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Spiral waves solutions in reaction-diffusion equations with symmetries. Analysis through specific models

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Abstract. Symmetries of nonlinear reaction-diffusion equations determine the existence of regular rotating spiral waves. They are only a consequence of kinetics processes and molecular diffusion. We prove the existence of these waves as invariant solutions of reaction-diffusion models with appropiate Lie point symmetries.

1. Introduction

Nonlinear reaction-diffusion equations have been widely studied. These equations arise naturally as description models of many evolution problems in the real world, as in chemistry [1], biology [2], ecology [3], etc. Sometimes models not related directly to nature have been proposed with the main object of finding various kinds of cooperative behaviour, as is the case of the so-called tri-molecular model of Lefever and Prigogine [4].

Among the structures which can be found in systems modelled by reaction-diffusion equations we must mention regular rotating spiral patterns. They occur in a wide variety of biological, physiological and chemical contexts, and the Belousov–Zhabotinskii reaction [5–7] provides a classic example.

The mathematical study of spirals in excitable media has followed several approachs. Some authors [8–12] have found asymptotic solutions which represent spiral waves far from a fixed origin. The far field of the spiral is viewed as a modification of periodic plane waves, but no analysis is given to show that these asymptotic spirals correspond to solutions that are smooth at the origin. In this approach, arguments have been advanced that additional kinematic rules must be present to produce and possibly maintain spiral waves in the core of the spiral.

For a certain class of models some authors [13] have demonstrated that rotating logarithmic spiral waves can be maintained by reaction and diffusion alone. They proved the existence via the Schauder fixed point theorem applied to a certain class of the $\lambda - \omega$ systems introduced by Kopell and Howard [14]. The importance of these models lies in the fact that they arise naturally as the dominant part in the asymptotic analysis of many general reaction-diffusion systems [13].

One-armed Archimedian spiral waves were obtained by Greenberg [15], and Hagan [16] obtained both one-armed and multi-armed Archimedian spiral waves. They used formal asymptotic expansions methods for two different classes of $\lambda - \omega$ systems.

Another aproach [17, 18] makes use of singular perturbation theory to examine the detailed motion of spiral interfaces between excited and recovering regions. The analysis results in a free-boundary problem for the shape and frequency of rotation of the spiral. Although this reduced problem is still unsolved in general, recently Keener [19] has solved this free-boundary problem in the special case where the spiral is rotationally symmetric. These spirals arise when the excitable medium is described by antisymmetric dynamics.

This paper deals with the application of Lie group theory to nonlinear reaction-diffusion systems of $\lambda-\omega$ type. Although group analysis of differential equations has been applied a great deal in many fields of mathematical physics [20–24], much less analysis has been used in connection with reaction-diffusion systems. In a previous paper [25] we have applied Lie theory of transformation groups to the study of $\lambda-\omega$ reaction-diffusion systems in two-dimensional media. Our study proves that they are invariant with respect to a five-parameter symmetry group. Multiple types of invariant solutions with physical interest are possible and spiral waves are among them.

A $\lambda-\omega$ model that has been extensively studied was proposed by Smoes and Dreitlein [26] with the aim of finding the local dissipative structures observed in the Belousov–Zhabotinskii reaction. For this model, some analytic solutions were obtained for the case of one spatial variable [26, 27]. These results were later obtained using Lie group theory as special cases of more general solutions by Steeb and Strampp [28]. Previously, numerical solutions were obtained for the case of two spatial variables [29], showing some classes of dissipative structures.

With the use of Lie group theory for the case of two spatial variables we demonstrate the existence of different types of rotating spiral waves . We can prove analytically the regularity at the origin for some of them. A similar method has been used by Greenberg [15] with a different system. We have also studied numerically the regularity of other solutions.

The regularity of the solutions implies that there is no need to add additional kinetics to produce the waves in the core of the spiral. Also, the potentiality of the method suggests the application to many models of reaction-diffusion systems.

2. Reaction–diffusion models

We consider models described by systems of partial differential equations (SPDE) of the form

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= D\nabla^2 u_1 + f(u_1,u_2)\\ \frac{\partial u_2}{\partial t} &= D\nabla^2 u_2 + g(u_1,u_2)\end{aligned} \tag{1}$$

where $u_1 = u_1(x, y, t)$, $u_2 = u_2(x, y, t)$ represent, for example, relative concentrations, i.e. deviations with respect to mean values, of two chemical reactants or relative populations of two biological species, and can take negative values. They diffuse through the plane (x, y) and react with kinetics given by the nonlinear functions $f(u_1, u_2)$ and $g(u_1, u_2)$. D represents the diffusion coefficient, which can be made equal to 1 after a suitable rescaling of (x, y).

The Dreitlein–Smoes model, which is analysed in some detail in this paper, is obtained as a consequence of a particular kinetics of some chemical reactants.

In a recent paper [25] the authors show that system (1) is a $\lambda - \omega$ system if, and only if, it is invariant under a Lie group of transformations [21], with characteristics

$$\begin{aligned} Q^1 &= a_1 u_{1_x} + a_2 u_{1_y} + a_3 u_{1_t} + a_4(x u_{1_y} - y u_{1_x}) + a_5 u_2 \\ Q^2 &= a_1 u_{2_x} + a_2 u_{2_y} + a_3 u_{2_t} + a_4(x u_{2_y} - y u_{2_x}) - a_5 u_1 \end{aligned} \tag{2}$$

where the set $\{a_i\}_{i=1}^5$ represents arbitrary constants.

The $\lambda - \omega$ reaction-diffusion systems with two reactants are described by systems (1) with f and g of the form $f(u_1, u_2) = \lambda(z)u_1 - w(z)u_2$; $g(u_1, u_2) = \omega(z)u_1 + \lambda(z)u_2$, where $z^2 = u_1^2 + u_2^2$.

Usually, $\lambda(z)$ is supposed to be a positive function of z for $0 \leqslant z < z_0$ and negative for $z > z_0$. Also, $\omega(z)$ is supposed to be a positive function of z, in order to assure that the model, without diffusion, has a limit cycle with amplitude z_0 and frequency $w(z_0)$. Thus, $\lambda - \omega$ systems have been proposed as models for chemical or biological systems which exhibit oscillating behaviour in homogeneous situations.

Each mono-parametric subgroup is associated with a set of determined constants $\{a_i\}_{i=1}^5$.

Invariant solutions with respect to different subgroups of the full group exhibit a great variety of patterns, and spiral wave patterns are among them.

In this paper we are interested in the question about the existence of regular rotating spiral waves. These waves are invariant with respect to rotations with phase shift and are also time-periodic. Then, we must look for the characteristics associated with these groups. It is easy to see that they correspond to the cases $a_1 = a_2 = a_3 = 0$, and $a_1 = a_2 = a_4 = 0$.

We denote these groups by $\mathcal{G}_{45}$ and $\mathcal{G}_{35}$, respectively. Their characteristics are given by

$$\begin{aligned} &Q_{45}^1 = a_4(x u_{1_y} - y u_{1_x}) + a_5 u_2 \qquad Q_{45}^2 = a_4(x u_{2_y} - y u_{2_x}) - a_5 u_1 \\ &Q_{35}^1 = a_3 u_{1_t} + a_5 u_2 \qquad Q_{35}^2 = a_3 u_{2_t} - a_5 u_1. \end{aligned} \tag{3}$$

It is clearly appropriate to use polar coordinates. That is, we introduce the set of variables (r, θ) and (z, ϕ), where

$$\begin{aligned} &x = r\cos(\theta) \qquad y = r\sin(\theta) \\ &u_1 = z\cos(\phi) \qquad u_2 = z\sin(\phi). \end{aligned}$$

Thus, z represents the *amplitude* and ϕ the *phase* of the wave. This change of variables gives rise to considerable simplification of the characteristics. We easily obtain

$$\begin{aligned} &Q_{45}^z = a_4 z_\theta \qquad Q_{45}^\phi = a_4\phi_\theta - a_5 \\ &Q_{35}^z = a_3 z_t \qquad Q_{35}^\phi = a_3\phi_t - a_5. \end{aligned} \tag{4}$$

These characteristics are determinated except for a multiplicative constant. Then, it is possible to choose $a_5 = 1$, and in these new coordinates, system (1) reads

$$\begin{aligned} &z_{rr} + \frac{z_r}{r} + \frac{z_{\theta\theta}}{r^2} - z\left(\phi_r^2 + \frac{\phi_\theta^2}{r^2}\right) + z\lambda(z) - z_t = 0 \\ &\phi_{rr} + \frac{\phi_r}{r} + \frac{\phi_{\theta\theta}}{r^2} + 2\phi_r\frac{z_r}{z} + 2\frac{\phi_\theta}{r^2}\frac{z_\theta}{z} + \omega(z) - \phi_t = 0. \end{aligned} \tag{5}$$

The characteristics associated to a group vanish for solutions which are invariant with respect to this group. Thus, invariant solutions with respect to $\mathcal{G}_{45}$ must have the form

$$u_1 = z(r,t)\cos\left(\frac{\theta}{a_4} + \beta(r,t)\right) \qquad u_2 = z(r,t)\sin\left(\frac{\theta}{a_4} + \beta(r,t)\right). \tag{6}$$

Since the concentrations u_i must be continuous at all points in space, they must satisfy $u_i(r,\theta) = u_i(r,\theta+2\pi)$. Hence, the possible values for a_4 are of the form $1/n$, with n an integer, and solutions (6) take the form

$$u_1 = z(r,t)\cos(n\theta + \beta(r,t)) \qquad u_2 = z(r,t)\sin(n\theta + \beta(r,t)). \tag{7}$$

These solutions are invariant with respect to phase rotations of amplitude $2\pi/n$. If, in addition, they are invariant with respect to the group $\mathcal{G}_{35}$, solutions take the form

$$u_1 = z(r)\cos(n\theta + \Omega t + \beta(r)) \qquad u_2 = z(r)\sin(n\theta + \Omega t + \beta(r)) \tag{8}$$

with $\Omega = 1/a_3$. Thus, Lie point theory of transformations permits us to establish in a somewhat straightforward way whether the model admits spiral waves solutions. There is no need to previously assume the existence of these type of solutions, as many authors do when they are looking for solutions of $\lambda - \omega$ systems. Of course there still remains the problem of studying the solutions of (9) with physical content.

Substitution of (8) into (5) yields

$$z_{rr} + \frac{z_r}{r} + z\left(\lambda(z) - \beta_r^2 - \frac{n^2}{r^2}\right) = 0 \tag{9a}$$

$$\beta_{rr} + \frac{\beta_r}{r} + 2\beta_r\frac{z_r}{z} + \omega(z) - \Omega = 0. \tag{9b}$$

The wavefronts corresponding to equations (8) are not defined as curves with u_1 and u_2 constant, because the function $z(r)$ could mask the shape of the spirals. They are better defined as curves of constant phase, which are steadily rotating waves with angular frequency $\omega_0 = 1/na_3$. In any case, both definitions coincide for certain phases; for example, we may consider the wavefronts, where $u_2 = 0$, with phase $2\pi m$:

$$\begin{aligned} x_m &= r\cos\left(-\frac{t}{na_3} - \frac{1}{n}\beta(r) + 2\pi\frac{m}{n}\right) \\ y_m &= r\sin\left(-\frac{t}{na_3} - \frac{1}{n}\beta(r) + 2\pi\frac{m}{n}\right) \end{aligned} \tag{10}$$

with $m = 0, 1, \ldots, n-1$.

Equation (9b) suggests the convenience of first considering the case where $\beta(r)$ is constant.

2.1. $\beta(r)$ constant

In this case the function $\omega(z)$ must also be a constant of value $\Omega = 1/a_3$. The wavefronts are straight lines. This shape may not be seen as a spiral in the conventional sense, but spiral waves are often defined as rotating, time-periodic, spatial structures [30], and this shape is usually included among spirals [16].

An interesting model of $\lambda - \omega$ type, with $\omega = -2$ constant is proposed by Smoes and Dreitlein [26] with $\lambda(z) = k - z^2$, where k is a real parameter. From (9b) we observe that $a_3 = -\frac{1}{2}$. Hence $\omega_0 = -\frac{2}{n}$.

The reduced equation is

$$z_{rr} + \frac{z_r}{r} + z\left(k - \frac{n^2}{r^2} - z^2\right) = 0. \tag{11}$$

This equation is invariant with respect to the transformation

$$Z_\epsilon = \sqrt{\epsilon}Z \qquad r_\epsilon = \frac{r}{\sqrt{\epsilon}} \qquad k_\epsilon = \epsilon k. \tag{12}$$

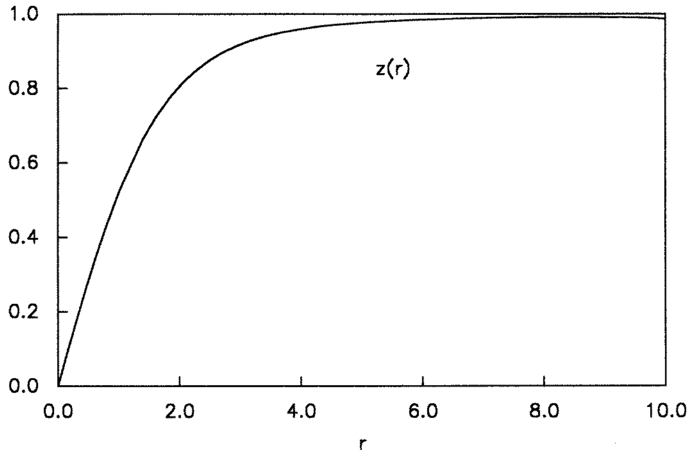

Figure 1. Plot of $z(r)$, in the case $\beta(r)$ constant, obtained by means of numerical integration of equation (13) with $\alpha_c = 0.583\,1746$.

Thus, the solutions for any value of $k > 0$ may be obtained from the solutions for $k = 1$. Hence, when $\beta(r)$ is constant it is sufficient to study the equation

$$z_{rr} + \frac{z_r}{r} + z\left(1 - \frac{n^2}{r^2} - z^2\right) = 0. \tag{13}$$

The solutions are rotating spirals with phase $\Phi = nt + \Omega t + \beta$.

In the appendix we investigate the existence of one-armed spiral waves ($n = 1$), such that $z(r)$ is analytical in $r = 0$, and bounded for $r \to \infty$.

We are able to demonstrate analytically the existence of these types of solutions. The following boundary conditions must be satisfied by solutions of equation (13) which are regular at the origin

$$\lim_{r\to 0} z(r) = 0 \qquad \lim_{r\to\infty} z(r) = 1. \tag{14}$$

For the case $\beta(r)$ constant, we have found numerically the value $\alpha_c = 0.583\,1746$. In figure 1 we plot $z(r)$ obtained by means of the numerical integration of equation (13). The associated solution $u_1 = z(r)\cos(\theta - 2t)$ with $t = 0$ is represented in figure 2, and we can see that the wavefronts are straight lines.

2.2. $\beta(r)$ non-constant

The case $\beta(r)$ non-constant is clearly more interesting from a physical point of view. A monotonous function $\beta(r)$ will produce the shape of conventional spiral waves observed in many physical systems, as, for example, the Belousov–Zhabotinskii reaction. However, it is not easy to find the appropriate $\lambda(z)$ and $\beta(z)$, which allow bounded solutions for both $z(r)$ and $\beta_r(r)$. It is easy to see that a necessary condition is that $w(z(\infty)) = \Omega$, and that if $w(z) =$ constant, as in the Dreitlein–Smoes model, there are no possible bounded solutions for $\beta(r)$ non-constant [30]. However, divergent solutions for $r \to \infty$ should not be disregarded. Many systems are proposed to model local dissipative structures and are not necessarily supposed to be valid when $r \to \infty$. That is the reason why we have done some numerical integration for $\omega(z) = \Omega = -2$. This integration also poses some problems. First, equation (9a) is singular in $r = 0$, where the values $z(0) = 0$ and $\beta_r(0) = 0$ are

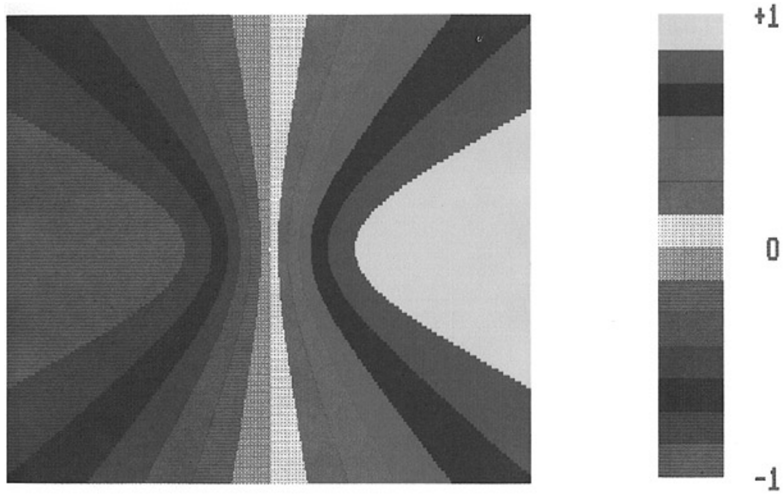

Figure 2. Spatial pattern of the concentration u_1, in the case $\beta(r)$ constant, obtained by means of numerical integration of equation (13) with $\alpha_c = 0.583\,1746$. The horizontal and the vertical axes are from $-5\sqrt{2}$ to $+5\sqrt{2}$.

(This figure can be viewed in colour in the electronic version of the article; see `http://www.iop.org/EJ/welcome`)

known. The procedure has been chosen to substitute a power series for $z(r)$ and $\beta_r(r)$ in a neighbourhood of $r = 0$ in equations (9), and to find the relations among the first coefficients up to order 3, using them to specify the initial conditions in function of $\alpha = z_r(0)$. Second, as suggested by some numerical integration and the demonstration in the appendix for the case $\beta(r)$ constant, there are two different types of solutions. Divergent solutions for α greater than certain value α_c, and oscillating solutions for smaller values of α. We look for solutions that are well behaved for relatively large r, although not for all r. Thus, we look for initial conditions very near the critical value α_c. The results shown in figure 3 are for $\beta(0) = 0$, $\beta_r(0) = 1$, $k = 1$ and $\Omega - \omega = -0.1$. We have numerically found the value $\alpha_c = 0.579\,889$.

The associated solutions u_1, u_2 are obtained through equations (8). Figure 4 represents $u_1(x, y)$ with $t = 0$.

It is possible to find some systems with $\omega(z)$ non-constant in the literature, which allow bounded solutions for all r, as, for example, Greenberg [15], for $\lambda(z) = 1 - z$ and $\omega(z) = 1 + \omega_1(z - 1)$, or Hagan [16], for $\lambda(z) = 1 - z^2$ and $\omega(z) = qz^2$.

It is not the aim of this paper to find the general conditions for $\omega(z)$ and $\lambda(z)$ which lead to bounded solutions, and this problem remains open.

3. Conclusions

The main aim of this article is to demonstrate the relation of certain Lie point symmetries with the existence of rotating spiral waves in reaction-diffusion systems. Application of Lie group theory to the study of nonlinear reaction-diffusion models may help to establish if these kinds of structures are a consequence of only the kinetics processes and of molecular diffusion, or if additional kinematics mechanisms must be present to produce them. Invariant solutions of subgroups of the full symmetry group, i.e. partially invariant solutions [31], as

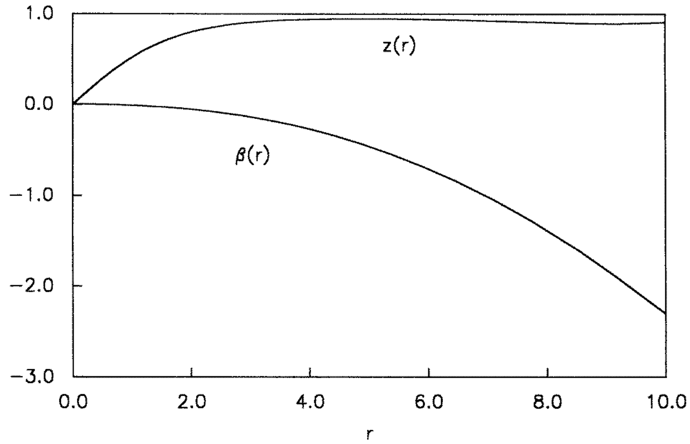

Figure 3. Plot of $z(r)$ and $\beta(r)$, obtained by means of numerical integration of system (9) with $\beta(0) = 0$, $\beta_r(0) = 1$, $k = 1$, $\Omega - \omega = -0.1$ and $\alpha_c = 0.579\,889$.

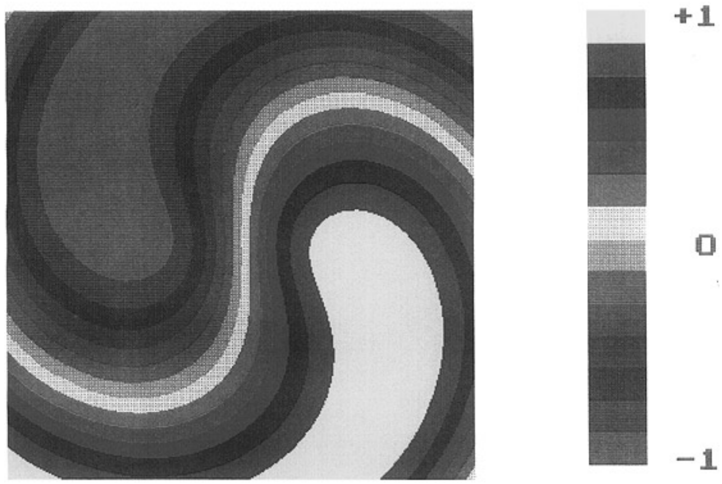

Figure 4. Spatial pattern of the concentration u_1, obtained by means of numerical integration of system (9) with $\beta(0) = 0$, $\beta_r(0) = 1$, $k = 1$, $\Omega - \omega = -0.1$ and $\alpha_c = 0.579\,889$. The horizontal and the vertical axes are from $-5\sqrt{2}$ to $+5\sqrt{2}$.

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the spiral waves studied in this paper, are usually of great interest, not only because of the richness of patterns exhibited but also because, as they have a lower degree of symmetry than the system, they are probably the emerging solutions in a spontaneous symmetry breaking process [32].

Appendix A

In this appendix we briefly sketch, without technical details, the method used for obtaining the characteristics of $\lambda - \omega$ systems. A complete reference can be found in [21].

Group of transformations

Let G be a local Lie Group, $x = (x_1, x_2, \ldots, x_n)$ the set of independent variables, and $u = (u^1, u^2, \ldots, u^m)$ the set of dependent variables, in a space of functions $u = u(x)$. A local Lie group of transformations in the space (x, u) is given by the set of equations

$$x^\epsilon = X(x, u, \epsilon) \qquad u^\epsilon = U(x, u, \epsilon) \tag{A1}$$

where ϵ is a continuous parameter of a local group, being $\epsilon = 0$ the value of the parameter for the identity element. The expression local means that the group properties are valid at least in some neighbourhood of $\epsilon = 0$. If the functions X and U depend not only on x and u but also on some derivatives, the transformations (A1) have no geometrical interpretation, and must be seen as transformations in the space of functions $u(x)$. In this case they are called generalized transformations.

Infinitesimals

For every transformation (A1) there is an infinitesimal transformation given by

$$\delta x = \xi(x, u)\epsilon \qquad \delta u = \eta(x, u)\epsilon \tag{A2}$$

with ϵ small enough; $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \ldots, \eta^m)$ are called the infinitesimals of the transformation and are given by

$$\xi = \left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0} \qquad \eta = \left(\frac{\partial U}{\partial \epsilon}\right)_{\epsilon=0}. \tag{A3}$$

Characteristics

The characteristic of the transformation group is defined as $Q = \eta - \xi^i u_i$. An equivalent transformation [21] to (A1) that leaves the x variables invariant is given infinitesimally by

$$\delta u = Q(x, u, \{u_i\})\epsilon \qquad \text{where } Q = \left(\frac{\partial U}{\partial \epsilon}\right)_{\epsilon=0}. \tag{A4}$$

This is a generalized transformation which has an equivalent geometrical transformation. The expression $\{u_i\}$ represents the set of derivatives $\partial u^\alpha/\partial x^i$ with $\alpha = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$.

We represent by $\{u_I\}$, where $I = (i_1, i_2, \ldots, i_n)$ is a multi-index, the set of derivatives, given explicitly by the expressions

$$\{u_I\} \rightarrow \frac{\partial^{|I|} u^\alpha}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}} \qquad \alpha = 1, 2, \ldots, m \qquad |I| = \sum_{j=1}^{n} i_j > 0.$$

The infinitesimal transformation for u_I is given by

$$\delta u_I = (D_I Q)\epsilon$$

where D_I is the total derivative operator

$$D_I = \frac{\partial}{x^I} + u_I \frac{\partial}{\partial u} + \sum_J u_{J,I} \frac{\partial}{\partial u_J} \qquad |J| > 0$$

with

$$\frac{\partial}{\partial x^I} = \frac{\partial^{|I|}}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}}.$$

Invariant functions

A function $u(x)$ is said to be invariant if it is left unchanged by the action of the transformation group, that is $\partial u^{\epsilon}/\partial\epsilon = 0$, or equivalently

$$Q(x, u, \{u_i\}) = 0. \tag{A5}$$

Symmetry Group

A system of partial differential equations,

$$F(x, u, \{u_J\}) = 0 \tag{A6}$$

is said to be invariant under a transformation group if every solution u is transformed by the group into another solution u^{ϵ}, that is, $F(x, u^{\epsilon}, \{u_I^{\epsilon}\}) = 0$. The corresponding infinitesimal condition is

$$Q\frac{\partial F}{\partial u} + D_I(Q)\frac{\partial F}{\partial u_I} = 0 \qquad |I| > 0 \tag{A7}$$

whenever u is a solution of the SPDE.

Invariant Solutions

Invariant solutions are solutions of the SPDE that are invariant with respect to a symmetry group. Then they must be solutions of equations (A5) and (A6). When the SPDE models a physical system, invariant solutions are very often functions that exhibit interesting patterns with physical interest.

Procedure

In order to find a symmetry group of a SPDE we first substitute the partial differential equations into (A7). The resulting equations are treated as forms in the derivatives of u, whose coefficients depend on (u, x, t) and the infinitesimals (η, ξ). After the substitution we collect together the coefficients of like derivative terms in u and set all of them equal to zero. The resulting equations are called the determining equations of the group. In practice these equations are solvable and thus the infinitesimals and characteristics of the group are determined. The subsequent study is clearly shown in this paper.

Mathematical Packages

These calculations, though not difficult in themselves, are clearly complicated as the order of the SPDE and the number of equations increase, so a software for symbolic mathematics becomes useful. To our knowledge, the best package for these kind of calculations is Macsyma. Programs written by the authors in Macsyma 4.0, running in a Convex, have been used to get the results shown in this paper.

Appendix B

In order to solve the boundary value problem (13), (14), a 'shooting' technique is used. A similar technique has been used by Greenberg [15] to study a different $\lambda - \omega$ system with $\lambda(z) = 1 - z$.

We study the solutions of (13) which satisfy initial conditions of the form

$$\lim_{r\to 0} z(r) = 0 \qquad \lim_{r\to 0} z_r(r) = \alpha \tag{B1}$$

where $\alpha > 0$ is a constant. We have to demonstrate that for a certain value of α, namely α_c, the corresponding solution of (13), is a solution of the problem (13), (14).

The first stage for our proof is the following: *If $z = z(r, \alpha)$ is a solution of (13), (B1) there exists a value $r_1 > 0$, such that $z_r(r_1) = 0$.*

We denote by B the set of values $r_0 \geqslant 0$ with $z_r(r_0) = 0$ and we shall divide the proof in several steps.

(1) B is not empty.

For r small enough z_{rr} is negative. This is easily seen by substituting the first terms of the power series of z in (13). If z_{rr} never vanish, then $0 \leqslant z_r < \alpha$, for every $r > 0$, and

$$\begin{aligned} 0 < z_r &= \frac{1}{r}\int_0^r s z(s)\left(\frac{1}{s^2} - 1 + z(s)^2\right) \mathrm{d}s \\ &\leqslant \frac{1}{r}\int_0^r s\alpha s\left(\frac{1}{s^2} - 1 + \alpha^2 s^2\right) \mathrm{d}s = \alpha - \frac{\alpha}{3}r^2 + \frac{\alpha^3}{5}r^4. \end{aligned} \tag{B2}$$

The discriminant of this polynomial is $\frac{\alpha^2}{9} - \frac{4}{5}\alpha^4$, which is positive for small enough α. Hence, the polynomial in (A2) is negative for some values of r and, for small enough α, z_r always vanish for some r_0.

(2) B is bounded below by 0, and the infimum of B must be different from 0.

If that infimum is 0 then $\lim z_r(r) = 0$ for $r \to 0$ and $r \in B$, the solution $z \equiv 0$ would be obtained. We shall denote $r_1 = r_1(\alpha) \equiv \inf(B) > 0$.

(3) $r_1 > 1$.

Equation (13) may be written as

$$(rz_r)_r + rz\left(1 - \frac{1}{r^2} - z^2\right) = 0. \tag{B3}$$

The mean value theorem implies that there exists a value $r' = r'(r)$ with the property

$$z_r(r) = \frac{1}{r}\int_0^r t z(t)\left(\frac{1}{t^2} - 1 + z(t)^2\right) \mathrm{d}t = \frac{r'}{r} z(r')\left(\frac{1}{r'^2} - 1 + z(r')^2\right) \tag{B4}$$

where $0 < r' < r$. For $r = r_1$, $z_r = 0$, this equation reads

$$z(r_1')\left(\frac{1}{r_1'^2} - 1 + z(r_1')^2\right) = 0 \tag{B5}$$

where $0 < r_1' < r_1$. If $z(r_1') = 0$, Rolle's theorem implies that there exists $r'' < r_1' < r_1$ such that $z_r(r'') = 0$, which is in contradiction with the definition of r_1. Hence

$$\frac{1}{r_1'^2} - 1 + z(r_1')^2 = 0. \tag{B6}$$

This equation implies that $r_1' > 1$ and, therefore, $r_1 > 1$.

(4) $0 < z(r) < 1$ for $0 \leqslant r \leqslant 1$.

According to the definition of r_1, $z(r)$ is an increasing function in $[0, r_1]$.

Hence, $z_{rr}(r_1) \leqslant 0$, and from equation (B3) we obtain

$$z_{rr}(r_1)\left(\frac{1}{r_1^2} - 1 + z(r_1)^2\right) < 0. \tag{B7}$$

Since $r_1 > 1$, this equation implies that

$$0 < z(r) < 1 \qquad \text{for } 0 \leqslant r \leqslant r_1. \tag{B8}$$

(5) r_1 grows with α.

We prove that $\mathrm{d}r_1/\mathrm{d}\alpha \geqslant 0$. By definition, $z_r(r_1(\alpha), \alpha) \equiv 0$. Differentiating this expression with respect to α we obtain

$$z_{rr}(r_1(\alpha), \alpha)\frac{\mathrm{d}r_1(\alpha)}{\mathrm{d}\alpha} + \frac{\partial z_r}{\partial \alpha}(r_1(\alpha), \alpha) = 0. \tag{B9}$$

Since $z_{rr}(r_1(\alpha), \alpha) < 0$, it is enough to prove that $\frac{\partial z_r}{\partial \alpha}(r_1(\alpha), \alpha) > 0$.

Let us consider the function $A(r) = \frac{\partial z}{\partial \alpha}(r, \alpha)$, where $z = z(r, \alpha)$ satisfies (13), (B1). $A(r)$ is a solution of the initial condition problem obtained by differentiating the first member of (B3) with the initial conditions (B1)

$$\begin{aligned} &(rA_r)_r + rA\left(1 - \frac{1}{r^2} - 3z^2\right) = 0 \\ &A(0) = 0 \qquad A_r(0) = 0. \end{aligned} \tag{B10}$$

If we prove that $A_r(r_1) > 0$, then we would obtain $dr_1(\alpha)/\mathrm{d}\alpha > 0$, and our assertion would be verified. Although this equation is not linear, our proof may be carried out by using Sturn's technique on the study of the distribution of zeros in solutions of second-order linear differential equations. We multiply equation (B3) by $A(r)$ and equation (B10) by $z(r)$ and subtract the resulting equations.

Since $(rz_r)_r - (rA_r)_r \equiv [r(z_rA - zA_r)]_r$, we obtain the following relation:

$$[r(z_rA - zA_r)]_r + rAz(2z^2) = 0. \tag{B11}$$

Let r_2 be the first positive zero of the function $A(r)$. Then $A(r) > 0$ in $(0, r_2)$, and $A'(r_2) \neq 0$. If we have that $z(r) > 0$ in $(0, r_2)$ then, from (B11), we would obtain $[rz_rA - zA_r)]r < 0$ in $(0, r_2)$. The integration of the first term of this relation over $(0, r_2)$ yields the relation

$$r_2[z_r(r_2)A(r_2) - z(r_2)A_r(r_2)] = -r_2z(r_2)A_r(r_2) < 0 \tag{B12}$$

and then $z(r_2) < 0$, which is in contradiction with $z(r) > 0$ in $(0, r_2)$. Hence, z_r changes its sign in $(0, r_2)$, and there exists a r_0 in $(0, r_2)$ where z_r vanish. This proves that $r_1 = r_1(\alpha) < r_2$, and $A(r_1) = \frac{\partial z}{\partial \alpha}(r_1) > 0$. Since the second additive term in (B11) is strictly positive in $(0, r_1)$, the first one must be strictly negative. The integration of this over $(0, r_1)$ yields

$$r_1z_r(r_1)A(r_1) - z(r_1)A_r(r_1) < 0. \tag{B13}$$

Hence, since $z_r(r_1) = 0$, $z(r_1)A_r(r_1) > 0$ and $A(r_1) > 0$. Therefore, $\frac{dr_1(\alpha)}{d\alpha} > 0$.

(6) In this step we proceed to study the system with boundary conditions (14).

Since r_1 increases with α, we may consider the set $R = \{\alpha : r_1(\alpha) < \infty\}$. We shall prove below that R is bounded above. In this case, if $\alpha_s = \sup(R)$:

$$\lim_{\alpha\to\alpha_s^+} r_1(\alpha) = 0 \qquad \lim_{\alpha\to\alpha_s^+} r_2(\alpha) = \infty$$

and, as a consequence of (B8), the following relations are satisfied when $r > 0$:

$$0 < z(r, \alpha_s) < 1 \qquad 0 < z_r(r, \alpha_s) \qquad \lim_{r\to\infty} z_r(r, \alpha_s) = 0. \tag{B14}$$

Considering equations (B3) for $r \to \infty$, we obtain $\lim_{r\to\infty} z_{rr}(r, \alpha_s) = 0$. Therefore, since

$$\lim_{r\to\infty} z(r, \alpha_s) = 1 \tag{B15}$$

R is bounded above and the function $z = z(r, \alpha_s)$ is the solution of (13), (14). In order to prove that R is bounded above a separation technique is used.

Equation (12) may be compared with the Bessel equation:

$$[rJz_r - rzJ_r]_r = rJz^3. \tag{B16}$$

Solutions of (12) are positive in $(0, j_1)$, where j_1 the first positive zero of J.

Since z vanishes in that interval, J will change its sign in it. This proves that the second term of (B4) is positive in $(0, j_1)$ and, therefore, for $r \in (0, j_1)$ we have

$$\int_0^r [sJ(s)z_r(s) - sz(s)J_r(s)]\,\mathrm{d}s = rJz_r - rzJ_r > 0. \tag{B17}$$

Hence $Jz_r - zJ_r$ is positive in $(0, j_1)$ and, as a consequence, $(z/J)' > 0$, i.e. z/J is positive in $(0, j_1)$. Since $\lim_{r\to 0} \frac{z(r)}{J(r)} = \alpha$, J increases in $(0, j_2)$, where j_2 is the first positive zero of J_r. The conclusion is that for great enough α, $z(r)$ can be arbitrarily large and z_r never vanish. Therefore, for great enough α, z is a function monotonical divergent to $+\infty$. We have demonstrated that there exist solutions $z(r, \alpha)$, of (13), (14) which are both monotonical increasing and bounded.

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