nearrow & & \searrow{\scriptstyle \phi_s} \\ \beta_{s-\delta} & \longrightarrow & \beta_{s+\delta} \end{array} \tag{5.2}$$

*in $\mathbf{Vec^2}$ commute for all $s < t$.*

The interleaving distance between ladder modules is defined in an analogous manner as

$$\mathrm{d_I}(\alpha,\beta) = \inf\{\delta \geq 0 \mid \alpha,\beta \text{ are } \delta\text{-interleaved}\} \tag{5.3}$$

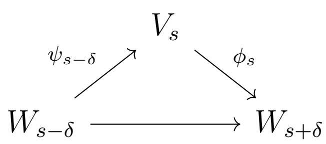
where we set $\mathrm{d_I}(\alpha,\beta) = \infty$ if $\alpha$ and $\beta$ are not $\delta$-interleaved for any $\delta \geq 0$. These definitions of $\delta$-interleaved ladder modules and interleaving distance are special cases of interleaving of diagrams [11, Def.3.1, Def.3.2], setting their arbitrary category $\mathbf{D}$ to be $\mathbf{Vec^2}$. (5.3) fits also in the definition of interleaving distance given for generalized persistence modules in [21].

As noted in [11, Sec.3], $\mathrm{d_I}$ fails to be a metric because it can take the value $\infty$ and $\mathrm{d_I}(\alpha,\beta) = 0$ does not imply $\alpha \cong \beta$. Notice that if $\alpha$ and $\beta$ are 0-interleaved, then $\alpha \cong \beta$. However $\mathrm{d_I}(\alpha,\beta) = 0$ only implies that $\alpha$ and $\beta$ are $\delta$ interleaved for all $\delta > 0$. This does not imply that $\alpha \cong \beta$. To see why this is the case, consider the next example.

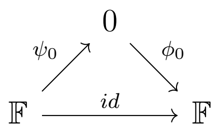
*Example* 5.2.7. Consider $\mathbb{V}, \mathbb{W} \in \mathbf{Vec}^{(\mathbb{R},\leq)}$, where $V_s = 0$ for all $s \in \mathbb{R}$ and $W_s$ is the ground field for $s = 0$ but is otherwise 0.

$$\begin{array}{ll} \mathbb{V} & \cdots \to 0 \to 0 \to 0 \to \cdots \\ \mathbb{W} & \cdots \to 0 \to \mathbb{F} \to 0 \to \cdots \end{array}$$

For all $\delta > 0$, the four diagrams commute so $\mathbb{V}$ and $\mathbb{W}$ are $\delta$-interleaved. However, replacing $\delta = 0$ and $s = 0$, the diagram



becomes



which doesn't commute, thus they are not 0-interleaved, because in fact, $\alpha \ncong \beta$.

Notice that $\alpha_i \colon V_i \to W_i$ and $\beta_i \colon V'_i \to W'_i$ are arrows between vector spaces, which individually form persistence modules $\mathbb{V} = (V_i)_{i\in\mathbb{R}}$, $\mathbb{V}' = (V'_i)_{i\in\mathbb{R}}$, $\mathbb{W} = (W_i)_{i\in\mathbb{R}}$, $\mathbb{W}' = (W'_i)_{i\in\mathbb{R}}$, in $\mathbf{Vec}^{(\mathbb{R},\leq)}$. By Corollary 2.0.24, $\alpha$ and $\beta$ are isomorphic to objects of $(\mathbf{Vec}^{(\mathbb{R},\leq)})^\mathbf{2}$, so

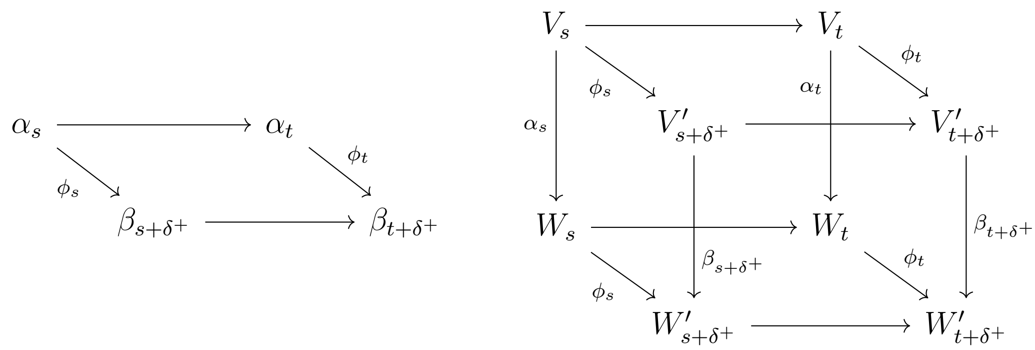
$$\alpha \in \mathrm{Hom}(\mathbb{V},\mathbb{W}), \qquad \beta \in \mathrm{Hom}(\mathbb{V}',\mathbb{W}').$$

We next give a lower bound of the interleaving distance between ladder modules.

*Proposition* 5.2.8. Let $\mathbf{C}$ be any category, and let $\mathbb{V}, \mathbb{V}', \mathbb{W}, \mathbb{W}' \in \mathbf{C}^{(\mathbb{R},\leq)}$. Given two ladder modules $\alpha \in \operatorname{Hom}(\mathbb{V}, \mathbb{W})$ and $\beta \in \operatorname{Hom}(\mathbb{V}', \mathbb{W}')$, their interleaving distances are lower bounded by

$$\max\{\mathrm{d_I}(\mathbb{V}, \mathbb{V}'), \mathrm{d_I}(\mathbb{W}, \mathbb{W}')\} \leq \mathrm{d_I}(\alpha, \beta). \tag{5.4}$$

*Proof.* $\alpha$ and $\beta$ are objects of $(\mathbf{C^2})^{(\mathbb{R},\leq)}$. Let $\mathrm{d_I}(\alpha, \beta) = \delta$. This implies there are maps $\Phi$ and $\Psi$ that define a $\delta^+$-interleaving: for any $\epsilon > 0$, the four diagrams (5.1) commute for $\delta^+ = \delta + \epsilon$ for all $s \leq t$. These diagrams in $\mathbf{C^2}$ can be written as diagrams in $\mathbf{C}$, like shown:




(5.5)

and similarly for the other three diagrams. Inspecting the lower and upper planes, one can observe that $\alpha$ and $\beta$ being $\delta^+$-interleaved implies $\mathbb{W}$ and $\mathbb{W}'$ are at most $\delta^+$-interleaved (lower plane), and $\mathbb{V}$ and $\mathbb{V}'$ too (upper plane):

$$\mathrm{d_I}(\mathbb{V}, \mathbb{V}') \leq \mathrm{d_I}(\alpha, \beta) \qquad\qquad \mathrm{d_I}(\mathbb{W}, \mathbb{W}') \leq \mathrm{d_I}(\alpha, \beta)$$

∎

### 5.2.2 Stability and interleavings of filtrations

Let $K \in \mathbf{SCpx}$ and $f_K$, $g_K$ two filtering functions. The distance between $f_K$ and $g_K$ is measured by the $L^\infty$ distance:

$$\|f_K - g_K\|_\infty = \sup_{\sigma \in K} \|f_K(\sigma) - g_K(\sigma)\|.$$

From $f_K$ one obtains a filtered complex $F_K$ applying the sublevelset construction (Example 3.4.4), and similarly from $g_K$, $G_K$. The following is a stability theorem of the interleaving distance between filtered complexes with respect to the $L^\infty$ distance between their filtering functions, and is sometimes called the *functional stability theorem.*

**Theorem 5.2.9.**

$$\mathrm{d_I}(F_K, G_K) \leq \|f_K - g_K\|_\infty$$

*Proof.* This theorem is the first part of [11, Th.5.1].

There is also a notion of stability of filtration functors with respect to the underlying metric space. As mentioned in the introduction of this chapter, the 'closeness' between two metric spaces is given by the Gromov-Hausdorff distance, defined here in terms of correspondences.

**Definition 5.2.10 (Correspondence [12, Def.7.3.17]).** *Let $X$ and $Y$ be two sets. The set $\mathcal{M} \subset X \times Y$ is a* correspondence *between $X$ and $Y$ if*

- *for all $x \in X$ there exists some $y \in Y$ with $(x, y) \in \mathcal{M}$, and*
- *for all $y \in Y$ there exists some $x \in X$ with $(x, y) \in \mathcal{M}$.*

**Definition 5.2.11 (Gromov-Hausdorff distance [12, Th.7.3.25]).** *The* Gromov-Hausdorff distance *between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is*

$$\mathrm{d_{GH}}(X, Y) = \frac{1}{2} \inf_{\mathcal{M}} \sup_{\substack{(x_1,y_1)\in\mathcal{M} \\ (x_2,y_2)\in\mathcal{M}}} |d_X(x_1, x_2) - d_Y(y_1, y_2)|$$

*where the infimum is taken over all correspondences $\mathcal{M}$.*

Given two datasets with similar geometric structure, quantified by the Gromov-Hausdorff distance, a filtration is stable if the filtrations of both datasets, quantified by the bottleneck distance, is also similar. The next theorem states that the Vietoris-Rips filtration is stable.

**Theorem 5.2.12 (Stability of Rips filtration [17, p.2]).** *For $(X, d_X)$ and $(Y, d_Y)$ finite metric spaces and all positive integers $k$,*

$$\mathrm{d_B}\Big(\operatorname{Pers} H_k\mathcal{VR}(X), \operatorname{Pers} H_k\mathcal{VR}(Y)\Big) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

The definition of interleaving distance can be used also for Vietoris-Rips filtrations as objects of $\mathbf{SCpx}^{(\mathbb{R},\leq)}$, before using the simplicial homology functor $H_k\colon \mathbf{SCpx} \to \mathbf{Vec}$. A shifted morphism $\Phi$ shifts the value of the parameter:

$$\Phi = (\phi_r\colon \mathcal{VR}_r(X) \to \mathcal{VR}_{r+\delta}(Y) \mid r, \delta \in \mathbb{R}) \in \mathrm{Hom}^\delta(\mathcal{VR}(X), \mathcal{VR}(Y))$$

such that the corresponding diagram commutes.

Two Vietoris-Rips filtrations are $\delta$-interleaved similarly to regular persistence modules and ladder modules.

*Remark* 5.2.13. The existence of a shifted morphism between Vietoris-Rips filtrations can be specified as a relation between the metric spaces (or alternatively, between the distance matrices of the underlying point clouds) by Corollary 3.4.22. Moreover, this allows to define a $\delta$-interleaving analogous to the one defined for persistence modules (Definition 5.2.2) and ladder modules (Definition 5.2.6) and an interleaving distance.

This interleaving distance can be used as an upper bound, as proposed next.

*Proposition* 5.2.14. The interleaving distance between two Vietoris-Rips filtrations $\mathcal{VR}(X)$, $\mathcal{VR}(Y)$ and the interleaving distance of their persistence modules (obtained by applying the simplicial homology functor $H_k$) satisfy:

$$\mathrm{d_I}\left(H_k(\mathcal{VR}(X)), H_k(\mathcal{VR}(Y))\right) \leq \mathrm{d_I}\left(\mathcal{VR}(X), \mathcal{VR}(Y)\right).$$

*Proof.* Recall $(\mathbb{R}, \leq)$ is a preordered set as a category (Example 2.0.15). Apply Proposition 5.2.4 to $\mathcal{VR}(X), \mathcal{VR}(Y) \in \mathbf{SCpx}^{(\mathbb{R},\leq)}$ and $H_k \colon \mathbf{SCpx} \to \mathbf{Vec}$. ∎

This last proposition gives an upper bound to the interleaving distance between two persistence modules obtained from Vietoris-Rips filtrations, without having to compute the homology of the filtrations. This may be interesting from an applied viewpoint, as a direct implementation of classical homology computation by linear algebra methods yields an $\mathcal{O}(N^3)$ algorithm [37].

# 6 | Conclusions and future work

This thesis has helped me introduce myself into the fields of Category Theory and Topological Data Analysis. We have reviewed the theory around persistence modules, their decomposition, comparison and stability, all from the perspective of category theory. In our bibliographical effort, we have unified the notation from multiple sources, which has resulted not to be an easy task.

Along the way, we have identified several gaps that could be filled and we added our small contributions. As any process through time, we realised along this journey that certain topics are approached in a more elegant way using different tools. Some of them happened to be minor tweaks, like our result about distance matrices better expressed using metric spaces. Others, like considering ladder modules from the perspective of matchings, involve more meaningful changes and we opted to leave them for future work, which we list next. We wanted to present, at least partially, general directions that serve as a starting point for future research.

- Extend results of Section 3.4 to filtrations in **Top**. This works more naturally, as $\mathbb{R}$ can be given the standard topology and filtrations correspond to objects in a slice category. This is the course taken in [21, 3.10]. Conditions of dominance can then be stated in terms of commuting diagrams. We realize afterwards that categories of filtrations and quotients are discussed in [23, 1.].
- Consider the category of matchings, which frees up the condition of ladder modules having one-way simplicial maps $\varphi : K \to L$, and instead considers relations giving to $K$ and $L$ the same importance [7]. Consider more generally dagger categories [39].
- Express Corollary 3.4.21 in terms of Gromov-Hausdorff distances. [15] developes the idea further and gives a geometric stability result [15, Th.3.1]. [10, p.8] tackles the concept of stability viewing the processes of obtaining a filtration from data, a persistence module from filtrations and barcodes from persistence modules as a 1-Lipschitz map (short map).
- Review solutions to noise and outliers in filtrations by using DTM-based filtrations [2, 5] and multiparameter persistence modules [35].
- Explore (Quillen) model categories and the traditional homotopy approach to persistence modules. Also, review cohomology and relate its categorical formula-

tion to the computational speedup that it brings, as first noted in [20] and also in [8], [22], [24].

This work and suggested continuations are by no means exhaustive. Even so, it can be observed that there is still a long way to go to get a complete categorical view of persistence modules and their potential applications.

# List of Symbols

| | |
|---|---|
| $\mathbf{C}$ | A generic category 3 |
| $\mathbf{C}_0$ | Class of objects of $\mathbf{C}$ 3 |
| $\mathbf{C}_1$ | Class of morphisms of $\mathbf{C}$ 3 |
| $\mathbb{1}$ | An identity morphism 3 |
| **Set** | Category of sets and functions 3 |
| **Ab** | Category of abelian groups and homomorphisms 4 |
| **Vec** | Category of vector spaces and linear maps 4 |
| **FinVec** | Category of finite vector spaces and linear maps 4 |
| **SCpx** | Category of simplicial complexes and simplicial maps 4 |
| **Top** | Category of topological spaces and continuous maps 4 |
| **P** | A preordered set as a category 6 |
| $(\mathbb{R}, \leq)$ | The reals as a preorder category 6 |
| **n** | A finite set of $n$ elements as a preorder category 6 |
| $K$ | A simplicial complex 17 |
| $K^{(0)}$ | The vertex set of $K$ 17 |
| $H_k$ | The $k$-th homology group 19 |
| $\partial_k$ | The $k$-th boundary operator 19 |
| $b_k$ | The $k$-th Betti number 19 |
| $\mathbb{F}$ | A field 19 |
| $\mathbb{V}$ | A persistence module 22 |
| $Q$ | A quiver (directed graph) 23 |
| $\boldsymbol{\tau}$**Mod** | Category of $\boldsymbol{\tau}$-modules 25 |
| $\mathcal{P}(S)$ | The power set of a set $S$ 28 |
| $f_K$ | Filtering function of a simplicial complex $K$ 28 |
| $F_K$ | Filtration of a simplicial complex $K$ 29 |
| $\mathbb{K}$ | A filtered simplicial complex 30 |
| $\mathbf{V}$ | A symmetric monoidal category 36 |
| $\boldsymbol{\tau}$ | An orientation of a $\mathbb{A}_n$ quiver 40 |
| $\mathbb{I}$ | An interval persistence module 40 |
| $\mathrm{d_B}$ | Bottleneck distance 53 |
| $\mathrm{d_I}$ | Interleaving distance 56 |
| $\mathcal{M}$ | A correspondence between sets 60 |
| $\mathrm{d_{GH}}$ | Gromov-Hausdorff distance 60 |

# Bibliography


[1] Adámek, J., Herrlich, H., and Strecker, G. E. *Abstract and concrete categories: the joy of cats.* Dover books on mathematics. Dover Publications, Mineola, New York, 2009.

[2] Anai, H., Chazal, F., Glisse, M., Ike, Y., Inakoshi, H., Tinarrage, R., and Umeda, Y. DTM-based Filtrations. *arXiv:1811.04757 [cs, math]* (May 2020).

[3] Asashiba, H., Escolar, E. G., Hiraoka, Y., and Takeuchi, H. Matrix method for persistence modules on commutative ladders of finite type. *Japan Journal of Industrial and Applied Mathematics 36*, 1 (2019), 97–130.

[4] Awodey, S. *Category theory*, 2nd ed ed. No. 52 in Oxford logic guides. Oxford University Press, Oxford, 2010.

[5] Baas, N. A., Carlsson, G. E., Quick, G., Szymik, M., and Thaule, M., Eds. *Topological Data Analysis: The Abel Symposium 2018*, vol. 15 of *Abel Symposia*. Springer International Publishing, Cham, 2020.

[6] Bauer, U., and Lesnick, M. Induced Matchings and the Algebraic Stability of Persistence Barcodes. *Proceedings of the Annual Symposium on Computational Geometry* (Nov. 2013), 355–364.

[7] Bauer, U., and Schmahl, M. The Structure of Morphisms in Persistent Homology, I. Functorial Dualities. *arXiv:2012.12881 [cs, math]* (Jan. 2021).

[8] Boissonnat, J. D., Dey, T. K., and Maria, C. The Compressed Annotation Matrix: An Efficient Data Structure for Computing Persistent Cohomology. *Algorithmica 73*, 3 (2015), 607–619.

[9] Borceux, F. *Handbook of categorical algebra*, vol. 1. Cambridge University Press, Cambridge, 1994.

[10] Bubenik, P., de Silva, V., and Scott, J. Metrics for Generalized Persistence Modules. *Foundations of Computational Mathematics 15*, 6 (2015), 1501–1531.

[11] Bubenik, P., and Scott, J. A. Categorification of Persistent Homology. *Discrete and Computational Geometry 51*, 3 (2014), 600–627.

[12] Burago, D., Burago, I. D., and Ivanov, S. *A course in metric geometry.* No. v. 33 in Graduate studies in mathematics. American Mathematical Society, Providence, RI, 2001.

[13] Carlsson, G., and de Silva, V. Zigzag Persistence. *Foundations of Computational Mathematics 10*, 4 (2010), 367–405.

[14] Chazal, F., Cohen-Steiner, D., Glisse, M., Guibas, L. J., and Oudot, S. Y. Proximity of persistence modules and their diagrams. In *Proceedings of the 25th annual symposium on Computational geometry - SCG '09* (Aarhus, Denmark, 2009), ACM Press, p. 237.

[15] Chazal, F., Cohen-Steiner, D., Guibas, L. J., Mémoli, F., and Oudot, S. Y. Gromov-Hausdorff Stable Signatures for Shapes using Persistence. *Computer Graphics Forum 28*, 5 (July 2009), 1393–1403.

[16] Chazal, F., de Silva, V., Glisse, M., and Oudot, S. *The Structure and Stability of Persistence Modules.* SpringerBriefs in Mathematics. Springer International Publishing, Cham, 2016.

[17] Chazal, F., de Silva, V., and Oudot, S. Persistence stability for geometric complexes. *arXiv:1207.3885 [cs, math]* (Nov. 2013).

[18] Chazal, F., and Michel, B. An introduction to Topological Data Analysis: fundamental and practical aspects for data scientists. *arXiv:1710.04019 [cs, math, stat]* (Feb. 2021).

[19] Crawley-Boevey, W. Decomposition of pointwise finite-dimensional persistence modules. *Journal of Algebra and its Applications 14*, 5 (July 2014), 1–8.

[20] de Silva, V., Morozov, D., and Vejdemo-Johansson, M. Dualities in persistent (co)homology. *Inverse Problems 27*, 12 (Dec. 2011), 124003.

[21] de Silva, V., Munch, E., and Stefanou, A. Theory of interleavings on categories with a flow. *arXiv 33*, 21 (2017), 583–607.

[22] de Silva, V., and Vejdemo-Johansson, M. Persistent Cohomology and Circular Coordinates. *Proceedings of the 25th annual symposium on Computational geometry - SCG '09* (2009), 227.

[23] Deligne, P. Théorie de Hodge, II. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques 40* (1970), 5–57.

[24] Dey, T. K., Fan, F., and Wang, Y. Computing topological persistence for simplicial maps. *Proceedings of the Annual Symposium on Computational Geometry* (2014), 345–354.

[25] Edelsbrunner, Letscher, and Zomorodian. Topological Persistence and Simplification. *Discrete & Computational Geometry 28*, 4 (Nov. 2002), 511–533.

[26] Edelsbrunner, H., and Harer, J. *Computational topology: an introduction*. American Mathematical Society, Providence, R.I, 2010.

[27] Escolar, E. G., and Hiraoka, Y. Persistence Modules on Commutative Ladders of Finite Type. *Discrete and Computational Geometry 55*, 1 (2016), 100–157.

[28] Fong, B., and Spivak, D. I. Seven Sketches in Compositionality: An Invitation to Applied Category Theory. *arXiv:1803.05316 [math]* (Oct. 2018).

[29] Gabriel, P. Unzerlegbare Darstellungen I. *Manuscripta mathematica 6*, 1 (Mar. 1972), 71–103.

[30] Ghrist, R. Barcodes: The persistent topology of data. *Bulletin of the American Mathematical Society 45*, 01 (Oct. 2007), 61–76.

[31] Hatcher, A. *Algebraic Topology*. Cambridge University Press, Cambridge ; New York, 2002.

[32] Lawvere, F. W. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano 43*, 1 (1973), 135–166.

[33] Lesnick, M. The Theory of the Interleaving Distance on Multidimensional Persistence Modules. *Foundations of Computational Mathematics 15*, 3 (June 2015), 613–650.

[34] Mac Lane, S. *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1978.

[35] McCleary, A., and Patel, A. Positivity of Multiparameter Persistence Diagrams and Bottleneck Stability. *arXiv:1905.13220 [math]* (Oct. 2020).

[36] Memoli, F. A Distance Between Filtered Spaces Via Tripods. *arXiv:1704.03965 [cs, math]* (Dec. 2017).

[37] Morozov, D. Persistence algorithm takes cubic time in worst case. *BioGeometry News, Dept. Comput. Sci., Duke Univ 2* (2005).

[38] Munkres, J. R. *Elements of algebraic topology*, 12. ed. Perseus Books, Cambridge, Mass, 2018.

[39] nLab authors. Dagger category. http://ncatlab.org/nlab/revision/dagger+category/42, May 2021.

[40] nLab authors. Filtered object. http://ncatlab.org/nlab/revision/filtered+object/15, May 2021.

[41] nLab authors. Limits and colimits by example. http://ncatlab.org/nlab/revision/limits+and+colimits+by+example/47, May 2021.

[42] nLab authors. Quiver. http://ncatlab.org/nlab/revision/quiver/46, May 2021.

[43] Riehl, E. *Category theory in context*. Aurora: Dover modern math originals. Dover Publications, Mineola, New York, 2016.

[44] Tausz, A., and Carlsson, G. Applications of Zigzag Persistence to Topological Data Analysis. *arXiv:1108.3545 [cs]* (Aug. 2011).

[45] Xia, K., and Wei, G.-W. Persistent homology analysis of protein structure, flexibility, and folding. *International Journal for Numerical Methods in Biomedical Engineering 30*, 8 (Aug. 2014), 814–844.