# Guaranteed State Estimation and Fault Detection based on Zonotopes for Differential-Algebraic-Equation Systems

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*Abstract*— **This paper presents a set-membership approach based on zonotopes for differential-algebraic-equation (DAE) systems with unknown-but-bounded disturbances and noise, which can be subsequently used for guaranteed state estimation and fault detection. Complex systems are usually modeled by differential and algebraic equations, where differential equations describe system dynamics and additionally, algebraic equations represent the static relations. The proposed algorithm provides a way to propagate a zonotopic set that contains the system states not only consistent with the measurement outputs but also constrained with their static relations. Finally, a real application has been presented to verify the proposed approach.**

## I. INTRODUCTION

Complex systems in many engineering applications are sometimes mathematically modeled by means of differential and algebraic equations. The control-oriented model of the differential-algebraic-equation (DAE) system, for instance [1], contains differential equations that describe system dynamics (system state evolutions) and algebraic equations that represent static relations between some certain system variables.

The general set-membership approach is established by propagating a polytopic inclusion of system states with the assumptions of an uncertain parametric model and unknown-but-bounded description of noise that is not assumed to follow a particular probabilistic distribution [2]. The set-membership approach for ordinary-differential-equation (ODE) systems can be used for state estimation and fault detection by means of checking the consistency of system states and measurement outputs [3], [4], [5], [6]. A zonotopic set is a convex symmetric polytope including all the possible system states. Compared to the other polytopes (i.e. ellipsoids), zonotopes are widely used to represent uncertainties (disturbances and noise) due to their flexibility, reduced complexity and computational load in a series of linear operators, for instance the Minkowski sum.

Guaranteed state estimation for ODE systems without static relations has been presented in [4]. All possible system states that are consistent with measurement at the same sampling time are characterized as a compact set that is found from a posterior intersection between a prior zonotope and a strip for single-output measurement or a polytope for multiple-output measurement. The intersection may be over-approximated by using a zonotope as well. The size of this zonotope can be minimized by a number of optimization-based methods, such as the minimization of the segment, volume and P-radius of the zonotope as proposed in [4], [5], [7]. Besides, fault detection based on a consistency-based state estimation test is presented in [6], [8]. When the corresponding system state set from a measurement is found to be inconsistent with the possible system state set, a fault is detected.

The paper proposes a set-membership approach for DAE systems with the assumption of known algebraic variables. The DAE systems can be also formulated in a linear time/parameter-varying (LTI/LPV) form with known scheduling parameters. The system states following the system dynamics are included into a compact zonotopic set while system states constrained by the static relations are inside a compact polytopic set. First of all, the consistency of system states in feasible dynamic state set is guaranteed by the output measurements. The measurement state set is characterized by a strip (for single-output case) or a polytope assembled by several strips (for multiple-output case). The intersection between the two sets is over-approximated as a zonotopic set for the purpose of simplicity. Then, another intersection between the constrained state set and previous intersection can be found. Furthermore, fault detection can be achieved by checking system consistency with the measurement outputs.

This paper starts with preliminaries in Section II and the problem formulation in Section III. The set-membership approach based on zonotopes for DAE systems is discussed in Section IV. Guaranteed state estimation and fault detection for DAE systems are introduced by using the proposed algorithm in Section V and VI, respectively. A real application of a portion of the Riera Blanca catchment in the Barcelona sewer network is presented in Section VII. Conclusions are addressed in Section VIII.

## II. PRELIMINARIES

Before presenting the approach in this paper, some preliminary set definitions and properties are briefly introduced as follows:

*Definition 1:* An interval $[a, b]$ is defined as the set $\{x \mid a \leq x \leq b\}$. Moreover, the unitary interval $\mathbf{B}^1$ is defined as $\mathbf{B}^1 = [-1, 1]$.

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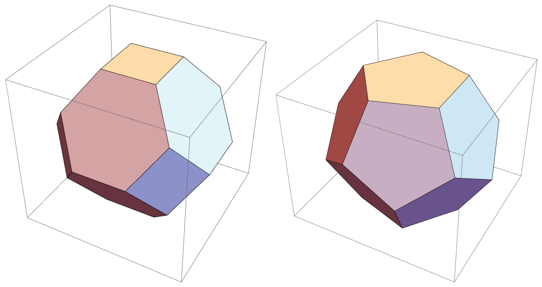


(a) Truncated Octahedron (b) Dodecahedron

Fig. 1. Some specific zonohedrons

*Definition 2:* A unitary box $\mathbf{B}^n \in \mathbb{R}^n$ is a box composed of $n$ unitary intervals.

*Definition 3:* The Minkowski sum of two sets $\mathcal{X}$ and $\mathcal{Y}$ is defined by $\mathcal{X} \oplus \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$.

*Definition 4:* A polyhedron set $\mathcal{P} \in \mathbb{R}^n$ is defined as an intersection of a finite number of half-spaces in a form $\mathcal{P} = \{x \in \mathbb{R}^n \mid Qx \leq b\}$ with $Q \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If $\mathcal{P}$ is bounded, then $\mathcal{P}$ is a polytope.

A zonohedron is a convex polyhedron, where every surface is a polygon with point symmetry as some zonohedrons shown in Figure 1. In general, a zonotope can be defined by means of the Minkowski sum of a matrix including line segments with a center.

*Definition 5:* An $m$-zonotope $\mathcal{Z} \in \mathbb{R}^n$ $(m \geq n)$ is defined by hypercube affine projection with the center $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n \times m}$ as $\mathcal{Z} = p \oplus H\mathbf{B}^m$.

*Definition 6:* A strip $S$ is defined as $S = \{x \in \mathbb{R}^n \mid |Cx - d| \leq \sigma\}$.

*Property 1:* The Minkowski sum of two zonotopes $\mathcal{Z}_1 = p_1 \oplus H_1\mathbf{B}^{n_1}$ and $\mathcal{Z}_2 = p_2 \oplus H_2\mathbf{B}^{n_2}$ is still a zonotope as $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2 = (p_1 + p_2) \oplus [H_1 \;\; H_2]\,\mathbf{B}^{n_1+n_2}$.

*Property 2:* The image of a zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^m$ by a linear mapping matrix $K$ can be calculated as $K\mathcal{Z} = Kp \oplus KH\mathbf{B}^m$.

*Property 3:* Consider a zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^m$, the smallest box (Interval Hull) containing the zonotope is given by $\square\mathcal{Z} = p \oplus rs(H)\mathbf{B}^m$, where $rs(H)$ is a diagonal matrix such that $rs(H)_{i,j} = \sum_{j=1}^{m} |H_{i,j}|$.

*Property 4 (Intersection between zonotope and strip [4]):* Given a zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^r$ and a strip $S = \{x \in \mathbb{R}^n \mid |Cx - d| \leq \sigma\}$, there exist a vector $\lambda \in \mathcal{R}^n$, the zonotopic over-approximation $\widetilde{\mathcal{Z}}$ of the intersection is defined as $\widetilde{\mathcal{Z}} = \widetilde{p}(\lambda) \oplus \widetilde{H}(\lambda)\mathbf{B}^{r+1}$, where $\widetilde{p}(\lambda) = p + \lambda\,(d - Cp)$ and $\widetilde{H}(\lambda) = [(I - \lambda C)H \;\; \sigma\lambda]$.

*Property 5 (Intersection between zonotope and polytope [9]):* Given a zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^r$ and a polytope $\mathcal{P} = \{x \in \mathbb{R}^n \mid |Qx - d| \leq b\}$ where $Q \in \mathbb{R}^{m \times n}$ and $b = [b_1, b_2, \ldots, b_m]^T$ with $b_i \in \mathbb{R}_+, i = 1, 2, \ldots, m$ , there exists a matrix $\Lambda \in \mathbb{R}^{n \times m}$, the zonotopic over-approximation $\widetilde{\mathcal{Z}}$ of the intersection is defined as $\widetilde{\mathcal{Z}} = \widetilde{p}(\Lambda) \oplus \widetilde{H}(\Lambda)\mathbf{B}^{r+1}$, where $\widetilde{p}(\Lambda) = p + \lambda\,(d - Qp)$ and

$$\widetilde{H}(\Lambda) = [(I - \Lambda Q)H \;\; \Lambda\Sigma] \text{ with } \Sigma = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix}.$$

## III. Problem Formulation

The generalized uncertain discrete-time control system can be formulated by using a DAE as follows:

$$x_{k+1} = f\left(x_k, u_k, z_k, \omega_k\right), \tag{1a}$$
$$0 = g\left(x_k, u_k, z_k, \sigma_k\right), \tag{1b}$$
$$y_k = h\left(x_k, u_k, z_k, \upsilon_k\right), \tag{1c}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the vector of system states that are differential variables, $z_k \in \mathbb{R}^{n_z}$ denotes the vector of algebraic variables and $y_k \in \mathbb{R}^{n_y}$ denotes the measurement output vector at time instant $k \in \mathbb{N}$. $u_k \in \mathbb{R}^{n_u}$ represents the vector of control inputs at time instant $k$. $f(\cdot)$ and $g(\cdot)$ are linear or nonlinear mapping functions describing system dynamics and static relations, respectively. $h(\cdot)$ is system measurement mapping function. Moreover, $\omega_k \in \mathbb{R}^p$, $\sigma_k \in \mathbb{R}^q$ and $\upsilon_k \in \mathbb{R}^r$ represent the system disturbances, static disturbances and measurement noise at time instant $k$, respectively.

*Assumption 1:* The vector of algebraic variables $z$ is assumed to be known at each time instant. In fact, the DAE system in (1) is equivalent to the dynamic system described in (1b) subject to a static constraint (1c).

*Assumption 2:* All the uncertainties including $\omega_k, \sigma_k$ and $\upsilon_k$ are unknown but bounded in compact sets: $\omega_k \in W, \sigma_k \in \Phi, \upsilon_k \in V$. The initial state is also bounded in a compact set: $x_0 \in \mathcal{X}_0$.

A LPV model is a linear representation of a given nonlinear or parameter dependent system. Therefore, a practical nonlinear DAE system can be reformulated as a LPV model by means of the automated generation method [10].

Therefore, a nonlinear DAE system can be transformed as

$$x_{k+1} = A\left(\theta_k\right)x_k + B\left(\theta_k\right)u_k + B_z\left(\theta_k\right)z_k + F\omega_k, \tag{2a}$$
$$0 = E_x\left(\theta_k\right)x_k + E_u\left(\theta_k\right)u_k + E_z\left(\theta_k\right)z_k + E\sigma_k, \tag{2b}$$
$$y_k = C\left(\theta_k\right)x_k + D\left(\theta_k\right)u_k + C_z\left(\theta_k\right)z_k + G\upsilon_k, \tag{2c}$$

where $A\left(\theta_k\right)$, $B\left(\theta_k\right)$, $B_z\left(\theta_k\right)$, $E_x\left(\theta_k\right)$, $E_u\left(\theta_k\right)$, $E_z\left(\theta_k\right)$, $C\left(\theta_k\right)$, $D\left(\theta_k\right)$ and $C_z\left(\theta_k\right)$ are continuous mapping corresponding to the time-dependent parameter vector $\theta_k$. $E$, $F$ and $G$ are linear time-invariant disturbance and noise transition matrices with suitable dimensions.

Before introducing the set-membership approaches, some sets are defined as follows:

*Definition 7 (Constrained Uncertain State Set):* Given a DAE system (1) containing system dynamics (1b) and static relations (1c), the constrained uncertain state set $\bar{\mathcal{X}}_k$ at time instant $k$ is defined as $\bar{\mathcal{X}}_k = \hat{\mathcal{X}}_k \cap \hat{\mathcal{X}}_k^s$, where $\hat{\mathcal{X}}_k$ denotes the system dynamics set and $\hat{\mathcal{X}}_k^s$ denotes the static relations set.

*Definition 8 (Consistent State Set [4]):* Given a DAE system (1) and a measurement output $y_k$, the consistent state set at time instant $k$ is defined as

$$\mathcal{X}_{y_k} = \left\{x \in \mathbb{R}^{n_x} \mid y_k \in h\left(x, u, V\right)\right\}. \tag{3}$$

*Definition 9 (Exact Uncertain State Set [4]):* Consider a DAE system (1), the exact uncertain state set $\hat{\mathcal{X}}_k$ is equal to constrained system states that are consistent with measurement outputs $y_1, y_2, \ldots, y_n$ with $n \in \mathbb{Z}_+$ and initial state set $\mathcal{X}_0$, which is defined by

$$\widetilde{\mathcal{X}}_k = \bar{\mathcal{X}}_k \cap \mathcal{X}_{y_k} = \hat{\mathcal{X}}_k \cap \mathcal{X}_{y_k} \cap \hat{\mathcal{X}}_k^s, \quad k \geq 1. \tag{4}$$

Note that the order of intersections among three state sets is forced to be followed since occurrence time of the three sets are different.

## IV. Set-membership Approach for DAE Systems

Considering the LPV representation of the DAE system in (2), the set-membership approach leads to a recursive calculation of a set including all the possible values of system states $x_k$ under unknown-but-bounded perturbations at each sampling time $k$. The set-membership approach for DAE systems has the following four steps: *Prediction*, *Measurement*, *Consistency Correction* and *Constrained* steps. As shown in Figure 2, the general set-membership for DAE systems is geometrical presented. For the sake of stating the algorithm, the following assumptions are considered at time instant $k$.

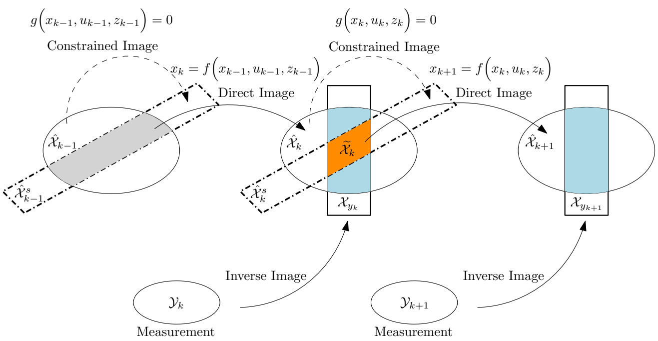



Fig. 2. Set-membership approach for DAE systems

*Assumption 3:* $x_{k-1}$ is assumed as known in a zonotopic set $\mathcal{X}_{k-1} = \mathcal{X}_{k-1}^c \oplus H_{\mathcal{X}_{k-1}} \mathbf{B}^{r_0}$ as initial condition and a sampled output measurement $y_k$ is available at time instant $k$. Moreover, disturbances and noise are bounded by geometrical sets as follows:

$$\omega_k = \omega^c \oplus H_\omega \mathbf{B}^{n_\omega},$$
$$|\sigma_k| \leq \sigma > 0,$$
$$|v_k| \leq v > 0.$$

where $\omega^c$ and $H_\omega$ are the center and segment matrix of the zonotope for system disturbances. $\sigma$ and $v$ are vectors of bounds of static disturbances and measurement noise, respectively. All of these parameters for describing bounds of disturbances and noise are assumed as known.

**Set-membership Approach for DAE systems:**

- *Prediction* step: Given DAE system in (2), compute a predicted zonotope of system states at the time instant $k$ as a propagated zonotope $\hat{\mathcal{X}}_k = \hat{\mathcal{X}}_k^c \oplus H_{\hat{\mathcal{X}}_k} \mathbf{B}^{r_1}$ with

$$\hat{\mathcal{X}}_k^c = A(\theta_k) \mathcal{X}_{k-1}^c + B(\theta_k) u_k + B_z(\theta_k) z_k + F\omega^c,$$
$$\hat{H}_{\mathcal{X}_k} = \left[A(\theta_k) H_{\mathcal{X}_{k-1}} \quad FH_\omega\right].$$

- *Measurement* step: Compute the measurement consistent state set $\mathcal{X}_{y_k}$ by means of the measurement output $y_k$, which is a strip for single output and a polytope clustering by a group of strips for multiple outputs.
- *Consistency Correction* step: Test whether there exists an intersection between $\hat{\mathcal{X}}_k$ and $\mathcal{X}_{y_k}$. If it exists, and then compute an outer approximation $\check{\mathcal{X}}_k = \check{\mathcal{X}}_k^c \oplus H_{\check{\mathcal{X}}_k} \mathbf{B}^{r_2}$.
- *Constrained* step: Find the constrained state set $\hat{\mathcal{X}}_k^s$ that might correspond to the control input $u_k$ and compute an outer approximation $\widetilde{\mathcal{X}}_k = \widetilde{\mathcal{X}}_k^c \oplus H_{\widetilde{\mathcal{X}}_k} \mathbf{B}^{r_3}$ of the intersection between $\check{\mathcal{X}}_k$ and $\hat{\mathcal{X}}_k^s$.

## V. Guaranteed State Estimation for DAE Systems

Based on the set-membership approach presented in the previous section, the trajectory and its bounds of DAE systems can be estimated by finding the zonotopic outer approximation of the exact uncertain state set.

First of all, the intersection between $\hat{\mathcal{X}}_k$ and $\mathcal{X}_{y_k}$ can be approximated by the zonotope inclusion $\bar{\mathcal{X}}_k$ as presented in *Property 4* (for single output) and *Property 5* (for multiple outputs). The optimal vectors $\lambda$ and $\Lambda$ can be computed by minimizing the volume of the zonotope inclusion whose performance has been proved better than the method by minimizing segment matrix [7].

*Definition 10:* The volume of a zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^m \in \mathbb{R}^n$ is given by

$$\mathrm{Vol}(\mathcal{Z}) = 2^n \sum_{i=1}^{N(n,m)} \left|\det\left[H_{s_1(i)}\ H_{s_2(i)}\ \cdots\ H_{s_n(i)}\right]\right|,$$

where $N(n,m)$ denotes the total number of different selections of choosing $n$ elements from $m$. $H_i$ denotes $i$-th column of $H$. Integers $s_j(i)$ denote each one of different ways of choosing $n$ elements from $m$, where satisfy $1 \leq s_1(i) < s_2(i) < \cdots < s_n(i) \leq m$.

The optimal vectors $\bar{\lambda}$ and $\bar{\Lambda}$ can be found by minimizing the volume formulas of $\bar{\mathcal{X}}_k(\bar{\lambda})$ and $\bar{\mathcal{X}}_k(\bar{\Lambda})$ as follows [4], [9]:

$$\begin{aligned}\mathrm{Vol}(\bar{\mathcal{X}}_k(\bar{\lambda})) =& 2^n \sum_{i=1}^{N(n,r)} \left|1 - C(\theta_k)\bar{\lambda}\right| \left|\det(M_i)\right| \\ &+ 2^n \sum_{i=1}^{N(n-1,r)} \sigma \left|\det[N_i \ \ q_i]\right| \left|q_i^T \bar{\lambda}\right|,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathrm{Vol}(\bar{\mathcal{X}}_k(\bar{\Lambda})) =& 2^n \sum_{i=1}^{N(n,r)} \left|I - C(\theta_k)\bar{\Lambda}\right| \left|\det(M_i)\right| \\ &+ 2^n \sum_{i=1}^{N(n-1,r)} \Sigma \left|\det[N_i \ \ q_i]\right| \left|q_i^T \bar{\Lambda}\right|,\end{aligned} \tag{8}$$

where $M_i$ is each of the different matrices of choosing $n$ columns of $H_{\bar{\mathcal{X}}_k}$, $N_i$ is each of the different matrices of choosing $n-1$ columns of $H_{\bar{\mathcal{X}}_k}$ and $q_i$ is the image of $N_i$ with the properties of $q_i^T q_i = 1$ and $q_i^T N_i = 0$.

Then, the outer approximation $\widetilde{\mathcal{X}}_k$ in the *Constrained* step is computed in a similar way. The optimal vectors $\widetilde{\lambda}$ and $\widetilde{\Lambda}$ can be found by minimizing the following convex formulas:

$$\mathrm{Vol}(\widetilde{\mathcal{X}}_k(\widetilde{\lambda})) = 2^n \sum_{i=1}^{N(n,r)} \left|1 - E_x(\theta_k)\,\bar{\lambda}\right| |\det(M_i)| + 2^n \sum_{i=1}^{N(n-1,r)} \sigma\,|\det[N_i \;\; q_i]| \left|q_i^T \widetilde{\lambda}\right|, \quad (9)$$

and

$$\mathrm{Vol}(\widetilde{\mathcal{X}}_k(\widetilde{\Lambda})) = 2^n \sum_{i=1}^{N(n,r)} \left|I - E_x(\theta_k)\,\widetilde{\Lambda}\right| |\det(M_i)| + 2^n \sum_{i=1}^{N(n-1,r)} \Sigma\,|\det[N_i \;\; q_i]| \left|q_i^T \widetilde{\Lambda}\right|. \quad (10)$$

Finally, the zonotope inclusion of $\widetilde{\mathcal{X}}_k = \widetilde{\mathcal{X}}_k^c \oplus H_{\widetilde{\mathcal{X}}_k}\mathbf{B}^{r_3}$ is found as follows:

$$\widetilde{\mathcal{X}}_k^c(\widetilde{\lambda}) = \bar{\mathcal{X}}_k^c - \widetilde{\lambda}(P_k + E_x(\theta_k)\,\bar{\mathcal{X}}_k^c), \quad (11a)$$

$$H_{\widetilde{\mathcal{X}}_k}(\widetilde{\lambda}) = \left[(I - \widetilde{\lambda}E_x(\theta_k))H_{\bar{\mathcal{X}}_k} \;\; \sigma\widetilde{\lambda}\right], \quad (11b)$$

or

$$\widetilde{\mathcal{X}}_k^c(\widetilde{\Lambda}) = \bar{\mathcal{X}}_k^c - \widetilde{\Lambda}(P_k + E_x(\theta_k)\,\bar{\mathcal{X}}_k^c), \quad (12a)$$

$$H_{\widetilde{\mathcal{X}}_k}(\widetilde{\Lambda}) = \left[(I - \widetilde{\Lambda}E_x(\theta_k))H_{\bar{\mathcal{X}}_k} \;\; \Sigma\widetilde{\Lambda}\right], \quad (12b)$$

with $P_k = E_u(\theta_k)\,u_k + E_z(\theta_k)\,z_k$.

Therefore, the trajectory of estimated states in DAE systems are the center $\widetilde{\mathcal{X}}_k^c$ of the zonotope inclusion and bounds of its orbits can be found by using interval hull method presented in *Property 3*.

*Example*

In order to illustrate the guaranteed state estimation algorithm, a numerical example is considered with the following description:

$$x_{k+1} = \begin{bmatrix} 0.8 & -0.3 \\ 0.01 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 0.01 & 1 \end{bmatrix} u_k + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} z_k + \omega_k, \quad (13a)$$

$$\begin{bmatrix} 0.05 & 0.1 \end{bmatrix} x_k + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} u_k + \begin{bmatrix} -1 & -1 \end{bmatrix} z_k + \sigma_k = 0, \quad (13b)$$

$$y_k = \begin{bmatrix} 0.5 & 0.3 \end{bmatrix} x_k + v_k \quad (13c)$$

The control inputs may be found by means of a simple controller based on Model Predictive Control (MPC). As shown in Figure 3, the measurement state set is presented as a strip in the dotted dashed line and static relation in (13c) is shown as another strip in dashed line. The zonotopic outer approximation of the intersection can be found by means of the aforementioned volume-based optimization method shown as the red real line in Figure 3. The smallest box of this zonotope is found by interval hull method to compute the bounds of the state estimation.

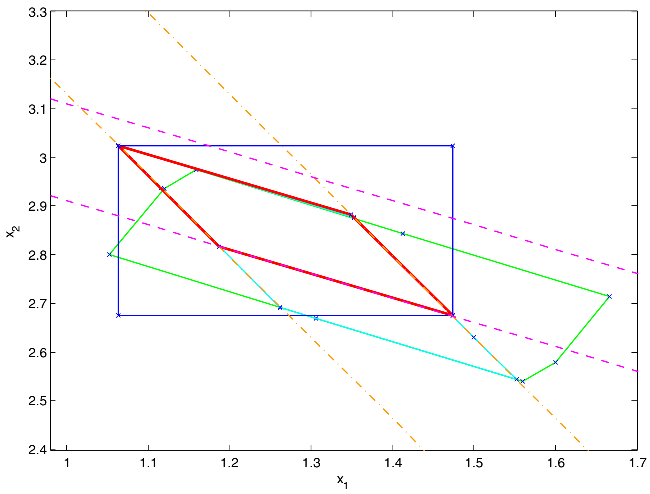



Fig. 3. Intersection between zonotope and two strips: the red zonotope is $\widetilde{\mathcal{X}}_k^c$ and a smallest box is in blue.

## VI. Fault Detection for DAE Systems

The use of the set-membership approach for DAE systems in fault detection is straightforward. Assume that a DAE system is operated in non-faulty conditions. The initial conditions are given in the set $\mathcal{X}_0$ including all the admissible system states. The existing faults can be detected based on the aforementioned set-membership approach for DAE systems. A fault is said to have occurred if the intersection set is empty.

$$\check{\mathcal{X}}_k = \hat{\mathcal{X}}_k \cap \mathcal{X}_{y_k} = \varnothing. \quad (14)$$

As defined in [8], there exists a support hyperplane of $\hat{\mathcal{X}}_k$, which is defined as

$$S_k \triangleq C(\theta_k) \cdot \hat{\mathcal{X}}_k = \left\{ x_k \in \mathbb{R}^{n_x} \mid \underline{q}_j(\theta_k) \leq c_j(\theta_k)\,x_k \leq \overline{q}_j(\theta_k), j = 1, \ldots, n_y \right\},$$

where $c_j(\theta_k)$ denotes the $j$-th row of matrix $C(\theta_k)$. $\underline{q}_j(\theta_k)$ and $\overline{q}_j(\theta_k)$ can be computed by the following equations:

$$\underline{q}_j(\theta_k) = c_j(\theta_k)\,\hat{\mathcal{X}}_k^c - \left\| H_{\hat{\mathcal{X}}_k}^T c_j(\theta_k)^T \right\|_1, \quad (15a)$$

$$\overline{q}_j(\theta_k) = c_j(\theta_k)\,\hat{\mathcal{X}}_k^c + \left\| H_{\hat{\mathcal{X}}_k}^T c_j(\theta_k)^T \right\|_1, \quad (15b)$$

where $\|\cdot\|$ denotes the 1-norm of a vector.

Similarly, the measurement set $\mathcal{X}_{y_k}$ can be transformed into a support hyperplane defined as

$$F_k \triangleq C(\theta_k) \cdot \mathcal{X}_{y_k} = \{x_k \in \mathbb{R}^{n_x} \mid -v \leq y_{k,j} - c_j(\theta_k)\,x_k \leq v, j = 1, \ldots, n_y\},$$

where $y_{k,j}$ denotes the $j$-th element of the vector of the measurement output signals $y_k$. The general fault detection algorithm for DAE systems are proposed in *Algorithm 1*.

**Algorithm 1** Fault Detection for DAE Systems

$\mathcal{X}_{k-1} \Longleftarrow \mathcal{X}_0$
**for** $k := 1$ to $N$ **do**
  Compute the *Prediction State* set $\hat{\mathcal{X}}_k$
  Compute the *Measurement State* set $\mathcal{X}_{y_k}$
  **if** $S_k \cap F_k = \varnothing$ **then**
    $fault_k = 1$ **(Fault Detected)**
  **else**
    $fault_k = 0$
  **end if**
  Compute the *Constrained State* set $\bar{\mathcal{X}}_k = \hat{\mathcal{X}}_k \cap \hat{\mathcal{X}}_k^s$
**end for**

## VII. Application: Riera Blanca Sewer Network

### A. System Description

The Riera Blanca pilot catchment in the Barcelona sewer network is presented in Figure 4. A portion of the Riera Blanca catchment network including three virtual tanks ($V8, V9$ and $V10$) and one node are selected as the case study in this paper in order to demonstrate the proposed set-membership approach. According to [11] and [12], the discrete-time control-oriented model of the Riera Blanca catchment network can be built by means of a sequence of nonlinear DAEs, which can be subsequently reformulated in a LPV form as follows:

$$x_{k+1} = A(\theta_k)x_k + Bu_k + B_d d_k + \omega_k, \tag{16a}$$

$$E_x(\theta_k)x_k + E_u u_k + E_d d_k + \sigma_k = 0, \tag{16b}$$

$$y_k = C(\theta_k)x_k + \upsilon_k, \tag{16c}$$

where $x_k$ represents the vector of water volumes in virtual tanks $V8$, $V9$ and $V10$ at time instant $k$, $u_k$ denotes the vector of control inputs found by MPC as presented in [11], $d_k$ denotes the vector of rain intensity associated to the virtual tanks with the relationship of the absorption coefficient $\varepsilon_i$ and surface of catchment $S_i$, which are given in Table I. $\omega_k, \sigma_k, \upsilon_k$ are system disturbances and measurement noise. Furthermore, $A(\theta_k)$, $E_x(\theta_k)$ and $C(\theta_k)$ are parameter-varying matrices with respect to the parameter $\theta_k$. $B$, $B_d$, $E_u$ and $E_d$ are linear time-invariant matrices of appropriate dimensions.

The configurations of the LPV model are presented in the following:

$$A(\theta_k) = \begin{bmatrix} 1-\theta_{1_k} & 0 & 0 \\ 0 & 1-\theta_{2_k} & 0 \\ 0 & 0 & 1-\theta_{3_k} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta t, B_d = \begin{bmatrix} \varepsilon_8 S_8 & 0 & 0 \\ 0 & \varepsilon_9 S_9 & 0 \\ 0 & \varepsilon_{10} S_{10} & 1 \end{bmatrix} \Delta t,$$

$$E_x(\theta_k) = \begin{bmatrix} \theta_{1_k} & \theta_{2_k} & 0 \end{bmatrix}, E_u = \begin{bmatrix} 0 & 1 \end{bmatrix}, E_d = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix},$$

$$C(\theta_k) = \begin{bmatrix} \theta_{1_k} & 0 & 0 \end{bmatrix}.$$

The parameter $\theta_k$ is defined by

$$\theta_k = [\theta_{1_k} \quad \theta_{2_k} \quad \theta_{3_k}]^T = [cvc_{8_k} \quad cvc_{9_k} \quad cvc_{10_k}]^T, \tag{17}$$

where $cvc_{8_k}, cvc_{9_k}$ and $cvc_{10_k}$ are varying along the time horizon as shown in Figure 5.

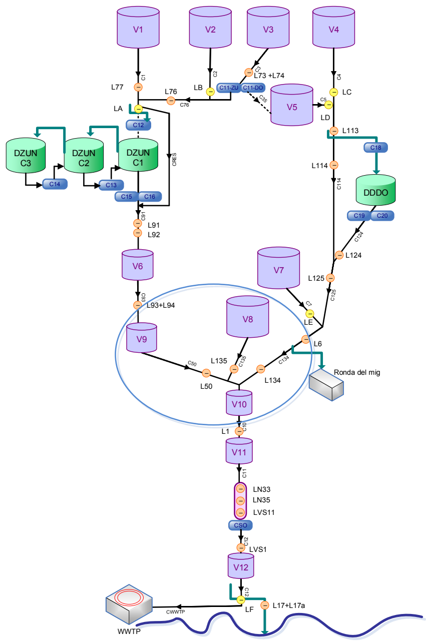



Fig. 4. Riera Blanca catchment conceptual model

TABLE I
PARAMETERS OF RIERA BLANCA PILOT CATCHMENT

| i | 8 | 9 | 10 |
|---|---|---|---|
| $\varepsilon_i$ | 0.584383 | 0.048593 | 0.9855 |
| $S_i$ | 769800 | 722500 | 149000 |
| $c_i$ | 0.000546 | 0.00052 | 0.0003 |

The initial conditions for applying the proposed set-membership approach are written in the following:

$$x_0 = \begin{bmatrix} 100 & 100 & 100 \end{bmatrix}^T, \omega^c = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$

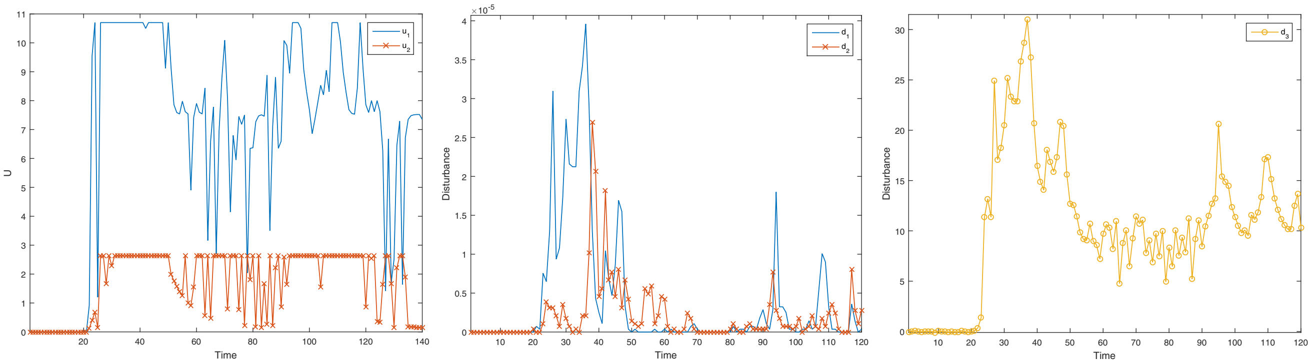
$$H_\omega = \begin{bmatrix} 70.9590 & 164.7320 & 270.6517 \end{bmatrix}^T,$$

$$\bar{\sigma} = 0.1, \bar{\upsilon} = 0.1.$$

### B. Results of Guaranteed State Estimation

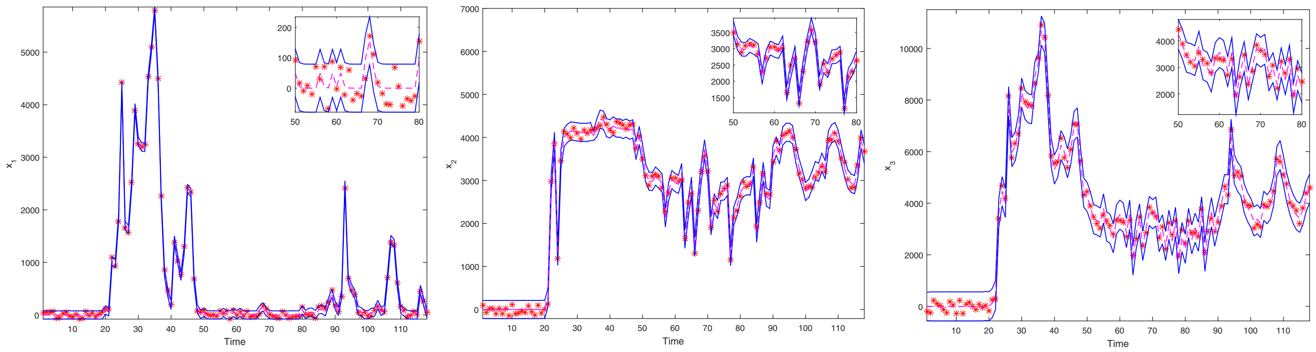
Control inputs found by an MPC controller presented in [11], measured disturbances (rainfall) and a measurement output are shown in Figure 6. By applying the proposed set-membership approach to the LPV model of the Riera Blanca pilot catchment, state estimation results are shown in Figure 7. The star points are sampled uncertain system states with the assumed description of bounded disturbances. It is obvious that all the sampled uncertain states are inside the sets (blue lines) obtained by the set-membership approach. The magenta dashed lines denote the nominal estimated states.

On the other hand, bounds for all the states are varying and all the testing states are located inside the estimated bounds by using the proposed set-membership approach.




(a) $u_1$, $u_2$ and $u_3$ (b) $d_1$ and $d_2$ (c) $d_3$

Fig. 6. Control inputs and measured disturbances of Riera Blanca pilot catchment




(a) $x_1$ (b) $x_2$ (c) $x_3$

Fig. 7. State estimation results of Riera Blanca pilot catchment

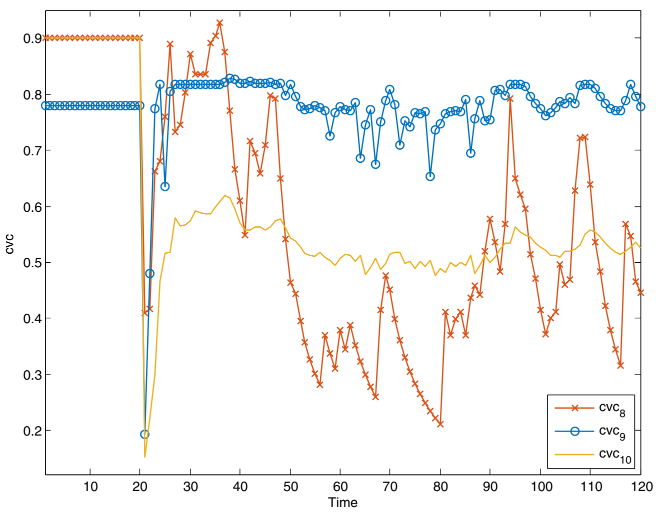



Fig. 5. Parameter values of the LPV model of Riera Blanca pilot catchment

### C. Results of Fault Detection

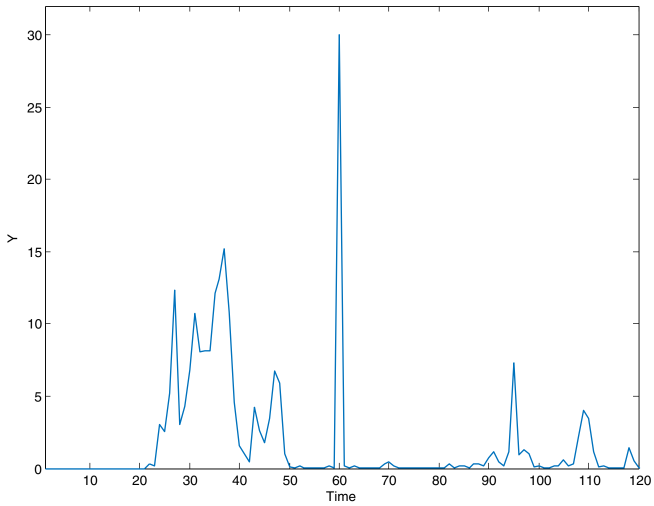
The measurement output of $x_1$ is plotted in Fig. 8(a). With this measurement output data, *Algorithm 1* is implemented. The fault detection result is shown in Fig. 8(b), where $fault = 1$ means a fault is detected. Through this result, a fault has been detected at time step 60.

It is noted that when the fault detection algorithm is applying. A suitable fault isolation result is assumed to be available. Hence, after a fault is detected. The intersection set is reconfigured and the procedure of fault detection can continue running.

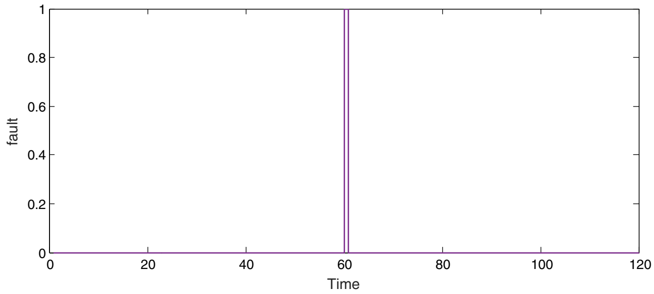
## VIII. Conclusion

This paper presents a set-membership approach of state estimation and fault detection for DAE systems. The proposed approach is suitable for not only linear DAE system but also nonlinear DAE system that can be reformulated as a LPV form. It is assumed that the DAE system have bounded description of parametric disturbances and measurement noise. Moreover, the algebraic variables in DAE system are assumed to be known. All the possible system states found by the set-membership approach are not only consistent with measurement outputs but also constrained with the system static relations. The zonotope is selected for applying the set-membership approach because its calculation complexity is acceptable. The outer approximation of the intersection is also based on zonotopes and the reduction of the size of zonotope is obtained by solving a convex optimization problem to minimize the volume of the zonotope inclusion. Besides, faults in the DAE system can be detected by means of the proposed fault detection algorithm based on set-membership approach for DAE systems.

As the future work, the set-membership approach for the DAE system with unknown algebraic variables will be deeply




(a) Measurement from sensor




(b) Detected fault

Fig. 8. Fault detection results of Riera Blanca pilot catchment

investigated. The set-membership state estimation will be used for observing the algebraic variables and checking the consistency between algebraic variables and measurement outputs.

## ACKNOWLEDGMENT

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