

# Probabilistic reachable and invariant sets for linear systems with correlated disturbance

Mirko Fiacchini<sup>a</sup>, Teodoro Alamo<sup>b</sup>,

<sup>a</sup>*Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France*

<sup>b</sup>*Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Sevilla 41092, Spain*

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## Abstract

In this paper a constructive method to determine and compute probabilistic reachable and invariant sets for linear discrete-time systems, excited by a stochastic disturbance, is presented. The samples of the disturbance signal are not assumed to be uncorrelated, only a bound on the correlation matrices is supposed to be known. The concept of correlation bound is introduced and employed to determine probabilistic reachable sets and probabilistic invariant sets. Constructive methods for their computation, based on convex optimization, are given.

*Key words:* Probabilistic sets, Correlated disturbance, Stochastic systems, Predictive control

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## 1 Introduction

The recent interest in the characterization and computation of probabilistic reachable sets and probabilistic invariant sets is mostly due to the growing popularity of stochastic Model Predictive Control (SMPC), see [13]. Indeed, as for deterministic and robust predictive techniques, several desirable features can be ensured also in the stochastic context by appropriately employing reachable and invariant sets to ensure probabilistic guarantees, for instance, of constraints satisfaction, recursive feasibility and some stability properties.

The stochastic tube-based approaches, for example, make a wide use of probabilistic invariant or reachable sets to pose deterministic constraints in the nominal prediction such that chance constraints are satisfied, see [5, 9]. Also in [6], probabilistic invariant sets are employed to handle probabilistic state constraints and a method for computing probabilistic invariant ellipsoids is presented.

Concerning the computation of reachable and invariant sets for deterministic systems and for robust control, i.e. in the worst-case disturbance context, several well-established results are present in the literature, for linear [3, 12] and non-linear systems [7]. In the recent years, some results have

been appearing also on probabilistic reachable and invariant sets. The work [11] is completely devoted to the problem of computing probabilistic invariant sets and ultimately bounds for linear systems affected by additive stochastic disturbances. Also the paper [8] presents a characterization of probabilistic sets based on the invariance property in the robust context, whereas [10] employ scenario-based methods to design them.

In most of the works concerning probabilistic reachable and invariant sets computation and SMPC, however, the stochastic disturbance is modelled by an independent sequence of random variables. The assumption of independence, and thus uncorrelation, in time between disturbances, though, is often unrealistic. In this paper, we consider the problem of characterizing and computing, via convex optimization, outer bounds of probabilistic reachable sets and probabilistic invariant ellipsoids for linear systems excited by disturbances whose realizations are correlated in time. Only bounds on covariance and correlation matrices are required to be known, even stationarity is not necessary. Based on these bounds, the called correlation bound is defined and then employed to determine constructive conditions for computing probabilistic reachable and invariance ellipsoidal sets. The method, resulting in convex optimization problems, is then illustrated through numerical examples.

Notation: The set of integers and natural numbers are denoted with  $\mathbb{Z}$  and  $\mathbb{N}$ , respectively. The spectral radius of  $A \in \mathbb{R}^{n \times n}$  is  $\rho(A)$ . The set of symmetric matrices in  $\mathbb{R}^{n \times n}$  is denoted  $\mathbb{S}^n$ . With  $\Gamma \succ 0$  ( $S \succeq 0$ ) it is denoted that  $\Gamma$  is a def-

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*Email addresses:* mirko.fiacchini@gipsa-lab.fr (Mirko Fiacchini), teodoroalamo@gmail.com (Teodoro Alamo).

inite (semi-definite) positive matrix. If  $\Gamma \succeq 0$  then  $\Gamma^{\frac{1}{2}}$  is the matrix satisfying  $\Gamma^{\frac{1}{2}}\Gamma^{\frac{1}{2}} = \Gamma$ . For all  $\Gamma \succeq 0$  and  $r \geq 0$  define  $\mathbb{B}(\Gamma, r) = \{x = \Gamma^{1/2}z \in \mathbb{R}^n : z^\top z \leq r\}$ ; if moreover  $\Gamma \succ 0$ , then  $\mathbb{B}(\Gamma, r) = \{x \in \mathbb{R}^n : x^\top \Gamma^{-1}x \leq r\}$ . Given two sets  $Y, Z \subseteq \mathbb{R}^n$ , their Minkowski set addition is  $Y + Z = \{y + z \in \mathbb{R}^n : y \in Y, z \in Z\}$ , their difference is  $Y - Z = \{x \in \mathbb{R}^n : x + Z \subseteq Y\}$ . The Gaussian (or normal) distribution with mean  $\mu$  and covariance  $\Sigma$  is denoted  $\mathcal{N}(\mu, \Sigma)$ , the  $\chi$  squared cumulative distribution function of order  $n$  is denoted  $\chi_n^2(x)$ .

## 2 Correlation bound

Consider the discrete-time system

$$x_{k+1} = Ax_k + w_k, \quad (1)$$

where  $x_k \in \mathbb{R}^n$  is the state and  $w_k \in \mathbb{R}^n$  an additive disturbance given by a sequence of random variables that are supposed to be correlated in time.

**Remark 1** *In this paper, no assumption on  $\{w_k\}_{k \in \mathbb{N}}$  is posed other than the existence of a bound on the covariance and correlation matrices. Neither stationarity is required. This aspect might be crucial in practice, as no exact knowledge of the matrices nor guarantee of stationarity are often available.*

The following definition of correlation bound encloses the key concept that permits to characterize and compute probabilistic reachable and invariant sets for linear systems affected by correlated disturbance.

**Definition 1 (Correlation bound)** *The random sequence  $\{w_k\}_{k \in \mathbb{Z}}$  is said to have a correlation bound  $\Gamma_w$  for matrix  $A$  if the recursion  $z_{k+1} = Az_k + w_k$  with  $z_0 = 0$ , satisfies*

$$\text{AE}\{z_k w_k^\top\} + \text{E}\{w_k z_k^\top\} A^\top + \text{E}\{w_k w_k^\top\} \preceq \Gamma_w, \quad (2)$$

or, equivalently

$$\text{E}\{z_{k+1} z_{k+1}^\top\} \preceq \text{AE}\{z_k z_k^\top\} A^\top + \Gamma_w,$$

for all  $k \geq 0$ .

### 2.1 Computation of a correlation bound

As it will be shown in the subsequent sections, a correlation bound permits to determine sequences of probabilistic reachable sets and probabilistic invariant sets. For this, it is necessary to provide a condition and a method to obtain a correlation bound. Such a condition is presented in the following proposition.

**Proposition 1** *Given the system (1) with  $\rho(A) < 1$ , let  $\{w_k\}_{k \in \mathbb{Z}} \subseteq \mathbb{R}^n$  be a random sequence such that*

$$\Gamma_{i,j} \tilde{\Gamma}^{-1} \Gamma_{i,j}^\top \preceq (\alpha + \beta \gamma^{j-i}) \tilde{\Gamma}, \quad \forall i \leq j, \quad (3)$$

where

$$\text{E}\{w_k w_k^\top\} = \Gamma_{k,k} \preceq \tilde{\Gamma}, \quad \forall k \in \mathbb{N}, \quad (4)$$

$$\Gamma_{i,j} = \text{E}\{w_i w_j^\top\}, \quad \forall i, j \in \mathbb{N}, \quad (5)$$

with  $\tilde{\Gamma} \succ 0$  and  $\alpha, \beta, \gamma \in \mathbb{R}$  such that  $\alpha \geq 0$ ,  $\beta \geq 0$  and  $\gamma \in (0, 1)$ . Given  $\eta \in [\rho(A)^2, 1)$  and  $\varphi \geq 1$  such that there is  $S \in \mathbb{S}^n$  satisfying

$$S \preceq \tilde{\Gamma} \preceq \varphi S, \quad ASA^\top \preceq \eta S \quad (6)$$

and  $p \in (\eta, 1)$ , then  $\Gamma_w \in \mathbb{S}^n$  with  $\Gamma_w \succ 0$  is a correlation bound for the sequence  $\{w_k\}_{k \in \mathbb{Z}}$  and matrix  $A$  if it satisfies

$$\left( \alpha \varphi \frac{\eta}{p - \eta} + \beta \varphi \frac{\gamma \eta}{p - \gamma \eta} + \frac{p}{1 - p} + 1 \right) \tilde{\Gamma} \preceq \Gamma_w. \quad (7)$$

*Proof:* From the definition of correlation bound and the equality  $z_k = \sum_{i=0}^{k-1} A^{k-1-i} w_i$ , matrix  $\Gamma_w$  must satisfy

$$\begin{aligned} \text{AE}\left\{\left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right) w_k^\top\right\} + \text{E}\left\{w_k \left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right)^\top\right\} A^\top \\ + \text{E}\{w_k w_k^\top\} \preceq \Gamma_w \end{aligned}$$

for all  $k \in \mathbb{N}$ . From condition (3) and

$$\begin{aligned} 0 \preceq & \left( \frac{A^{j-i} \Gamma_{i,j} \tilde{\Gamma}^{-\frac{1}{2}}}{p^{\frac{j-i}{2}}} - p^{\frac{j-i}{2}} \tilde{\Gamma}^{\frac{1}{2}} \right) \left( \frac{A^{j-i} \Gamma_{i,j} \tilde{\Gamma}^{-\frac{1}{2}}}{p^{\frac{j-i}{2}}} - p^{\frac{j-i}{2}} \tilde{\Gamma}^{\frac{1}{2}} \right)^\top \\ = & p^{-(j-i)} A^{j-i} \Gamma_{i,j} \tilde{\Gamma}^{-1} \Gamma_{i,j}^\top (A^{j-i})^\top + p^{j-i} \tilde{\Gamma} - A^{j-i} \Gamma_{i,j} - \Gamma_{i,j}^\top (A^{j-i})^\top \end{aligned}$$

for every  $i, j \in \mathbb{N}$  with  $i \leq j$  and  $p \neq 0$ , it follows that

$$\begin{aligned} A^{j-i} \Gamma_{i,j} + \Gamma_{i,j}^\top (A^{j-i})^\top \\ \preceq (\alpha p^{-(j-i)} + \beta (\gamma p^{-1})^{j-i}) A^{j-i} \tilde{\Gamma} (A^{j-i})^\top + p^{j-i} \tilde{\Gamma}. \end{aligned}$$

Therefore, for every  $k \in \mathbb{N}$  it holds

$$\begin{aligned} \text{AE}\left\{\left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right) w_k^\top\right\} + \text{E}\left\{w_k \left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right)^\top\right\} A^\top \\ + \text{E}\{w_k w_k^\top\} \preceq \sum_{i=0}^{k-1} A^{k-i} \text{E}\{w_i w_i^\top\} + \sum_{i=0}^{k-1} \text{E}\{w_k w_i^\top\} (A^{k-i})^\top + \tilde{\Gamma} \\ = \left( \sum_{i=0}^{k-1} A^{k-i} \Gamma_{i,k} + \Gamma_{i,k}^\top (A^{k-i})^\top \right) + \tilde{\Gamma} \\ \preceq \left( \sum_{i=0}^{k-1} (\alpha p^{-(k-i)} + \beta (\gamma p^{-1})^{k-i}) A^{k-i} \tilde{\Gamma} (A^{k-i})^\top + p^{k-i} \tilde{\Gamma} \right) + \tilde{\Gamma}. \end{aligned}$$

From (6), it follows that

$$A^j \tilde{\Gamma} (A^j)^\top \preceq \varphi A^j S (A^j)^\top \preceq \varphi \eta^j S \preceq \eta^j \tilde{\Gamma} \quad (8)$$

for all  $j \in \mathbb{N}$ , and then

$$\begin{aligned}
& AE\left\{\left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right) w_k^\top\right\} + E\left\{w_k \left(\sum_{i=0}^{k-1} A^{k-1-i} w_i\right)^\top\right\} A^\top + E\left\{w_k w_k^\top\right\} \\
& \preceq \sum_{i=0}^{k-1} \alpha \varphi (\eta p^{-1})^{k-i} \tilde{\Gamma} + \sum_{i=0}^{k-1} \beta \varphi (\gamma \eta p^{-1})^{k-i} \tilde{\Gamma} + \sum_{i=0}^{k-1} p^{k-i} \tilde{\Gamma} + \tilde{\Gamma} \\
& = \left( \sum_{j=1}^k \alpha \varphi (\eta p^{-1})^j + \sum_{j=1}^k \beta \varphi (\gamma \eta p^{-1})^j + \sum_{j=1}^k p^j \right) \tilde{\Gamma} + \tilde{\Gamma} \\
& = \left( \alpha \varphi (\eta p^{-1}) \frac{1 - (\eta p^{-1})^k}{1 - \eta p^{-1}} + \beta \varphi (\gamma \eta p^{-1}) \frac{1 - (\gamma \eta p^{-1})^k}{1 - \gamma \eta p^{-1}} \right. \\
& \quad \left. + p \frac{1 - p^k}{1 - p} \right) \tilde{\Gamma} + \tilde{\Gamma}. \tag{9}
\end{aligned}$$

Two possibilities exist,  $\eta$  can be either positive or zero. If  $\eta > 0$  then  $0 < \gamma \eta < \eta < p < 1$ , and all the terms in the summation in (9) are positive and monotonically increasing with  $k$ . If  $\eta = 0$  the first two terms in (9) are null and the third one, i.e.  $p(1 - p^k)/(1 - p)$ , is positive and monotonically increasing with  $k$ , since  $0 = \eta < p < 1$ . In both cases the supremum is finite and attained for  $k \rightarrow +\infty$  and then condition (7) implies that  $\Gamma_w$  is a correlation bound for  $A$ . ■

Note that condition (3) is the reasonable assumption of a correlation that exponentially vanishes with time. For one dimensional systems and  $\alpha = 0$ , for instance, it means that the correlation function of  $w_i$  and  $w_j$  is exponentially vanishing as  $|j - i|$  grows.

**Remark 2** Notice moreover that only an upper bound on the covariance  $\tilde{\Gamma}$ , ensuring the satisfaction of (3), is necessary to be known. This is also reasonable, since the exact values of  $\Gamma_{i,j}$  for all  $i, j \in \mathbb{N}$  are often not available, in practice.

The result of Proposition 1 is used hereafter to design an optimization based procedure to compute the tightest correlation bound. To obtain the sharper bound through (7), the parameter multiplying  $\tilde{\Gamma}$ , has to be minimized. Note first that such parameter is monotonically increasing with  $\varphi$  and  $\eta$ , for  $\varphi \geq 1$  and  $\eta \in [\rho(A)^2, 1)$ . Nevertheless, the minimizing pair  $\varphi$  and  $\eta$  is not evident, even for a given  $p$ , due to the constraint (6). One possibility is to grid the interval  $[\rho(A)^2, 1)$  of  $\eta$  and then obtain, for every value of  $\eta$  on the grid, the optimal  $\varphi$  and  $p$ . To do so, one should first fix  $\eta$  and then solve the semidefinite programming problem

$$\begin{aligned}
(\varphi^*, S^*) &= \min_{\varphi, S} \varphi \\
\text{s.t.} \quad & \tilde{\Gamma} \preceq S \preceq \varphi \tilde{\Gamma} \\
& ASA^\top \preceq \eta S.
\end{aligned}$$

Note now that, once  $\eta$  and  $\varphi$  are given, condition (7) is a convex constraint in  $p$  and then in  $\Gamma_w$ . In fact,  $a/(p - a)$  is zero if  $a = 0$  and it is finite, convex and decreasing for

$p \in (a, +\infty)$  if  $a > 0$ , whereas  $p/(1 - p)$  is finite, convex and increasing for  $p \in (-\infty, 1)$ . Then, the minimum of the function multiplying  $\tilde{\Gamma}$  exists and is unique in  $(\eta, 1)$ . This means that, once  $\varphi$  and  $\eta$  are fixed, the value of  $p$  that minimizes the parameter multiplying  $\tilde{\Gamma}$  at the lefthand-side of (7) can be computed by solving the following convex optimization problem in a scalar variable:

$$\begin{aligned}
p^*(\eta, \varphi) &= \min_p \alpha \varphi \frac{\eta}{p - \eta} + \beta \varphi \frac{\gamma \eta}{p - \gamma \eta} + \frac{p}{1 - p} \\
\text{s.t.} \quad & \eta < p < 1.
\end{aligned}$$

Finally,  $\Gamma_w$  can be computed by using in (7) the minimal value of the parameter multiplying  $\tilde{\Gamma}$  over the optimal ones obtained for the different  $\eta$  on the grid and then minimizing a measure of  $\Gamma_w$ , or, even, get  $\Gamma_w$  by imposing the equality to hold in (7).

**Remark 3** Note that  $\gamma$  could also be bigger than or equal to 1: this would lead to an (although non realistic) increasingly correlated disturbance. The limit would exist provided that  $\eta$  is smaller than the inverse of  $\gamma$ , for all  $p \in (\gamma \eta, 1)$ . The case of  $\gamma = 1$  is realistic, for instance for the case of constant disturbances, and can modelled by the constant term  $\alpha$ .

The dependence of the bound (7) on the parameter  $\varphi$  can be removed by avoiding using the bound  $S \preceq \varphi \tilde{\Gamma}$  as in (8). The corollary below, providing a potentially less conservative correlation bound, follows straightforwardly.

**Corollary 1** Under the hypothesis of Proposition 1, given  $p \in (\eta, 1)$ ,  $\Gamma_w$  is a correlation bound for matrix  $A$  if it satisfies

$$\left( \frac{\alpha \eta}{p - \eta} + \frac{\beta \gamma \eta}{p - \gamma \eta} \right) S + \left( \frac{p}{1 - p} + 1 \right) \tilde{\Gamma} \preceq \Gamma_w. \tag{10}$$

Condition (10) provides a further degree of freedom, i.e. the matrix  $S$ , that can be used to improve the bound.

### 3 Probabilistic reachable and invariant sets

Based on the correlation bound, conditions for computing probabilistic reachable and invariant sets are presented. First, two properties are given that are functional to the purpose.

**Property 1** For every  $r > 0$  and every  $\tilde{\Gamma}, \Sigma \in \mathbb{S}^n$  such that  $\tilde{\Gamma} \succeq 0$  and  $\Sigma \succ 0$ , it holds

$$\mathbb{B}(A \tilde{\Gamma} A^\top + \Sigma, r) \subseteq A \mathbb{B}(\tilde{\Gamma}, r) + \mathbb{B}(\Sigma, r). \tag{11}$$

*Proof:* Notice first that  $A\tilde{\Gamma}A^\top + \Sigma \succ 0$  and then

$$\begin{aligned} \mathbb{B}(A\tilde{\Gamma}A^\top + \Sigma, r) &= \{x \in \mathbb{R}^n : x^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \leq r\} \\ A\mathbb{B}(\tilde{\Gamma}, r) + \mathbb{B}(\Sigma, r) &= \{x = A\tilde{\Gamma}^{1/2}y + \Sigma^{1/2}w \in \mathbb{R}^n : \\ &\quad y^\top y \leq r, w^\top w \leq r\}. \end{aligned} \quad (12)$$

For a given  $x \in \mathbb{B}(A\tilde{\Gamma}A^\top + \Sigma, r)$ , the vectors  $y$  and  $w$  defined

$$y = \tilde{\Gamma}^{1/2}A^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x, \quad w = \Sigma^{1/2}(A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \quad (13)$$

are such that

$$A\tilde{\Gamma}^{1/2}y + \Sigma^{1/2}w = A\tilde{\Gamma}A^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x + \Sigma(A\tilde{\Gamma}A^\top + \Sigma)^{-1}x = x.$$

Moreover,

$$\begin{aligned} y^\top y &= x^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}A\tilde{\Gamma}A^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \\ &\leq x^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \leq r \end{aligned}$$

since  $A\tilde{\Gamma}A^\top \preceq A\tilde{\Gamma}A^\top + \Sigma$  and  $x \in \mathbb{B}(A\tilde{\Gamma}A^\top + \Sigma, r)$ . Analogously

$$\begin{aligned} w^\top w &= x^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}\Sigma(A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \\ &\leq x^\top (A\tilde{\Gamma}A^\top + \Sigma)^{-1}x \leq r \end{aligned}$$

from  $\Sigma \preceq A\tilde{\Gamma}A^\top + \Sigma$ . Hence, given  $x \in \mathbb{B}(A\tilde{\Gamma}A^\top + \Sigma, r)$ , two vectors  $y$  and  $w$  exist, as defined in (13), such that  $x = A\tilde{\Gamma}^{1/2}y + \Sigma^{1/2}w$  and  $y^\top y \leq r$  and  $w^\top w \leq r$ , which means that  $x \in A\mathbb{B}(\tilde{\Gamma}, r) + \mathbb{B}(\Sigma, r)$ , from (12). Thus (11) is proven. ■

The result in Property 1 is used in the following one, to characterize bounds on the covariance matrices and probabilities of the system trajectory.

**Property 2** *Suppose that the random sequence  $\{w_k\}_{k \in \mathbb{N}}$  has a correlation bound  $\Gamma_w \succ 0$  for matrix  $A$  with  $\rho(A) < 1$ . Given  $r > 0$ , consider the system  $z_{k+1} = Az_k + w_k$  with  $z_0 = 0$  and the recursion*

$$\Gamma_{k+1} = A\Gamma_k A^\top + \Gamma_w \quad (14)$$

with  $\Gamma_0 = 0 \in \mathbb{R}^{n \times n}$ . Then,

- (i)  $\mathbb{E}\{z_k z_k^\top\} \preceq \Gamma_k, \quad \forall k \geq 0,$
- (ii)  $\Pr\{z_k \in \mathbb{B}(\Gamma_k, r)\} \geq 1 - \frac{n}{r}, \quad \forall k \geq 1,$
- (iii)  $\mathbb{B}(\Gamma_k, r) \subseteq \mathbb{B}(\Gamma_{k+1}, r) \subseteq A\mathbb{B}(\Gamma_k, r) + \mathbb{B}(\Gamma_w, r), \quad \forall k \geq 1.$

*Proof:* The claims are proved successively.

- (i) Suppose that  $\mathbb{E}\{z_k z_k^\top\} \preceq \Gamma_k$  with  $\Gamma_k$  recursively defined

through (14). Then

$$\begin{aligned} \mathbb{E}\{z_{k+1} z_{k+1}^\top\} &= \mathbb{E}\{Az_k z_k^\top A^\top + Az_k w_k^\top + w_k z_k^\top A^\top + w_k w_k^\top\} \\ &= A\mathbb{E}\{z_k z_k^\top\}A^\top + A\mathbb{E}\{z_k w_k^\top\} + \mathbb{E}\{w_k z_k^\top\}A^\top + \mathbb{E}\{w_k w_k^\top\} \\ &\preceq A\mathbb{E}\{z_k z_k^\top\}A^\top + \Gamma_w \preceq A\Gamma_k A^\top + \Gamma_w = \Gamma_{k+1}, \end{aligned}$$

where the first inequality follows from the definition of correlation bound.

- (ii) This result is based on the Chebyshev inequality, [14, 15]. From Markov's inequality, [1, 2], a nonnegative random variable  $x$  with expected value  $\mu$ , satisfies  $\Pr\{x > r\} \leq \mu/r$  for all  $r > 0$ . From  $\Gamma_w \succ 0$ , it follows that  $\Gamma_k \succ 0$  and  $\Gamma_k^{-1} \succ 0$  for all  $k \geq 1$  and then there exists  $D_k \in \mathbb{R}^{n \times n}$  such that  $\Gamma_k^{-1} = D_k^\top D_k$  for all  $k \geq 1$ . Thus

$$\begin{aligned} \mathbb{E}\{z_k^\top \Gamma_k^{-1} z_k\} &= \mathbb{E}\{z_k^\top D_k^\top D_k z_k\} = \mathbb{E}\{\text{tr}\{z_k^\top D_k^\top D_k z_k\}\} \\ &= \mathbb{E}\{\text{tr}\{D_k z_k z_k^\top D_k^\top\}\} = \text{tr}\{D_k \mathbb{E}\{z_k z_k^\top\} D_k^\top\} \\ &\leq \text{tr}\{D_k \Gamma_k D_k^\top\} = \text{tr}\{\Gamma_k D_k^\top D_k\} = \text{tr}\{I\} = n \end{aligned}$$

and then, by applying the Markov's inequality, one gets  $\Pr\{z_k^\top \Gamma_k^{-1} z_k > r\} \leq n/r$  and hence  $\Pr\{z_k^\top \Gamma_k^{-1} z_k \leq r\} \geq 1 - n/r$ , for all  $k \geq 1$ .

- (iii) From the definition of  $\Gamma_k$ , it follows  $\Gamma_k = \sum_{i=0}^{k-1} A^i \Gamma_w (A^i)^\top$  for  $k \geq 1$  and then

$$\Gamma_{k+1} = A^k \Gamma_w (A^k)^\top + \sum_{i=0}^{k-1} A^i \Gamma_w (A^i)^\top = A^k \Gamma_w (A^k)^\top + \Gamma_k \succeq \Gamma_k.$$

This implies  $\Gamma_{k+1}^{-1} \preceq \Gamma_k^{-1}$  and hence,  $\mathbb{B}(\Gamma_k, r) \subseteq \mathbb{B}(\Gamma_{k+1}, r)$  for all  $k \geq 1$ . The inclusion  $\mathbb{B}(\Gamma_{k+1}, r) \subseteq A\mathbb{B}(\Gamma_k, r) + \mathbb{B}(\Gamma_w, r)$  follows by applying Property 1 with the definition of  $\Gamma_{k+1}$  as in (14). ■

### 3.1 Probabilistic reachable sets

The simplest confidence regions are ellipsoids, that have been widely used in the context of MPC, see, for example, [5, 9]. The definition of probabilistic reachable sets is recalled.

**Definition 2 (Probabilistic reachable set)** *It is said that  $\Omega_k \subseteq \mathbb{R}^n$  with  $k \in \mathbb{N}$  is a sequence of probabilistic reachable sets for system (1), with violation level  $\varepsilon \in [0, 1]$ , if  $x_0 \in \Omega_0$  implies  $\Pr\{x_k \in \Omega_k\} \geq 1 - \varepsilon$  for all  $k \geq 1$ .*

A condition for a sequence of sets to be a probabilistic reachable sets is presented, in terms of correlation bound. The analogous result for uncorrelated disturbance is in [8].

**Proposition 2** *Suppose that the random sequence  $\{w_k\}_{k \in \mathbb{N}}$  has a correlation bound  $\Gamma_w \succ 0$  for matrix  $A$  with  $\rho(A) < 1$ .*

Given  $r > 0$ , consider the system (1) and the recursion (14) with  $x_0 = 0 \in \mathbb{R}^n$ ,  $\Gamma_0 = 0 \in \mathbb{R}^{n \times n}$ . Then the sets defined as

$$\mathcal{R}_{k+1} = A\mathcal{R}_k + \mathbb{B}(\Gamma_w, r), \quad (15)$$

for all  $k \in \mathbb{N}$ , and  $\mathcal{R}_0 = \{0\}$  are probabilistic reachable sets with violation level  $n/r$  for every  $r > 0$ .

*Proof:* It will be firstly proved that  $\mathbb{B}(\Gamma_k, r) \subseteq \mathcal{R}_k$ , for all  $k \geq 1$ . Note first that  $\Gamma_1 = A\Gamma_0A^\top + \Gamma_w = \Gamma_w$  and  $\mathcal{R}_1 = A\mathcal{R}_0 + \mathbb{B}(\Gamma_w, r)$ . Thus,  $\mathbb{B}(\Gamma_1, r) = \mathbb{B}(\Gamma_w, r) = \mathcal{R}_1$  and hence the claim is satisfied for  $k = 1$ . It suffice now to prove that  $\mathbb{B}(\Gamma_k, r) \subseteq \mathcal{R}_k$  implies  $\mathbb{B}(\Gamma_{k+1}, r) \subseteq \mathcal{R}_{k+1}$ . Supposing  $\mathbb{B}(\Gamma_k, r) \subseteq \mathcal{R}_k$  implies

$$\mathbb{B}(\Gamma_{k+1}, r) \subseteq A\mathbb{B}(\Gamma_k, r) + \mathbb{B}(\Gamma_w, r) \subseteq A\mathcal{R}_k + \mathbb{B}(\Gamma_w, r) = \mathcal{R}_{k+1},$$

where the first inclusion follows from (iii) of Property 2. From this and the second claim of Property 2, it follows

$$\Pr\{x_k \in \mathcal{R}_k\} \geq \Pr\{x_k \in \mathbb{B}(\Gamma_k, r)\} \geq 1 - \frac{n}{r},$$

which implies that  $\mathcal{R}_k$  with  $k \in \mathbb{N}$  is a sequence of probability reachable sets with violation level  $n/r$ . ■

### 3.2 Probabilistic invariant sets

The concept of probabilistic invariant sets, as defined and used in [8, 11], is recalled.

**Definition 3 (Probabilistic invariant set)** *The set  $\Omega \subseteq \mathbb{R}^n$  is a probabilistic invariant set for the system (1), with violation level  $\varepsilon \in [0, 1]$ , if  $x_0 \in \Omega$  implies  $\Pr\{x_k \in \Omega\} \geq 1 - \varepsilon$  for all  $k \geq 1$ .*

A first condition for a set to be probabilistic invariant, analogous to that proved in [8] for uncorrelated disturbances, is given below.

**Property 3** *Suppose that the random sequence  $\{w_k\}_{k \in \mathbb{N}}$  has a correlation bound  $\Gamma_w \succ 0$  for matrix  $A$ . If  $W \in \mathbb{S}^n$  and  $r > 0$  are such that  $W \succ 0$  and*

$$A\mathbb{B}(W, 1) + \mathbb{B}(\Gamma_w, r) \subseteq \mathbb{B}(W, 1), \quad (16)$$

then  $\mathbb{B}(W, 1)$  is a probabilistic invariant set with violation probability  $n/r$ .

*Proof:* By definition, it is sufficient to show that  $x_0 \in \mathbb{B}(W, 1)$  implies  $\Pr\{x_k \in \mathbb{B}(W, 1)\} \geq 1 - n/r$ , for all  $k \geq 0$ . The state  $x_k$  can be written as the sum of a nominal term  $\bar{x}_k$  and a random vector  $z_k$  that depends on the past realizations of the uncertainty. That is,  $x_k = \bar{x}_k + z_k$ , where  $\{\bar{x}_k\}_{k \geq 0}$  and  $\{z_k\}_{k \geq 0}$  are given by the recursions

$$\bar{x}_{k+1} = A\bar{x}_k, \quad z_{k+1} = Az_k + w_k, \quad (17)$$

for all  $k \geq 0$ , with  $\bar{x}_0 = x_0$  and  $z_0 = 0$ . Below it is first proved that  $x_0 \in \mathbb{B}(W, 1)$  and (16) imply

$$\bar{x}_k + \mathcal{R}_k \subseteq \mathbb{B}(W, 1), \quad \forall k \geq 0, \quad (18)$$

with  $\mathcal{R}_k$  as in (15). Since  $\mathcal{R}_0 = \{0\}$ , the inclusion is trivially satisfied for  $k = 0$ . Supposing that  $\bar{x}_k + \mathcal{R}_k \subseteq \mathbb{B}(W, 1)$  yields

$$\begin{aligned} \bar{x}_{k+1} + \mathcal{R}_{k+1} &= A\bar{x}_k + (A\mathcal{R}_k + \mathbb{B}(\Gamma_w, r)) \\ &= A(\bar{x}_k + \mathcal{R}_k) + \mathbb{B}(\Gamma_w, r) \\ &\subseteq A\mathbb{B}(W, 1) + \mathbb{B}(\Gamma_w, r) \subseteq \mathbb{B}(W, 1), \end{aligned}$$

and then (18) holds. Condition (18) implies

$$\Pr\{x_k \in \mathbb{B}(W, 1)\} = \Pr\{\bar{x}_k + z_k \in \mathbb{B}(W, 1)\} \geq \Pr\{z_k \in \mathcal{R}_k\} \geq 1 - \frac{n}{r}$$

for all  $k \geq 0$ , where the last inequality follows from Proposition 2. ■

Property 3 implies that the existence of a correlation bound provides a condition for probabilistic invariance that has the same structure as the one corresponding to robust invariance. In case of ellipsoidal invariant, (16) results in a bilinear condition, see [4], that can be solved, for instance, by gridding the space of the Lagrange multiplier and solving an LMI for every value.

An additional novel condition for a set to be probabilistic invariant, employing the correlation bound of the correlated random sequence  $w_k$ , follows.

**Proposition 3** *Suppose that the random sequence  $\{w_k\}_{k \in \mathbb{Z}}$  has a correlation bound  $\Gamma_w \succ 0$  for matrix  $A$ . If  $W \in \mathbb{S}^n$  and  $\lambda \in [0, 1)$  are such that  $W \succ 0$  and*

$$AWA^\top + \Gamma_w \preceq W \quad (19)$$

and

$$AWA^\top \preceq \lambda W \quad (20)$$

then  $\mathbb{B}(W, \rho)$  is a probabilistic invariant set with violation probability  $n/(1 - \lambda)\rho$ . If, moreover,  $w_k$  has normal distribution, then  $\mathbb{B}(W, \rho)$  is a probabilistic invariant set with violation probability  $1 - \chi_n^2((1 - \lambda)\rho)$ .

*Proof:* As in the proof of Property 3, denote  $x_k = \bar{x}_k + z_k$ , where  $\{\bar{x}_k\}_{k \geq 0}$  and  $\{z_k\}_{k \geq 0}$  are given by the recursions (17) for all  $k \geq 0$ , with  $\bar{x}_0 = x_0$  and  $z_0 = 0$ . Then, from  $x_0 \in \mathbb{B}(W, \rho)$  and (20), it follows that  $\bar{x}_k \in \mathbb{B}(W, \lambda^k \rho)$  for all  $k \in \mathbb{N}$ . First it is proved that  $E\{z_k z_k^\top\} \preceq W$  implies  $E\{z_{k+1} z_{k+1}^\top\} \preceq W$  for every  $k \in \mathbb{N}$ . Since  $z_0 = 0$ , the inequality  $E\{z_0 z_0^\top\} = 0 \preceq W$  is trivially satisfied. Suppose now that

$E\{z_k z_k^\top\} \preceq W$ , then

$$\begin{aligned} E\{z_{k+1} z_{k+1}^\top\} &= E\{(Az_k + w_k)(Az_k + w_k)^\top\} \\ &= E\{Az_k z_k^\top A^\top + Az_k w_k^\top + w_k z_k^\top A^\top + w_k w_k^\top\} \\ &= AE\{z_k z_k^\top\}A^\top + AE\{z_k w_k^\top\} + E\{w_k z_k^\top\}A^\top + E\{w_k w_k^\top\} \\ &\preceq AWA^\top + \Gamma_w \preceq W \end{aligned}$$

where the first inequality follows from the definition of correlation bound and the second from (19). Note now that

$$\begin{aligned} \Pr\{x_k \in \mathbb{B}(W, \rho)\} &= \Pr\{\bar{x}_k + z_k \in \mathbb{B}(W, \rho)\} \\ &\geq \Pr\{z_k \in \mathbb{B}(W, \rho) - \mathbb{B}(W, \lambda^k \rho)\} \end{aligned}$$

since  $\bar{x}_k \in \mathbb{B}(W, \lambda^k \rho)$  with probability 1. It follows that  $\mathbb{B}(W, (1 - \lambda)\rho) = \mathbb{B}(W, \rho) - \mathbb{B}(W, \lambda\rho) \subseteq \mathbb{B}(W, \rho) - \mathbb{B}(W, \lambda^k \rho)$  and, from  $E\{z_k z_k^\top\} \preceq W$  and the Chebyshev inequality (see, for example, proof of claim (ii) of Property 2), then

$$\begin{aligned} \Pr\{x_k \in \mathbb{B}(W, \rho)\} &\geq \Pr\{z_k \in \mathbb{B}(W, (1 - \lambda)\rho)\} \\ &= \Pr\{z_k W^{-1} z_k \leq (1 - \lambda)\rho\} \geq 1 - \frac{n}{(1 - \lambda)\rho}. \end{aligned}$$

The results for  $w_k$  with normal distribution follow directly from the definition of the  $\chi$  squared cumulative distribution, that is  $\Pr\{y^\top y \leq r\} = \chi_n^2(r)$  for  $y \in \mathbb{R}^n$  with standard normal distribution and  $r > 0$ , see [1, 2]. ■

The proposition below proves that the convex condition (19) is less conservative than (16).

**Proposition 4** *Suppose that the random sequence  $\{w_k\}_{k \in \mathbb{N}}$  has a correlation bound  $\Gamma_w \succ 0$  for matrix  $A$ . If  $W$  is such that condition (16) holds for  $r \geq 1$ , then also (19) is satisfied.*

*Proof:* First note that condition (19) is equivalent to  $\mathbb{B}(AWA^\top + \Gamma_w, 1) \subseteq \mathbb{B}(W, 1)$ . From claim (iii) of Property 1 it follows

$$\mathbb{B}(AWA^\top + \Gamma_w, 1) \subseteq A\mathbb{B}(W, 1) + \mathbb{B}(\Gamma_w, 1) \subseteq A\mathbb{B}(W, 1) + \mathbb{B}(\Gamma_w, r)$$

since  $r \geq 1$ , and thus that (16) implies (19). ■

Therefore, condition (19) can be used to efficiently determine probabilistic invariant ellipsoids and is less conservative than (16).

## 4 Numerical examples

Two examples are presented, concerning different bounds on the correlation matrices of the random sequence  $w_k$  that affects the system (1) with  $A = \begin{bmatrix} 0.25 & 0 \\ 0.1 & 0.3 \end{bmatrix}$ .

### 4.1 Exponentially decaying bound

Consider the case in which the bound (3) holds with  $\alpha = 0$ ,  $\beta = 1$  and  $\gamma < 1$ . This means that the correlation between the samples of  $w_k$  and  $w_{k-l}$  is assumed to be exponentially decreasing with  $l$  and it represents the systems for which the dependence between samples fades with time.

To validate the presented results, it is necessary to generate a random sequence satisfying the bound (3). This can be obtained by feeding an asymptotically stable discrete-time system with an i.i.d. random process with zero mean and a given constant covariance matrix. Consider the i.i.d. process  $u_k$  with  $k \in \mathbb{N}$  such that

$$E\{u_k\} = 0 \quad E\{u_k u_k^\top\} = U,$$

for all  $k \in \mathbb{N}$  and

$$w_{k+1} = Hw_k + Fu_k. \quad (21)$$

Then  $w_k = \sum_{j=-\infty}^{k-1} H^{j-k+1} Fu_j$  and it can be proved that

$$E\{w_k\} = 0, \quad E\{w_k w_k^\top\} = \tilde{\Gamma}, \quad E\{w_{k+l} w_k^\top\} = H^l \tilde{\Gamma}$$

for all  $k, l \in \mathbb{N}$ , where  $\tilde{\Gamma} \in \mathbb{S}^n$  is the unique solution of  $\tilde{\Gamma} = H\tilde{\Gamma}H^\top + FUF^\top$ . Then, the bound (3) holds with  $\alpha = 0$ ,  $\beta = 1$  and  $\gamma$  solution of the following optimization problem:

$$\min_{\gamma} \{\gamma : H\tilde{\Gamma}H^\top \preceq \gamma\tilde{\Gamma}\}. \quad (22)$$

Therefore, given  $\tilde{\Gamma}$  and  $\gamma$ , a random sequence satisfying (3) for this values can be obtained by appropriately designing  $H$  and  $F$  or, viceversa, given  $U$ ,  $H$  and  $F$ , the matrix  $\tilde{\Gamma}$  and  $\gamma$  can be obtained.

An i.i.d. random sequence with distribution  $\mathcal{N}(0, U)$ , with  $U = \text{diag}(1.5, 0.26)$ , has been used to feed system (21) with

$$H = \begin{bmatrix} 0.75 & -0.2 \\ 0 & 0.6 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 0.5 & -3 \end{bmatrix}$$

giving a correlated sequence  $w_k$  with null mean and covariance matrix

$$\tilde{\Gamma} = \begin{bmatrix} 7.8381 & -2.3983 \\ -2.3983 & 4.2422 \end{bmatrix}.$$

The bound (3) holds with  $\alpha = 0$ ,  $\beta = 1$  and  $\gamma = 0.676$ . The correlation bound  $\Gamma_w$ , computed using (10), and matrix  $W$  from (19) are

$$\Gamma_w = \begin{bmatrix} 19.5198 & -5.9726 \\ -5.9726 & 10.5646 \end{bmatrix}, \quad W = \begin{bmatrix} 20.8211 & -5.8942 \\ -5.8942 & 11.4496 \end{bmatrix}$$

with  $\lambda = 0.1221$  from (20). Using the fact that  $w_k$  has normal distribution, the set  $\mathbb{B}(W, \rho)$  is a probabilistic invariant set with violation probability of  $p_v$  with  $\chi^2_2((1-\lambda)\rho) = 1 - p_v$ . Different values of violation probability  $p_v$  have been tested, in particular  $p_v = 0.1, 0.2, 0.3, 0.4, 0.5$ . For every  $p_v$ ,  $N = 1000$  initial states  $x_0$  have been uniformly generated on the boundary of  $\mathbb{B}(W, \rho)$  and assumed independent on  $w_k$ . For each  $x_0$ , a sequence  $w_k$  has been generated and applied. For every  $k = 1, \dots, 100$ , the set of states  $x_k$  are computed and the number of violation  $v_k$  of the constraint  $x_k \in \mathbb{B}(W, \rho)$  have been computed. The frequencies of violation  $v_k/N$ , for every  $p_v$  and  $k \in 1, \dots, 100$ , are depicted in Fig. 1, that shows that the bound is always satisfied.

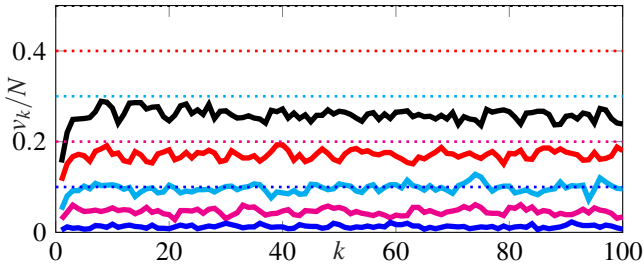


Fig. 1. Frequency of violations  $v_k/N$  of  $x_k \in \mathbb{B}(W, \rho)$  for  $k = 1, \dots, 100$ , with  $\alpha = 0$  and  $\beta = 1$ , obtained for violation probability of: 50% in black; 40% in red; 30% in cyan; 20% in magenta; 10% in blue.

#### 4.2 Constant bound

A case for which the bound (3) holds with  $\alpha = 1$  and  $\beta = 0$  is considered here. Supposing the that  $w_k = w$  for all  $k \in \mathbb{N}$ , such that  $E\{ww^T\} \preceq \tilde{\Gamma}$ , would lead (3) to hold with  $\alpha = 1$  and  $\beta = 0$ .

The constant value of the disturbance  $w$  has been generated according to  $\mathcal{N}(0, \tilde{\Gamma})$  with  $\tilde{\Gamma}$  randomly generated:

$$\tilde{\Gamma} = \begin{bmatrix} 0.4785 & -0.7254 \\ -0.7254 & 1.5215 \end{bmatrix}.$$

The correlation bound as in (10) and related probabilistic invariant are given by the matrices

$$\Gamma_w = \begin{bmatrix} 1.1877 & -1.8007 \\ -1.8007 & 3.7767 \end{bmatrix}, \quad W = \begin{bmatrix} 1.2669 & -1.9125 \\ -1.9125 & 4.0380 \end{bmatrix}.$$

with  $\lambda = 0.0921$  from (20). As for the case of decaying bound, the violation probabilities  $p_v = 0.1, 0.2, 0.3, 0.4, 0.5$  are considered and 1000 initial states  $x_0$  are uniformly distributed on the boundary of  $\mathbb{B}(W, \rho)$  to check the violation frequencies. For every  $x_0$ , a constant sequence  $w_k = w$ , with  $k \in \mathbb{N}$ , is generated with distribution  $\mathcal{N}(0, \tilde{\Gamma})$  for  $w$  and the number of the set inclusion  $x_k \in \mathbb{B}(W, \rho)$  violations  $v_k$  are obtained for all  $k = 1, \dots, 100$ . See the results in Fig 2, the probability violation bound is satisfied.

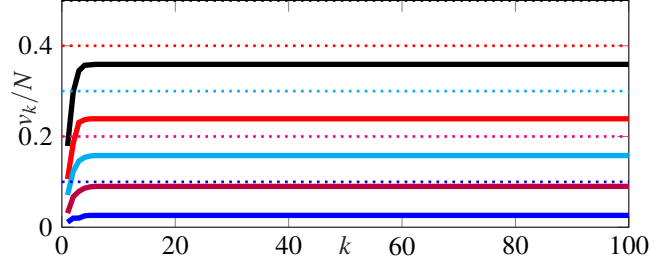


Fig. 2. Frequency of violations  $v_k/N$  of  $x_k \in \mathbb{B}(W, \rho)$  for  $k = 1, \dots, 100$ , with  $\alpha = 1$  and  $\beta = 0$ , obtained for violation probability of: 50% in black; 40% in red; 30% in cyan; 20% in magenta; 10% in blue.

## 5 Conclusions

This paper presented methods, based on convex optimization, to compute probabilistic reachable and invariant sets for linear systems fed by a stochastic disturbance correlated in time. From the knowledge of some bound on the correlation matrices, the characterization of the called correlation bound is given and then employed for obtaining the reachable and invariant sets.

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