



LINEAR PROGRAMMING FOR THE ANALYSIS AND VIRTUAL RECREATION OF HISTORICAL EVENTS: THE ALLOCATION OF THE ARTILLERY DURING THE SIEGE OF BILBAO IN 1874

PROGRAMACIÓN LINEAL PARA EL ANÁLISIS Y LA RECREACIÓN VIRTUAL DE EPISODIOS HISTÓRICOS: LA DISTRIBUCIÓN DE LA ARTILLERÍA DURANTE EL SITIO DE BILBAO EN 1874

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Highlights:

- Geographic Information Systems (GIS) allow studying past events through the recreation of the geographical space and the interactions between the elements.
- Linear programming can be a suitable option to include actors' reasoning as a part of the modelling process.
- The usefulness of the system models also enables the identification of critical issues, testing alternative scenarios and sharing information.

Abstract:

The current digital technologies development makes it possible to apply new forms of studying historical events considering the geographical point of view. They rely on the location and the relationships among the different elements that took part in them over a recreated space (e.g. relief, roads, rivers); once these elements have been laid out on the virtual space, Geographic Information Systems (GIS) can be used to analyse several factors, such as distances, visibility, connectivity and so on. Nevertheless, the development of the actions was also driven by the aims, needs and beliefs (either wise or misguided) of the people/actors involved in those situations; therefore, some ways of including reasoning would significantly improve the actual recreation and understanding of the episodes. In this sense, "linear programming" is a very versatile tool for system modelling and optimization that is broadly used in many fields (e.g. industry, transports, agriculture, etc.). Likewise, this technique can also be applied to past scenarios to simulate dynamics and cross-check sources. In this text, two models regarding the distribution and the allocation of supplies during the siege of Bilbao, in the framework of the Third Carlist War (1872-1876), from both parties —beleaguerer and besieged— were established based on the war front textual reports. In these models, the scenario is recreated through the *system variables* (which define the alternatives that can be or could have been taken) and the *constraints* (which limit the range of action); moreover, the actors' goals that guided the course of events are defined by the *objective*. Despite the simplification in the modelling, the results show very interesting hints about the dynamics involved during the processes and are able to highlight some critical issues that significantly conditioned the final results. Besides, the modelling process itself proved to be an opportunity for collaboration between historians and computer scientists.

Keywords: archaeology of conflict; battlefield archaeology; cyber-archaeology; Geographic Information System (GIS); nineteenth-century wars

Resumen:

El desarrollo de las tecnologías digitales ha posibilitado nuevas formas de estudio de los sucesos históricos desde la perspectiva geográfica. Estos métodos se basan en la localización (sobre un espacio que incluye el relieve, las vías de comunicación, los ríos, etc.) y el establecimiento de las relaciones entre los diferentes elementos que intervinieron en dichos sucesos. Una vez que toda esta información ha sido representada en el espacio virtual, es posible recurrir a los Sistemas de Información Geográfica (SIG) con el fin de analizar diversos factores como las distancias, la visibilidad, la conectividad, etc. Sin embargo, resulta evidente que el desarrollo de los acontecimientos también estuvo condicionado por las intenciones, las necesidades y las impresiones (tanto correctas como equivocadas) de las personas/actores que intervinieron en ellos; por lo tanto, resulta oportuno pensar que la recreación del desarrollo de los eventos históricos, así

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como su correcta comprensión, mejorará sustancialmente si se incorpora algún método para simular el razonamiento de los actores implicados. En esta línea, la “programación lineal” es una opción versátil para el modelado y la optimización de sistemas que cuenta con una amplia experiencia en diversos campos como la industria, los transportes, la agricultura, etc. Asimismo, esta técnica de modelado también es aplicable a escenarios históricos con el fin de realizar simulaciones de las dinámicas que se establecieron y como método de validación de las fuentes. En el presente texto, se desarrollan —con base a los informes del frente de guerra— dos modelos relativos a la distribución de suministros durante el sitio de la villa de Bilbao —que tuvo lugar en el contexto de la Tercera Guerra Carlista (1872-1876)— que corresponden a ambas partes (es decir, a los sitiadores y a los sitiados). En los modelos, el escenario se recrea a través de las *variables del sistema* (las cuales definen las alternativas que pueden tomarse) y las *restricciones* (que limitan el rango de acción), por otro lado, las metas que guiaron el curso de los acontecimientos se definen mediante el *objetivo*. A pesar de la simplificación que implica el proceso de modelado, los resultados ofrecen interesantes indicaciones sobre las dinámicas que intervinieron en el desarrollo de los procesos y son capaces de identificar aspectos críticos que, efectivamente, condicionaron los resultados finales. Además, el propio proceso de modelado resulta ser una oportunidad de colaboración entre historiadores y expertos informáticos.

Palabras clave: arqueología del conflicto; arqueología del campo de batalla; ciberarqueología; Sistemas de Información Geográfica (SIG); guerras del siglo XIX

1. Introduction

The 19th century in Europe was characterized by continuous and violent struggles between Liberal revolutions and Absolutist counter-revolutions. On many occasions, these antagonisms resulted in armed confrontations and civil wars, such as the three Carlist Wars (1833-1839, 1846-1849 and 1872-1876) in Spain, the main stages of which took place in the North (Basque provinces and Navarre), Catalonia and Centre (provinces of Castellón and Teruel).

In this series of clashes, the city of Bilbao played a noteworthy role because of its tough and successful resistance against Carlist troops. Indeed —due to its importance in economic, commercial and propagandistic terms— the population found itself facing up to four sieges over the successive confrontations. The city resisted all of them, which was the seed of a memory and collective exaltation of Liberalism.

The historiography concerning the Carlist Wars is firmly settled on an important number of historic sources: on the one hand, archives retain original operational debriefings, telegrams, plans and sketches of the fortifications and battles; in addition, there are written biographies of the main protagonists and descriptions of relevant battles; moreover, newspaper and periodicals libraries contain testimonies of war correspondents as well as illustrative engravings and, finally, during the Last Carlist War, the photography —albeit timidly and especially focused on portraits— made its appearance.

On the other hand, although forts, batteries, trenches and so on were wiped out a long time ago, many material remains are still present and recognizable for the careful observer, such as buildings that were at the core of some major events (e.g. churches that were fortified and, even today, show impact marks, train stations...). Nevertheless, the relentless urban development of the cities has erased or transformed substantially most of them.

Finally, in recent years, archaeological excavations have become fruitful sources of information. In particular, the so-called *archaeology of conflict* —research of human events, behaviours and cultural patterns which includes different archaeological sites such as fortifications, detention facilities, bunkers, battlefields, defence towers, camps, and others (Scott & McFeater, 2011)— is providing innovative and complementary perspectives since it is based on pieces of evidence not recorded in

the written sources (Roldán, Martín, & Escribano, 2019). In fact, there are several archaeological studies concerning the military actions within the Northern Front of the Carlist Wars for the area of Navarre (Roldán & Escribano, 2015; Roldán & Escribano 2017) and Bilbao (Martín, 2017; Martín, 2019); furthermore, in 2019, a research project focused on the battlefields of this period was funded by the Basque Government, the results of which are under process of publishing¹.

Having at hand several sources of information helps to analyse the historical events from a multifaceted approach but, at the same time, it implies the challenge to combine them in order to create an integrated and multidisciplinary discourse. In addition, historians need to pay attention to the limitations of the data employed, for instance: (1) many textual sources were written at a later time with the aim of justifying the decisions taken by one or another protagonist, (2) newspapers have a predefined editorial line with a clear ideology biased towards one of the sides of the conflict and (3) the maps showing the fronts of the battle and the movement of the troops were usually made long afterwards by mixing different events and times; moreover, gaps and transcription mistakes in names and locations are not uncommon (Martín, 2019).

Anyhow, GIS and scientific simulation might be suitable tools to deal with the aforementioned issues:

- The former (GIS), because it permits merging large sets of data through georeferenced tables and provides graphic resources to show the layout of the different elements involved in each event (forts, trenches, bell towers, mountains, routes, rivers...) on a map. Moreover, the link between archaeological artefacts and their contexts allows learning more about the events that took place at a site.
- As for the latter (scientific simulation), the mathematical models —which are abstract representations based on the recording, analysis and partial capture of a complex reality that have a structure formed by a group of elements and their relationships (Suárez & Sancho-Caparrini, 2016)—

¹ Call for research projects aimed at the protection of the Basque cultural heritage. Project entitled: “*Carlist war battlefields. Proposal for Identification, characterization and study*”, file number: 003-KOI-2019, directed by Prof. Escribano.

can help us understand some unclear aspects, since it is possible to start with the available information and the conditioning factors so as to figure out the dynamics and the inherent logic that guided the establishment of the fortified landscape and the interactions during the different happenings. To do this, archaeological research develops theoretical models with the aim of explaining the events and testing hypotheses; these models are useful to represent a real occurrence because they simplify it by selecting an essential part that will be analysed as a representation of the whole.

Let us see both tools in more detail.

1.1. Geographic Information Systems

Recent applications of GIS to historical research have made significant progress on many topics, such as:

- 1) Documenting, managing and integrating geo-historical data (Rinaudo & Devoti, 2013; Ferreira-Lopes, 2015; Trapero, 2016; Licerias-Garrido, Favila-Vázquez, Bellamy, Murrieta-Flores, Jiménez Badillo, & Martins, 2019).
- 2) Understanding the historical events as processes that are dynamic and not static like a snapshot (Cuca, Brumana, Scaioni, & Oreni, 2011; Crespo Solana, 2014).
- 3) Identifying and interrelating data from different sources and disciplines in order to discover patterns (Prieto, Ortiz, Macías-Bernal, Chávez, & Ortiz, 2020).
- 4) Recognising the importance of the relationship between environment and human behaviour, their transformations and interplay. (Murrieta-Flores, 2012; Verhagen, Nuninger, Tourneux, Bertonecello, & Jeneson, 2013; Bevan & Wilson, 2013).
- 5) Exploring historical corpora, also known as *literary GIS*, (Cooper & Gregory, 2011; Rupp, Rayson, Gregory, Hardie, Joulain, & Hartmann, 2014; Alves & Queiroz, 2015).

In any case, when it comes to explaining the existence and structure of a defensive system, several factors need to be considered in order to be an efficient “opposition against the enemy” (Blanco-Rotea, 2015). Obviously, the topography of the area will play a key role, as well as the control and proximity to natural resources and human settlements and facilities (Quesada-García & Romero-Vergara, 2019).

Beginning with the topography, Digital Terrain Models (DTM) —apart from the most direct representations of height, slope and orientation— can be employed for analysing the visibility —either between specific points or to show complete viewsheds from a network of watchtowers (e.g. Llobera, 2007; Rua, Gonçalves, & Figueiredo, 2013; Deidda, Musa, & Vacca, 2015; Earley-Spadoni, 2015, Ferreira-Lopes & Molina, 2018; Murphy, Gittings, & Crow, 2018)—, as well as for the obtention of the minimum cost paths and, hence, to retrace lost routes (Canosa-Betés, 2016) or to represent different groups of transit points which can explain episodes that may have happened during the journeys (Ferreira-Lopes & Pinto, 2018). Another possibility is to define an index of “defensiveness” based on the relief of the surrounding terrain that can explain the location of defensive

structures and settlements. In particular, Martindale & Supernant (2009) employed four parameters: visibility, elevation advantage, accessibility and site area, all of them evaluable from the DTM (Bocinsky, 2014).

In other studies, GIS —in conjunction with satellite data and historical documents— has been applied to the identification of suspected military forts (Jahjah, Olivieri, Invernizzi, & Parapetti, 2007; Luo, Wang, & Cai, 2014; Bachagha, Wang, Luo, Li, Khatteli, & Lasaponara, 2020). Further valuable sources of information can be combined with GIS, such as geostatistics, aerial photos, and Ground Penetrating Radar (GPR) (Barone, 2019) or 3D models (de Kleijn, de Hond, & Martinez-Rubi, 2016; Richards-Risseto, 2017) among others.

1.2. Scientific simulation

In any case, although computational archaeology has resulted in a breakthrough in the study of historical facts, there is some criticism concerning its use in a too deterministic manner. Indeed, these tools should not be employed assuming a strong cause-and-effect relationship; on the contrary, it would be more appropriate to use them to examine alternatives with varying parameters and assign levels of confidence to our different hypotheses (Whitley, 2017). With that in mind, it would be interesting to carry out the shift between “spatial analysis” —where the methodology for data processing and the obtention of analytical results precede and define the interpretation— and “spatial narrative” —where analysis and interpretation are integrated— (Lock & Pouncett, 2017).

In recent years, extensive research has been devoted to the investigation of movements from a landscape perspective (including positions, dynamic processes and flows related to historical events), these approaches reveal how network models arising from graph theory may be employed. In particular, the most usual way to visualize and represent network data is the *network graph*, a mathematical tool used to explore the pairwise relationship among entities (Peeples, 2019). Here are the two aspects that must be considered: the relationships between entities (nodes that could represent people, objects, ideas, physical elements, etc.) and the patterns that arise from them (Brughmans, 2013). Systems in which there are different types of relationships between elements are of particular interest, as well as when specific properties that characterize these relationships need to be considered.

The development of a definite network model for a historical study depends on the availability of, firstly, solid historical/archaeological data and, secondly, suitable tools/technologies to process them. Likewise, research questions have to be raised and then, in accordance with these questions, the model’s scheme will be created, i.e. the nodes and the types of relationships among them will be defined.

Network analysis could be used to determine the shortest routes that attackers and defenders took to reach defenceless villages or battlefields (Caldwell, 2019). Mullins (2016) applied GIS and Complex Network Analysis (CNA) to link identified visual network structures and simulate their relationships with patterns of political authority and warfare in the study of ancient settlements. In other cases, graph models were created

in order to work with fragmentary data from a wide variety of sources focused on particular people/groups, for instance, Düring (2016) analysed the links between helpers and refugees in Berlin during the Second World War to study their associations and applied centrality measures to analyse the potentially influential actors.

Although network analysis is still a tool not widely used in archaeology and historical research, *network thinking* provides a new paradigm that allows formal and quantitative exploration and interaction of heterogeneous data sources, including social structures, relationships with the physical environment and properties that are not possible with other tools. In addition, it offers a new way of posing questions that would be difficult to elaborate and/or visualize with traditional methods.

In the same vein, it can be interesting to introduce “linear programming” as a mathematical tool for finding optimal solutions of systems that are expressed according to a series of rules. For the purpose of this study, these rules will be broadly summarized as follows:

- *System variables* ($x_1, x_2, x_3\dots$) will show how the available resources are distributed. Indeed, the algorithm will find the optimal solution by selecting from the alternative ways of allocating the resources among the *variables* (by definition, *variables* cannot have negative values).
- The system functioning is ruled by an *objective*, which is formulated by means of a linear expression of the *variables* that needs to be maximized (or minimized).
- In addition, several *constraints* (conditions) will be added in form of linear relationships among the *variables* (either equalities or inequalities). The fulfilment of these conditions delimits the feasible region of the problem, i.e. the range of values for the *variables* which provides valid solutions.

These systems are presented as follows (bold lowercase letters stand for column vectors and uppercase letters for matrixes):

$$\text{Maximize (or minimize): } \mathbf{c}^T \mathbf{x} \quad (1)$$

$$\text{subject to: } A \mathbf{x} \leq \mathbf{b}$$

$$\text{and: } \mathbf{x} \geq \mathbf{0}$$

A positive aspect of this approach is that linear programming offers a simple and flexible mode of creating models that can simulate a line of reasoning for decision making. In addition, once the problem has been stated in the aforementioned form, there are several methods that can be applied to obtain the optimal solution (that is to say: the optimal values of the *variables*), such as the *simplex* algorithm.

The rest of this paper is organized as follows: the objective is explained in Section 2 —testing the validity of linear programming as a modelling tool for historical events, by means of the generation of two models describing the supply chains during the siege of Bilbao—, next, in Section 3, a description of the historical context concerning the warfare from both conflict parties will be presented, as well as how these scenarios were modelled. The availability of the solving method is demonstrated in Section 4; afterwards, a discussion

about the method and its use in a school assignment is presented in Section 5. To conclude, in the last section, the work is summarized and some last remarks on the usefulness of linear programming to simulate different hypothesis regarding historical events are made.

2. Objectives

In the present text, two different scenarios of the siege of Bilbao of 1874 —one of the key moments of the Last Carlist War (1872-1876)— will be represented by means of “linear programming” models.

These models will exemplify how the knowledge about historic events can be expressed through a set of linear equations, variables and parameters. Moreover, they will allow seeing to what extent the modelling process and the outcomes that are obtained with the generated models can contribute to better understand the dynamics of the occurrences, unveil unexpected relationships and cross-check the validity of sources and hypotheses.

3. Materials and methods

This study focuses on the siege of Bilbao (administrative centre of the province of Biscay and main port open to the Cantabrian Sea, it is located 14 km ashore in the estuary of the Nervión river) which took place in 1874. By the end of 1873, the civil war-hardened in the north, the Carlist army controlled all the Basque Country and Navarre, except for the capitals and saw itself strong enough to take over the city of Bilbao. With this movement, Carlists looked for greater international credibility that, eventually, would generate new funding sources and supplies they were in imperative need of. As a previous step, they sieged and defeated the Liberal defence of Portugalete (surrendered on 22 January) and, thus, landlocked the city, closing the main communication path between Bilbao and the rest of the territory controlled by the Liberal government (Martín 2019). By doing so, the assailing forces completed the encirclement and set up the artillery on the hills that dominate the city. It has to be mentioned that Bilbao is a position very difficult to defend since it is located at the bottom of a narrow valley and surrounded by a series of heights that, once taken by the attackers, make the defences very vulnerable (Fig. 1). The first bomb was fired on 4 February and was the start of a three-month siege.

Finally, on 28 April 1874, the Liberal troops commanded by General Concha broke down the mountain pass of Las Muñecas (to the west of the city) and, in fear of being flanked by the enemy, the Carlist forces retreated from all their positions on the night of 1-2 May, bringing the end of this episode of the war.

Based on the previous description, the first model tries to simulate the logistics of the attacking forces during this period of warfare.

The map (Fig. 2) shows the positions of six main topographical heights that were employed by the beleaguers during the bombing, these points are marked with the letter “V”. Moreover, two bases for the provisioning of bombs are situated on both banks of the river and denoted with the letter “B”. Finally, dotted lines represent the routes for transporting the supplies.



Figure 1: Panoramic view of Bilbao in 1874, with the positions of defenders and attackers during the siege (source: Álbum del Sitio de Bilbao, Archivo Histórico Foral de Bizkaia, AL00011-0001).

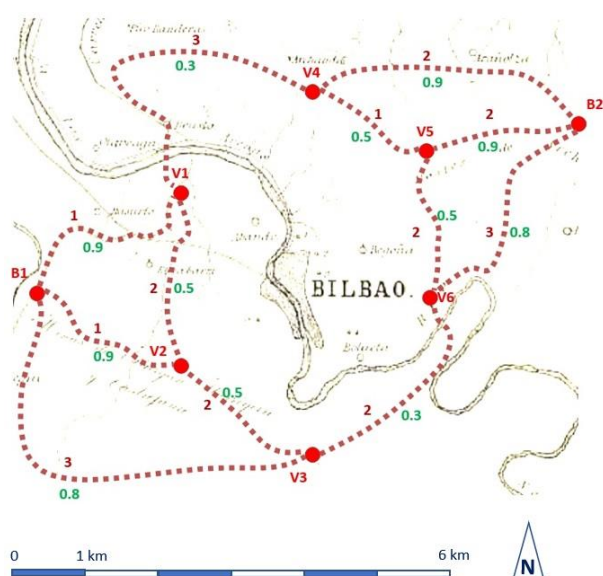


Figure 2: Map showing beleaguerees' network of supply and attack (background-image: section of the map entitled *Croquis del teatro de la Guerra*, available in the Hispanic Digital Library: <http://bdh.bne.es/bnearch/detalle/bdh0000021989>).

Let us see now some of the possibilities for setting up the model:

- The number of bombs that will be available at B_1 and B_2 at the beginning of the day can be defined.
- Likewise, there can be a limit in the number of projectiles shot from each position "V".

Many parameters can be associated with the routes (dotted lines) in order to model the travelling. In particular, two values were added to this model: a cost and a loss. The former (the "cost", numbers in red in Figure 2) is a cumulative value that is used to limit the distance that the supplies can be moved from the origin (say, a determined distance in kilometres or a maximum

time in hours). As for the latter (the "loss", figures in green in Figure 2) is a coefficient that indicates the part of the materials that does not arrive at the end of the path (in this case, the range of values goes from zero: "complete loss" to one: "no loss at all").

Some of the values and parameters for setting up the model can be obtained with a high degree of certainty from the war diaries (such as the number of bombs fired over the city each day); some others (such as the loss coefficient associated with each path), however, will be estimated intuitively and adjusted by means of successive approximations, that is to say, the model will be run once and again with new sets of parameters until its operation and results are in line with the sources.

Regarding the *objective* of the model, several options are also possible. For instance, the computation can find the way of reaching the maximum number of bombs fired over the city: alternatively, if it is considered that an efficient attack has to exert similar pressure on the whole perimeter, the pursued objective can be modified and redrafted in terms like: "the maximum number of bombs fired, provided that a similar amount of projectiles are shot from all the points".

Next, we will move on to the second situation that will be modelled, in this case, from the defenders' standpoint. In charge of the defence was general Ignacio María del Castillo who, on 5 February, counts on 1020 men manning 33 sites with a total of 37 pieces of ordnance of different calibre (*Cuerpo del Estado Mayor del Ejército, 1885: volume IV, pp. 180 and following*). According to the circumstances of the siege, the distribution of the forces and equipment was changing; anyhow, the main positions were the forts of San Agustín, Mallona, El Morro, Begoña and Miravilla (Fig. 3), complemented by 6 advanced bastions—heavily armed semi-permanent and almost autonomous positions, in many occasions manors or farmhouses on the outskirts that were used as observation points and the first line of

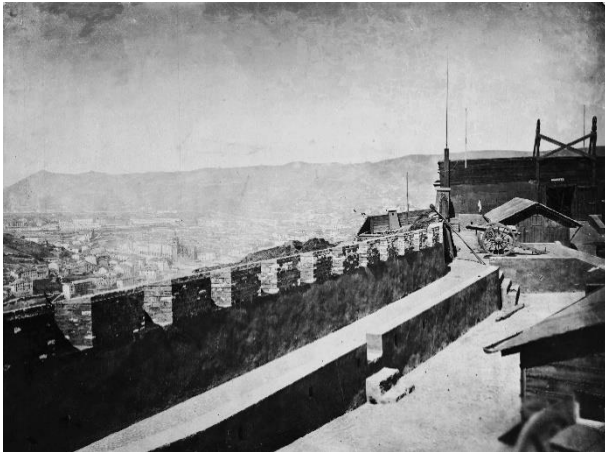


Figure 3. Fort of El Morro during the siege (source: Álbum del Sitio de Bilbao, Archivo Histórico Foral de Bizkaia, AL00011-0033).

containment— and 17 artillery batteries, i.e. scarcely armed provisional positions, but highly versatile and movable so they could be rapidly translated when circumstances so required.

The chosen option to model this situation started by dividing the plan of the city into twelve districts²: five for the populated area —on the inside—, another five forming an external ring and two more along the river (Fig. 4).

The resource to be distributed among the districts in an optimal way are the pieces of artillery that were available. For the construction of the model, some *constraints* can be defined so as to require a minimum or to limit the maximum number of pieces that each district can have.

In any case, the success of the defence is based on the consideration that the system operates as a whole and that neighbouring areas can support one another; hence, the connectivity and the adjacency between districts should also be taken into account. Therefore, a new quantity —the “defensive value”— was defined for every district as the sum of the pieces of artillery stationed in it plus the 50% of the pieces stationed in the neighbouring districts.

Regarding the *objective* of the model and bearing in mind that the city is under attack from all fronts, a sensible approach would be to obtain a situation where no weak points exist³.

² In fact, the ordnance was not allocated in areas but in strategic places (fixed points) or gunboats (movable along the river). However, resorting to the “districts” to model the configuration of the defence was considered a useful trick for the purpose of this model, since it gave continuity to the space and allowed defining the “adjacency” (a feature that will be employed for the computation of the defensive value of each part of the city). Of course, this is not the only possible option.

³ This objective is defined here for illustrative purposes. In fact, although the city was completely surrounded, the real bombings mainly focused around the church of Begonia (district E2).

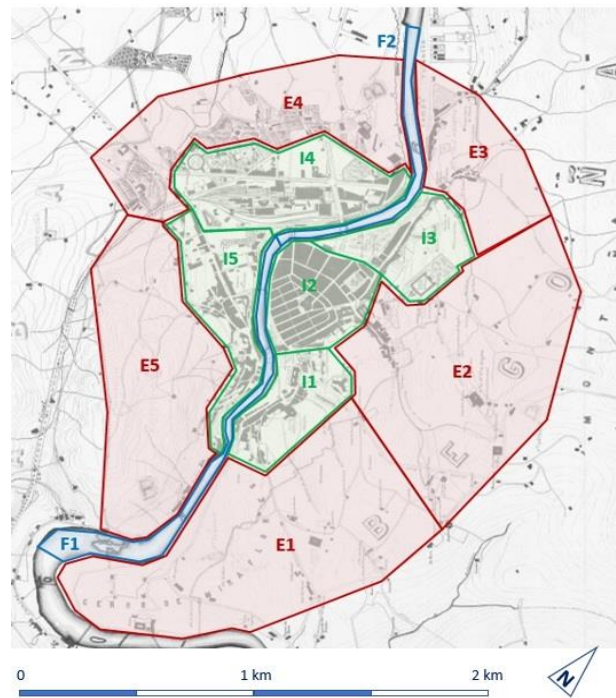


Figure 4. Division of the city into 12 districts (5 internal, 5 external and 2 fluvial) for the allocation of the defensive ordnance (background-image: Spanish Ministry of Defence, Instituto de Historia y Cultura Militar. Archivo General Militar de Madrid. BI-03-10).

4. Results

The first (attackers’) model can be represented by means of a graph showing a modified version of an “unbalanced transportation problem” (a well-defined case of “linear programming”). The following graph (Fig. 5) shows a possible proposal, after the values for the supplies from B₁ and B₂, and the maximum number of bombs that can be shot from the “V” points have been established. As can be seen, the *variables* (x) are the number of projectiles that are sent from each supply point to the different attacking positions.

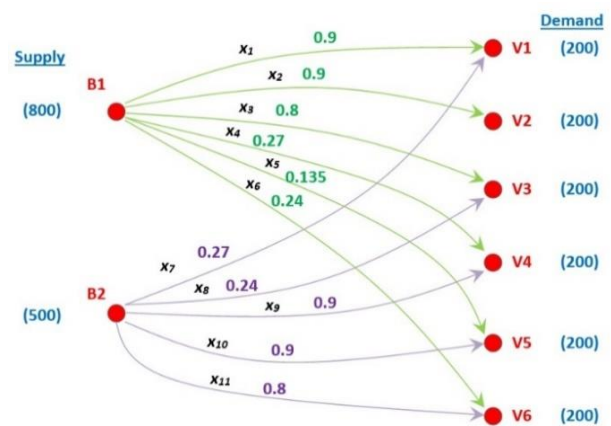


Figure 5: The attack expressed as a “transportation graph”. Only optimal routes from the supply points to the attacking positions are considered; moreover, position “V₂” is too distant from “B₂” so this specific supply line is not considered.

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Next, we define an auxiliary value to construct the model: the “firepower from each attacking position” (f), which is defined as the quantity of bombs that arrived regardless the origin —bearing in mind that part of the material sent from each supply point (x) is lost on the way.

$$f_{V1} = 0.9 x_1 + 0.27 x_7 \quad (2)$$

$$f_{V2} = 0.9 x_2$$

$$f_{V3} = 0.8 x_3 + 0.24 x_8$$

$$f_{V4} = 0.27 x_4 + 0.9 x_9$$

$$f_{V5} = 0.135 x_5 + 0.9 x_{10}$$

$$f_{V6} = 0.24 x_6 + 0.8 x_{11}$$

The aim of “having a similar pressure from all the attacking positions” can be formulated by maximizing the firepower from one selected point (for instance “ f_{V1} ”) and adding *constraints* in order to force that the rest of the positions will be, at least, as intense as the reference one (e.g. $f_{V2} \geq f_{V1}$). Therefore, the complete model will be formulated as a problem of “linear programming” as follows:

$$\text{Maximize: } f_{V1} \rightarrow \text{Maximize: } 0.9 x_1 + 0.27 x_7 \quad (3)$$

subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 800$$

$$x_7 + x_8 + x_9 + x_{10} + x_{11} \leq 500$$

$$f_{V1} \leq 200 \rightarrow 0.9 x_1 + 0.27 x_7 \leq 200$$

$$f_{V2} \geq f_{V1} \rightarrow 0.9 x_2 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V3} \geq f_{V1} \rightarrow 0.8 x_3 + 0.24 x_8 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V4} \geq f_{V1} \rightarrow 0.27 x_4 + 0.9 x_9 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V5} \geq f_{V1} \rightarrow 0.135 x_5 + 0.9 x_{10} \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V6} \geq f_{V1} \rightarrow 0.24 x_6 + 0.8 x_{11} \geq 0.9 x_1 + 0.27 x_7$$

$$\text{and: } x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11} \geq 0$$

In the second (defenders’) model, the *variables* are the quantity of ordnance allocated in each district. In order to make clearer the meaning of each equation, instead of denoting them with the letter (x), letters (“e”, “i” and “f”) will be employed.

As in the previous model, it will be useful to define an ancillary quantity: the “defensive value” (v). This value is equal to the number of pieces of artillery allocated in the district plus the 50% of the pieces allocated in the adjacent neighbouring ones.

$$v_{E1} = e_1 + 0.5 (e_2) + 0.5 (i_1) + 0.5 (f_1) \quad (4)$$

$$v_{E2} = e_2 + 0.5 (e_1 + e_3) + 0.5 (i_1 + i_2 + i_3)$$

$$v_{E3} = e_3 + 0.5 (e_2) + 0.5 (i_3) + 0.5 (f_2)$$

$$v_{E4} = e_4 + 0.5 (e_5) + 0.5 (i_4 + i_5) + 0.5 (f_2)$$

$$v_{E5} = e_5 + 0.5 (e_4) + 0.5 (i_5) + 0.5 (f_1)$$

$$v_{I1} = i_1 + 0.5 (e_1 + e_2) + 0.5 (i_2) + 0.5 (f_1)$$

$$v_{I2} = i_2 + 0.5 (e_2) + 0.5 (i_1 + i_3) + 0.5 (f_1 + f_2)$$

$$v_{I3} = i_3 + 0.5 (e_2 + e_3) + 0.5 (i_2) + 0.5 (f_2)$$

$$v_{I4} = i_4 + 0.5 (e_4) + 0.5 (i_5) + 0.5 (f_2)$$

$$v_{I5} = i_5 + 0.5 (e_4 + e_5) + 0.5 (i_4) + 0.5 (f_1)$$

So, let us imagine that, as *objective*, it is taken that none of the external districts should have a “defensive value” less than 10 and that we want to know the minimum number of pieces of artillery necessary to guarantee this situation. A formulation of the problem according to the “linear programming” will be:

$$\text{Minimize: } e_1 + e_2 + e_3 + e_4 + e_5 + i_1 + i_2 + i_3 + i_4 + i_5 + f_1 + f_2 \quad (5)$$

subject to:

$$e_1 + 0.5 (e_2) + 0.5 (i_1) + 0.5 (f_1) \geq 10$$

$$e_2 + 0.5 (e_1 + e_3) + 0.5 (i_1 + i_2 + i_3) \geq 10$$

$$e_3 + 0.5 (e_2) + 0.5 (i_3) + 0.5 (f_2) \geq 10$$

$$e_4 + 0.5 (e_5) + 0.5 (i_4 + i_5) + 0.5 (f_2) \geq 10$$

$$e_5 + 0.5 (e_4) + 0.5 (i_5) + 0.5 (f_1) \geq 10$$

$$\text{and: } e_1, e_2, e_3, e_4, e_5, i_1, i_2, i_3, i_4, i_5, f_1, f_2 \geq 0$$

Both models (equations 3 and 5) can be reformulated in standard form (see the annexe of this document for more details) and, then, be solved by applying any of the existing algorithms. In general, the solution will inform us of the following aspects:

- a) The algorithm will provide an optimal solution if it exists. Otherwise, the model will be *infeasible* (i.e. there exists no solution that satisfies all *constraints*), in such a case, the limit values of the *constraints*, parameters, number of resources, etc., need to be revised.
- b) The optimal value for the *objective* will be provided. In the analysed models, this value will be: (1) the maximum number of bombs fired from the attacking position “V₁” (and, by extension, the number of bombs fired from any attacking position) and (2) the minimum quantity of pieces of artillery that are needed to achieve, at least, a defensive value of 10 points in all the external districts.
- c) The values for all the *variables* that generate the optimal situation.

The study does not finish with the calculation of the optimal situation. On the contrary, one of the most powerful features of “linear programming” is the possibility of carrying out very detailed *sensitivity analyses*. These analyses permit foreseeing what would happen with the model in case some of the initial values change.

5. Discussion

Over the last few years, the number of studies focused on the space-time relationships between the different entities of a given historical event is increasing. In the case of GIS, their implementation has grown fast and, today, we have at our disposal a varied set of application examples to historical occurrences.

To a great extent, these researches aimed at experimenting with different options and proposals that can conform to past episodes. Nevertheless, these approximations usually work with heterogeneous data from manifold sources (including many of them that may be incomplete, biased or imprecise) and have to be considered permanently open to new sources and

perspectives; therefore, they can never be seen as closed answers. In fact, communication and discussion with other specialists are some of the best ways to improve the models.

Likewise, the very fact of modelling historical events demands a transdisciplinary approach beyond the realms of History and Archaeology. Take as an example the establishment of routes and the simulation of people's movements and transports during historic events, which is an active area of interest both for archaeologists and for computer science developers. However, most of the approaches place the emphasis on defining optimal connections based on a different definition of costs, while the strategy that is behind the action is usually relegated (Verhagen, Nuninger, & Groenhuijzen, 2019).

Nonetheless, mathematical models —as the ones employed in “linear programming”— can be used to: describe, explain, predict and/or prescribe (Villalba & Bueno, 2012). The procedure starts with the gathering of the base information about the occurrence, the establishment of the hypotheses concerning the dynamics and underlying principles and the obtention of results. These results predicted by the models can be compared with the real outcomes so, by this mean, the soundness of the available knowledge about the data sources and the unfolding of the events can be cross-checked.

On a different note, it is clear that modelling requires an iterative approach, in which the communication between archaeologists/historians —who provide the source data and the context of the events, propose the ideas to be tested and analyse the results—, on the one hand, and mathematicians/computer scientists —who formulate and solve the successive versions of the models—, on the other hand, become essential.

In order to explore the possibilities to establish this multidisciplinary link, as well as deepen the relationship between research and learning, the two models explained in this text were proposed to students of the Degree on Management Informatics and Information Systems at the Engineering School of Vitoria-Gasteiz (UPV/EHU) as part of the evaluation of a four-month course on Operational Research (2nd year).

They had around four hours to develop individually (although with the possibility of consulting their study notes) one of the two available scenarios: commanding the attacking force or be in charge of the sieged army. The exercise consisted of three sequential blocks: modelling, computing and graphic representation.

a) The activity concerning the modelling corresponded, approximately, to the development that was presented in Section 4. Results of this work. In any case, students had leeway to interpret the wording of the exercises and propose alternative ways of modelling ill-defined ideas such as: “similar pressure from all the attacking positions” or “no weak points in the defence system”. For the evaluation of this block, sound reasoning during the construction was as important as the obtained set of mathematical expressions.

b) Due to the limitation in time, students were asked to calculate a simplified version of the problem. For instance, Figure 6 shows the selected part for the first model. Thus, in addition to obtaining a reduced version of the model prepared for the previous block, students had to solve the mathematical problem and obtain the optimal result.

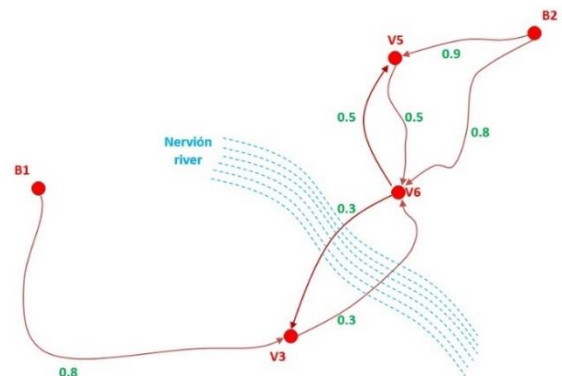


Figure 6: Selected part of the graph which was used for the computation during the exercise (attackers' version).

The graph represented in the figure shows —with numbers in green— the ratio of pieces of artillery that are lost in each path. As can be seen, the river crossing is a critical part (as was indeed the case in the real battle); therefore, students were asked to perform a *sensitivity analysis* in order to see how the variation of the loss rate in this particular zone would affect the optimality of the solution.

c) Another interesting feature of “linear programming” is that problems with only two *variables* can be represented and solved graphically. This quality allows visualizing the meaning of the different parts of the technique (*variable, feasible region, objective, constraint*, etc.) and, therefore, helps to understand the procedure and the justification of the method. For instance, starting from Fig. 6, a more simplified version of the problem will be considered, in particular, the one which analyses the supply of a single position (“V₆”). In such a case, the model would have just two variables: “x₁” (bombs sent from B₁) and “x₂” (bombs sent from B₂) (Fig. 7).

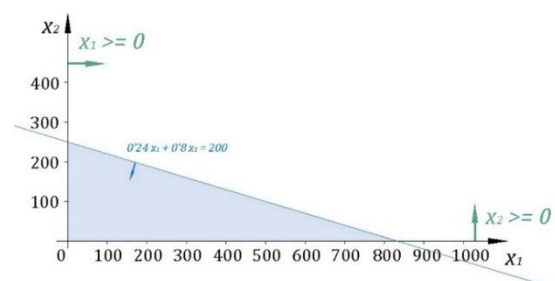


Figure 7: Graphic representation of the *feasible region* (shaded in blue), that is to say: possible combination of values of “x₁” and “x₂” which make that the number of bombs arrived at “V₆” equal to or less than 200.

The following image (Fig. 8), shows what students had to analyse, i.e. how the variations in the loss coefficient for the crossing of the river would modify the *feasible region*.

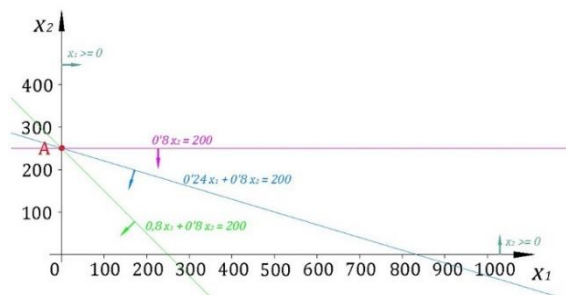


Figure 8: Changes in the loss coefficient due to the crossing of the river would greatly modify the *feasible region* of the problem as the graphic representation shows.

It is important to note that, although the questions were formulated in mathematical terms, there is a direct link with the meaning in the framework of the real situation. For instance, the changes in the value of the loss coefficient that students were asked to analyse also imply which one of the two parties of the armed conflict was controlling the river.

Consequently, every mathematical solution can be also expressed in textual form. As can be seen, the communication between historians and mathematicians must go two-way: firstly, the sources, ideas and hypotheses need to be formalised in mathematical expressions during the modelling and, afterwards, the numeric results have to be translated and interpreted in textual mode.

Summing up, linear programming is a flexible option for creating a mathematical model from a textual context, a set of hypothesis and constraints; in addition, the calculation procedure is well established and affordable with limited computational resources and the variation of the results due to changes of the initial data can be also estimated. With regard to all these considerations, it can be said that linear programming is a very interesting tool for including reasoning and improving the geographical analysis for historical purposes.

Evidently, linear programming is not the only mathematical resource included in the Operational Research toolbox. For instance, complementary interesting approaches can be generated from game theory, in which the strategies of both parts (attacker's and defender's) are considered simultaneously (Garrec, 2019).

6. Conclusions

This text exemplifies how the knowledge about a historical event (in this case a siege) that is scattered in different sources can be formalised by means of linear programming.

The modelling goes far away from a simple portrayal of facts; conversely, a wide range of objectives, limitations, relationships, conditions... can be included. Hence, the events can be simulated dynamically. Moreover, the procedure can cope with imprecise ideas (such as “defence without weak-points” or “similar intensity of the attack in all fronts”).

On the other hand, *sensitivity analyses* allow study of alternative scenarios and venture answers to questions such as “what might have happened if...”. Likewise, the models can be continuously updated with new insights from other sources and see how the simulations and estimations change.

In any case, it must be taken into account that the representativeness of the models usually encompasses the general dynamics of the events, but they can hardly be effective to reproduce each step in detail. Indeed, the historical truth is extremely complex and the grounds of a victory or a defeat cannot be simply extracted from a set of equations, regardless of their extension. We cannot pretend to be aware of all the causes, conditions and nuances which influenced the decision-making and the development of the events.

Concerning the use of these two scenarios as part of the evaluation on Management Informatics and Information Systems, the students' satisfaction enquiries about the course were very high (4.6 out of 5) and, in spite of a few participants that were rather puzzled by the subject of the exercise and did not know how to address it, more than two-thirds of the 36 students were able to tackle the exam successfully (which was a better ratio than the one obtained the previous year, based on a list of separate exercises about the different topics covered by the collection of themes). On the other side, some students even acknowledged that they liked very much the challenge of working out such a complex issue.

Moreover, this kind of open exercises where many students have the possibility of making models of their own invention is very useful to see alternative ways of acting —some of them, very imaginative and effective ones— and check for incorrect approaches, including errors and ambiguities in the wording of the exercise itself. So, overall, it constitutes a most helpful testbed for validating and improving models concerning the recreation of the past.

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Annexe

In this annexe, the models which describe the strategies of both the attackers' and the defenders' armies are solved by means of the *simplex* algorithm.

First of all, we return to the model for the distribution of the bombs from the supply bases to the hills around the city (equation 3).

$$\text{Maximize: } f_{V1} \rightarrow \text{Maximize: } 0.9 x_1 + 0.27 x_7 \quad (3)$$

subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 800$$

$$x_7 + x_8 + x_9 + x_{10} + x_{11} \leq 500$$

$$f_{V1} \leq 200 \rightarrow 0.9 x_1 + 0.27 x_7 \leq 200$$

$$f_{V2} \geq f_{V1} \rightarrow 0.9 x_2 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V3} \geq f_{V1} \rightarrow 0.8 x_3 + 0.24 x_8 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V4} \geq f_{V1} \rightarrow 0.27 x_4 + 0.9 x_9 \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V5} \geq f_{V1} \rightarrow 0.135 x_5 + 0.9 x_{10} \geq 0.9 x_1 + 0.27 x_7$$

$$f_{V6} \geq f_{V1} \rightarrow 0.24 x_6 + 0.8 x_{11} \geq 0.9 x_1 + 0.27 x_7$$

$$\text{and: } x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11} \geq 0$$

This set of inequations needs to be reformulated according to the so-called *standard form*, in which all the constraints are expressed as equalities. To do so, new non-negative variables has to be added:

- Each constraint of type (\leq) will be transformed in ($+ s_i =$), being "s" a *slack* variable.
- Each constraint of type (\geq) will be transformed in ($- t_i =$), being "t" a *surplus* variable.

For the symbolic formulation, all the variables are placed in the left-hand side of the constraints and the independent term in the right-hand side, all values of this latter have to be non-negative.

$$\text{Maximize: } \mathbf{c}^T \mathbf{x} \quad (6)$$

$$\text{subject to: } A \mathbf{x} = \mathbf{b}$$

$$\text{and: } \mathbf{x}, \mathbf{b} \geq \mathbf{0}$$

That is to say:

$$\text{Maximize: } 0.9 x_1 + 0.27 x_7 \quad (7)$$

subject to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + s_1 = 800$$

$$x_7 + x_8 + x_9 + x_{10} + x_{11} + s_2 = 500$$

$$0.9 x_1 + 0.27 x_7 + s_3 = 200$$

$$0.9 x_1 - 0.9 x_2 + 0.27 x_7 + s_4 = 0$$

$$0.9 x_1 - 0.8 x_3 + 0.27 x_7 - 0.24 x_8 + s_5 = 0$$

$$0.9 x_1 - 0.27 x_4 + 0.27 x_7 - 0.9 x_9 + s_6 = 0$$

$$0.9 x_1 - 0.135 x_5 + 0.27 x_7 - 0.9 x_{10} + s_7 = 0$$

$$0.9 x_1 - 0.24 x_6 + 0.27 x_7 - 0.8 x_{11} + s_8 = 0$$

$$\text{and: } x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \geq 0$$

This system can be arranged in form of a *preliminary table* (Table 1). This table shows an admissible solution (i.e. a solution which complies with all the constraints) when it includes an identity matrix of size equal to the number of constraints (in this model 8), as is the case here with all the variables (s). When that happens, the bold letter (**b**) at the top right corner is changed by the label (**x_B**) so as to indicate that what can be read in the right column corresponds with the variable with the "1" in the identity matrix (i.e. $s_1=800, s_2=500, s_3=200, s_4=0, s_5=0, s_6=0, s_7=0, s_8=0$). The variables included in the identity matrix are called *basic variables*, the rest are *non-basic variables* and their value is equal to zero.

LINEAR PROGRAMMING FOR THE ANALYSIS AND VIRTUAL RECREATION OF HISTORICAL EVENTS: THE ALLOCATION OF THE ARTILLERY DURING THE SIEGE OF BILBAO IN 1874

Table 1: Preliminary table corresponding to the attackers' model. It is also an initial table for the *simplex* algorithm.

X ₁	X ₂	X ₃	X ₄	X ₅	X ₆	X ₇	X ₈	X ₉	X ₁₀	X ₁₁	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈	b
1	1	1	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	800
0	0	0	0	0	0	1	1	1	1	1	0	1	0	0	0	0	0	0	500
0.9	0	0	0	0	0	0.27	0	0	0	0	0	0	1	0	0	0	0	0	200
0.9	-0.9	0	0	0	0	0.27	0	0	0	0	0	0	0	1	0	0	0	0	0
0.9	0	-0.8	0	0	0	0.27	-0.24	0	0	0	0	0	0	0	1	0	0	0	0
0.9	0	0	-0.27	0	0	0.27	0	-0.9	0	0	0	0	0	0	0	1	0	0	0
0.9	0	0	0	-0.135	0	0.27	0	0	-0.9	0	0	0	0	0	0	0	1	0	0
0.9	0	0	0	0	-0.24	0.27	0	0	0	-0.8	0	0	0	0	0	0	0	1	0
-0.9	0	0	0	0	0	-0.27	0	0	0	0	0	0	0	0	0	0	0	0	0

The meaning of this first solution with all the (x) variables equal to zero is that no bomb is distributed from the supply bases. As said before, this situation is consistent with the constraints but, obviously, it is not optimal.

This is the point from which the *simplex* algorithm starts optimizing the solution. Broadly speaking the algorithm swaps a variable that is outside the identity matrix (*entering variable*) for one that is inside (*leaving variable*), the *entering variable* is selected by the most negative value on the lower row (these coefficients are called the *improvement indicators*) and the leaving variable is selected depending on the coefficients obtained by dividing —one by one— the values of the (x_b) column by the values of the *entering column* (only positive values in the *entering column* generate valid coefficients, among these coefficients, the smallest one indicates the variable that will leave). This algorithm is iterative and finishes when there are no more negative *improvement indicators*.

Here is the result (Table 2) after applying successively the algorithm to Table 1:

Table 2: Results obtained by the application of the *simplex* algorithm (attackers' model).

X ₁	X ₂	X ₃	X ₄	X ₅	X ₆	X ₇	X ₈	X ₉	X ₁₀	X ₁₁	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈	x_b
0	0	0	1	0.88	1	-1	-1	0	0	0	0.77	-0.77	0	0.85	0.96	-0.85	-0.85	-0.96	230.77
0	0	0	0	0.04	0.3	0.25	0.25	0	0	1	0.08	0.28	0	0.09	0.10	0.31	0.31	-0.90	204.92
0	0	0	0	-0.03	0	-0.20	-0.20	0	0	0	-0.07	-0.22	1	-0.07	-0.08	-0.25	-0.25	-0.28	36.06
1	0	0	0	0.04	0	0.52	0.22	0	0	0	0.07	0.25	0	0.08	0.09	0.27	0.27	0.31	182.15
0	1	0	0	0.04	0	0.22	0.22	0	0	0	0.07	0.25	0	-1.03	0.09	0.27	0.27	0.31	182.15
0	0	1	0	0.04	0	0.25	0.55	0	0	0	0.08	0.28	0	0.09	-1.15	0.31	0.31	0.35	204.92
0	0	0	0	-0.23	-0.3	0.52	0.52	1	0	0	-0.16	0.48	0	-0.17	-0.20	-0.58	0.53	0.60	112.92
0	0	0	0	0.19	0	0.22	0.22	0	1	0	0.07	0.25	0	0.08	0.09	0.27	-0.84	0.31	182.15
0	0	0	0	0.03	0	0.20	0.20	0	0	0	0.07	0.22	0	0.07	0.08	0.25	0.25	0.28	163.94

As can be seen in the table, the algorithm computes the results with decimal digits which —in this case— do not are adequate; therefore, the results are obtained after rounding the values:

- x₁ (from B₁ to V₁) = 182 bombs
- x₂ (from B₁ to V₂) = 182 bombs
- x₃ (from B₁ to V₃) = 205 bombs
- x₄ (from B₁ to V₄) = 231 bombs
- x₉ (from B₂ to V₄) = 113 bombs
- x₁₀ (from B₂ to V₅) = 182 bombs
- x₁₁ (from B₂ to V₆) = 205 bombs

Finally, the optimal value for the objective function appears in the bottom right. In this case, this value (163 bombs) stands for the maximum firepower that can be obtained in all the positions simultaneously.

Moving on to the model related to the defence of the city, the preliminary approach was defined in equation 5:

Minimize: $e_1 + e_2 + e_3 + e_4 + e_5 + i_1 + i_2 + i_3 + i_4 + i_5 + f_1 + f_2$ (5)

subject to:

$$e_1 + 0.5 (e_2) + 0.5 (i_1) + 0.5 (f_1) \geq 10$$

$$e_2 + 0.5 (e_1 + e_3) + 0.5 (i_1 + i_2 + i_3) \geq 10$$

$$e_3 + 0.5 (e_2) + 0.5 (i_3) + 0.5 (f_2) \geq 10$$

$$e_4 + 0.5 (e_5) + 0.5 (i_4 + i_5) + 0.5 (f_2) \geq 10$$

$$e_5 + 0.5 (e_4) + 0.5 (i_5) + 0.5 (f_1) \geq 10$$

and: $e_1, e_2, e_3, e_4, e_5, i_1, i_2, i_3, i_4, i_5, f_1, f_2 \geq 0$

Which, according to the standard form, is:

Maximize: $-e_1 - e_2 - e_3 - e_4 - e_5 - i_1 - i_2 - i_3 - i_4 - i_5 - f_1 - f_2$ (8)

subject to:

$$e_1 + 0.5 (e_2) + 0.5 (i_1) + 0.5 (f_1) - t_1 = 10$$

$$e_2 + 0.5 (e_1 + e_3) + 0.5 (i_1 + i_2 + i_3) - t_2 = 10$$

$$e_3 + 0.5 (e_2) + 0.5 (i_3) + 0.5 (f_2) - t_3 = 10$$

$$e_4 + 0.5 (e_5) + 0.5 (i_4 + i_5) + 0.5 (f_2) - t_4 = 10$$

$$e_5 + 0.5 (e_4) + 0.5 (i_5) + 0.5 (f_1) - t_5 = 10$$

and: $e_1, e_2, e_3, e_4, e_5, i_1, i_2, i_3, i_4, i_5, f_1, f_2, t_1, t_2, t_3, t_4, t_5, \geq 0$

The *preliminary table* of this system is (Table 3):

Table 3: Preliminary table corresponding to the defenders' model.

e ₁	e ₂	e ₃	e ₄	e ₅	i ₁	i ₂	i ₃	i ₄	i ₅	f ₁	f ₂	t ₁	t ₂	t ₃	t ₄	t ₅	b
1	0.5	0	0	0	0.5	0	0	0	0	0.5	0	-1	0	0	0	0	10
0.5	1	0.5	0	0	0.5	0.5	0.5	0	0	0	0	0	-1	0	0	0	10
0	0.5	1	0	0	0	0	0.5	0	0	0	0.5	0	0	-1	0	0	10
0	0	0	1	0.5	0	0	0	0.5	0.5	0	0.5	0	0	0	-1	0	10
0	0	0	0.5	1	0	0	0	0	0.5	0.5	0	0	0	0	0	-1	10
1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0

Unlike Table 1 for the previous model, in this case (Table 3), there is not an identity matrix. Therefore, prior to running the *simplex* algorithm, the values have to be processed according to the Gauss-Jordan elimination, including the lower row (the one for the *improvement indicators*) which will end with all the values equal to zero under the *basic variables*. Moreover, the values of the right column have to remain non-negative at all times.

For example, below is presented a possible initial table for the algorithm (Table 4), in which there are 10 pieces allocated in both (e₁) and (e₂) and 20 more in (i₅), with a total of 40 ordnance pieces distributed (in this table, the value of the *objective function*—bottom right cell— appears as negative since we are computing a minimum).

Table 4: Initial table for the simplex algorithm (defenders' model).

e ₁	e ₂	e ₃	e ₄	e ₅	i ₁	i ₂	i ₃	i ₄	i ₅	f ₁	f ₂	t ₁	t ₂	t ₃	t ₄	t ₅	X_B
1	0	0	0	0	0.25	-0.5	-0.25	0	0	0.75	0.25	-1.5	1	-0.5	0	0	10
0	1	0	0	0	0.5	1	0.5	0	0	-0.5	-0.5	1	-2	1	0	0	0
0	0	1	0	0	-0.25	-0.5	0.25	0	0	0.25	0.75	-0.5	1	-1.5	0	0	10
0	0	0	1	-1	0	0	0	1	0	-1	1	0	0	0	-2	2	0
0	0	0	1	2	0	0	0	0	1	1	0	0	0	0	0	-2	20
0	0	0	-1	0	0.5	1	0.5	0	0	0.50	-0.50	1	0	1	2	0	-40

LINEAR PROGRAMMING FOR THE ANALYSIS AND VIRTUAL RECREATION OF HISTORICAL EVENTS: THE ALLOCATION OF THE ARTILLERY DURING THE SIEGE OF BILBAO IN 1874

As said before, from any admissible solution, the *simplex* algorithm progresses until the optimal one (Table 5):

Table 5: Results obtained by the application of the *simplex* algorithm (defenders' model).

e ₁	e ₂	e ₃	e ₄	e ₅	i ₁	i ₂	i ₃	i ₄	i ₅	f ₁	f ₂	t ₁	t ₂	t ₃	t ₄	t ₅	X _B
1	0	0	0	0	0,25	-0,5	-0,25	0	0	0,75	0,25	-1,5	1	-0,5	0	0	10
0	1	0	0	0	0,5	1	0,5	0	0	-0,5	-0,5	1	-2	1	0	0	0
0	0	1	0	0	-0,25	-0,5	0,25	0	0	0,25	0,75	-0,5	1	-1,5	0	0	10
0	0	0	1	0	0	0	0	0,67	0,33	-0,33	0,67	0	0	0	-1,33	0,67	6,67
0	0	0	0	1	0	0	0	-0,33	0,33	0,67	-0,33	0	0	0	0,67	-1,33	6,67
0	0	0	0	0	0,5	1	0,5	0,67	0,33	0,17	0,17	1	0	1	0,67	0,67	-33,3

In a similar way to the attackers' model, the final values need to be rounded in order to provide results coherent with the context of the problem.

e₁ = 10 pieces allocated

e₃ = 10 pieces allocated

e₄ = 7 pieces allocated

e₅ = 7 pieces allocated

In this case, the optimal value for the *objective function* is 34 artillery pieces.

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