Singularly nonautonomous semilinear evolution equations with almost sectorial operators

(Ecuaciones de evolución semilineales singularmente no autónomas con operadores casi sectoriales)

Memoria escrita por

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Abstract

In this work we consider the singularly nonautonomous semilinear parabolic problem

$$\begin{aligned} &u_t + A(t)u = F(u), \quad t > \tau, \\ &u(\tau) = u_0, \end{aligned}$$

in a Banach space X, where $A(t)$, $t \in \mathbb{R}$, is a family of uniformly almost sectorial operators. The term singularly nonautonomous express the fact that the linear part of the equation, $A(t) : D \subset X \to X$, is time-dependent and the almost sectoriality of the family $A(t)$ comes from a deficiency in its resolvent estimate. For this semilinear problem in the abstract setting we study local well-posedness, regularity of the solution and the asymptotic dynamics of the problem.

To illustrate the ideas developed for the abstract initial value problem, we consider a singularly nonautonomous reaction-diffusion equation in a domain with a handle. This type of domain consists in a subset of $\mathbb{R}^N$, $\Omega_0 = \Omega \cup R_0$, where Ω is an open set of $\mathbb{R}^N$ and R_0 is diffeomorphic to a subset $(0,1) \subset \mathbb{R}$. The "handle" refers to this line segment R_0 attached to Ω. In Ω_0 we consider the following reaction-diffusion equation

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1). \end{cases}$$

This equation generates a singularly nonautonomous evolution equation with almost sectorial operator and local well-posedness, existence of strong solution and existence of pullback attractor are studied, in the lights of the abstract theory developed. In particular, in order to obtain existence of attractors, the system above will be decouple, originating two evolution equations: one with Neumann homogeneous boundary condition in Ω and another with nonhomogeneous and time-dependent Dirichlet boundary conditions in R_0. The properties of those two decoupled equations are thoughtfully studied and from them, estimates on the pullback attractor are obtained.

Key words: Singularly nonautonomous parabolic problems, almost sectorial operators, regularization, asymptotic dynamics, pullback attractor.

Resumen

En este trabajo consideramos el problema parabólico semilineal singularmente no autónomo

$$u_t + A(t)u = F(u), \quad t > \tau,$$
$$u(\tau) = u_0,$$

en un espacio de Banach X, donde $A(t)$, $t \in \mathbb{R}$, es una familia de operadores uniformemente casi sectoriales. El término singularmente no autónomo expresa el hecho de que la parte lineal de la ecuación, $A(t) : D \subset X \to X$, es dependiente del tiempo y la casi sectorialidad de la familia $A(t)$ proviene de una deficiencia en la estimación para la resolvente. Para este problema semilineal en el contexto abstracto, estudiamos el buen planteamento local de la ecuación, la regularidad de la solución y la dinámica asintótica del problema.

Para ilustrar las ideas desarrolladas para el problema con valor inicial abstracto, consideramos una ecuación de reacción-difusión singularmente no autónoma en un dominio con una asa. Este dominio consiste en un subconjunto de $\mathbb{R}^N$, $\Omega_0 = \Omega \cup R_0$, donde Ω es un conjunto abierto de $\mathbb{R}^N$ y R_0 es difeomórfico a un subconjunto $(0,1) \subset \mathbb{R}$. La "asa" se refiere a este segmento de línea R_0 unido a Ω. En Ω_0 consideramos la siguiente ecuación de reacción-difusión

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1). \end{cases}$$

Esta ecuación genera una ecuación de evolución singularmente no autónoma con operador casi sectorial y el buen plantemento local, la existencia de solución fuerte y existencia de atractores *pullback* son estudiados, a la luz de la teoría abstracta desarrollada. En particular, para obtener la existencia de atractores, el sistema anterior será desacoplado, originando dos ecuaciones de evolución: una con condición de frontera de Neumann homogénea en Ω y otra con condiciones de frontera de Dirichlet no homogéneas y dependientes del tiempo en R_0. Las propiedades de esas dos ecuaciones desacopladas son estudiadas y, a partir de ellas, se obtienen estimativas del atractor *pullback*.

Key words: Problema parabólico semilineal singularmente no autónomo, operadores casi sectoriales, regularización, dinámica asintótica, atractor *pulback*.

Resumo

Neste trabalho consideramos o problema parabólico semilinear singularmente não autônomo

$$
\begin{aligned}
&u_t + A(t)u = F(u), \quad t > \tau,\\
&u(\tau) = u_0,
\end{aligned}
$$

em um espaço de Banach X, onde $A(t)$, $t \in \mathbb{R}$, é uma família de operadores uniformemente quase setoriais. O termo singularmente não autônomo expressa o fato de que a parte linear da equação, $A(t) : D \subset X \to X$, é dependente do tempo e a quase setorialidade da família $A(t)$ vem de uma deficiência na estimativa do resolvente deste operador. Para este problema semilinear no contexto abstrato, estudamos a boa postura local, a regularidade da solução e a dinâmica assintótica do problema.

Para ilustrar as idéias desenvolvidas para o problema de valor inicial abstrato, consideramos uma equação de reação-difusão singularmente não autônoma em um domínio com uma alça. Este tipo de domínio consiste em um subconjunto de $\mathbb{R}^N$, $\Omega_0 = \Omega \cup R_0$, onde Ω é um aberto de $\mathbb{R}^N$ e R_0 é difeomórficos a $(0,1) \subset \mathbb{R}$. A " alça "refere-se a este segmento de linha R_0 anexado a Ω. Em Ω_0 consideramos a seguinte equação de reação-difusão

$$
\begin{cases}
w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau,\\
\frac{\partial w}{\partial n} = 0, & x \in \partial\Omega,\\
v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau,\\
v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1). &
\end{cases}
$$

Esta equação gera uma problema de evolução singularmente não autônomo com operador quase setorial e boa postura local, existência de solução forte e existência de atrator *pullback* são estudados à luz da teoria abstrata desenvolvida. Em particular, para obter a existência de atrator, o sistema acima será desacoplado, originando duas equações de evolução: uma com condição de fronteira de Neumann homogênea em Ω e outra com condições de fronteira de Dirichlet não homogênea e dependente do tempo em R_0. As propriedades dessas duas equações desacopladas são cuidadosamente estudadas e, a partir delas, estimativas do atrator *pullback* são obtidas.

Key words: Problemas parabólicos singularmente não autônomos, operadores quase setoriais, regularização, dinâmica assintótica, atrator *pullback*.

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List of symbols used in the abstract theory

Symbol	Meaning	Location
α	Sectoriality of $A(t)$ in X	(P.2)
δ	Hölder exponent of $A(t)$	(P.3)
ρ	Polynomial growth of F	(G)
ω	Sectoriality of $A^Y(t)$	(P.4)
β	Compatibility condition between $A(t)$ and Y	(P.4)

Auxiliary constants	Meaning	Location
η	Hölder continuity of $t \mapsto \varphi_1(t, \cdot)$, $t \mapsto \Phi(t, \cdot)$ in $\mathcal{L}(X)$	Lemma 3.12
ω	Hölder continuity of $t \mapsto \varphi_1(t, \cdot)$, $t \mapsto \Phi(t, \cdot)$ in $\mathcal{L}(Y)$	Corollary 3.14
μ	Hölder continuity in $\mathcal{L}(X)$ of $h \mapsto T_{-A(\tau)}(t + h)$	Lemma 1.6
ν	Hölder continuity in $\mathcal{L}(Y)$ of $h \mapsto T_{-A(\tau)}(t + h)$	Lemma 3.4

List of symbols used for the application

Symbol	Meaning	Location
Ω	Open subset of $\mathbb{R}^N$	
R_0	Line segment	
Ω_0	Domain with a handle $\Omega_0 = \Omega \cup R_0$	
x	Variable that takes values in Ω	
r	Variable that takes values in R_0	
$t, s, \tau \in \mathbb{R}$	Time variables	
$X = U_p^0$	Phase space $U_p^0 = L^p(\Omega) \times L^p(0,1)$	
$Y = U_q^0$	Phase space $U_q^0 = L^q(\Omega) \times L^q(0,1)$	
$\alpha \in (0, \alpha^+)$	$0 < \alpha < 1 - \frac{N}{2p} := \alpha^+$	(2.11)
$\beta \in (0, \beta^+)$	$0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) := \beta^+$	(2.14)
$\omega \in (0, \omega^+)$	$0 < \omega < 1 - \frac{N}{2q} := \omega^+$	(2.13)
M, γ	dissipativeness condition on f	(7.3)
$k_0 \geq \log_2 p$	k_0 positive integer such that $L^{2^{k_0}} \hookrightarrow L^p$	(7.12)

Introduction

A great variety of phenomena that happens in nature and in complex systems relies on the theory of differential equations to be properly modeled and explained. Physics, chemistry, mechanics and several other applied sciences contribute to, and benefit from, the development of this branch of Mathematics.

Inside this subject called differential equations, many subclassifications appear, depending on different features of the problem being studied and on which of those features one wishes to emphasize.

In this work we focus on a specific class of partial differential equations named *singularly nonautonomous semilinear evolution equations with almost sectorial operator*. Each term that appears in this nomenclature refers to one characteristic of the problem, as we specify next. The abstract model for this equation is given by

$$\begin{cases} u_t + A(t)u = F(u), \quad t > \tau, \\ u(\tau) = u_0 \in X, \end{cases} \tag{1}$$

where X is a Banach space, $A(t)$, $t \in \mathbb{R}$, is a family of closed linear operators defined on a fixed dense subspace D of X and F is a nonlinearity defined in X. The term *singularly nonautonomous* is used to express the fact that the linear operator $A(t)$ is time-dependent whereas the term *semilinear* designates the fact that F depends on the function u. *Evolution equation* refers to the fact that the equation models a state in the Banach space X that evolves as the time goes by, represented by u_t in the equation. Finally, *almost sectorial operator* express a certain property that the family $A(t), t \in \mathbb{R}$, possesses.

As we discuss throughout this work, this almost sectoriality property allows us to define two families of linear operators in X that help the construction of solution (in an appropriate sense) for (1). Those families are called *semigroup of growth* $1-\alpha$, denoted by $T_{-A(\tau)}(t)$, and *linear process of growth* $1-\alpha$, denoted by $U(t,\tau)$.

Just as there are several types of differential equations, there are also many different features that we can choose to study whenever we are analyzing a problem like (1).

To study its *local well-posedness* means to prove that the problem, for any initial time τ and initial condition $u_0 \in X$, has a unique solution $u = u(t)$ defined at least for a small interval of time and behaving continuously for small changes in the initial condition.

In case there exists this local solution for the problem, we can study the *regularity* of the solution, which consists in analyzing the properties (concerning continuity, differentiability and integrability) that $u = u(t)$ possesses. The problem (1) considered in this work has strong regularization properties for its solutions, even when the initial condition u_0 lacks regularity. It belongs to a class of problem called *parabolic problems*.

It is interesting in many cases to study the *global well-posedness* for the equation, which is the existence of the solution at any time $t > \tau$. This is specially interesting if we wish to understand the long-time behavior of the problem.

The global well-posedness is usually hitched to finding *estimates for the solutions* as well as sets in the phase space X that has the property of attracting this solution to them. Those sets are called *attractors* and they are an important tool to study the *asymptotic dynamics* of the problem.

This work is organized in a way to attend the topics mentioned above. In order to facilitate the comprehension of the topics presented and illustrate the ideas developed, we will intercalate abstract theory with an application of it to a *singularly nonautonomous reaction-diffusion equation in domain with a handle*. The presentation is structured in the following manner: Chapter P is dedicated to preliminaries results necessary to the development of the theory. We then divided the work in three parts and in each one of them we address a different aspect of the problem studied.

Part I of this works deals with local well-posedness of singularly nonautonomous evolution equations with almost sectorial operators. It consists of two chapters.

- Chapter 1: we present an abstract approach developed in [19] to obtain local well-posedness for (1). The treatment and notation we use in this part slightly diverge from the treatment provided in [19]. In there, the authors studied local solvability by understanding the relation that the nonlinearity F has with the domain of fractional powers of the operators $A(t)$. We choose to pose the problem and the results in a different setting, which seemed more suitable to the application we had in mind.

- Chapter 2: the reaction-diffusion equation in a domain with a handle is rigorously presented at this point. This application is a nonautonomous version of the autonomous problem presented in the series of articles [7, 8, 9]. The general properties of the problem are posed and the theory developed in Chapter 1 is applied to achieve local well-posedness for the problem.

The definition of solution established in Part I lacks any property of differentiability so far. As a matter of fact, the local solution obtained there is what we call *mild solution* and might not be a solution in the "usual" sense (by usual we understand a function that is differentiable in time t and satisfies the equation $u'(t) = -A(t)u + F(u), t > \tau$).

Part II is dedicated to prove the differentiability property of the solution, as well as other results on improving the properties of $u = u(t)$, which, as mentioned before, is called *regularization*. This part is organized in three chapters:

- Chapter 3: is dedicated to obtain Hölder continuity properties for several functions that feature in the problem studied.

- Chapter 4: the results on regularization for abstract singularly nonautonomous evolution equations with almost sectorial operators are presented. The theory developed in this part complements the theory of regularization provided, for instance, by [25, 37] for the autonomous setting where A is sectorial, by [31] for the autonomous case with almost sectorial operator and by [21, 55] for the nonautonomous case where the family $A(t), t \in \mathbb{R}$, is sectorial. To the best of our knowledge, there were no preview studies on regularization properties of local mild solution for the case considered in this work.

- Chapter 5: this effect of regularization is studied at an example, a reaction-diffusion equation in a domain with a handle.

The results in *Part II* were presented in two articles: [15] which deals with regularizing properties of semilinear autonomous equations with almost sectorial operators and [14] which concerns the singularly nonautonomous case.

The content in the two first parts of this work prepares for the final one, *Part III*. This part deals with the global dynamics of the problem:

- Chapter 6: the abstract theory on pullback attractors is briefly presented, following the approach adopted at [20] and some results on fractional powers of sectorial operators are also introduced.
- Chapter 7: global well-posedness and existence of pullback attractor for the singularly nonautonomous reaction-diffusion equation are studied. To approach this problem, we will develop an iteration procedure (inspired in the ideas presented by Moser-Alikakos [2, 28, 29]) in order to obtain L^p and L^∞ estimates for the solution. Then, the smoothing effect of the equation (which is the property of improving the regularity of u and u_t) allows us to extend those estimates to more regular spaces and finally obtain a compact set that pullback attracts.

The new results established in Chapter 7 (alongside with local well-posedness established in Chapter 2) were presented in [12].

We also reserve a chapter to final considerations and further discussion. A particular topic discussed is how the theory developed in this work connects with the usual approach on parabolic problems (via fractional powers).

There is also an Appendix Section (Appendix A) that deals with the smoothing effect that singularly nonautonomous differential parabolic equations has on the derivative u_t of the solution. We proved in this appendix that u_t also belongs to regular spaces. The content of this appendix is crucial to the develop of Chapter 7 and also to treat the matter of classical solution and global existence for any other singularly nonautonomous parabolic equations. This content was presented in [13].

Before we advance to the theory, we would like to introduce, in a non rigorous exposition at first, the problem of reaction-diffusion equation in a domain with a handle that sets the tone of this work.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain to which a line segment R_0 is attached at. We denote $\Omega_0 = \Omega \cup R_0$ and, in order to simplify the description of this set, we will assume that Ω is formed by two disjoint components, one at the left side of R_0 and one at the right side, while the line segment R_0

is given by $R_0 = \{(r, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}; r \in (0, 1)\}$. Furthermore, Ω and R_0 are connected by the points $(0, 0) \in \mathbb{R} \times R^{N-1}$ and $(1, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, as illustrated in Figure 1.

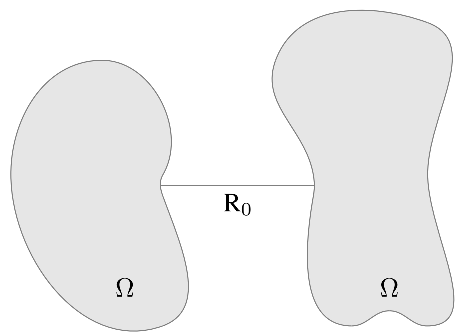


Figure 1: Domain with a handle

In this domain, we consider the coupled system of reaction-diffusion equation:

$$\begin{cases} w_t - div(a(t, x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ v_t - \frac{1}{g}\partial_r(g(r)a(t, r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1), \end{cases} \tag{2}$$

where p_0 and p_1 are the end points of R_0, $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ is a positive function that represents variations of the rate of diffusion at different points of Ω_0 and at different times $t \in \mathbb{R}$. The function f denotes a nonlinear function defined in $\mathbb{R}$. A brief glance at the equation allows us to point out some considerations:

1. First we note the presence of a function g inside the differential term in the second equation. This function g appears as a consequence of the way the problem is obtained. In physical terms, this problem could represent, for instance, two isolated tanks (the left and right side of the set Ω), at certain temperature distribution, to which we attach a cable connecting them.

   It might be of interest to determine how the heat flow occurs in this new domain, formed by to sets in $\mathbb{R}^3$ (the two tanks) and the line segment, R_0, represented by the cable. However, even though the cable has proportions much smaller than the tanks, it is actually a set in $\mathbb{R}^3$, not a line segment. The correct representation of the domain being studied should be given as in Figure 2 (and it is called dumbbell domain), where R_ε represents the cable with diameter proportion of a small order ε. The limit case, where $\varepsilon = 0$ and the channel becomes the line R_0, could be seen as an approximation of the problem.

   This limiting procedure gives rise to equation (2) and the function g appears as a consequence of the geometry of channel R_ε and how it collapses to R_0 as $\varepsilon \to 0^+$.

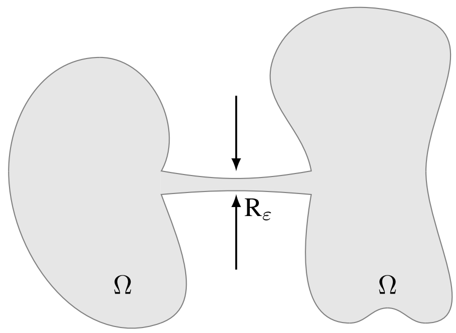


Figure 2: Dumbbell domain

This type of analysis considering differential equations on thin domains, that is, domains in which the size in one direction is much smaller than the size in other directions, is an interesting topic and have been studied by several authors. Hale and Raugel [35, 36] made important contributions to the development of this field.

The dumbbell domains belong to this class of thin domains and have also been studied by several authors, for instance Jimbo in a series of works [38, 39, 40, 41], Arrieta et. al. ([7, 8, 9]) in the articles that inspired this work and Carvalho et. al. [19] that considered a nonautonomous version of the problem earlier posed by Arrieta et. al.

The studies on this type of constricted domain emerged in the literature as a counterpart of working with convex domains. One of the reasons to work with them is: if we consider an autonomous reaction-diffusion equation in a convex domain, stable equilibria for the equation are constant in the domain (see, for instance, [23, 45]). In order to obtain stable equilibria that are not spatially constant, we can not allow the domain to be convex. This is when the dumbbell domains appeared as prototype of nonconvex domains. Its constriction prevents the diffusion phenomena to take place properly, creating the possibility of existence of stable equilibrium that is not spatially constant.

2. The system (2) is a *weakly coupled system*, which means that the differential equations are coupled in only one direction. Indeed, the first equation given in terms of w is independent of the second equation (in terms of v), whereas the second one depends on the values assumed by w at the junction points.

   Returning to the example of the tanks and the cable, this would mean that the temperature distribution of the tanks is insensible to what happens in the cable. On the other hand, the temperature of the cable would be given in terms of the temperature of the tanks in the points where its extremes are attached.

3. The conditions at p_0 and p_1 only make sense if $w \in \mathcal{C}(\overline{\Omega})$. This implies restrictions on the space of function that we will pose the problem.

4. The linear operator associated to Problem (2) has the desired property of being almost sectorial and represents the class of evolutions equations we focus in this work.

Further properties and analysis of the system above will appear throughout the work as needed.

CHAPTER P

Preliminaries

P.1 Integral in Banach spaces

Let Z be an arbitrary Banach space, $h : (t_1, t_2) \to Z$ a continuous function and $A : D(A) \subset Z \to Z$ a closed linear operator. Several results in this work involves analyzing the convergence and obtaining estimates for $\int_{t_1}^{t_2} h(t)dt$, as well as $\int_{t_1}^{t_2} A(t)h(t)dt$. To aid this task, we present in the sequel some remarks on integration of functions in Banach spaces that are necessary throughout the text.

The convergence of $\int_{t_1}^{t_2} h(t)dt$ is strictly connected with the convergence of $\int_{t_1}^{t_2} \|h(t)\|dt$: one will converge if and only if the other does. Therefore, tools on convergence of integrals of real functions will be handy at several moments, in special the ability of recognizing a *beta function* whenever it appears in the calculations. *Beta function* is the mapping $\mathcal{B} : (0,\infty) \times (0,\infty) \to \mathbb{R}$ given by

$$\mathcal{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$$

and it takes only finite values. A simple change of variable turns this integral to a form that shows up frequently in the calculations:

Lemma P.1. *If* $a, b > 0$ *and* $\tau < t$*, then*

$$\int_\tau^t (t-s)^{a-1}(s-\tau)^{b-1}ds = (t-\tau)^{a+b-1}\mathcal{B}(a,b).$$

Another well known function involving integral is the *Gamma function*, $\Gamma : (0,\infty) \to \mathbb{R}$, given by

$$\Gamma(a) = \int_0^\infty e^{-u}u^{a-1}du. \tag{P.1}$$

Integrability properties of a function $h : (t_1, t_2) \to Z$ are listed below.

Proposition P.2. *[25, Proposition 2.1.1] Let $h : (t_1, t_2) \to Z$ be a continuous function. If h is integrable on (t_1, t_2), then*

$$\int_{t_1}^{t_2} h(s)ds = \lim_{(\tau,t)\to(t_1,t_2)} \int_\tau^t h(s)ds.$$

Furthermore, if $h \in \mathcal{C}([t_1, t_2], Z) \cap \mathcal{C}^1((t_1, t_2), Z)$, then

$$h(t_2) - h(t_1) = \int_{t_1}^{t_2} h'(s)ds.$$

In any closed interval $[\tau, t] \subset (t_1, t_2)$, h will be bounded and $\int_\tau^t h(t)dt$ can be given by the classical Riemann approximations. This allows to prove the following proposition.

Proposition P.3. *Let $A : D(A) \subset Z \to Z$ be a closed linear operator and $h : [\tau, t] \to Z$ a continuous function with image in $D(A)$. If $Ah : [\tau, t] \to Z$ is continuous, then $\int_\tau^t h(s)ds \in D(A)$ and*

$$A \int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds.$$

Proof. Let $P = \{s_0, .., s_n\}$ be a partition of $[\tau, t]$ and s_i^* be any value between $[s_{i-1}, s_i]$. If $\Delta s_i = s_i - s_{i-1}$ and $|P|$ denotes the maximum length of the Δs_i intervals, then both integrals $\int_\tau^t h(s)ds$ and $\int_\tau^t Ah(s)ds$ (which are known to exists due to the continuity) can be written as a limit of Riemann sums, that is

$$\sum_i h(s_i^*)\Delta s_i \overset{|P|\to 0}{\longrightarrow} \int_\tau^t h(s)ds$$
$$A\left(\sum_i h(s_i^*)\Delta s_i\right) = \sum_i Ah(s_i^*)\Delta s_i \overset{|P|\to 0}{\longrightarrow} \int_\tau^t Ah(s)ds.$$

From the closedness of A follows that $\int_\tau^t h(s)ds \in D(A)$ and $A \int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds$. □

Corollary P.4. *Let $A : D(A) \subset Z \to Z$ be a closed linear operator, $h : [\tau, t) \to Z$ continuous with image in $D(A)$ and $Ah : [\tau, t) \to Z$ also continuous. Assume that $\int_\tau^t h(s)ds$ and $\int_\tau^t Ah(s)ds$ exist. Then, $\int_\tau^t h(s)ds \in D(A)$ and*

$$A \int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds.$$

Proof. Given any $0 < \rho < t - \tau$, we have from Proposition P.3 that $A \int_\tau^{t-\rho} h(s)ds = \int_\tau^{t-\rho} Ah(s)ds$. We also have

$$\int_\tau^{t-\rho} h(s)ds \overset{\rho\to 0}{\longrightarrow} \int_\tau^t h(s)ds$$
$$A \int_\tau^{t-\rho} h(s)ds = \int_\tau^{t-\rho} Ah(s)ds. \overset{\rho\to 0}{\longrightarrow} \int_\tau^t Ah(s)ds$$

From the closedness of A follows that $\int_\tau^t h(s)ds \in D(A)$ and $A \int_\tau^t h(s)ds = \int_\tau^t Ah(s)ds$.

□

Remark P.5. *It is important to distinguish existence of $A\int h$ from existence of $\int Ah$. The first can exist while the second does not. In other words, if the first term $A\int h$ exists, it does not mean that we can switch the operator with the integral, since $\int Ah$ might not exist. Moreover, a situation where $A\int_\tau^t h(s)ds$ and $A\int_\tau^{t-\rho} h(s)ds$ exist for a given $\rho > 0$, does not ensure that $A\int_\tau^{t-\rho} h(s)ds \to A\int_\tau^t h(s)ds$ when $\rho \to 0$, as we will see in Lemma 4.12 and Corollary 4.14.*

The next lemma is helpful if one wishes to differentiate under the integral sign.

Lemma P.6. *Let $f : [a,b]\times[a,b] \to Z$, $\rho \geq 0$ and $a \leq \tau < t-\rho \leq b$. Suppose that f is continuously differentiable in $(\tau, t-\rho]$ in the first variable and that $\int_\tau^{t-\rho} \|f_t(t,\xi)\|_X d\xi$ exists. Then, we have*

$$\frac{d}{dt}\int_\tau^{t-\rho} f(t,\xi)d\xi = f(t,t-\rho) + \int_\tau^{t-\rho} f_t(t,\xi)d\xi.$$

Proof. If $h > 0$, we have

$$\frac{1}{h}\left[\int_\tau^{t-\rho+h} f(t+h,\xi)d\xi - \int_\tau^{t-\rho} f(t,\xi)d\xi\right] = \frac{1}{h}\int_{t-\rho}^{t-\rho+h} f(t+h,\xi)d\xi + \frac{1}{h}\int_\tau^{t-\rho}[f(t+h,\xi)-f(t,\xi)]d\xi.$$

From the continuity of f near $t-\rho$ it follows that the first integral approaches $f(t,t-\rho)$. From the differentiability of f and from the fact that

$$\int_\tau^{t-\rho} \|f(t+h,\xi)-f(t,\xi)\|_X d\xi \leq K,$$

the Dominated Convergence Theorem ensures that the second term converges to $\int_\tau^{t-\rho} f_t(t,\xi)d\xi$.

□

As an illustration, if we wish to differentiate in t the function $\int_\tau^{t-\rho} e^{-a(\xi)(t-\xi)}d\xi$, where $a \in C^1(\mathbb{R},\mathbb{R})$, then we would have $\frac{d}{dt}\int_\tau^{t-\rho} e^{-a(\xi)(t-\xi)}d\xi = e^{-a(t-\rho)(\rho)} + \int_\tau^{t-\rho} -a(\xi)e^{-a(\xi)(t-\xi)}d\xi$.

The Gronwall's Lemma is important tool that provides estimates for inequalities involving integrals. We present two versions of it in the sequel.

Lemma P.7. *[37, p.190] If a, b are positive real constants, $0 \leq \alpha, \beta < 1$, $\tau < T < \infty$ and*

$$u(t) \leq a(t-\tau)^{-\alpha} + b\int_\tau^t (t-s)^{-\beta}u(s)ds, \quad t \in (\tau, T),$$

then there exists a constant $C(\beta, b, T) < \infty$ such that $u(t) \leq \frac{a(t-\tau)^{-\alpha}}{1-\alpha}C(\beta,b,T)$.

Lemma P.8. *[37, p.190] If a, b are positive real constants, $0 < \alpha, \beta, \gamma$ satisfying $\beta+\gamma-1 > 0$ and $\alpha+\gamma-1 > 0$, and*

$$u(t) \leq a(t-\tau)^{\alpha-1} + b\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{\gamma-1}u(s)ds, \quad t \in (\tau, T),$$

then

$$u(t) \leq a(t-\tau)^{\alpha-1}C(\beta, \alpha+\gamma-1, \beta+\gamma-1).$$

We will also use the following results that allows us to obtain differentiability properties of a function by analyzing its right-side derivative

Lemma P.9. *[52, p.43] Let $\phi : [a, b) \to Z$ be continuous and differentiable from the right on $[a, b)$. If $\frac{d}{dt}^{+}\phi$ is continuous in $[a, b)$, then ϕ is continuously differentiable in $[a, b)$.*

P.2 Spectrum of a closed linear operator

Let $A : D \subset Z \to Z$ be a closed and densely defined linear operator. The *resolvent of* A is a subset of the complex plane given by

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda - A) : D \to Z \text{ is a bijection}\}.$$

The complementary set of $\rho(A)$ in $\mathbb{C}$ is called *spectrum of* A and it is denoted by $\sigma(A)$. In [62, Section 1.7], several properties of those are presented. We mention in the sequel the ones we will use in this work.

Proposition P.10. *[62, Theorem 1.7.2] Let $A : D \subset Z \to Z$ be a closed linear operator. Then $\rho(A)$ is an open set in $\mathbb{C}$ and the function $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathcal{L}(Z)$ is analytic in each connected component of $\rho(A)$. Moreover,*

$$\frac{d^n}{d\lambda^n}(\lambda - A)^{-1} = (-1)^n n!(\lambda - A)^{-n-1}.$$

Proposition P.11. *[62, Theorem 1.7.3] Let $A, B : D \subset Z \to Z$ be a closed linear operator.*

1. *If $\lambda, \mu \in \rho(A)$, then*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}$$

and we refer to this property as first resolvent equality.

2. *If $\lambda \in \rho(A) \cap \rho(B)$, then*

$$(\lambda - A)^{-1} - (\lambda - B)^{-1} = (\lambda - A)^{-1}(A - B)(\lambda - B)^{-1}$$

and we refer to this property as second resolvent equality.

P.3 Some technical results

In the sequel we present two technical lemmas that play a leading role in the estimates obtained in this work. The first one is an interpolation inequality due to Nirenberg-Gagliardo that can be found in [25, Theorem 1.2.2]

Lemma P.12. *[Nirenberg-Gagliardo's inequality] Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ be a bounded domains with $\mathcal{C}^m$ smooth boundary $\partial\Omega$. Let $v \in W^{m,r}(\Omega) \cap L^q(\Omega)$ with $1 \leq r, q < \infty$. Then, for any integer j, $0 \leq j < m$, and any number $\theta \in \left[\frac{j}{m}, 1\right)$, define*

$$\frac{1}{p} = \frac{j}{n} + \theta \left(\frac{1}{r} - \frac{m}{N} \right) + (1-\theta)\frac{1}{q}.$$

If $m - j - \frac{n}{r}$ is not a nonnegative integer, then

$$\|D^j u\|_{L^p(\Omega)} \leq C \|u\|^{\theta}_{W^{m,r}(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}, \tag{P.2}$$

where D^j denotes any partial derivative of order j and C depends on Ω, r, q, m, j, θ. If $m - j - \frac{n}{r}$ is a nonnegative integer, then (P.2) *holds with $\theta = \frac{j}{m}$.*

The second result is the generalized Young Inequality.

Lemma P.13. *[25, Lemma 1.2.2] Let $\varepsilon > 0$, $a, b \geq 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q}.$$

Part I: Local well-posedness

We consider the local well-posedness of the singularly nonautonomous evolution equation

$$\begin{aligned} &u_t + A(t)u = F(u),\ t > 0, \\ &u(0) = u_0 \in X, \end{aligned} \tag{P.3}$$

where the family of linear operators $A(t), t \in \mathbb{R}$, is time-dependent and uniformly almost sectorial, and F is a nonlinearity defined in $\mathbb{R} \times X$.

As we shall see during this part, almost sectorial operator fails to generate C_0-semigroups. Nevertheless, it generates a special type of integrated semigroups, called semigroup of growth $1 - \alpha$. The theory of semigroups of growth is presented in Chapter 1 as well as its application on solving autonomous evolution equations.

To the family $A(t), t \in \mathbb{R}$, we also associate a two parameter family of linear operators called *linear process of growth* $1 - \alpha$, that inherits several of the properties of the semigroup of growth $1 - \alpha$ and can be associated to the solution of a singularly nonautonomous evolution equation.

The usual strategy to approach (P.3), adopted for instance in [19, 21, 55], consists in "breaking" the problem into less complex problems, in order to solve initially the simplest one and then use its solution to solve the more complex ones. Chapter 1 is structured accordingly to this partition and we fix the nomenclature that will appear throughout this entire work.

1. *Autonomous linear evolution equation*: in this case we fix $\tau \in \mathbb{R}$ and consider $A(\tau)$, one single operator of the family $A(t), t \in \mathbb{R}$. The operator $A(\tau)$ is almost sectorial and associated to it there exists a parabolic equation

$$u_t + A(\tau)u = 0,\ t > 0; \quad u(0) = u_0 \in X. \tag{P.4}$$

2. *Singularly nonautonomous and homogeneous linear evolution equation*: we consider the equation with the nonautonomous linear operator

$$u_t + A(t)u = 0,\ t > \tau; \quad u(\tau) = u_0 \in X. \tag{P.5}$$

3. *Singularly nonautonomous and nonhomogeneous linear evolution equation*: this case differs from the previous one by a perturbation of the problem with a nonlinearity depending only on time t, that is,

$$u_t + A(t)u = G(t),\ \tau < t < \tau + T; \quad u(\tau) = u_0 \in X. \tag{P.6}$$

4. *Singularly nonautonomous semilinear evolution equation*: this is the semilinear problem (1), where the nonlinearity depends on u:

$$u_t + A(t)u = F(u),\ \tau < t < \tau + T; \quad u(\tau) = u_0 \in X. \tag{P.7}$$

The local well-posedness for the semilinear abstract problem provided in Theorem 1.24 slightly differs from a version given in [19] and it is more suitable to the application we are dealing. This alternative result on local well-posedness was compiled in [12].

CHAPTER 1

Local well-posedness for singularly nonautonomous evolution equations

Let X be a Banach space with norm given by $\|\cdot\|_X$. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators $S : X \to X$, which is also a Banach space with the norm $\|S\|_{\mathcal{L}(X)} = \sup_{\|x\|_X \leq 1} \|Sx\|_X$.

If $S(t) \in \mathcal{L}(X)$, $t \in [0,T]$, is a one parameter family of linear operators, we say that the function $[0,T] \ni t \to S(t)$ is *continuous in the uniform topology* if the map $[0,T] \ni t \to S(t) \in \mathcal{L}(X)$ is continuous or, equivalently, $S(\cdot) \in \mathcal{C}([0,T], \mathcal{L}(X))$.

We say that $[0,T] \ni t \to S(t)$ is *strongly continuous* if for each $x \in X$, $[0,T] \ni t \to S(t)x \in X$ is continuous or, equivalently, $S(\cdot)x \in \mathcal{C}([0,T], X)$.

Continuity in the uniform topology implies strong continuity, but the reverse is not true. Also, uniform continuity and strong continuity could be stated for t in any interval in $\mathbb{R}$ (open or closed, finite or infinite) or any other metric space where the parameter lies. The same nomenclature also applies to functions in two parameters $(t,s) \mapsto S(t,s) \in \mathcal{L}(X)$.

To carry out the program proposed in Introduction, we assume that $A(t), t \in \mathbb{R}$, is a family of linear operators satisfying:

(P.1) $A(t) : D(A(t)) \subset X \to X$ is closed, densely defined and $D(A(t)) = D = X^1$, for all $t \in \mathbb{R}$.

(P.2) There exist constants $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, $C > 0$ and $\alpha \in (0,1)$, independent of $t \in \mathbb{R}$, such that, if Σ_φ represents the sector $\Sigma_\varphi := \{\lambda \in \mathbb{C}; |arg\lambda| \leq \varphi\}$, then $\Sigma_\varphi \cup \{0\} \subset \rho(-A(t))$ (see Figure 1.1) and

$$\left\|(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|^\alpha}, \quad \forall \lambda \in \Sigma_\varphi. \tag{1.1}$$

We refer to this property as the family $A(t), t \in \mathbb{R}$, being *uniformly almost sectorial* or $\alpha-$*uniformly almost sectorial* if we intend to emphasize the constant. We refer to α as the *constant of almost sectoriality*.

(P.3) There are constants $C > 0$ and $0 < \delta \leq 1$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{\mathcal{L}(X)} \leq C|t-\tau|^{\delta}. \tag{1.2}$$

To express this fact we say that the function $\mathbb{R} \ni t \mapsto A(t)A^{-1}(s) \in \mathcal{L}(X)$ is *uniformly Hölder continuous* or $\delta-$*uniformly Hölder continuous* if we intend to emphasize the constant. We refer to δ as the *Hölder exponent*.

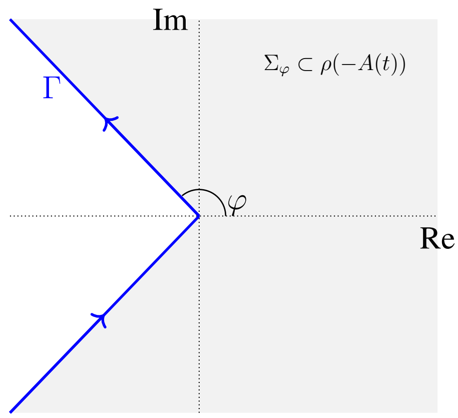


Figure 1.1: Sector Σ_φ and its contour Γ

Some immediate consequences of the properties (P.1) - (P.3) are: if $\tau = s$ in (1.2), then

$$\left\|A(t)A(s)^{-1}\right\|_{\mathcal{L}(X)} \leq 1 + C|t-s|^{\delta} \tag{1.3}$$

and $A(t)A(s)^{-1}$ is a bounded linear operator in X. In future calculations, (t,s) will be located in a compact set $K \subset \mathbb{R}^2$ and we will use (1.3) as $\|A(t)A(s)^{-1}\|_{\mathcal{L}(X)} \leq C$.

From the fact that $0 \in \rho(-A(t))$ and from the continuity of the resolvent map $\rho(-A(t)) \ni \lambda \mapsto (\lambda + A(t))^{-1} \in \mathcal{L}(X)$ in the uniform topology, the inequality (1.1) is equivalent to

$$\left\|(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{1+|\lambda|^{\alpha}}, \quad \forall \lambda \in \Sigma_\varphi \cup \{0\}.$$

Still from (1.1) and the resolvent's equality, we deduce

$$\left\|A(t)(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(X)} \leq 1 + C|\lambda|^{1-\alpha}, \quad \forall \lambda \in \Sigma_\varphi \cup \{0\}. \tag{1.4}$$

Whenever we fix $\tau \in \mathbb{R}$, the linear operator $A(\tau)$ enjoys the properties (P.1) and (P.2) stated above. The constant $\alpha \in (0,1)$ that features in estimate (1.1) prevents us from concluding that $-A(\tau)$ generates a C_0-semigroup, since Hille-Yosida's necessary conditions are not fulfilled (see [52, Theorem 1.3.1]). However, this *almost sectorial* operator (we drop the "uniform" from the name when we fix one time $\tau \in \mathbb{R}$) generates a special type of semigroup, called *semigroup of growth* $1-\alpha$, which we introduce in the next section.

1.1 Autonomous linear evolution equation

Almost sectorial operators have a close connection with generation of semigroups of growth. These semigroups were first introduced by Da Prato in [26], where the growth considered was given by a positive integer n. Later on, this concept was generalized to semigroups of growth β, for any $\beta > 0$, and its properties were studied by several authors, like [50, 51, 56, 63]. The definition of this type of semigroup is presented in the sequel.

Definition 1.1. *[51, Definition 1.1] Let X be a Banach space and $\alpha \in (0,1)$. A family $\{T(t) \in \mathcal{L}(X) : t \geq 0\}$ is a* semigroup of growth $1-\alpha$ *if*

1. *$T(0) = I$ and $T(t)T(s) = T(t+s)$, for all $t, s > 0$.*
2. *There exists $M, \gamma > 0$ such that $\|t^{1-\alpha}T(t)\|_{\mathcal{L}(X)} \leq M$, for all $0 \leq t \leq \gamma$.*
3. *If $T(t)x = 0$ for every $t > 0$, then $x = 0$.*
4. *$X_0 = \bigcup_{t>0} T(t)[X]$ is dense X.*

The connection between almost sectorial operators and semigroups of growth $1-\alpha$ was then explored in several works. It was proved in [19] that, for a fixed $\tau \in \mathbb{R}$, the operator $-A(\tau)$ generates a family of linear operators $T_{-A(\tau)}(t)$, $t > 0$, given by

$$T_{-A(\tau)}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda, \tag{1.5}$$

where Γ is the contour of Σ_{φ} (see Figure 1.1), that is, $\Gamma = \{re^{-i\varphi} : r > 0\} \cup \{re^{i\varphi} : r > 0\}$ and it is oriented with increasing imaginary part. This family satisfies the following properties.

Proposition 1.2. *If $T_{-A(\tau)}(t), t > 0$, is the family defined in* (1.5)*, then:*

1. *Each operator $T_{-A(\tau)}(t)$ is bounded and*
$$\left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}, \quad \forall t > 0. \tag{1.6}$$
2. *$T_{-A(\tau)}(t) : X \to D$, $A(\tau)T_{-A(\tau)}(t)$ is a bounded linear operator and*
$$\left\|A(\tau)T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X)} \leq Ct^{\alpha-2}, \quad \forall t > 0. \tag{1.7}$$
3. *There exists $\xi > 0$ (independent of τ) such that $T_{-A(\tau)}(t)$ has an exponential decay*
$$\left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}e^{-\xi t}, \quad \forall t > 0. \tag{1.8}$$

Proof. We briefly mention the ideas used to obtain those estimates. For a detailed description of the proof, we recommend [8, Section 2]. Inequality (1.6) follows from the resolvent estimate (1.1). Indeed, parameterizing the part of Γ with positive imaginary part by $\lambda = re^{i\varphi}$, $r \in [0, \infty)$, where φ is a fixed value in $(\frac{\pi}{2}, \pi)$, and doing the analogous for the negative imaginary part, we have

$$\|T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \leq \frac{1}{2\pi} 2 \int_0^\infty e^{r\cos(\varphi)t} \frac{C}{r^\alpha} dr \leq t^{\alpha-1} \frac{1}{\pi} \int_0^\infty e^{\cos(\varphi)u} \frac{C}{u^\alpha} du = Ct^{\alpha-1}.$$

From (1.4), we obtain $\int_\Gamma e^{\lambda t} A(\tau)(\lambda + A(\tau))^{-1} d\lambda$ converges and from the fact that $A(\tau)$ is closed, (1.7) follows.

As for the exponencial decay in (1.8), since $0 \in \rho(-A(\cdot))$ and the resolvent is an open set, we can slightly shift the sector Σ_φ to the left by a positive constant $\xi > 0$. In this case, $\xi I - A(\tau)$ is also almost sectorial with $\|T_{\xi I - A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}$, for a (possibly different) constant C. This implies that

$$e^{\xi t}\|T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} = \|T_{\xi I - A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}.$$

□

Up to this point, we have been calling $T_{-A(\tau)}(t), t > 0$, just a family of bounded linear operators. The next result states that $u(t) = T_{-A(\tau)}(t)u_0$ is a classical solution of the autonomous problem

$$u_t + A(\tau)u = 0, \quad t > 0; \quad u(0) = u_0 \in X,$$

which allows us to conclude, from the uniqueness of the solution, that $T_{-A(\tau)}(t)T_{-A(\tau)}(s) = T_{-A(\tau)}(t+s)$ for any $t, s > 0$. Since conditions 3 and 4 of Definition 1.1 are readily verified, we conclude that $T_{-A(\tau)}(t)$ is a semigroup of growth $1 - \alpha$.

Lemma 1.3. *([8, Lemma 2.1 and Lemma 2.4]) Let $T_{-A(\tau)}(t)$ be the linear operator defined in* (1.5)*. The mapping $T_{-A(\tau)}(t) : (0, \infty) \to \mathcal{L}(X)$ is differentiable and*

$$\frac{d}{dt} T_{-A(\tau)}(t) = -A(\tau)T_{-A(\tau)}(t) = \frac{1}{2\pi i} \int_\Gamma \lambda e^{\lambda t} (\lambda + A(\tau))^{-1} d\lambda.$$

That is, for $u_0 \in X$,

$$\frac{d}{dt} T_{-A(\tau)}(t)u_0 + A(\tau)T_{-A(\tau)}(t)u_0 = 0, \quad \forall t > 0,$$

and $u(t) = T_{-A(\tau)}(t)u_0$ is a classical solution of (P.4).

As a consequence of the previous lemma, we obtain the following result.

Proposition 1.4. *For $\tau \in \mathbb{R}$, the family $T_{-A(\tau)}(t), t > 0$, defined in* (1.5)*, satisfies:*

$$\|A(\tau)^2 T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-3}, \quad t > 0. \tag{1.9}$$

Proof. It follows from Lemma 1.3 that

$$A(\tau)^2 T_{-A(\tau)}(t) = -A(\tau)\frac{d}{dt}T_{-A(\tau)}(t) = -\frac{1}{2\pi i}A(\tau)\int_\Gamma \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda.$$

If $\lambda = re^{i\varphi}$, $r \in [0,\infty)$, is the parametrization of the branch of Γ with positive imaginary part and $\lambda = re^{-i\varphi}$ the parametrization of the negative branch, we obtain

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}d\lambda\right\|_{\mathcal{L}(X)} &\leq C\int_0^\infty e^{r\cos(\varphi)t} r\|A(\tau)(\lambda + A(\tau))^{-1}\|_{\mathcal{L}(X)}dr \\ \leq C\int_0^\infty e^{r\cos(\varphi)t}r^{2-\alpha}dr &\leq C\int_0^\infty e^{-u}\frac{u^{2-\alpha}}{t^{2-\alpha}\cos^{2-\alpha}(\varphi)}\frac{1}{\cos(\varphi)t}du \leq Ct^{\alpha-3}\Gamma(3-\alpha) \\ \leq Ct^{\alpha-3},\end{aligned}$$

where we used the Gamma function defined in (P.1).

Therefore, from the closedness of $A(\tau)$ added to the existence of the integral estimated above,

$$A(\tau)\int_\Gamma \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda = \int_\Gamma \lambda e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}d\lambda$$

and (1.9) follows. □

The semigroups of growth $1-\alpha$ are not necessarily continuous at $t = 0$ and their singular behavior distinguishes them from the usual C_0-semigroups. However, for any initial condition in D, the continuity at $t = 0$ holds, as we will see next. Moreover, if $x \in D^2$, then $-A(\tau)$ satisfies a property (item 3 below) that resembles the definition of infinitesimal generator for C_0-semigroups.

Lemma 1.5. *Let $T_{-A(\tau)}(t)$, $t > 0$, be the semigroup of growth $1-\alpha$ obtained by $-A(\tau)$.*

1. *If $x \in D$, then $\left\|T_{-A(\tau)}(t)x - x\right\|_X \to 0$ when $t \to 0^+$.*
2. *If $x \in D$, then $A(\tau)T_{-A(\tau)}(t)x = T_{-A(\tau)}(t)A(\tau)x$.*
3. *If $x \in D^2$, then $\lim_{t\to 0^+}\dfrac{T_{-A(\tau)}(t)x - x}{t} = -A(\tau)x$.*
4. *If $x \in D^2$, then $T_{-A(\tau)}(t)x$ is continuously differentiable in $[0,\infty)$ (including $t = 0$) and*

$$\frac{d}{dt}T_{-A(\tau)}(t)x = \begin{cases} -A(\tau)T_{-A(\tau)}(t)x, & \text{if } t > 0, \\ -A(\tau)x, & \text{if } t = 0. \end{cases}$$

5. *Given any $x \in X$ and $0 < s_1 < s_2$,*

$$T_{-A(\tau)}(s_2)x - T_{-A(\tau)}(s_1)x = -\int_{s_1}^{s_2} A(\tau)T_{-A(\tau)}(s)xds. \tag{1.10}$$

If $s_1 = 0$, then equality holds only for $x \in D^2$.

Proof. First statement was proved in [8, Proposition 2.6]. For the second one, if $x \in D$, it follows from the closedness of $A(\tau)$ that

$$\begin{aligned} A(\tau)T_{-A(\tau)}(t)x &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\tau)(\lambda + A(\tau))^{-1}x d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1}A(\tau)x d\lambda \\ &= T_{-A(\tau)}(t)A(\tau)x. \end{aligned}$$

The proof of the third statement is given in [8, Proposition 2.7]. We then use this information to prove the fourth statement. If $x \in D^2$,

$$\frac{d}{dt}T_{-A(\tau)}(t)x = \begin{cases} -A(\tau)T_{-A(\tau)}(t)x, & t > 0, \\ -A(\tau)x, & t = 0. \end{cases}$$

The continuity for $t > 0$ is already known. To prove the continuity at $t = 0$, we note that

$$\frac{d}{dt}T_{-A(\tau)}(t)x = -A(\tau)T_{-A(\tau)}(t)x = -T_{-A(\tau)}(t)A(\tau)x \to -A(\tau)x,$$

since $x \in D^2$ and $A(\tau)x \in D$.

Last statement follows from the fact that $(0,\infty) \ni t \mapsto \frac{d}{dt}T_{-A(\tau)}(t)x = -A(\tau)T_{-A(\tau)}(t)x$ is continuous. If $x \in D^2$, then this map is continuous including at $t = 0$.

□

The semigroup of growth $1 - \alpha$ also presents a certain type of Hölder continuity when we consider $h \mapsto T_{-A(\tau)}(t + h)$, for $t > 0$.

Lemma 1.6. *Let $T_{-A(\tau)}(t)$, $t > 0$, be the semigroup of growth $1 - \alpha$ obtained by $-A(\tau)$. Given any $0 < \mu < \alpha^2$, for $t, h > 0$, we have*

$$\|T_{-A(\tau)}(t + h) - T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \le Ch^\mu t^{\alpha - 1 - \frac{\mu}{\alpha}}, \tag{1.11}$$

and $\alpha - 1 - \frac{\mu}{\alpha} \in (-1, 0)$.

Proof. Note that

$$\begin{aligned} &\|T_{-A(\tau)}(t + h)x - T_{-A(\tau)}(t)x\|_X = \|T_{-A(\tau)}(h)T_{-A(\tau)}(t)x - T_{-A(\tau)}(t)x\|_X \\ &= \left\| \int_0^h \frac{d}{d\xi}T_{-A(\tau)}(\xi)T_{-A(\tau)}(t)x d\xi \right\|_X \le \left(\int_0^h \|T_{-A(\tau)}(\xi)\|_{\mathcal{L}(X)} d\xi \right) \|A(\tau)T_{-A(\tau)}(t)x\|_X \\ &\le C\left(\int_0^h \xi^{\alpha-1} d\xi \right) t^{\alpha-2}\|x\|_X = Ch^\alpha t^{\alpha-2}\|x\|_X. \end{aligned}$$

A positive exponent for h appeared, but at the downside t has a power in the negative interval $(-2, -1)$, which is not convenient when convergence of integrals is being considered. However, we already know that $\|T_{-A(\tau)}(t + h) - T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \le (t + h)^{\alpha-1} + t^{\alpha-1} \le Ct^{\alpha-1}$.

In order to improve the estimate, we interpolate the estimates already obtained

$$\|T_{-A(\tau)}(t+h)x - T_{-A(\tau)}(t)x\|_{\mathcal{L}(X)} \leq Ch^{\alpha}t^{\alpha-2}$$
$$\|T_{-A(\tau)}(t+h)x - T_{-A(\tau)}(t)x\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1},$$

with coefficients $\frac{\mu}{\alpha}$ and $(1-\frac{\mu}{\alpha})$, $0 \leq \mu \leq \alpha$, resulting in

$$\|T_{-A(\tau)}(t+h) - T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ch^{\mu}t^{\alpha-1-\frac{\mu}{\alpha}}. \tag{1.12}$$

The exponent of t will be in the interval $(-1,0)$ provided that $0 < \mu < \alpha^2$. □

The results up to now explored the properties of the semigroup $T_{-A(\tau)}(t)$ for $\tau \in \mathbb{R}$ fixed and we only required the uniformly sectoriality of the family $A(t), t \in \mathbb{R}$. This is enough to deal with the autonomous problem (P.4). However, to treat the singularly nonautonomous case (P.5), other estimates are necessary.

For instance, we can transfer the Hölder continuity of the family $A(t)$, $t \in \mathbb{R}$, to the semigroup generated by this family.

Lemma 1.7. *[19, Lemma 2.2] Let $A(t), t \in \mathbb{R}$, be a family satisfying (P.1),(P.2) and (P.3). Given $t, s \in \mathbb{R}$, we have*

$$\|T_{-A(t)}(\tau) - T_{-A(s)}(\tau)\|_{\mathcal{L}(X)} \leq C\tau^{-2+2\alpha}(t-s)^{\delta}, \quad \tau > 0. \tag{1.13}$$

In other words, the function $\mathbb{R} \ni t \mapsto T_{-A(t)}(\cdot)$ is Hölder continuous with exponent δ.

This transference of the Hölder continuity of the family $A(t)$, $t \in \mathbb{R}$, to the semigroup will play an essential role in Chapter 3. We will also see in this chapter that other families associated to $A(t), t \in \mathbb{R}$, inherit the Hölder continuity of this family.

In order to facilitate future calculations, we gather in the sequel properties of continuity for certain maps. The results on continuity in the uniform topology are proved in [19, Lemma 2.3 and Corollary 2.1], whereas the strong continuity of the functions below follows from the results and estimates already mentioned in this section.

Proposition 1.8. *[Continuity in the uniform topology] The following maps are continuous in the uniform topology:*

$$(0,\infty) \times \mathbb{R} \ni (\tau, s) \mapsto T_{-A(s)}(\tau) \in \mathcal{L}(X),$$
$$(0,\infty) \times \mathbb{R} \times \mathbb{R} \ni (\tau, t, s) \mapsto A(t)T_{-A(s)}(\tau) \in \mathcal{L}(X),$$
$$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto T_{-A(\tau)}(t-\tau) \in \mathcal{L}(X),$$
$$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto T_{-A(t)}(t-\tau) \in \mathcal{L}(X),$$
$$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau) \in \mathcal{L}(X),$$
$$\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto [A(\tau) - A(t)]T_{-A(t)}(t-\tau) \in \mathcal{L}(X).$$

Note that all continuities pointed above avoid the initial time, as it is expected. We would only have continuity of the first map in time $t = 0$ at the uniform topology if $A(t)$, $t \in \mathbb{R}$, were a continuous family of bounded linear operator (and $T_{-A(\tau)}(t)$ a uniformly continuous semigroup), for example.

However, we can expect strong continuity at $t = 0$ provided that $x \in D$.

Corollary 1.9. *[Continuity in the strong topology] Let $x \in D$. The following maps are strongly continuous:*

$$
\begin{aligned}
&[0,\infty) \times \mathbb{R} \ni (\tau, s) \mapsto T_{-A(s)}(\tau)x \in \mathcal{L}(X),\\
&[0,\infty) \times \mathbb{R} \times \mathbb{R} \ni (\tau, t, s) \mapsto A(t)T_{-A(s)}(\tau)x \in \mathcal{L}(X)\\
&\{(t,\tau) \in \mathbb{R}^2; t \geq \tau\} \ni (t,\tau) \mapsto T_{-A(\tau)}(t-\tau)x \in \mathcal{L}(X),\\
&\{(t,\tau) \in \mathbb{R}^2; t \geq \tau\} \ni (t,\tau) \mapsto T_{-A(t)}(t-\tau)x \in \mathcal{L}(X),\\
&\{(t,\tau) \in \mathbb{R}^2; t \geq \tau\} \ni (t,\tau) \mapsto [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau)x \in \mathcal{L}(X),\\
&\{(t,\tau) \in \mathbb{R}^2; t \geq \tau\} \ni (t,\tau) \mapsto [A(\tau) - A(t)]T_{-A(t)}(t-\tau)x \in \mathcal{L}(X).
\end{aligned}
$$

Remark 1.10. *For C_0-semigroups, all continuities in Corollary 1.9 hold for any $x \in X$.*

1.2 The nonautonomous linear associated problem

Consider the singularly nonautonomous equation (P.5) and the associated family $A(t), t \in \mathbb{R}$. Unlike the previous case, in the present situation we search for a two parameter family of linear operator $U(t,\tau)$ that, in some sense, is connected to the solution of the evolution equation. Ideally, if $u(t,\tau,u_0)$ is the local solution for (P.5), we would be searching for a family $U(t,\tau)$ such that $U(t,\tau)u_0 = u(t,\tau,u_0)$.

This two parameter family of linear operators $U(t,\tau)$ replaces in the nonautonomous case the one parameter semigroup $T_{-A(\tau)}(t)$ and it inspires the following definition:

Definition 1.11. *Let X be a Banach space and $\alpha \in (0,1)$. A family $\{U(t,s) \in \mathcal{L}(X); t > s\}$ is a* process of growth $1-\alpha$ *if*

1. *$U(t,t) = I$ and $U(t,\tau)U(\tau,s) = U(t,s)$, for all $s < \tau < t$.*
2. *There exists $M > 0$ such that $\|(t-s)^{1-\alpha}U(t,s)\|_{\mathcal{L}(X)} \leq M$, for all $t > s$.*
3. *$(t,s,x) \to U(t,s)x$ is continuous for $t > s$ and for all $x \in X$.*

Kato in [42, 43, 44] was the first to prove the existence of this process $\{U(t,\tau); t \geq \tau\}$ associated to the family $A(t), t \in \mathbb{R}$. However, the family $A(t), t \in \mathbb{R}$, considered was a hyperbolic family of linear operators and $U(t,\tau)$ was obtained through an approximation procedure.

The parabolic problem where each operator $A(t)$ is sectorial (that is, the family $A(t), t \in \mathbb{R}$ is uniformly sectorial) was studied simultaneously by Sobolevskii in [55] and Tanabe in [59, 58, 60]. They proved existence of a family $U(t,\tau)$ associated to $A(t)$ satisfying (P.1), (P.2) and (P.3). Both authors constructed a much more amiable procedure to obtain the process $U(t,\tau)$, which involved an argument of fixed point.

The strategy of obtaining the family $U(t,\tau)$ as a fixed point of a contraction map was successfully replied for the almost sectorial case (see [19]). We briefly motivate the formal computation that inspire the definition of the contraction map considered.

Suppose $U(t,\tau)$ is a family satisfying (P.5), that is, $\partial_t U(t,\tau) = -A(t)U(t,\tau)$. Also, assume that there exists another family $\Phi(t,\tau)$ such that $U(t,\tau)$ is obtained trough the integral equation below

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds. \tag{1.14}$$

Differentiating in t, adding $A(t)U(t,\tau)$ on both sides and using $\partial_t U(t,\tau) + A(t)U(t,\tau) = 0$, we obtain

$$0 = \Phi(t,\tau) - [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau) - \int_\tau^t [A(s) - A(t)]T_{-A(s)}(t-s)\Phi(s,\tau)ds.$$

If we denoted

$$\varphi_1(t,\tau) = [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau), \tag{1.15}$$

then $\Phi(t,\tau)$ would have to satisfy

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds \tag{1.16}$$

and it would be a fixed point of the map $S(\Psi)(t) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Psi(s)ds$.

If we had a family $\Phi(t,\tau)$ that satisfied (1.16), then we could proceed in the reverse way to obtain $U(t,\tau)$. This is what the authors in [19, Section 2] did and we enunciate in the sequel.

Lemma 1.12. *The family $\{\varphi_1(t,\tau) \in \mathcal{L}(X); t > \tau\}$ given by* (1.15) *satisfies:*

1. $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \varphi_1(t,\tau) \in \mathcal{L}(X)$ *is continuous in the uniform topology.*
2. *Its norm can be estimated by*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(X)} \le C(t-\tau)^{\alpha+\delta-2},$$

where α is the constant of almost sectoriality and δ the Hölder exponent.

Proof. It follows from Proposition 1.8, (1.2) and (1.7). □

Theorem 1.13. *[19, Section 2] Let $A(t), t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) - (P.3) and assume $\alpha + \delta > 1$, then there exists a unique two parameters family $\{\Phi(t,\tau) \in \mathcal{L}(X); t > \tau\}$ satisfying* (1.16) *with the following properties:*

1. $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \Phi(t,\tau) \in \mathcal{L}(X)$ *is continuous in the uniform topology.*

2. *Its norm satisfies*
$$\|\Phi(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha+\delta-2}, \tag{1.17}$$
where α is the constant of almost sectoriality and δ the Hölder exponent.

Corollary 1.14. *Under the conditions of Theorem 1.13, there exists a unique two parameter family $U(t,\tau)$ given by*
$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds$$
with the following properties:

1. $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto U(t,\tau) \in \mathcal{L}(X)$ *is continuous in the uniform topology.*
2. $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$.

Remark 1.15. *The family $U(t,\tau)$ obtained in Corollary 1.14 cannot be called a linear process of growth $1-\alpha$ so far, since we have not proved that*
$$U(t,s)U(s,\tau) = U(t,\tau), \quad \tau < s < t.$$

This will follow from the results in Chapter 4, when we prove that $u(t) = U(t,\tau)u_0$ is a strong solution of (P.5). *For now, we will avoid using the term linear process of growth $1-\alpha$ to refer to $U(t,\tau)$.*

Also, the existence of such family depends on the condition $\alpha + \delta > 1$. In the sectorial case this is trivially satisfied ($\alpha = 1$). In the almost sectorial case, this condition restricts the values that α can assume and it will play an essential part in the local well-posedness of the semilinear problem.

As it happens for the semigroup $T_{-A(\tau)}(t)$, the family $U(t,\tau)$ is also strongly continuous at the instant $t = \tau$ for elements in the domain of the operators, D.

Proposition 1.16. *If $x \in D$ and $\alpha + \frac{\delta}{2} > 1$, then $U(t,\tau)x \xrightarrow{t\to\tau^+} x$.*

Proof. The formulation for $U(t,\tau)$ implies
$$\|U(t,\tau)x - x\|_X \leq \left\|T_{-A(\tau)}(t-\tau)x - x\right\|_X + \left\|\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)xds\right\|_X.$$

The first term on the right side of the inequality approaches zero as $t \to \tau^+$, as a consequence of Lemma 1.5. For the second term, using inequalities (1.6) and (1.17), we have
$$\begin{aligned}\left\|\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)xds\right\|_X &\leq C\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\alpha+\delta-2}ds \\ &\leq C\mathcal{B}(\alpha,\alpha+\delta-1)(t-\tau)^{2\alpha+\delta-2} \xrightarrow{t\to\tau^+} 0.\end{aligned}$$

□

Other properties concerning the strong differentiability of $U(t,\tau)$ will be studied in Chapter 4.

The Hölder continuity observed for $h \mapsto T_{-A(\tau)}(t+h)$, $t>0$ obtained in Lemma 1.6 is transferred to the linear process $t \mapsto U(t,\tau)$, $t>\tau$, as we see next.

Proposition 1.17. *Suppose $\alpha+\delta>1$ and let $U(t,\tau)$ be the linear process associated to $A(t), t \in \mathbb{R}$. Then, given any $0<\mu<\alpha^2$, we have*

$$\|U(t+h,\tau)-U(t,\tau)\|_{\mathcal{L}(X)} \leq Ch^{\mu}[(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}+(t-\tau)^{\alpha+\delta-2}], \quad \textit{for } t>\tau, h>0.$$

Proof. Using the characterization available for the process in Corollary 1.14, we have

$$\begin{aligned}
U(t+h,\tau)-U(t,\tau) &= T_{-A(\tau)}(t+h-\tau)-T_{-A(\tau)}(t-\tau)\\
&\quad+\int_{\tau}^{t+h} T_{-A(s)}(t+h-s)\Phi(s,\tau)ds-\int_{\tau}^{t} T_{-A(s)}(t-s)\Phi(s,\tau)ds\\
&= T_{-A(\tau)}(t+h-\tau)-T_{-A(\tau)}(t-\tau)+\int_{t}^{t+h} T_{-A(s)}(t+h-s)\Phi(s,\tau)ds\\
&\quad+\int_{\tau}^{t}[T_{-A(s)}(t+h-s)-T_{-A(s)}(t-s)]\Phi(s,\tau)ds.
\end{aligned}$$

It follows from (1.11) that, for any $0<\mu<\alpha^2$,

$$\|T_{-A(\tau)}(t+h-\tau)-T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)} \leq h^{\mu}(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}.$$

From (1.17), we obtain

$$\begin{aligned}
&\left\|\int_{t}^{t+h} T_{-A(s)}(t+h-s)\Phi(s,\tau)ds\right\|_{\mathcal{L}(X)} \leq \int_{t}^{t+h} \|T_{-A(s)}(t+h-s)\|_{\mathcal{L}(X)}\|\Phi(s,\tau)\|_{\mathcal{L}(X)}ds\\
&\quad\leq C\int_{t}^{t+h}(t+h-s)^{\alpha-1}(s-\tau)^{\alpha+\delta-2}ds \leq C(t-\tau)^{\alpha+\delta-2}\int_{t}^{t+h}(t+h-s)^{\alpha-1}ds\\
&\quad\leq Ch^{\alpha}(t-\tau)^{\alpha+\delta-2}.
\end{aligned}$$

One more time, it follows from (1.11) that

$$\begin{aligned}
&\left\|\int_{\tau}^{t}[T_{-A(s)}(t+h-s)-T_{-A(s)}(t-s)]\Phi(s,\tau)ds\right\|_{\mathcal{L}(X)}\\
&\quad\leq \int_{\tau}^{t}\|[T_{-A(s)}(t+h-s)-T_{-A(s)}(t-s)]\|_{\mathcal{L}(X)}\|\Phi(s,\tau)\|_{\mathcal{L}(X)}du\\
&\quad\leq C\int_{\tau}^{t} h^{\mu}(t-s)^{\alpha-1-\frac{\mu}{\alpha}}(s-\tau)^{\alpha+\delta-2}ds\\
&\quad\leq Ch^{\mu}(t-\tau)^{(\alpha-1-\frac{\mu}{\alpha})+(\alpha+\delta-1)}\mathcal{B}(\alpha-\tfrac{\mu}{\alpha},\alpha+\delta-1)\\
&\quad\leq Ch^{\mu}(t-\tau)^{(\alpha-1-\frac{\mu}{\alpha})+(\alpha+\delta-1)},
\end{aligned}$$

and note that $\alpha-1-\frac{\mu}{\alpha} \in (-1,0)$, whereas $\alpha+\delta-1>0$.

From the estimates obtained above, we conclude that

$$\begin{aligned}\|U(t+h,\tau)-U(t,\tau)\|_{\mathcal{L}(X)} &\leq C[h^{\mu}+h^{\alpha}]\left[(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}+(t-\tau)^{\alpha+\delta-2}+(t-\tau)^{(\alpha-1-\frac{\mu}{\alpha})+(\alpha+\delta-1)}\right]\\ &\leq Ch^{\mu}[1+h^{\alpha-\mu}][(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}+(t-\tau)^{\alpha+\delta-2}+(t-\tau)^{-1+(\alpha+\delta-1)}]\\ &\leq Ch^{\mu}[(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}+(t-\tau)^{\alpha+\delta-2}],\end{aligned}$$

where we selected the smallest positive exponent for h (in order to obtain Hölder contintuity for this term) and the most negative exponent for $(t-\tau)$.

□

1.3 Existence of local solution for the semilinear problem

Consider the semilinear problem

$$u_t + A(t)u = F(u),\ t>\tau;\quad u(\tau)=u_0\in X,$$

where the family $A(t), t\in\mathbb{R}$, satisfies properties (P.1), (P.2) and (P.3), $\alpha\in(0,1)$ is the constant of almost sectoriality and $\delta\in(0,1]$ is the Hölder exponent of the family.

We assume that the nonlinearity F added to the equation has a certain growth that decreases the regularity of the elements in X. This growth condition is expressed below:

(G). There exists a Banach space Y in which X is embedded ($X\hookrightarrow Y$) and constants $C>0$, $\rho\geq 1$, such that $F: X\to Y$ and, for every $u,v\in X$,

$$\begin{aligned}\|F(u)-F(v)\|_Y &\leq C\,\|u-v\|_X\left(1+\|u\|_X^{\rho-1}+\|v\|_X^{\rho-1}\right)\\ \|F(u)\|_Y &\leq C(1+\|u\|_X^{\rho}).\end{aligned}$$

We will refer to the property above as the nonlinearity F having a *polynomial growth of order* ρ. For instance, a situation like above happens when $X=L^p(\Omega)$ and $F(u)=|u|^2$. In this case, F take elements of $L^p(\Omega)$ to the less regular space $L^{\frac{p}{2}}(\Omega)$ (Ω a bounded domain in $\mathbb{R}^N$).

We will also assume that the family $A(t), t\in\mathbb{R}$, and the space Y can be related somehow. This relation is expressed next.

(P.4) The realization of the family $A(t), t\in\mathbb{R}$, in Y, denoted by $A^Y(t): D^Y = D(A^Y(t))\subset Y\to Y$, satisfies properties (P.1), (P.2) and (P.3) in the Banach space Y, possibly with different exponent of almost sectoriality, $\omega\in(0,1)$. Moreover, the resolvent of $-A(t)$ satisfies: $(\lambda+A(t))^{-1}: Y\to X$ (which means that $D^Y\hookrightarrow X$) and there exists $\beta\in(0,1)$ such that

$$\left\|(\lambda+A(t))^{-1}\right\|_{\mathcal{L}(Y,X)}\leq\frac{C}{|\lambda|^{\beta}+1},\qquad\forall\lambda\in\Sigma_{\varphi}\cup\{0\}.\tag{1.18}$$

We will refer to the property above as the *compatibility between* $A(t)$ *and* Y.

Remark 1.18. *To be precise, inequality* (1.18) *should be posed as* $\|(\lambda + A^Y(t))^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{C}{|\lambda|^\beta+1}$. *However, we denote by* $A(t)$ *the operator acting in* X *or* Y*, and the only distinction between them appears in the resolvent estimate, that is*

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|^\alpha + 1},$$
$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|^\omega + 1},$$
$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Y,X)} \leq \frac{C}{|\lambda|^\beta + 1}.$$

This convention extends to the semigroup generated by $-A(\tau)$*, that is, we say* $T_{-A(\tau)}(t)$ *is an element of* $\mathcal{L}(X)$*,* $\mathcal{L}(Y)$ *or* $\mathcal{L}(Y,X)$*. Same holds for the families* $U(t,\tau)$*,* $\varphi_1(t,\tau)$ *and* $\Phi(t,\tau)$*. They can be seen acting on* $\mathcal{L}(X)$*,* $\mathcal{L}(Y)$ *or* $\mathcal{L}(Y,X)$*. In Lemma 1.21 we discuss some properties of those operators in* $\mathcal{L}(Y,X)$ *and in Chapter 3 we obtain further results on estimates and Hölder continuity on those different spaces.*

To help fix the ideas above, we illustrate (P.4) in a simple case (a sectorial case).

Example 1.19. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded smooth domain and* $A(t) = A_p = -\Delta_p^{\mathcal{N}} : D(\Delta_p^{\mathcal{N}}) \subset L^p(\Omega) \to L^p(\Omega)$*, where* $\Delta_p^{\mathcal{N}}$ *is the Laplacian acting in* $L^p(\Omega)$ *with Neumann boundary condition. The domain of* A_p *is*

$$D(\Delta_p^{\mathcal{N}}) = W^{2,p}_{\mathcal{N}} := \{u \in W^{2,p}(\Omega); \partial_n u = 0\}$$

and this operator is known to be sectorial ($\alpha = 1$*), see [52, Section 7.3]. Let* $X = L^p(\Omega)$ *and suppose* $\frac{N}{2} < q < p$*. If we denote* $Y = L^q(\Omega)$*, then we can consider the Laplacian acting now on* Y*, that is,* $A_q : W^{2,q}_{\mathcal{N}} \subset L^q(\Omega) \to L^q(\Omega)$*. Furthermore,* $X \hookrightarrow Y$ *and* $W^{2,q}_{\mathcal{N}} \hookrightarrow L^\infty(\Omega) \hookrightarrow X$*.*

As in Remark 1.18, we denote both A_p *and* A_q *as the same operator* A*, and we only distinguish them in the space they act. From the considerations above,* $(\lambda + A)^{-1}$ *can be seen as an operator in* $\mathcal{L}(X)$*,* $\mathcal{L}(Y)$ *or* $\mathcal{L}(Y,X)$

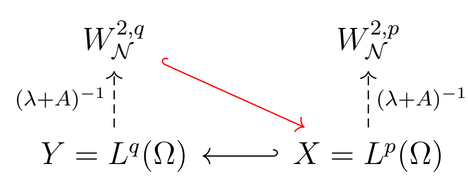


Figure 1.2: Diagram of embeddings
$\hookrightarrow$: embed $\dashrightarrow$: action of the operator

In Chapter 2, Proposition 2.5, we present a similar situation for a family of uniformly almost sectorial. In that case, α*,* β *and* ω *are all different from* 1 *and the operator is almost sectorial.*

Remark 1.20. *At this point the reader might have realized that we opted not to use the regular approach for parabolic problems (as in [5, 21, 37]) in which one works with a scale of fractional power spaces* $\{X^\xi\}_{0\leq\xi\leq 1+\varepsilon}$ *and the growth of* F *is given as* $F: X^\gamma \to X^\theta$, $0<\gamma-\theta<1$.

In this fractional power approach, a good knowledge of the spaces X^ξ*, as well as their embed in the* L^p-*spaces, are required. To the application we have in mind, this characterization of the spaces* $X^\xi = D(A(t)^\xi)$ *is not available, preventing us from following this way.*

On the other hand, working with the spaces X *and* Y *as in (G) and (P.4) allows us to treat cases where a different scale of Banach space is available, rather than the fractional power scale. Moreover, this approach, with* X *and* Y *spaces, incorporates the usual one with fractional power spaces, but in the downside, it is limited if we want to study smoothing effects of the differential equation. We discuss this with more details in Chapter 8.*

1.3.1 Estimates in $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$

Inequality (1.18) can be read as the ability of the operator $(\lambda+A(t))^{-1}$ to regularize elements of Y back to the space X. This feature is transmitted to the semigroup $T_{-A(\tau)}(t)$ and to the linear process $U(t,\tau)$. We have the following properties for them:

Lemma 1.21. *Let* X *and* Y *be Banach spaces as in (G) and assume* (1.18) *holds, with constants* β *and* ω *given in (P.4). Then,*

(1) The linear operators $A(t), t\in\mathbb{R}$*, satisfies, for all* $\lambda\in\Sigma_\varphi\cup\{0\}$*,*

$$\|A(t)(\lambda+A(t))^{-1}\|_{\mathcal{L}(Y,X)} \leq 1+C|\lambda|^{1-\beta},$$
$$\|A(t)(\lambda+A(t))^{-1}\|_{\mathcal{L}(Y)} \leq 1+C|\lambda|^{1-\omega}.$$

(2) The semigroup $T_{-A(\tau)}(t)$ *obtained by the family* $A(t)$*,* $t\in\mathbb{R}$*, satisfies the following estimates:*

$$\left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(Y,X)} \leq Ct^{\beta-1}, \qquad \left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(Y)} \leq Ct^{\omega-1},$$
$$\left\|A(\tau)T_{-A(\tau)}(t)\right\|_{\mathcal{L}(Y,X)} \leq Ct^{\beta-2}, \qquad \left\|A(\tau)T_{-A(\tau)}(t)\right\|_{\mathcal{L}(Y)} \leq Ct^{\omega-2}.$$

(3) The family $\varphi_1(t,\tau)$*, defined in* (1.15)*, satisfies*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \leq C(t-\tau)^{\omega+\delta-2}.$$

Moreover, if $\omega+\delta>1$*, then* $\Phi(t,\tau)$ *defined in* (1.16) *satisfies*

$$\|\Phi(t,\tau)\|_{\mathcal{L}(Y)} \leq C(t-\tau)^{\omega+\delta-2}.$$

(4) If $\omega+\delta>1$ *(for both estimates in* $\mathcal{L}(Y,X)$ *and* $\mathcal{L}(Y)$*), then the linear process* $U(t,\tau)$ *satisfies*

$$\|U(t,\tau)\|_{\mathcal{L}(Y,X)} \leq C(t-\tau)^{\beta-1},$$
$$\|U(t,\tau)\|_{\mathcal{L}(Y)} \leq C(t-\tau)^{\omega-1}.$$

Proof. Item (1) follows from the resolvent estimates in $\mathcal{L}(Y, X)$ and $\mathcal{L}(Y)$, respectively. For the semigroups in item (2), all those estimates are obtained in the same way as they were in Proposition 1.2, when we considered $\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{1+|\lambda|^\alpha}$.

For the linear family $\varphi_1(t,\tau)$, it follows from its formulation in (1.15), that is,

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \leq \|[A(\tau) - A(t)]A(\tau)^{-1}\|_{\mathcal{L}(Y)} \|T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(Y)} \leq c(t-\tau)^{\delta+\omega-2}.$$

For $\Phi(t,\tau)$, note that condition $\omega + \delta > 1$ is necessary and it replaces $\alpha + \delta > 1$ in Theorem 1.13, as we see next:

$$\begin{aligned}\|\Phi(t,\tau)\|_{\mathcal{L}(Y)} &\leq \|\varphi_1(t,\tau)\|_{\mathcal{L}(Y)} + \int_\tau^t \|\varphi_1(t,s)\|_{\mathcal{L}(Y)} \|\Phi(s,\tau)\|_{\mathcal{L}(Y)} ds \\ &\leq C(t-\tau)^{\omega+\delta-2} + \int_\tau^t C(t-s)^{\omega+\delta-2} \|\Phi(s,\tau)\|_{\mathcal{L}(Y)} ds,\end{aligned}$$

since $\omega + \delta - 2 > -1$, we can apply a generalized Gronwall inequality (see Lemma P.7) and obtain

$$\|\Phi(t,\tau)\|_{\mathcal{L}(Y)} \leq C(t-s)^{\delta+\omega-2}.$$

Finally, formulation for $U(t,\tau)$ given in (1.14), implies

$$\begin{aligned}\|U(t,\tau)\|_{\mathcal{L}(Y,X)} &\leq \|T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(Y,X)} + \int_\tau^t \|T_{-A(s)}(t-s)\|_{\mathcal{L}(Y,X)} \|\Phi(s,\tau)\|_{\mathcal{L}(Y)} ds \\ &\leq C(t-\tau)^{\beta-1} + \int_\tau^t C(t-s)^{\beta-1}(s-\tau)^{\omega+\delta-2} ds \\ &\leq C(t-\tau)^{\beta-1} + C(t-\tau)^{\beta-1+\omega+\delta-1}\mathcal{B}(\beta, \omega+\delta-1).\end{aligned}$$

By hypothesis $\omega + \delta - 1 > 0$ and, consequently, $\|U(t,\tau)\|_{\mathcal{L}(Y,X)} \leq C(t-\tau)^{\beta-1}$. The estimate for $\|U(t,\tau)\|_{\mathcal{L}(Y)}$ is identical to the estimate in $\|U(t,\tau)\|_{\mathcal{L}(X)}$ with α replaced by ω. □

1.3.2 Local mild solution

The type of solution for the semilinear problem that we explore in this section is a generalized one, called *mild solution*, which we define next.

Definition 1.22. *A function $u : (\tau, \tau + t_0] \to X$ is a* mild solution *in the interval $(\tau, \tau + t_0]$ for* (P.7) *if*

1. $u(\cdot) \in \mathcal{C}((\tau, \tau + t_0], X)$ *and* $\lim_{t\to\tau^+} \|u(t) - U(t,\tau)u_0\|_X = 0$.
2. $u(\cdot)$ *is given by the variation of constants formula*

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds, \quad \forall t \in (\tau, \tau + t_0]. \tag{1.19}$$

Remark 1.23. *The condition* $\lim_{t\to\tau^+} \|u(t) - U(t,\tau)u_0\|_X = 0$ *states that the mild solution replies the same type of discontinuity that the process $U(t,\tau)$, $t > \tau$, might posses as $t \to \tau^+$.*

Given $\tau \in \mathbb{R}$ and $u_0 \in X$, we will search for mild solutions in the following space

$$K(t_0,u_0) = \left\{v \in C((\tau,\tau+t_0],X); \sup_{t\in(\tau,\tau+t_0]} \|v(t) - U(t,\tau)u_0\|_X \leq \mu\right\},$$

where μ is a positive constant and t_0 will be suitably chosen later. $K(t_0,u_0)$ is a Banach space with norm $\|\xi\|_K = \sup_{t\in(\tau,\tau+t_0]} \|\xi(t) - U(t,\tau)u_0\|_X$. Note that

$$\begin{aligned}(t-\tau)^{1-\alpha}\|v(t)\|_X &\leq (t-\tau)^{1-\alpha}\|v(t)-U(t,\tau)u_0\|_X + (t-\tau)^{1-\alpha}\|U(t,\tau)u_0\|_X \\ &\leq t_0^{1-\alpha}\mu + C\|u_0\|_X,\end{aligned}$$

and in case u_0 is in a bounded set $B \subset X$, we can obtain uniform estimates for $(t-\tau)^{1-\alpha}\|v(t)\|_X$, that is,

$$(t-\tau)^{1-\alpha}\|v(t)\|_X \leq k, \text{ where } k := t_0^{1-\alpha}\mu + C\sup_{u_0\in B}\|u_0\|_X. \tag{1.20}$$

The following theorem proves the existence of mild solution for the problem studied. It is similar to Theorem 3.1 in [19]. However, the authors there posed the problem in a different setting (using fractional power spaces, as mentioned in Remark 1.20). We rewrite the proof using the setting established in this section (with X and Y spaces). This alternate version is presented in [12].

Theorem 1.24. *Let X, Y be Banach spaces with $X \hookrightarrow Y$. Suppose $A(t), t \in \mathbb{R}$, is $\alpha-$uniformly sectorial and $\delta-$uniformly Hölder continuous. Additionally, we assume that $A(t)$, $t \in \mathbb{R}$ satisfies the condition in (P.4). If $\alpha + \delta > 1$, $\omega + \delta > 1$ and $F : X \to Y$ is a nonlinearity satisfying (G) with*

$$1 \leq \rho < \frac{\beta}{1-\alpha}, \tag{1.21}$$

then, for every $u_0 \in X$, there exists $t_0 > 0$, such that the initial value problem

$$u_t + A(t)u = F(u), t > \tau; \quad u(\tau) = u_0 \in X,$$

has a unique mild solution defined in $(\tau,\tau+t_0]$. This t_0 depends on u_0, but can be chosen uniformly for u_0 in bounded sets of X. Furthermore, we can extend this mild solution to a maximal interval $(\tau,\tau_M(u_0))$.

Proof. The conditions $\alpha+\delta > 1$ and $\omega+\delta > 1$ guarantees the existence of the family $U(t,\tau)$ in $\mathcal{L}(X)$ and $\mathcal{L}(Y,X)$ (see Corollary 1.14 and Lemma 1.21). We can consider the operator

$$(Tv)(t) := U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds, \quad t \in (\tau,\tau+t_0],$$

defined in $K(t_0,u_0)$. From the continuity of $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto U(t,\tau) \in \mathcal{L}(X)$ stated in Corollary 1.14, it follows readily that $T : K(t_0,u_0) \to C((\tau,\tau+t_0],X)$. We prove that, for small values of t_0, T is a contraction in $K(t_0,u_0)$. The Banach Fixed Point Theorem will ensure then the existence of a unique fixed point for T in $K(t_0,u_0)$.

Given $v \in K(t_0, u_0)$, in order for $Tv(\cdot)$ to be in $K(t_0, u_0)$, we must have $\|Tv(t) - U(t,\tau)u_0\|_X \leq \mu$, for all $t \in (\tau, \tau + t_0)$. Using the estimates available for the process, for the nonlinearity and for $v \in K(t_0, u_0)$ in (1.20), we have

$$\begin{aligned}
\|Tv(t) - U(t,\tau)u_0\|_X &\leq \int_\tau^t \|U(t,s)F(v(s))\|_X \, ds \leq \int_\tau^t \|U(t,\tau)\|_{\mathcal{L}(Y,X)} \|F(v(s))\|_Y \, ds \\
&\leq C\int_\tau^t (t-s)^{\beta-1}(1 + \|v(s)\|_X^\rho) ds \leq C\int_\tau^t (t-s)^{\beta-1}(1 + k^\rho (s-\tau)^{(\alpha-1)\rho}) ds \\
&\leq C\int_\tau^t (t-s)^{\beta-1} ds + Ck^\rho \int_\tau^t (t-s)^{\beta-1}(s-\tau)^{(\alpha-1)\rho} ds \\
&\leq C(t-\tau)^\beta + C(t-\tau)^{\beta-(1-\alpha)\rho}\mathcal{B}(\beta, 1-(1-\alpha)\rho).
\end{aligned}$$

Condition (1.21) implies $\beta - (1-\alpha)\rho > 0$ and also $1 - (1-\alpha)\rho > 0$. Therefore,

$$\|Tv(t) - U(t,\tau)u_0\|_X \leq C[t_0^\beta + t_0^{\beta-(1-\alpha)\rho}] \leq \mu$$

for t_0 small enough. Moreover, $\|Tv(t) - U(t,\tau)v_0\|_X \xrightarrow{t\to\tau^+} 0$.

Also,

$$\begin{aligned}
\|Tv(t) - Tw(t)\|_X &\leq \int_\tau^t \|U(t,s)\|_{\mathcal{L}(Y,X)} \|F(v(s)) - F(w(s))\|_Y \, ds \\
&\leq \int_\tau^t C(t-s)^{\beta-1} \|v(s) - w(s)\|_X (1 + \|v(s)\|_X^{\rho-1} + \|w(s)\|_X^{\rho-1}) ds \\
&\leq C\int_\tau^t (t-s)^{\beta-1} \left(1 + 2k^{\rho-1}(s-\tau)^{-(1-\alpha)(\rho-1)}\right) ds \, \|v-w\|_K \\
&\leq C\left[\int_\tau^t (t-s)^{\beta-1} + \int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(1-\alpha)(\rho-1)} ds\right] \|v-w\|_K \\
&\leq C\left[(t-\tau)^\beta + (t-\tau)^{\beta-(1-\alpha)(\rho-1)}\mathcal{B}(\beta, 1-(1-\alpha)(\rho-1))\right] \|v-w\|_K
\end{aligned}$$

and from condition (1.21), we have $\rho - 1 < \frac{\beta}{1-\alpha}$, which implies $\beta - (1-\alpha)(\rho-1) > 0$ and, consequently, $1 - (1-\alpha)(\rho-1) > 0$. Therefore, for t_0 sufficiently small and any $t \in (\tau, \tau + t_0)$,

$$\|Tv(t) - Tw(t)\|_X \leq C\left[t_0^\beta + t_0^{\beta-(1-\alpha)(\rho-1)}\right] \|v-w\|_K \leq \frac{1}{2}\|v-w\|_K.$$

If v is the fixed point of the contraction T, then it is a mild solution for the semilinear equation, since $\|v(t) - U(t,\tau)v_0\|_X \xrightarrow{t\to\tau^+} 0$ and

$$v(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(v(s))ds.$$

The uniformity of t_0 in bounded sets $B \subset X$ follows from the fact that k in (1.20) can be chosen uniform in B.

□

Note that condition (1.21) indicates that for any $\rho < \frac{\beta}{1-\alpha}$ the problem can be solved, whereas for $\rho = \frac{\beta}{1-\alpha}$, it is impossible to ensure convergence of the integrals that appears in the proof of the theorem.

However, for the problem being studied, we have more than one condition that will impose restrictions on the growth of F. There are actually two sets of conditions required for the well-posedness to hold:

1. One related to the existence of the process $U(t,\tau)$ in $\mathcal{L}(X)$ and $\mathcal{L}(Y,X)$. Those are the conditions $\alpha+\delta>1$ and $\omega+\delta>1$.

2. The other related to the blow-up at initial time $t=\tau$: $(t-\tau)^{\alpha-1}$. F cannot have a growth ρ so large that this blow-up becomes incontrollable. This is expressed by condition (1.21).

We will distinguish them in order to emphasize where each condition is being required, as we see in Table 1.1.

Local well-posedness	
I. Conditions on the existence of $U(t,\tau)$	II. Conditions to ensure existence of mild solution
$\alpha+\delta>1$ (in $\mathcal{L}(X)$) $\omega+\delta>1$ (in $\mathcal{L}(Y,X)$)	$1 \leq \rho < \frac{\beta}{1-\alpha}$

Table 1.1: Conditions of Theorem 1.24

We prove next that the mild solution obtained depends continuously on initial data and satisfies an alternative blow-up.

Theorem 1.25. *Under the conditions of Theorem 1.24, for each $u_0 \in X$, the unique mild solution $u:(\tau,\tau_M(u_0)) \to X$, defined in its maximal interval, satisfies:*

1. $\tau_M(u_0)=+\infty$ *or* $\limsup_{t\to\tau_M(u_0)^-}\|u(t)\|_X=+\infty$.

2. *The solution is continuous with respect to the initial data in the following sense: given $u_0 \in X$ and $\delta>0$, there exists $\tau<t^*<\tau_M(u_0)$ sufficiently small, such that, for all $v_0 \in B_X[u_0,\delta]$, the mild solution $v(t,\tau,v_0)$ associated to v_0 is defined in $(\tau,t^*]$ and*

$$\|u(t,\tau,u_0)-v(t,\tau,v_0)\|_X \leq C(t-\tau)^{\alpha-1}\|u_0-v_0\|_X\,, t\in(\tau,t^*].$$

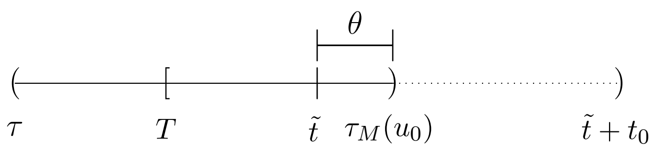
Proof. To prove the first claim, suppose $\tau_M(u_0)<\infty$. We show that $\limsup_{t\to\tau_M(u_0)}\|u(t,\tau,u_0)\|_X=\infty$. If that is not the case, then there exists $M>0$ such that $\limsup_{t\to\tau_M(u_0)}\|u(t,\tau,u_0)\|_X \leq M$. From the continuity of u in $(\tau,\tau_M(u_0))$, given any $T\in(\tau,\tau_M(u_0))$, it follows that there exists $K>0$ such that

$$\|u(t,\tau,u_0)\|_X \leq K, \quad \forall t\in[T,\tau_M(u_0))\,.$$

Let $B \subset X$ be the closed ball of radius K and centered in zero. From the uniformity in the time of existence of the mild solution obtained in Theorem 1.24, there exists $t_0 > 0$ such that, for every $v_0 \in B$, the problem

$$v_t + A(t)v = F(v), t > \tau; \quad v(\tau) = v_0,$$

admits a unique mild solution in the interval $(\tau, \tau + t_0)$.



Let $\tilde{t} = t_M(u_0) - \theta$, with $\theta = \min\left\{\frac{t_0}{4}, \frac{\tau_M(u_0)-T}{4}\right\}$. Then $\tilde{t} \in [T, \tau_M(u_0))$ and $\left\|u(\tilde{t})\right\|_X \leq K$. By taking $v_0 = u(\tilde{t})$, there exists $v : (\tau, \tau + t_0) \to X$ mild solution of

$$v_t + A(t)v = F(v) \quad t > \tau; \quad v(\tau) = u(\tilde{t}).$$

From the uniqueness of solution, $u(t + \tilde{t}, \tau, u_0) = v(\tau + t)$, for all $t \in (0, t_0)$ and we can extend u to $(\tau, t_0 + \tilde{t})$, which is larger than $(\tau, \tau_M(u_0))$, contradicting the maximality of the interval $(\tau, \tau_M(u_0))$. Therefore, $\limsup_{t \to \tau_M(u_0)} \|u(t, \tau, u_0)\|_X = \infty$.

As for the second item, we prove that solutions $u(t, \tau, u_0)$ and $v(t, \tau, v_0)$ strating arbitrarily close has a behavior similar to $U(t, \tau)(u_0 - v_0)$, that is, $\|u(t) - v(t)\|_X$ is similar to $\|U(t, \tau)(u_0 - v_0)\| \leq C(t - \tau)^{\alpha - 1}\|u_0 - v_0\|_X$, at least for t close to τ.

Let $(\tau, \tau + t_0]$ be an interval in which both u and v are defined. From (1.20), there exists $k > 0$ such that $\|u(t)\|_X \leq k(t - \tau)^{\alpha - 1}$ and $\|v(t)\|_X \leq k(t - \tau)^{\alpha - 1}$, for all $t \in (\tau, \tau + t_0]$. Let

$$\Psi(t) = \|u(t) - v(t) - U(t, \tau)(u_0 - v_0)\|_X$$

then, for every $t \in (\tau, \tau + t_0]$,

$$\begin{aligned}
\|\Psi(t)\|_X &= \|u(t) - v(t) - U(t, \tau)(u_0 - v_0)\|_X \\
&\leq \int_\tau^t \|U(t, s)\|_{\mathcal{L}(Y, X)} \|F(u(s)) - F(v(s))\|_Y \, ds \\
&\leq \int_\tau^t C(t - s)^{\beta - 1} \|u(s) - v(s)\|_X \left(1 + \|u(s)\|_X^{\rho - 1} + \|v(s)\|_X^{\rho - 1}\right) ds \\
&\leq \int_\tau^t C(t - s)^{\beta - 1} \left(1 + \|u(s)\|_X^{\rho - 1} + \|v(s)\|_X^{\rho - 1}\right) \|u(s) - v(s) - U(s, \tau)(u_0 - v_0)\|_X \, ds \\
&\quad + \int_\tau^t C(t - s)^{\beta - 1} \left(1 + \|u(s)\|_X^{\rho - 1} + \|v(s)\|_X^{\rho - 1}\right) \|U(s, \tau)(u_0 - v_0)\|_X \, ds \\
&\leq C \left\{ \int_\tau^t (t - s)^{\beta - 1} \left(1 + 2k^{\rho - 1}(s - \tau)^{-(1 - \alpha)(\rho - 1)}\right) \Psi(s) ds \right\} \\
&\quad + C \left\{ \int_\tau^t (t - s)^{\beta - 1} \left(1 + 2k^{\rho - 1}(s - \tau)^{-(1 - \alpha)(\rho - 1)}\right) (s - \tau)^{-(1 - \alpha)} ds \right\} \|u_0 - v_0\|_X
\end{aligned}$$

$$\leq C\left\{\int_\tau^t (t-s)^{\beta-1}\Psi(s)ds + \int_\tau^t (t-\tau)^{\beta-1}(s-\tau)^{-(1-\alpha)(\rho-1)}\Psi(s)ds\right\}$$
$$+ C\left\{(t-\tau)^{\alpha+\beta-1}\mathcal{B}(\beta,\alpha) + (t-\tau)^{\beta-\rho(1-\alpha)}\mathcal{B}(\beta, 1-\rho(1-\alpha))\right\}\|u_0 - v_0\|_X$$

Since all the exponents of $(t-\tau)$ in the last line are positive,

$$\Psi(t) \leq C\|u_0 - v_0\|_X + C\left\{\int_\tau^t (t-s)^{\beta-1}\Psi(s)ds + \int_\tau^t (t-\tau)^{\beta-1}(s-\tau)^{-(1-\alpha)(\rho-1)}\Psi(s)ds\right\}.$$

The functions $u(t)$ and $v(t)$ are mild solution, therefore $\|u(t) - U(t,\tau)u_0\|_X \stackrel{t\to\tau^+}{\longrightarrow} 0$ and $\|v(t) - U(t,\tau)v_0\|_X \stackrel{t\to\tau^+}{\longrightarrow} 0$. In this case, $\Psi(t)$ is bounded in $(\tau, \tau+t_0]$. For any $t \in (\tau, \tau+t_0]$, let

$$\Psi^*(t) = \sup_{s\in(\tau,t]} \|\Psi(s)\|_X.$$

We have,

$$\Psi(t) \leq C\|u_0 - v_0\|_X + C\left\{t_0^\beta + t_0^{\beta-(1-\alpha)(\rho-1)}\mathcal{B}(\beta, 1-(1-\alpha)(\rho-1)\right\}\Psi^*(\tau+t_0).$$

Taking the supreme on the left side (for $t \in (\tau, \tau+t_0])$ and choosing t_0 small enough so that $C\left\{t_0^\beta + t_0^{\beta-(1-\alpha)(\rho-1)}\mathcal{B}(\beta, 1-(1-\alpha)(\rho-1)\right\} < \frac{1}{2}$, we achieve

$$\frac{1}{2}\Psi^*(\tau+t_0] \leq C\|u_0 - v_0\|_X$$

which proves the continuous dependence.

□

CHAPTER 2

Domains with a handle: Existence of mild solution

In order to illustrate the ideas and results presented in Chapter 1, we consider the following example of a reaction-diffusion equation in a domain with a handle that will follow us through this entire work.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain formed by two disjoint components: $\Omega = \Omega_L \cup \Omega_R$, $\overline{\Omega_L} \cap \overline{\Omega_R} = \emptyset$. Attached to this Ω, consider the line segment R_0 given by $R_0 = \{(r,0) \in \mathbb{R} \times \mathbb{R}^{N-1}; r \in (0,1)\}$. We assume that Ω and R_0 are connected by the points $(0,0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and $(1,0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and that there exists a cylinder centered in the line segment R_0 that only intersects Ω in its bases, as illustrated in Figure 1.

We denote $\Omega_0 = \Omega \cup R_0$ and in this domain we consider the following one-sided coupled reaction-diffusion equation:

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1), \end{cases} \tag{2.1}$$

where $p_0 = (0,0,...,0) \in \mathbb{R}^N$ and $p_1 = (1,0,...,0) \in \mathbb{R}^N$ are the junction points between the sets Ω and R_0. We also assume that:

(A.1). $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary (C^2) formed by two smooth disjoint components: Ω_L and Ω_R, with $p_0 \in \partial\Omega_L$ and $p_1 \in \partial\Omega_R$.

(A.2). The function $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ is continuously differentiable, that is, $a \in \mathcal{C}^1(\mathbb{R} \times \overline{\Omega_0}, \mathbb{R}^+)$ and has its image in a closed interval $[a_0, a_1] \subset (0, \infty)$. Furthermore, we denote by $b(t,x) := \nabla_x a(t,x)$

the gradient function (in x) of $a(t,x)$ and we assume that $a'(t,x)$ and $b(t,x)$ are both bounded, that is, $a'(t,x), b(t,x) \in L^\infty(\Omega_0)$.

(A.3). Both functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are *Hölder continuous in the first variable* with same Hölder exponent $\delta \in (0,1]$:

$$|a(t,x)-a(s,x)| \leq C|t-s|^\delta, \quad |b(t,x)-b(s,x)| \leq C|t-s|^\delta. \tag{2.2}$$

(A.4). The nonlinearity f is continuously differentiable, $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ and satisfies a *polynomial growth condition*, that is,

$$|f'(\xi)| \leq C(1+|\xi|^{\rho-1}), \text{ for some } \rho \geq 1. \tag{2.3}$$

Remark 2.1. *To avoid misunderstands, we will fix the variables used for each situation. We will save x for the variable that takes values in Ω, r for the variable that takes values in R_0 and $t,s,\tau \in \mathbb{R}$ for variables representing a given instant of time. Note that r has the form $(z,0) \in \mathbb{R}\times\mathbb{R}^{N-1}$, with $z \in [0,1]$. As consequence, at some points, we will consider r as an element in the interval $[0,1]$ and treat $v_r(t,r)$ as the derivative of v in the real variable $r \in [0,1]$.*

Remark 2.2. *Equations* (2) *and* (2.1) *differ only by the presence of the function g. As mentioned at the Introduction, this function g comes from the geometry of the sequence of channels R_ε that collapses to R_0 as $\varepsilon \to 0^+$. In order to simplify the calculations, we assume that $g \equiv 1$. This would be the case if equation* (2.1) *was obtained as a limit case of reaction-diffusion equations $u_t - div(a(t,x)\nabla u) + u = f(u)$ in dumbbell domains $\Omega \cup R_\varepsilon$ where the channels R_ε were straight cylinders with diameter ε and collapsed to R_0 as $\varepsilon \to 0$ (see [7]).*

The phase space in which we consider this equation is given by $U_p^0 = L^p(\Omega) \times L^p(0,1)$ with norm

$$\|(w,v)\|_{U_p^0} = \|w\|_{L^p(\Omega)} + \|v\|_{L^p(0,1)}.$$

The space $(U_p^0, \|\cdot\|_{U_p^0})$ is Banach and (2.1) originates the following abstract singular semilinear evolution problem:

$$\begin{aligned} &(w,v)_t + A_0(t)(w,v) = F_0(w,v), \quad t > \tau,\\ &(w,v)(\tau) = (w_0,v_0) \in U_p^0, \end{aligned} \tag{2.4}$$

where $A_0(t) : D(A_0(t)) \subset U_p^0 \to U_p^0$ is the linear operator with time-independent domain $D = D(A_0(t))$ given by

$$D = \left\{(w,v) \in W^{2,p}(\Omega) \times W^{2,p}(0,1) : \partial_n w = 0 \text{ in } \partial\Omega \text{ and } v(p_i) = w(p_i), i = 0,1\right\}, \tag{2.5}$$

$$A_0(t)(w,v) = \left(-div(a(t,x)\nabla w) + w, -\partial_r(a(t,r)\partial_r v) + v\right), \quad \text{for } (w,v) \in D, \tag{2.6}$$

and the nonlinearity F_0 is given by

$$F_0(w,v)(x) = \begin{cases} f(w(x)), & x \in \Omega,\\ f(v(r)), & r \in R_0. \end{cases} \tag{2.7}$$

Remark 2.3. *The condition imposed at p_0 and p_1 in* (2.5) *only makes sense if $w \in C(\overline{\Omega})$. Therefore, the restriction on $p > \frac{N}{2}$ must be required at this point, which ensures that $W^{2,p}(\Omega) \hookrightarrow C(\Omega)$ [1, Theorem 5.4].*

Under those conditions, we have the following properties for the family $A_0(t), t \in \mathbb{R}$.

Lemma 2.4. *Let $A_0(t) : D \subset U_p^0 \to U_p^0$ be the family of linear operators defined in* (2.6)*. If condition* (2.2) *holds, then $\mathbb{R} \ni t \mapsto A_0(t)A_0(\tau)^{-1} \in \mathcal{L}(U_p^0)$ is Hölder continuous with exponent δ, that is,*

$$\|[A_0(t) - A_0(s)]A(\tau)^{-1}\|_{\mathcal{L}(U_p^0)} \leq C|t-s|^{\delta}, \quad \textit{for all } \tau, s, t \in \mathbb{R}.$$

Proof. For $(w, v) \in D$, we have

$$A_0(t)(w,v) - A_0(s)(w,v) = (-div\left([a(t,x) - a(s,x)]\nabla w\right), -\partial_r\left([a(t,r) - a(t,s)]\partial_r v\right))$$

and

$$\begin{aligned}
&\int_\Omega |div([a(t,x) - a(s,x)]\nabla w(x))|^p \, dx \\
&\quad = \int_\Omega |\nabla_x([a(t,x) - a(s,x)])\nabla w(x) + [a(t,x) - a(s,x)]\Delta w|^p \, dx \\
&\quad \leq (t-s)^{\delta p} \int_\Omega \left\{ \frac{|\nabla_x a(t,x) - \nabla_x a(s,x)|}{|t-s|^{\delta}} \right\}^p |\nabla w(x)|^p dx \\
&\qquad + (t-s)^{\delta p} \int_\Omega \left\{ \frac{|a(t,x) - a(s,x)|}{|t-s|^{\delta}} \right\}^p |\Delta w(x)|^p dx \\
&\quad \leq C(t-s)^{\delta p} \left\{ \|\nabla w\|^p_{L^p(\Omega)} + \|\Delta w\|^p_{L^p(\Omega)} \right\} \\
&\quad \leq C(t-s)^{\delta p} \|w\|^p_{W^{2,p}(\Omega)}.
\end{aligned}$$

The same calculation now on the line segment R_0 gives

$$\int_\Omega |\partial_r\left([a(t,r) - a(s,r)]\partial_r v(r)\right)|^p \, dx \leq C(t-s)^{\delta p} \|v\|^p_{W^{2,p}(0,1)}.$$

Therefore, $\|[A_0(t) - A_0(s)](w,v)\|^p_{U_p^0} \leq C|t-s|^{p\delta} \|(w,v)\|^p_D$, for all $(w,v) \in D$. Taking the $p-th$ roots on both sides and replacing (w,v) by $A_0(\tau)^{-1}(\tilde{w}, \tilde{v})$, we have

$$\|[A_0(t) - A_0(s)]A(\tau)^{-1}(\tilde{w},\tilde{v})\|_{U_p^0} \leq C|t-s|^{\delta} \|(\tilde{w},\tilde{v})\|_{U_p^0}, \quad \forall (\tilde{w},\tilde{v}) \in U_p^0.$$

□

In [19, Proposition 4.1] and [8, Proposition 3.1] several properties of the family $A_0(t), t \in \mathbb{R}$, are presented, including its almost sectoriality. We enunciate it in the sequel and in the last statement we provide information about the spectrum of $A_0(t)$ that can be found in [8, Section 3.2].

Proposition 2.5. *The family of linear operators $A_0(t), t \in \mathbb{R}$, satisfies:*

1. *$A_0(t)$ is a closed linear operator and it has a fixed dense domain D.*

2. *$A_0(t)$ has compact resolvent and the semigroup $T_{-A_0(t)}(s)$ is compact.*

3. *There exists $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $C > 0$ (independent of t) such that $\Sigma_\varphi \subset \rho(-A_0(t))$, for all $t \in \mathbb{R}$, and, for $\frac{N}{2} < q \leq p$, $\lambda \in \Sigma_\varphi \cup \{0\}$, we have*

$$\|(\lambda + A_0(t))^{-1}\|_{\mathcal{L}(U_q^0, U_p^0)} \leq \frac{C}{|\lambda|^\beta + 1},$$

for each $0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$. In particular, the case $p = q$ yields

$$\|(\lambda + A_0(t))^{-1}\|_{\mathcal{L}(U_p^0)} \leq \frac{C}{|\lambda|^\alpha + 1},$$
$$\|A_0(t)(\lambda + A_0(t))^{-1}\|_{\mathcal{L}(U_p^0)} \leq C(1 + |\lambda|^{1-\alpha}),$$

for $0 < \alpha < 1 - \frac{N}{2p} < 1$.

4. *The spectrum of $A_0(t)$ consists entirely of isolated eigenvalues, all of them positive and real. To be more precise, if $\{\mu_i(t)\}_{i=1}^\infty$ are the eigenvalues of*

$$\begin{aligned} &-div(a(t,x)\nabla w) + w = \lambda w, && x \in \Omega, \\ &\partial_n w = 0, && x \in \partial\Omega, \end{aligned}$$

and $\{\tau_i(t)\}_{i=1}^\infty$ the eigenvalues of

$$\begin{aligned} &-\partial_r(a(t,r)\partial_r v) + v = \lambda v, && s \in (0,1), \\ &v(0) = 0 = v(1), \end{aligned}$$

then $\sigma(A_0(t)) = \{\mu_i(t)\}_{i=1}^\infty \cup \{\tau_i(t)\}_{i=1}^\infty$. They can be arranged in a way such that

$$\sigma(A_0(t)) = \{\lambda_i(t) : 1 = \lambda_1(t) \leq \lambda_2(t) \leq ... \leq \lambda_n(t) \leq ...\}.$$

Remark 2.6. *The operator $A_0(t)$, $t \in \mathbb{R}$, given in* (2.6) *differs from the operators considered in [8, 19]. In [8] the authors work with an autonomous version of the linear operator given by $A_0(w,v) = (-\Delta + I, -\frac{d^2}{dr^2} + I)$, whereas [19] deals with a nonautonomous version $A_0(t)(w,v) = (-a(t,x)\Delta + I, -a(t,r)\frac{d^2}{dr^2} + I)$.*

Despite the difference, the proof of each statement above is exactly the same as the one presented in [8], since it only depends on the sectoriality of the operator $-\Delta + I$ in Ω, on the sectoriality of $-\frac{d^2}{dr^2} + I$ (with Dirichlet boundary condition) in R_0, and on Sobolev embeddings.

Even though the linear operators on Ω and on R_0 are sectorial, the condition at p_0 and p_1 imposes restriction on the estimate of the resolvent that culminates with $A_0(w,v) = (-\Delta + I, -\frac{d^2}{dr^2} + I)$ being almost sectorial.

Lemma 2.4 and Proposition 2.5 imply that $A_0(t), t \in \mathbb{R}$, satisfies conditions (P.1) - (P.3) posed in Chapter 1. We turn our attention to the nonlinearity f. The growth condition (2.3) and the mean value theorem imply the existence of a constant $C > 0$ such that

$$|f(\xi) - f(\psi)| \le C|\xi - \psi|(1 + |\xi|^{\rho-1} + |\psi|^{\rho-1}), \tag{2.8}$$

$$|f(\xi)| \le C(1 + |\xi|^{\rho}). \tag{2.9}$$

This polynomial growth of order ρ reflects on the operator F_0.

Lemma 2.7. *Let F_0 be the nonlinearity defined in* (2.7) *and suppose* (2.3) *is satisfied. Then F_0 take elements of U_p^0 to elements in U_q^0, that is, $F : U_p^0 \to U_q^0$, where $q = \frac{p}{\rho}$. Furthermore, for each $(w, v) \in U_p^0$, we have*

$$\|F_0(w, v) - F_0(\tilde{w}, \tilde{v})\|_{U_q^0} \le C\|(w, v) - (\tilde{w}, \tilde{v})\|_{U_p^0}(1 + \|(w, v)\|_{U_p^0}^{\rho-1} + \|(\tilde{w}, \tilde{v})\|_{U_p^0}^{\rho-1}),$$
$$\|F_0(t, (w, v))\|_{U_q^0} \le C(1 + \|(w, v)\|_{U_p^0}^{\rho}).$$

Proof. We only verify the first inequality. The second follows in a similar way. Note that

$$\|F_0(w, v) - F_0(\tilde{w}, \tilde{v})\|_{U_q^0} = \left[\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx\right]^{\frac{1}{q}} + \left[\int_0^1 |f(v(s)) - f(\tilde{v}(s))|^q ds\right]^{\frac{1}{q}}.$$

We consider the integrals separately. Firstly, we have

$$\begin{aligned}\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx &\le \int_\Omega C^q |w(x) - \tilde{w}(x)|^q (1 + |w(x)|^{q(\rho-1)} + |\tilde{w}(x)|^{q(\rho-1)}) dx \\ &\le C \left(\int_\Omega |w - \tilde{w}|^p\right)^{\frac{q}{p}} \left(\int_\Omega \left[1 + |w|^{q(\rho-1)} + |\tilde{w}|^{q(\rho-1)}\right]^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \\ &\le C \|w - \tilde{w}\|_{L^p(\Omega)}^q \left(1 + \int_\Omega |w|^{q(\rho-1)\frac{p}{p-q}} + |\tilde{w}|^{q(\rho-1)\frac{p}{p-q}}\right)^{\frac{p-q}{p}},\end{aligned}$$

we used that $q(\rho - 1)\frac{p}{p-q} = p$. Therefore

$$\begin{aligned}\left[\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx\right]^q &\le C \|w - \tilde{w}\|_{L^p(\Omega)} \left(1 + \|w\|_{L^p(\Omega)}^p + \|\tilde{w}\|_{L^p(\Omega)}^p\right)^{\frac{p-q}{pq}} \\ &\le C \|w - \tilde{w}\|_{L^p(\Omega)} \left(1 + \|w\|_{L^p(\Omega)}^{\rho-1} + \|\tilde{w}\|_{L^p(\Omega)}^{\rho-1}\right).\end{aligned}$$

For the second term, we obtain

$$\begin{aligned}&\int_0^1 |f(v(s)) - f(\tilde{v}(s))|^q ds \\ &\quad\le \int_0^1 C^q |v - \tilde{v}|^q (1 + |v|^{q(\rho-1)} + |\tilde{v}|^{q(\rho-1)}) ds \\ &\quad\le C \int_0^1 |v - \tilde{v}|^q (1 + |v|^{q(\rho-1)} + |\tilde{v}|^{q(\rho-1)}) ds\end{aligned}$$

$$\leq C\left[\int_0^1 |v-\tilde{v}|^p ds\right]^{\frac{q}{p}}\left[1+\left(\int_0^1 |v|^{q(\rho-1)\frac{p}{p-q}}ds\right)^{\frac{p-q}{p}}+\left(\int_0^1 |\tilde{v}|^{q(\rho-1)\frac{p}{p-q}}ds\right)^{\frac{p-q}{p}}\right]^{\frac{q}{p}}$$
$$\leq C\,\|v-\tilde{v}\|^q_{L^g(0,1)}\,(1+\|v\|^{p-q}_{L^p(0,1)}+\|\tilde{v}\|^{p-q}_{L^p(0,1)}).$$

Therefore,

$$\left[\int_0^1 |f(v(s))-f(\tilde{v}(s))|^q ds\right]^{\frac{1}{q}} \leq C\,\|v-\tilde{v}\|_{L^p(0,1)}\,(1+\|v\|^{\rho-1}_{L^p(0,1)}+\|\tilde{v}\|^{\rho-1}_{L^p(0,1)}).$$

Using the above inequalities,

$$\begin{aligned}
&\|F_0(w,v)-F_0(\tilde{w},\tilde{v})\|_{U^0_q}\\
&\quad\leq C(\|w-\tilde{w}\|_{L^p(\Omega)}+\|v-\tilde{v}\|_{L^p(0,1)})(1+\|w\|^{\rho-1}_{L^p(\Omega)}+\|v\|^{\rho-1}_{L^p(0,1)}+\|\tilde{w}\|^{\rho-1}_{L^p(\Omega)}+\|\tilde{v}\|^{\rho-1}_{L^p(0,1)})\\
&\quad\leq C(\|w-\tilde{w}\|_{L^p(\Omega)}+\|v-\tilde{v}\|_{L^p(0,1)})(1+\|(w,v)\|^{\rho-1}_{U^0_p}+\|(\tilde{w},\tilde{v})\|^{\rho-1}_{U^0_p})\\
&\quad\leq C\,\|(w,v)-(\tilde{w},\tilde{v})\|_{U^0_p}\,(1+\|(w,v)\|^{\rho-1}_{U^0_p}+\|(\tilde{w},\tilde{v})\|^{\rho-1}_{U^0_p}).
\end{aligned}$$

□

2.1 Local well-posedness and maximal growth

The conditions established above for the linear operators $A_0(t), t\in\mathbb{R}$, and the nonlinearity F_0 allow us to pose the problem in the same abstract setting developed in Chapter 1 and use the results presented there to obtain local well-posedness (in terms of mild solution) for (2.1).

As a starting point, in order for the problem to be well defined, we must have

$$\frac{N}{2} < q \leq p \tag{2.10}$$

(see Remark 2.3).

The phase space in which the initial data will be taken is $X = U^0_p$. In this space, the family $A_0(t), t\in\mathbb{R}$, is almost sectorial with constant of almost sectoriality α being any real number in the interval

$$0 < \alpha < 1 - \frac{N}{2p} =: \alpha^+, \tag{2.11}$$

where $\alpha^+ = 1-\frac{N}{2p}$ is the upper bound for this interval. In this case, each $-A(\tau)$ generates a semigroup of growth $1-\alpha$, $T_{-A_0(\tau)}(t)$, that satisfies the estimate $\|T_{-A_0(t)}(t)\|_{\mathcal{L}(X)} \leq Ct^{\alpha-1}$. Note that the closer α is to 1, the closer the semigroup is of being a C_0-analytic semigroup.

The nonlinearity F_0, which is known to have a growth of order ρ, will take elements of U^0_p and decrease its regularity to an element of U^0_q, where $q = \frac{p}{\rho}$.

We denote $Y = U_q^0$. Assume for now that $1 \le \rho < \rho_0$ is such that $q = \frac{p}{\rho} > \frac{N}{2}$ (later on we will calculate the range for which this situation can occur). In this case, we can consider the operator $A_0^Y(t) : D(A_0^Y(t)) \subset U_q^0 \to U_q^0$ given by

$$\begin{aligned} D^Y &= \left\{(w,v) \in W^{2,q}(\Omega) \times W^{2,q}(0,1) : \partial_n w = 0 \text{ in } \partial\Omega \text{ and } v(p_i) = w(p_i), i = 1,2\right\}, \\ A_0^Y(t)(w,v) &= (-div(a(t,x)\nabla w) + w, -\partial_r(a(t,r)\partial_r v) + v), \quad \text{for } (w,v) \in D^Y, \end{aligned} \tag{2.12}$$

and note that $D^Y \hookrightarrow U_p^0$, since $q > \frac{N}{2}$.

Taking into account Remark 1.18, we will not distinguish between $A_0(t)$ and $A_0^Y(t)$. Proposition 2.5 states that $A_0(t)$ is almost sectorial in $Y = U_q^0$ with constant of almost sectoriality ω in the interval

$$0 < \omega < 1 - \frac{N}{2q} =: \omega^+. \tag{2.13}$$

The connection between those two spaces X, Y and the family $A_0(t)$ is then established one more time via Proposition 2.5, which ensures the existence of a constant β in the interval

$$0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) =: \beta^+ \tag{2.14}$$

such that the resolvent of $-A_0(t)$ satisfies $\|(\lambda + A_0(t))^{-1}\|_{\mathcal{L}(U_q^0, U_p^0)} \le \frac{C}{|\lambda|^{\beta}+1}$. This means that the operator $(A_0(t))^{-1}$ (or any $(\lambda + A_0(t))^{-1}$ for $\lambda \in \Sigma_\varphi$) take elements in the less regular space Y back to X. In the same spirit of Example 1.19, we have the following diagram that illustrates the relation among the spaces:

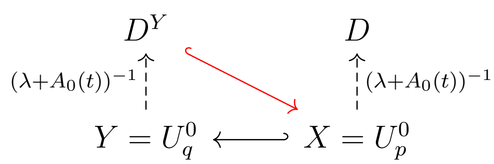


Figure 2.1: Diagram of embeddings

$\hookrightarrow$: embed $\dashrightarrow$: action of the operator

The embed $D^Y \hookrightarrow U_p^0$ is ensured by the fact that $q > \frac{N}{2}$. Moreover, the above conditions imply that the semigroup $T_{-A_0(\tau)}(t)$ satisfies the following estimates

$$\|T_{-A_0(\tau)}(t)\|_{\mathcal{L}(U_p^0)} \le Ct^{\alpha-1}, \quad \|T_{-A_0(\tau)}(t)\|_{\mathcal{L}(U_q^0, U_p^0)} \le Ct^{\beta-1},$$

with $\alpha \in (0, 1 - \frac{N}{2p})$ and $\beta \in (0, 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}))$.

The existence of a two parameter family $U_0(t,\tau)$ associated to $A_0(t), t \in \mathbb{R}$, and a local solution depends on the conditions established in Theorem 1.24 (summarized in Table 1.1).

2.1.1 Conditions associated to the existence of $U_0(t,\tau)$ and mild solution

Condition $\alpha+\delta>1$ provides a lower bound for the possible value of p, that is,

$$\alpha > 1-\delta \Leftrightarrow 1-\frac{N}{2p} > 1-\delta \Leftrightarrow p > \frac{N}{2\delta}$$

and in the same way, $\omega+\delta>1$ implies

$$q > \frac{N}{2\delta}.$$

Those two conditions are summarized in

$$\frac{N}{2\delta} < q \leq p. \tag{2.15}$$

Note that closer δ (Hölder exponent) is to zero, the harder it is to obtain existence of the process (in $\mathcal{L}(X)$ and $\mathcal{L}(Y,X)$) and, consequently, existence of mild solution. This value $\frac{N}{2\delta}$ will play an important role in future calculations for this example (to be precise, in Proposition 7.14, the fact that $\delta > \frac{N}{2q}$ will ensure estimates on the L^∞ norm of w_t, the derivative of the component in Ω of the mild solution).

Remark 2.8. *Condition* (2.15) *imposes restriction on the growth* $\rho = \frac{p}{q}$ *of* F_0. *The maximum value that* ρ *can achieve when* p *is given and* $\frac{N}{2\delta} < q \leq p$ *is*

$$\rho = \frac{p}{q} \leq \frac{p}{\frac{N}{2\delta}} = \frac{2\delta p}{N}.$$

We denote this value by

$$\rho_I = \frac{2\delta p}{N} \tag{2.16}$$

and refer to ρ_I *as the* maximal growth in order to ensure existence of $U_0(t,\tau)$, *that is,* $U_0(t,\tau)$ *exists in* $\mathcal{L}(U^0_{\frac{p}{\rho}}, U^0_p)$ *only if* F_0 *has a growth* $\rho < \rho_I$.

Remark 2.9. *As illustrate in the previous remark, any lower bound* $l = l(p)$ *for* q *creates a restriction of the type* $l(p) < q \leq p$, *which generates a maximal value for* ρ *given by* $\frac{p}{l(p)}$.

We also need to guarantee that the discontinuity at $t=\tau$ is controlled in order to obtain existence of mild solution (w,v). The next lemma provides conditions on p and q such that this is satisfied.

Lemma 2.10. *Let* $\frac{N}{2} < q \leq p$ *and* $\rho = \frac{p}{q}$. *There exist* $0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)$ *and* $0 < \alpha < 1 - \frac{N}{2p}$ *such that*

$$1 \leq \rho < \frac{\beta}{1-\alpha}$$

if and only if, for fixed $p > N$, *we have*

$$\frac{p(2N+1)}{2p+1} < q \leq p. \tag{2.17}$$

Proof. It is enough to obtain a condition on q such that $\frac{p}{q} = \rho < \frac{\beta^+}{1-\alpha^+}$, that is,

$$\frac{p}{q} < \frac{1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}{1 - \left(1 - \frac{N}{2p}\right)} \Leftrightarrow q > \frac{p(2N+1)}{2p+1}.$$

Also, the condition $\frac{p(2N+1)}{2p+1} < q \leq p$ will only make sense if $p > N$.

□

Inequality (2.17) allows us to calculate the largest growth F_0 can have so that the problem is still locally well-posed. We will denote this value as ρ_{II} and it is given by

$$\rho_{II} = \frac{p}{\frac{p(2N+1)}{2p+1}} = \frac{2p+1}{2N+1}. \tag{2.18}$$

We refer to (2.18) as *maximal growth in order to ensure existence of* (w, v).

2.1.2 Local well-posedness

The calculations established earlier are gathered in next proposition. It was compiled and presented in [12], alongside with the analysis of the maximal growth for F_0 in the sequel.

Conditions(2.10), (2.15) and (2.17) and Theorem 1.24 ensure the existence of local mild solution for the problem. Since $\delta \in (0, 1]$, (2.15) is more restrictive than (2.10). and we can state the local well-posedness as:

Proposition 2.11. *Assume that* $p > N$ *and* $\max\left\{\frac{N}{2\delta}, \frac{p(2N+1)}{2p+1}\right\} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ *satisfies (A.2) and (A.3) and* $f : \mathbb{R} \to \mathbb{R}$ *satisfies (A.4). Then* (2.4) *have a local mild solution* $(w, v)(\cdot) : (\tau, \tau_M(u_0)) \to U_p^0$ *given by*

$$(w, v)(t) = U_0(t, \tau)u_0 + \int_\tau^t U_0(t, s)F_0((w, v)(s))ds.$$

Note that it is only required to know N and δ in order to establish values of p and ρ ($\rho = \frac{p}{q}$) for which the problem can be locally solved. For instance, if $N = 3$ and $\delta = \frac{3}{4}$, we have the shaded region below that comprehends the possible values for p and ρ:

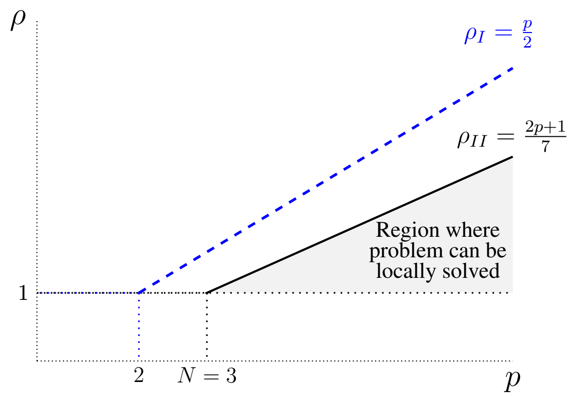


Figure 2.2: Maximal growth $N = 3$, $\delta = \frac{3}{4}$

In this case where $\delta = \frac{3}{4}$, the conditions associated to the discontinuity at $t = \tau$ are more restrictive. On the other hand, if $\delta = \frac{1}{4}$, for example, the conditions on existence of $U_0(t, \tau)$ become the ones to impose more restriction:

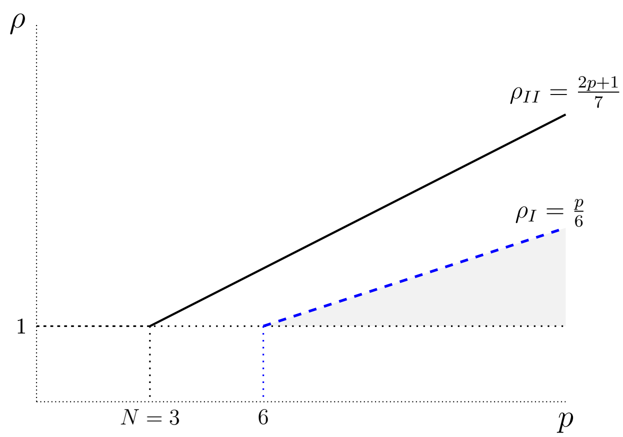


Figure 2.3: Maximal growth $N = 3$, $\delta = \frac{1}{4}$

To summarize the results in this chapter, we rewrite Table 1.1 with the relations obtained above. Note that ρ_I comes from the left column whereas ρ_{II} comes from the right one.

Local well-posedness: Domain with a handle	
I. Conditions on the existence of $U_0(t, \tau)$	II. Conditions on existence of the (w, v)
$\frac{N}{2\delta} < q \leq p$	$\frac{p(2N+1)}{2p+1} < q \leq p, \quad (p > N)$

Table 2.1: Conditions of Proposition 2.11

Part II: Regularization

Up to now, we were able to established the existence of a two parameter family of linear operators $U(t,\tau)$ associated to the family $A(t), t \in \mathbb{R}$. Using this family, a generalized notion of solution for the semilinear problem was given in (1.19), called *mild solution*.

However, the connection of the family $U(t,\tau)$ with equation (P.5) or the connection of the mild solution with the equation (P.7) is not yet established.

We dedicate this part to prove that the solutions obtained so far recovers a classical idea of solution for a partial differential equation. The mainly inspiration for the topics studied here were the work of Sobolevskii [55] and the series of papers by Tanabe [58, 59, 60]. The classical work by Henry [37] was also an inspiration in many of the underlying ideas.

The almost sectorial case and the discontinuity at the initial time $t = \tau$ that this case carries with it imply that some of the convergence arguments used in the works just mentioned fails to occur. In Chapter 4 we present a way to overcome this.

The ideas and results proved in this part were organized in two articles: [15] which is a compiled of the results developed in Chapter 3 and 4 but focused on the existence of strong solution for the autonomous case with almost sectorial operators (which was an open problem) and [14] which concerns the singularly nonautonomous case and the differentiability properties associated to the linear process $U(t,\tau)$.

CHAPTER 3

Hölder continuities

The mild solution $u(t, \tau, u_0) : (\tau, \tau_M(u_0)) \to X$ obtained in Theorem 1.24 satisfies the equation in the sense of Definition 1.22, that is, u satisfies the integral equation

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds.$$

It is of interest to know whether or not this solution u is differentiable and satisfies the equation in the usual sense. In order to obtain this differentiability, the differential quotient must be evaluated, that is,

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h}\Big\{U(t+h,\tau)u_0 - U(t,\tau)u_0 + \int_\tau^{t+h} U(t+h,s)F(u(s))ds - \int_\tau^t U(t,s)F(u(s)ds\Big\}.$$

In this chapter we present properties of Hölder continuity for several functions, with the purpose of using them in the next chapter to obtain the desired differentiability.

We call the attention for one point: whereas the mild solution is placed in the phase space X, the problem

$$u_t + A(t)u = F(u), t > \tau; \quad u(\tau) = u_0,$$

takes place in the less regular space Y, since $F : X \to Y$. Therefore, we must analyze what happens to $\left\|\frac{u(t+h)-u(t)}{h}\right\|_Y$ as $h \to 0$. This justify the results in this chapter, in which we present several of the estimates obtained in Chapter 1 for $T_{-A(\tau)}(t)$, $\varphi_1(t,\tau)$, $\Phi(t,\tau)$ and $U(t,\tau)$ in $\mathcal{L}(X)$, now for the spaces $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$. Some of those have already been introduced in Lemma 1.21.

Remark 3.1. *This type of analysis in which is necessary to obtain estimates for linear operators in $\mathcal{L}(Z_1, Z_2)$ is usual when we are dealing with scales of fractional power spaces. Usually, if $A(t)$ is sectorial, positive and generates fractional powers spaces, one has to estimate $\|e^{-A(\tau)(t)}\|_{\mathcal{L}(Z^\theta,Z^\gamma)}$, as well as $\|U(t,\tau)\|_{\mathcal{L}(Z^\theta,Z^\gamma)}$, in order to study the properties of the differential equation in which the family $A(t)$ features (see for instance [19, 21]). Those fractional power spaces and the estimates $\mathcal{L}(Z^\theta, Z^\gamma)$ would be the parallel to the spaces X, Y and the estimates $\mathcal{L}(X)$, $\mathcal{L}(Y)$ and $\mathcal{L}(Y,X)$ considered in this work (actually, this approach with X and Y incorporates the case of fractional powers, as discussed in Chapter 8).*

3.1 Estimates in $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$

Since $A(t), t \in \mathbb{R}$, acting in Y is an almost sectorial operator, all the estimates for $T_{-A(\tau)}(t)$, $\varphi_1(t,\tau)$, $\Phi(t,\tau)$ and $U(t,\tau)$ in the space $\mathcal{L}(Y)$ are the same as the estimates in $\mathcal{L}(X)$, with α replaced by ω (and a possibly correction of the constant). However, in order to obtain estimates in $\mathcal{L}(Y,X)$ a careful analysis must be done.

3.1.1 Hölder continuity of $t \mapsto T_{-A(t)}(\cdot)$ in $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$

We prove the result in Lemma 1.7 now for the spaces $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$.

Corollary 3.2. *Let $t,s \in \mathbb{R}$ and $\tau > 0$, then*

$$\|T_{-A(t)}(\tau) - T_{-A(s)}(\tau)\|_{\mathcal{L}(Y,X)} \leq C(t-s)^{\delta}\tau^{\alpha+\beta-2}, \tag{3.1}$$

$$\|T_{-A(t)}(\tau) - T_{-A(s)}(\tau)\|_{\mathcal{L}(Y)} \leq C(t-s)^{\delta}\tau^{2\omega-2}. \tag{3.2}$$

Proof. From the resolvent equality

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}(A(s) - A(t))(\lambda + A(s))^{-1}$$

we obtain

$$\begin{aligned} T_{-A(t)}(\tau) - T_{-A(s)}(\tau) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}[(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}]d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}[(\lambda + A(t))^{-1}(A(s) - A(t))(\lambda + A(s))^{-1}]d\lambda. \end{aligned}$$

Parameterizing the branch of Γ with positive imaginary part by $\lambda = re^{i\varphi}$, $r \in [0,\infty)$, where $\varphi \in (\frac{\pi}{2}, \pi)$, and doing the analogous for the negative imaginary part, it follows from the estimates on item (1) of Lemma 1.21 that

$$\begin{aligned} &\|T_{-A(t)}(\tau) - T_{-A(s)}(\tau)\|_{\mathcal{L}(Y,X)} \\ &\leq C\int_\Gamma |e^{\lambda\tau}|\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)}\|[A(t) - A(s)]A(s)^{-1}\|_{\mathcal{L}(X)}\|A(s)(\lambda + A(s))^{-1}\|_{\mathcal{L}(Y,X)}d|\lambda| \\ &\leq C\int_0^\infty e^{r\tau\cos\varphi}\frac{1}{r^\alpha}(t-s)^\delta(r^{1-\beta})dr \\ &\leq C(t-s)^\delta\tau^{\alpha+\beta-2}\Gamma(2-\alpha-\beta) \end{aligned}$$

and in the last inequality we used the fact that $\cos\varphi \in (-1,0)$. Inequality (3.2) follows exactly as (1.13) with α replaced by ω. □

Remark 3.3. *The appearance of β and α at the same time in* (3.1) *is caused by the chaining of the two norms $\|[A(t) - A(s)]A(s)^{-1}\|_{\mathcal{L}(X)}$ and $\|A(s)(\lambda + A(s))^{-1}\|_{\mathcal{L}(Y,X)}$.*

3.1.2 Hölder continuity of $h \mapsto T_{-A(\tau)}(t+h)$ in $\mathcal{L}(Y)$

In Lemma 1.6 it was established that, for $\tau \in \mathbb{R}$, $t > 0$ and $h > 0$, given any $0 < \mu < \alpha^2$,

$$\|T_{-A(\tau)}(t+h) - T_{-A(\tau)}(t)\|_{\mathcal{L}(X)} \leq Ch^{\mu} t^{\alpha-1-\frac{\mu}{\alpha}}.$$

By simply replacing α to ω, we can restate the result in $\mathcal{L}(Y)$.

Lemma 3.4. *Let $T_{-A(\tau)}(t)$, $t > 0$, be the semigroup obtained by $-A(\tau)$. Given any $0 < \nu < \omega^2$, we have*

$$\|T_{-A(\tau)}(t+h) - T_{-A(\tau)}(t)\|_{\mathcal{L}(Y)} \leq Ch^{\nu} t^{\omega-1-\frac{\nu}{\omega}}, \quad h > 0,$$

and $\omega - 1 - \frac{\nu}{\omega} \in (-1, 0)$.

Remark 3.5. *In Lemmas 1.6 and 3.4, μ and ν are auxiliary constants that establish a range of possible estimates for the difference $T_{-A(\tau)}(t+h) - T_{-A(\tau)}(t)$. They feature in several moments in this chapter and they play an essential role in Chapter 4, especially when studying the existence of regular solutions for the nonautonomous linear equation $u_t + A(t)u = G(t)$.*

There is a certain trade-off in choosing the value of those constants. For example, if μ in Lemma 1.6 is close to α^2, the exponent of Hölder continuity in h^{μ} is close to its maximum, but that causes the exponent of $t^{\alpha-1-\frac{\mu}{\alpha}}$ to be more negative. On the other hand, if μ is close to zero, the blow-up at $t = 0$ is more controlled, but we also have a smaller Hölder exponent for h.

The optimal choice for the value of μ will depend on other features of the problem being considered.

3.2 Hölder continuity of $t \mapsto F(u(t)) \in Y$

Assume that all the conditions on Theorem 1.24 are satisfied, that is, $A(t)$, $t \in \mathbb{R}$ is $\alpha-$uniformly almost sectorial, $\delta-$uniformly Hölder continuous, satisfies (P.4), $\alpha + \delta > 1$, $\omega + \delta > 1$ and $F : X \to Y$ is a nonlinearity satisfying (G), with $1 \leq \rho < \frac{\beta}{1-\alpha}$.

Then there exists a local mild solution $u : (\tau, \tau_M(u_0)) \to X$ for the semilinear problem (P.7). Let $(\tau, \tau_M(u_0)) \ni t \mapsto F(u(t)) \in Y$. We prove in this section that under certain conditions, this map is Hölder continuous. Before we do that, we need the following technical result:

Proposition 3.6. *Let $\tau < t$ and $h > 0$. Then, for any $0 \leq \eta < \omega + \delta - 1$, we have*

$$\|\varphi_1(t+h, \tau+h) - \varphi_1(t, \tau)\|_{\mathcal{L}(Y)} \leq Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta},$$
$$\|\Phi(t+h, \tau+h) - \Phi(t, \tau)\|_{\mathcal{L}(Y)} \leq Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta}.$$

Proof. Note that

$$\begin{aligned}&\varphi_1(t+h, \tau+h) - \varphi_1(t, \tau)\\&\quad = [A(\tau+h) - A(t+h)]T_{-A(\tau+h)}(t-\tau) - [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau)\end{aligned}$$

$$
\begin{aligned}
&= [A(\tau+h) - A(t+h) - A(\tau) + A(t)]T_{-A(\tau+h)}(t-\tau) \\
&\quad + [A(\tau) - A(t)][T_{-A(\tau+h)}(t-\tau) - T_{-A(\tau)}(t-\tau)] \\
&= [A(\tau+h) - A(\tau)]A(\xi)^{-1}A(\xi)T_{-A(\tau+h)}(t-\tau) \\
&\quad + [A(t) - A(t+h)]A(\xi)^{-1}A(\xi)T_{-A(\tau+h)}(t-\tau) \\
&\quad + [A(\tau) - A(t)]A(\xi)^{-1}A(\xi)[T_{-A(\tau+h)}(t-\tau) - T_{-A(\tau)}(t-\tau)].
\end{aligned}
$$

The differences above are estimated in $\mathcal{L}(Y,X)$ by $Ch^{\delta}(t-\tau)^{\omega-2}$, $Ch^{\delta}(t-\tau)^{\omega-2}$ and $C(t-\tau)^{\delta}h^{\delta}(t-\tau)^{2\omega+\delta-2}$, respectively, the last one follows from (3.2). Therefore,

$$
\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \le Ch^{\delta}(t-\tau)^{\omega-2}.
$$

On the other hand, this difference can be estimated by

$$
\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \le c(t-\tau)^{\omega+\delta-2}.
$$

Interpolating those two estimates with exponents $\frac{\eta}{\delta}$ and $1-\frac{\eta}{\delta}$, for $0 \le \eta < \delta$, we obtain

$$
\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \le Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta}.
$$

The last assertion follows from

$$
\begin{aligned}
&\|\Phi(t+h,\tau+h) - \Phi(t,\tau)\|_{\mathcal{L}(Y)} \\
&\quad \le \left\| \varphi_1(t+h,\tau+h) - \varphi_1(t,\tau) + \int_{\tau+h}^{t+h} \varphi_1(t+h,s)\Phi(s,\tau+h)ds - \int_{\tau}^{t} \varphi_1(t,s)\Phi(s,\tau)ds \right\|_{\mathcal{L}(Y)} \\
&\quad \le Ch^{\eta}(t-\tau)^{(\delta-\eta)-1} + \left\| \int_{\tau}^{t} [\varphi_1(t+h,s+h) - \varphi(t,s)]\Phi(s+h,\tau+h)ds \right\|_{\mathcal{L}(Y)} \\
&\qquad + \left\| \int_{\tau}^{t} \varphi_1(t,s)[\Phi(s+h,\tau+h) - \Phi(s,\tau)]ds \right\|_{\mathcal{L}(Y)} \\
&\le Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta} + C\int_{\tau}^{t} h^{\eta}(t-s)^{\omega+\delta-2-\eta}(s-\tau)^{\omega+\delta-2}ds \\
&\qquad + C\int_{\tau}^{t} (t-s)^{\omega+\delta-2}\|\Phi(s+h,\tau+h) - \Phi(s+h,\tau+h)\|_{\mathcal{L}(Y)}ds \\
&\le Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta} + C\int_{\tau}^{t} (t-s)^{\omega+\delta-2}\|\Phi(s+h,\tau+h) - \Phi(s,\tau)\|_{\mathcal{L}(Y)}ds.
\end{aligned}
$$

Applying Gronwall's inequality (Lemma P.7), we obtain

$$
\|\Phi(t+h,\tau+h) - \Phi(t,\tau)\|_{\mathcal{L}(X)} \le Ch^{\eta}(t-\tau)^{\omega+\delta-2-\eta}.
$$

□

With those properties for the families $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$ and the estimates obtained earlier, the Hölder continuity of $t \mapsto F(u(t))$ follows, and this property is essential to derive the regularity of the solution. However the almost sectoriality, given by α and ω causes several restrictions on the values of ρ in terms of α, β, ω. This is a consequence of the initial blow-up $(t-\tau)^{\alpha-1}$ that the solution has, which is amplified by the growth of F.

Proposition 3.7. *Assume that conditions of Theorem 1.24 are satisfied and let $u : (\tau, \tau_M(u_0)) \to X$ be the local mild solution obtained for the semilinear problem given by*

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s)), \quad t \in (\tau, \tau_M(u_0)).$$

If $0 < \mu < \min\{\alpha^2, \omega + \delta - 1\}$ and $1 \leq \rho < \min\{\frac{\beta}{1-\alpha} + 1, \frac{\delta}{1-\alpha}, \frac{\alpha-\mu}{\alpha(1-\alpha)}\}$, then

$$\|F(u(t+h)) - F(u(t))\|_Y \leq C h^{\min\{\mu, 1-\rho(1-\alpha)\}} (t-\tau)^{\min\left\{-\frac{\mu}{\alpha}, \delta-1, \beta-\alpha\right\}} (t-\tau)^{-\rho(1-\alpha)},$$

for any $t > \tau$ and $h > 0$. Moreover, the exponents for $(t-\tau)$ given by

$$\min\left\{-\tfrac{\mu}{\alpha} - \rho(1-\alpha), \delta - 1 - \rho(1-\alpha), \beta - \alpha - \rho(1-\alpha)\right\},$$

belong to $(-1, \infty)$.

Proof. From the growth condition on F we obtain

$$\|F(u(t+h)) - F(u(t))\|_Y \leq C\|u(t+h) - u(t)\|_X \left(1 + \|u(t)\|_X^{\rho-1} + \|u(t+h)\|_X^{\rho-1}\right).$$

We already know from (1.20) that there exists a constant k such that,

$$\|u(t+h)\|_X, \|u(t)\|_X \leq k(t-\tau)^{\alpha-1}.$$

Taking this into account, we obtain

$$\begin{aligned} &\|F(u(t+h)) - F(u(t))\|_Y \leq C\|u(t+h) - u(t)\|_X (t-\tau)^{-(\rho-1)(1-\alpha)} \\ &(t-\tau)^{(\rho-1)(1-\alpha)} \|F(u(t+h)) - F(u(t))\|_Y \leq C\|u(t+h) - u(t)\|_X. \end{aligned} \tag{3.3}$$

Let $\Psi(t) = (t-\tau)^{(\rho-1)(1-\alpha)} \|F(u(t+h)) - F(u(t))\|_Y$. Inequality (3.3) is rewritten as

$$\Psi(t) \leq C\|u(t+h) - u(t)\|_X. \tag{3.4}$$

We study in the sequel properties of the difference $\|u(t+h) - u(t)\|_X$ in order to obtain the desired result. Before we attend to it, let us point out that $\|F(u(t))\|_Y$ can be locally estimated from the local estimate we have for $u(t)$:

$$\|F(u(t))\|_Y \leq C\left(1 + \|u(t)\|_X^\rho\right) \leq C(1 + (t-\tau)^{-\rho(1-\alpha)}) \leq C(t-\tau)^{-\rho(1-\alpha)}.$$

From the variation of constant formula, we obtain

$$\begin{aligned} u(t+h)-u(t) &= U(t+h,\tau)u_0 - U(t,\tau)u_0 + \left(\int_\tau^{\tau+h} + \int_{\tau+h}^{t+h}\right) U(t+h,s)F(u(s))ds \\ &\quad - \int_\tau^t U(t,s)F(u(s))ds \\ &= [U(t+h,\tau)-U(t,\tau)]u_0 + \int_\tau^{\tau+h} U(t+h,s)F(u(s))ds \\ &\quad + \int_\tau^t [U(t+h,s+h)F(u(s+h)) - U(t,s)F(u(s))]ds \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned}$$

(see Remark 3.9 below for a discussion over the way we grouped the terms above).

We estimate each of the terms above in $\|\cdot\|_X$. From Proposition 1.17 and for any $0<\mu<\alpha^2$, we obtain

$$\|U(t+h,\tau)u_0 - U(t,\tau)u_0\|_X \le Ch^\mu \left[(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}} + (t-\tau)^{\alpha+\delta-2}\right].$$

The second term follows the same idea,

$$\begin{aligned} \left\|\int_\tau^{\tau+h} U(t+h,s)F(u(s))ds\right\|_X &\le \int_\tau^{\tau+h} \|U(t+h,s)\|_{\mathcal{L}(Y,X)}\|F(u(s))\|_Y ds \\ &\le C\int_\tau^{\tau+h} (t+h-s)^{\beta-1}(s-\tau)^{-\rho(1-\alpha)}ds \\ &\le Ch^{1-\rho(1-\alpha)}(t-\tau)^{\beta-1}. \end{aligned}$$

The last term requires more reasoning, as we see in the sequel. To perform the necessary calculations, we use formulation (1.14) for the process:

$$\begin{aligned} \mathcal{I}_3 &= \int_\tau^t U(t+h,s+h)F(u(s+h)) - U(t,s)F(u(s))ds \\ &= \int_\tau^t \left\{T_{-A(s+h)}(t-s)F(u(s+h)) - T_{-A(s)}(t-s)F(u(s))\right\}ds \\ &\quad + \int_\tau^t \left\{\int_{s+h}^{t+h} T_{-A(\xi)}(t+h-\xi)\Phi(\xi,s+h)F(u(s+h))d\xi\right\}ds \\ &\quad - \int_\tau^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)F(u(s))d\xi\right\}ds. \end{aligned}$$

Adding and subtracting $T_{-A(s)}(t-s)F(u(s+h))$ inside the integral on the first term and performing a simple change of variable in the second integral, we obtain

$$\mathcal{I}_3 = \int_\tau^t \left\{[T_{-A(s+h)}(t-s) - T_{-A(s)}(t-s)]F(u(s+h))\right\}ds$$

$$
\begin{aligned}
&+\int_\tau^t \left\{T_{-A(s)}(t-s)[F(u(s+h))-F(u(s))]\right\} ds \\
&+\int_\tau^t \left\{\int_s^t T_{-A(\xi+h)}(t-\xi)\Phi(\xi+h,s+h)F(u((s+h))d\xi\right\} ds \\
&-\int_\tau^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)F(u(s))d\xi\right\} ds \\
=&\int_\tau^t \left\{[T_{-A(s+h)}(t-s)-T_{-A(s)}(t-s)]F(u(s+h))\right\} ds \\
&+\int_\tau^t \left\{T_{-A(s)}(t-s)[F(u(s+h))-F(u(s))]\right\} ds \\
&+\int_\tau^t \left\{\int_s^t [T_{-A(\xi+h)}(t-\xi)-T_{-A(\xi)}(t-\xi)]\Phi(\xi+h,s+h)F(u(s+h))d\xi\right\} ds \\
&+\int_\tau^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)[\Phi(\xi+h,s+h)-\Phi(\xi,s)]F(u(s+h))d\xi\right\} ds \\
&+\int_\tau^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)[F(u(s))-F(u(s+h))]d\xi\right\} ds \\
=&\,\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5.
\end{aligned}
$$

We estimate each item separately due to their particularity. Initially, recall that $1 \leq \rho < \frac{\beta}{1-\alpha}$, which implies $\alpha+\beta-1>0$. From (3.1), we obtain

$$
\begin{aligned}
\|\mathcal{S}_1\|_X \leq &\int_\tau^t \|T_{-A(s+h)}(t-s)-T_{-A(s)}(t-s)\|_{\mathcal{L}(Y,X)}\|F(u(s+h))\|_Y ds \\
&C\int_\tau^t h^\delta (t-s)^{\alpha+\beta-2}(s-\tau)^{-\rho(1-\alpha)} ds \\
&\leq Ch^\delta (t-\tau)^{(\alpha-1)+(\beta-\rho(1-\alpha))}.
\end{aligned}
$$

For $\mathcal{S}_2$, using item (2) of Lemma 1.21, we have

$$
\begin{aligned}
\|\mathcal{S}_2\|_X &\leq \int_\tau^t \|T_{-A(s)}(t-s)\|_{\mathcal{L}(Y,X)}\|F(u(s+h))-F(u(s))\|_Y ds \\
&\leq C\int_\tau^t (t-s)^{\beta-1}\|F(u(s+h))-F(u(s))\|_Y ds \\
&\leq C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s) ds.
\end{aligned}
$$

Term $\mathcal{S}_3$ also follows from (3.1) and condition $\alpha+\beta-1>0$ (as in $\mathcal{S}_1$), as well as the estimate for $\Phi(\cdot,\cdot)$ given in item (3) of Lemma 1.21

$$
\begin{aligned}
\|\mathcal{S}_3\|_X &\leq \int_\tau^t \left\{\int_s^t \|[T_{-A(\xi+h)}(t-\xi)-T_{-A(\xi)}(t-\xi)\|_{\mathcal{L}(Y,X)}\|\Phi(\xi+h,s+h)\|_{\mathcal{L}(Y)}\|F(u(s+h))\|_Y d\xi\right\} ds \\
&\leq C\int_\tau^t \left\{\int_s^t h^\delta (t-\xi)^{\alpha+\beta-2}(\xi-s)^{\omega+\delta-2} d\xi\right\}(s-\tau)^{-\rho(1-\alpha)} ds
\end{aligned}
$$

$$\begin{aligned}
&\leq C\int_\tau^t h^\delta (t-s)^{(\alpha+\beta-1)+(\omega+\delta-1)-1}(s-\tau)^{-\rho(1-\alpha)}ds\\
&\leq Ch^\delta (t-\tau)^{(\alpha+\beta-1)+(\omega+\delta-1)-\rho(1-\alpha)}.
\end{aligned}$$

From Proposition 3.6, for any $0 \leq \eta < \omega+\delta-1$, we obtain

$$\begin{aligned}
\|\mathcal{S}_4\|_X &\leq \int_\tau^t \left\{\int_s^t \|T_{-A(\xi)}(t-\xi)\|_{\mathcal{L}(Y,X)}\|[\Phi(\xi+h,s+h)-\Phi(\xi,s)]\|_{\mathcal{L}(Y)}\|F(u(s+h))\|_Y d\xi\right\} ds\\
&\leq C\int_\tau^t \left\{\int_s^t (t-\xi)^{\beta-1}h^\eta(\xi-s)^{\omega+\delta-2-\eta}d\xi\right\}(s-\tau)^{-\rho(1-\alpha)}ds\\
&\leq Ch^\eta\int_\tau^t (t-s)^{\beta+(\omega+\delta-2-\eta)}(s-\tau)^{-\rho(1-\alpha)}ds\\
&\leq Ch^\eta(t-\tau)^{(\omega+\delta-1-\eta)+(\beta-\rho(1-\alpha))}.
\end{aligned}$$

For the last term, note that items (2) and (3) of Lemma 1.21 imply

$$\begin{aligned}
\|\mathcal{S}_5\|_X &\leq \int_\tau^t \left\{\int_s^t \|T_{-A(\xi)}(t-\xi)\|_{\mathcal{L}(Y,X)}\|\Phi(\xi,s)\|_{\mathcal{L}(Y)}\|[F(u(s))-F(u(s+h))]\|_Y d\xi\right\} ds\\
&\leq C\int_\tau^t \left\{\int_s^t (t-\xi)^{\beta-1}(\xi-s)^{\omega+\delta-2}d\xi\right\}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds\\
&\leq C\int_\tau^t (t-s)^{(\beta+\omega+\delta-1)-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.
\end{aligned}$$

Using the above estimates and selecting the terms with smallest exponents for h and the most negative for $(t-\tau)$, we obtain

$$\begin{aligned}
\|\mathcal{I}_3\|_X \leq\; &Ch^\delta(t-\tau)^{(\alpha-1)+(\beta-\rho(1-\alpha))}+Ch^\delta(t-\tau)^{(\alpha-1)+(\beta-\rho(1-\alpha))+(\omega+\delta-1)}\\
&+Ch^\eta(t-\tau)^{(\omega+\delta-1-\eta)+(\beta-\rho(1-\alpha))}\\
&+C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds\\
&+C\int_\tau^t (t-s)^{\beta-1+(\omega+\delta-1)}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds\\
\leq\; &Ch^\eta(t-\tau)^{(\alpha-1)+(\beta-\rho(1-\alpha))}+C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.
\end{aligned}$$

From $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, we conclude that, for any $0<\mu<\alpha^2$ and $0<\eta<\omega+\delta-1$,

$$\begin{aligned}
\|u(t+h)-u(t)\|_X \leq\; &Ch^\mu\left[(t-\tau)^{\alpha-1-\frac{\mu}{\alpha}}+(t-\tau)^{\alpha+\delta-2}\right]\\
&+Ch^{1-\rho(1-\alpha)}(t-\tau)^{\beta-1}\\
&+Ch^\eta(t-\tau)^{(\alpha-1)+(\beta-\rho(1-\alpha))}\\
&+C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds
\end{aligned}$$

$$\leq Ch^{\min\{\mu,1-\rho(1-\alpha),\eta\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\alpha+\delta-2,\beta-1\right\}} + C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.$$

and we used above $\alpha+\delta-2<\alpha-1+(\beta-\rho(1-\alpha))$ to group the terms. Indeed,

$$\alpha+\delta-2=\alpha-1+(\delta-1)<\alpha-1+(\beta-\rho(1-\alpha)),$$

since $\delta-1<0<\beta-\rho(1-\alpha)$.

Therefore, replacing the estimate obtained in (3.4), we obtain

$$\Psi(t)\leq Ch^{\min\{\mu,1-\rho(1-\alpha),\eta\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\alpha+\delta-2,\beta-1\right\}}+C\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-(\rho-1)(1-\alpha)}\Psi(s)ds.$$

In order to apply the generalized version of Gronwall inequality given in Lemma P.8, the following inequalities must hold:

$$\alpha-\frac{\mu}{\alpha}-(\rho-1)(1-\alpha)>0\Rightarrow\rho<\frac{\alpha-\mu}{\alpha(1-\alpha)},$$
$$\alpha+\delta-1-(\rho-1)(1-\alpha)>0\Rightarrow\rho<\frac{\delta}{1-\alpha},$$
$$\beta-(\rho-1)(1-\alpha)>0\Rightarrow\rho<\frac{\beta}{1-\alpha}+1.$$

In this case,

$$\Psi(t)\leq Ch^{\min\{\mu,1-\rho(1-\alpha),\eta\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\alpha+\delta-2,\beta-1\right\}},$$
$$\|F(u(t+h))-F(u(t))\|_Y\leq Ch^{\min\{\mu,1-\rho(1-\alpha),\eta\}}(t-\tau)^{\min\left\{\alpha-1-\frac{\mu}{\alpha},\alpha+\delta-2,\beta-1\right\}}(t-\tau)^{-(\rho-1)(1-\alpha)},$$
$$\|F(u(t+h))-F(u(t))\|_Y\leq Ch^{\min\{\mu,1-\rho(1-\alpha),\eta\}}(t-\tau)^{\min\left\{-\frac{\mu}{\alpha},\delta-1,\beta-\alpha\right\}}(t-\tau)^{-\rho(1-\alpha)},$$

and we can group the exponents μ and η together, requiring that $0<\mu<\min\{\alpha^2,\omega+\delta-1\}$. □

Remark 3.8. *If $A(t)$ were sectorial, then $\alpha=\omega=1$, all the conditions above would be trivially satisfied and $t\mapsto F(u(t))$ would be $\mu-$Hölder continuous for any $0\leq\mu<\min\{1^2,1+\delta-1\}=\delta$. This agrees with the results on [55, (2.84)]*

Remark 3.9. *The way we separate the integral $\int_\tau^{t+h}=\int_\tau^{\tau+h}+\int_{\tau+h}^{t+h}$ differs from the way we did in the previous results, as $\int_\tau^{t+h}=\int_\tau^t+\int_t^{t+h}$. This is done in order to force the difference $F(u(t+h))-F(u(t))$ to appear, which is useful later at the moment we apply Gronwall's inequality. The same procedure is used for the nonsingular and sectorial case ($A(t)=A$), but in that case the analysis is much simpler once $U(t+h,s+h)=T_{-A}(t+h-s-h)=T_{-A}(t-s)$. The existence of strong solution for the autonomous semilinear equations with almost sectorial is compiled and presented it in [15].*

Remark 3.10. *To avoid the cumbersomeness of exponents that features Proposition 3.7, we will use the following notation: a function $\phi \in \mathcal{C}^{\theta}_{\psi}((\tau,T),Y)$ for $\theta,\psi > 0$ if it is locally Hölder continuous with exponent θ and its norm close to $t=\tau$ satisfies (for small values of h)*

$$\|\phi(t+h)-\phi(t)\|_Y \le Ch^{\theta}(t-\tau)^{-\psi}.$$

In this case,

$$F(u(t)) \in \mathcal{C}^{\theta}_{\psi}((\tau,T),Y),$$

where $0<\theta<\min\{\mu, 1-\rho(1-\alpha)\}$, for any $0<\mu<\min\{\alpha^2, \omega+\delta-1\}$ and

$$-\psi = \min\left\{-\tfrac{\mu}{\alpha}-\rho(1-\alpha), \delta-1-\rho(1-\alpha), \beta-\alpha-\rho(1-\alpha)\right\}.$$

For now, we do not choose a specific value for μ. We will postpone this choice to the Chapter 4, where a minimum exponent of Hölder continuity for $F(u(t))$ will be required.

3.3 Hölder continuity of $\mathbb{R} \ni t \mapsto \varphi_1(t,\cdot)$ and $\mathbb{R} \ni t \mapsto \Phi(t,\cdot)$

At the results in the previous sections we saw how to obtain a Hölder continuity by exploiting properties of the evolution in time of the semigroup, that is, $h \mapsto T_{-A(\tau)}(t+h)$, with $t>0$. Now we study how the Hölder continuity of the family $A(t), t\in\mathbb{R}$, can be used to obtain certain estimates.

We already did such analysis at Lemma 1.7, where the Hölder continuity of the family $A(t), t\in\mathbb{R}$, implied

$$\|T_{-A(t)}(\tau)-T_{-A(s)}(\tau)\|_{\mathcal{L}(X)} \le C\tau^{-2+2\alpha}(t-s)^{\delta}, \quad \tau>0.$$

In the next two lemmas we use a similar idea to prove how the Hölder continuity of $A(t)$, $t\in\mathbb{R}$, reflects on the maps $\mathbb{R}\ni t\mapsto \varphi_1(t,\cdot)$ and $\mathbb{R}\ni t\mapsto\Phi(t,\cdot)$, defined in (1.15) and (1.16), respectively.

Remark 3.11. *Note how the results in the sequel differ from the results in Proposition 3.6. In the last one we estimated the difference $\varphi(t+h,\tau+h)-\varphi(t,\tau)$ when both initial and final time suffered and increase of $h>0$. Now, we fix the initial time and evaluate the difference $\varphi(t+h,\tau)-\varphi(t,\tau)$.*

Lemma 3.12. *Given any $0<\eta<\alpha(\alpha+\delta-1)$, $\tau<\theta<t$,*

$$\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)} \le C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}. \tag{3.5}$$

Furthermore, $\alpha+\delta-2-\frac{\eta}{\alpha}\in(-1,0)$.

Proof. From Lemma 1.12, it follows that

$$\begin{aligned}
&\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)}\\
&\quad\le \|[A(\tau)-A(t)]T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)} + \|[A(\tau)-A(\theta)]T_{-A(\tau)}(\theta-\tau)\|_{\mathcal{L}(X)}\\
&\quad\le (t-\tau)^{\delta+\alpha-2}+C(\theta-\tau)^{\delta+\alpha-2} \le C(\theta-\tau)^{\delta+\alpha-2}
\end{aligned}$$

and we have a first estimate for the difference that only acknowledges the blow-up close to the initial time τ:

(i). $\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)} \le C(\theta-\tau)^{\delta+\alpha-2}$.

On the other hand, by adding and subtracting $A(\theta)T_{-A(\tau)}(t-\tau)$ at the difference, we deduce

$$\begin{aligned}
&\varphi_1(t,\tau)-\varphi(\theta,\tau)\\
&\quad=[A(\tau)-A(t)]T_{-A(\tau)}(t-\tau)-[A(\tau)-A(\theta)]T_{-A(\tau)}(\theta-\tau)\\
&\quad=-[A(t)-A(\theta)]T_{-A(\tau)}(t-\tau)-[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)].
\end{aligned}\tag{3.6}$$

Note that the first term of (3.6) can be estimated by

$$\begin{aligned}
\big\|[A(t)-A(\theta)]T_{-A(\tau)}(t-\tau)\big\|_{\mathcal{L}(X)} &\le \big\|[A(t)-A(\theta)](A(\tau))^{-1}A(\tau)T_{-A(\tau)}(t-\tau)\big\|_{\mathcal{L}(X)}\\
&\le C(t-\theta)^{\delta}(t-\tau)^{\alpha-2},
\end{aligned}\tag{3.7}$$

and a positive power of $(t-\theta)$ emerges.

As for the second term an immediate estimate would be

$$\begin{aligned}
&\big\|[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)]}(\theta-\tau)\big\|_{\mathcal{L}(X)}\\
&\quad\le C(\theta-\tau)^{\delta}(t-\tau)^{\alpha-2}+C(\theta-\tau)^{\delta}(\theta-\tau)^{\alpha-2}\\
&\quad\le C(\theta-\tau)^{\alpha+\delta-2}.
\end{aligned}\tag{3.8}$$

Therefore, $[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}]$ is a bounded operator. We will provide an alternative estimate for this operator, one that features the difference $(t-\theta)$ with a positive exponent.

From Lemma 1.5, if $x\in D^2$, then $\xi\mapsto T_{-A(\tau)}(\xi)A(\tau)T_{-A(\tau)}(\theta-\tau)x$ is continuously differentiable in $[0,\infty)$, with derivative equals to $-T_{-A(\tau)}(\xi)A(\tau)^2T_{-A(\tau)}(\theta-\tau)x$. Hence, for any $x\in D^2$,

$$\begin{aligned}
&[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)]x\\
&\quad=[A(\theta)-A(\tau)]A(\tau)^{-1}[A(\tau)T_{-A(\tau)}(t-\tau)-A(\tau)T_{-A(\tau)}(\theta-\tau)]x\\
&\quad=[A(\theta)-A(\tau)]A(\tau)^{-1}\left\{[T_{-A(\tau)}(t-\theta)-I]A(\tau)T_{-A(\tau)}(\theta-\tau)]x\right\}\\
&\quad=[A(\theta)-A(\tau)]A(\tau)^{-1}\int_0^{t-\theta}\frac{d}{dt}\left\{T_{-A(\tau)}(\xi)A(\tau)T_{-A(\tau)}(\theta-\tau)x\right\}d\xi\\
&\quad=-[A(\theta)-A(\tau)]A(\tau)^{-1}\int_0^{t-\theta}T_{-A(\tau)}(\xi)A(\tau)^2T_{-A(\tau)}(\theta-\tau)xd\xi.
\end{aligned}$$

We obtain from (1.2), (1.6) and (1.9)

$$\begin{aligned}
&\|[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)]x\|_X\\
&\quad\le\|[A(\theta)-A(\tau)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\left\{\int_0^{t-\theta}\|T_{-A(\tau)}(\xi)\|_{\mathcal{L}(X)}d\xi\right\}\|A(\tau)^2T_{-A(\tau)}(\theta-\tau)x\|_X\\
&\quad\le C(\theta-\tau)^{\delta}\left\{\int_0^{t-\theta}\xi^{\alpha-1}d\xi\right\}(\theta-\tau)^{\alpha-3}\|x\|_X\\
&\quad\le C(t-\theta)^{\alpha}(\theta-\tau)^{\alpha+\delta-3}\|x\|_X.
\end{aligned}\tag{3.9}$$

The positive power of $(t-\theta)$ appeared, but at the downside $(\theta-\tau)$ has an exponent in the negative interval $(-2,-1)$, which is not fitted when convergence of integrals is being considered. If we interpolate the estimates (3.8) and (3.9) with the exponents $\psi \in [0,1]$ and $(1-\psi)$, we obtain

$$\|[A(\theta)-A(\tau)][T_{-A(\tau)}(t-\tau)-T_{-A(\tau)}(\theta-\tau)]\|_{\mathcal{L}(X)} \leq C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}. \tag{3.10}$$

Therefore, (3.7) and (3.10) implies

$$\begin{aligned}\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)} &\leq C(t-\theta)^{\delta}(t-\tau)^{\alpha-2}+C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}\\ &\leq C[(t-\theta)^{\delta}+(t-\theta)^{\alpha\psi}][(\theta-\tau)^{\alpha-2}+(\theta-\tau)^{\alpha-2+\delta-\psi}].\end{aligned} \tag{3.11}$$

Note that if ψ approaches 1, we have larger exponents for $(t-\theta)$, whereas $\alpha-2+\delta-\psi$ decreases. However, the improvement on the first term cannot exceed the power δ. Therefore, it is pointless to consider any $\psi > \frac{\delta}{\alpha}$, since it will not cause any improvement in the Hölder exponent of $(t-\theta)$. We assume $\psi \leq \frac{\delta}{\alpha}$ and rewrite (3.11), for any $\psi \in \left[0, \max\left\{1, \frac{\delta}{\alpha}\right\}\right]$, as

$$\|\varphi_1(t,\tau)-\varphi_1(\theta-\tau)\|_{\mathcal{L}(X)} \leq C(t-\theta)^{\alpha\psi}[(\theta-\tau)^{\alpha-2}+(\theta-\tau)^{\alpha-2+\delta-\psi}].$$

On the other hand, $(\theta-\tau)^{\alpha-2}$ delimits the improvement on the blow-up at initial time. Note that it is pointless to consider any $\psi \leq \delta$, since it will not cause any improvement on the term involving $(\theta-\tau)$, but it will decrease the exponent of $(t-\theta)^{\alpha\psi}$. Therefore, we restrict the possible values of ψ one more time and we obtain

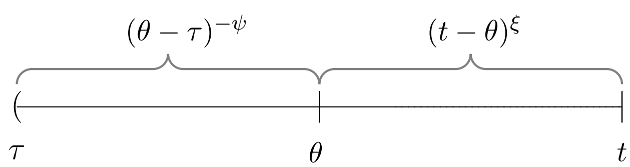
(ii). $\|\varphi_1(t,\tau)-\varphi_1(\theta-\tau)\|_{\mathcal{L}(X)} \leq C(t-\theta)^{\alpha\psi}(\theta-\tau)^{\alpha-2+\delta-\psi}$, for any $\psi \in \left[\delta, \max\left\{1, \frac{\delta}{\alpha}\right\}\right]$.

Finally, (I) and (II) provides two estimates for the difference $\varphi_1(t,\tau)-\varphi_1(\theta,\tau)$. An interpolation of them with exponents $\dfrac{\eta}{\alpha\psi}$ and $1-\dfrac{\eta}{\alpha\psi}$, $\eta \in [0,\alpha\psi]$, provides

$$\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\|_{\mathcal{L}(X)} \leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}.$$

Moreover, as $\alpha+\delta-2-\frac{\eta}{\alpha} > -1$ then $\eta < \alpha(\alpha+\delta-1)$. Since $\alpha(\alpha+\delta-1) < \alpha\delta \leq \alpha\psi$, it does not matter the value that ψ assumes on the interval $\left[\delta, \max\left\{1, \frac{\delta}{\alpha}\right\}\right]$. ☐

Remark 3.13. *Lemma 3.12 above states that if we consider the function $t \mapsto \varphi(t,\tau)x$ starting at a giving time τ, evolving to $\theta > \tau$ and then to $t > \theta$*



we can obtain $\xi > 0$ such that the evolution from θ to t is controlled by $(t-\theta)^{\xi}$, but in order to do this, we have to lose some of the control at initial time τ, represented by $(\theta-\tau)^{-\psi}$, where $-\psi < \alpha+\delta-2$.

Furthermore, with the estimates that we have so far, (3.5) is optimal in the sense that we can not obtain a pair of values $\xi, \psi \in (0,1)$ such that $\|\varphi_1(t,\tau)-\varphi_1(\theta,\tau)\| \leq (t-\theta)^{\xi}(\theta-\tau)^{-\psi}$, with $\xi \geq \eta$ and $\psi \leq 2+\frac{\eta}{\alpha}-\alpha-\delta$, unless $\xi = \eta$ and $\psi = 2+\frac{\eta}{\alpha}-\alpha-\delta$.

The same result holds in $\mathcal{L}(Y)$, but in this case the constant of almost sectoriality is ω.

Corollary 3.14. *Given any* $0 < \vartheta < \omega(\omega + \delta - 1)$, $\tau < \theta < t$,

$$\|\varphi_1(t,\tau) - \varphi_1(\theta,\tau)\|_{\mathcal{L}(Y)} \leq C(t-\theta)^{\vartheta}(\theta-\tau)^{\omega+\delta-2-\frac{\vartheta}{\omega}},$$

with $\omega + \delta - 2 - \frac{\vartheta}{\omega} \in (-1, 0)$.

As in Remark 3.5, those η and ϑ are auxiliary constants that provide us a range of possible estimates for the difference $\varphi_1(t,\tau) - \varphi_1(\theta,\tau)$. They play an essential role in Chapter 4, especially when proving differentiability of the linear process $U(t,\tau)$. In the same way that happens for μ or ν, there is a *trade-off* in choosing different values of η.

The Hölder continuity of $\mathbb{R} \ni t \mapsto \varphi_1(t,\cdot)$ is transferred to the map $\mathbb{R} \ni t \mapsto \Phi(t,\cdot)$, as we enunciate in next lemma.

Lemma 3.15. *Given any* $0 < \eta < \alpha(\alpha + \delta - 1)$ *and* $\tau < \theta < t$, *there exists a constant* $C > 0$ *such that*

$$\|\Phi(t,\tau) - \Phi(\theta,\tau)\|_{\mathcal{L}(X)} \leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}},$$

with $\alpha + \delta - 2 - \frac{\eta}{\alpha} \in (-1, 0)$.

Proof. Note that

$$\begin{aligned}\Phi(t,\tau) - \Phi(\theta,\tau) &= \varphi_1(t,\tau) - \varphi_1(\theta,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds - \int_\tau^\theta \varphi_1(\theta,s)\Phi(s,\tau)ds \\ &= [\varphi_1(t,\tau) - \varphi_1(\theta,\tau)] + \int_\theta^t \varphi_1(t,s)\Phi(s,\tau)ds + \int_\tau^\theta [\varphi_1(t,s) - \varphi_1(\theta,s)]\Phi(s,\tau)ds.\end{aligned}$$

Using (3.5) alongside with the properties of the families $\varphi_1(t,s)$ and $\Phi(t,s)$ obtained in Lemma 1.12 and Theorem 1.13, we have

$$\begin{aligned}\|\Phi(t,\tau) - \Phi(\theta,\tau)\|_{\mathcal{L}(X)} &\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} + C\int_\theta^t (t-s)^{\alpha+\delta-2}(s-\tau)^{\alpha+\delta-2}ds \\ &\quad + \int_\tau^\theta C(t-\theta)^{\eta}(\theta-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}(s-\tau)^{\alpha+\delta-2}ds. \\ &\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} + C(\theta-\tau)^{\alpha+\delta-2}\int_\theta^t (t-s)^{\alpha+\delta-2}ds \\ &\quad + C(t-\theta)^{\eta}\int_\tau^\theta (\theta-s)^{(\alpha+\delta-1-\frac{\eta}{\alpha})-1}(s-\tau)^{(\alpha+\delta-1)-1}ds. \\ &\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} + C(\theta-\tau)^{\alpha+\delta-2}(t-\theta)^{\alpha+\delta-1} \\ &\quad + C(t-\theta)^{\eta}(\theta-\tau)^{2\alpha+2\delta-2-\frac{\eta}{\alpha}-1}\mathcal{B}(\alpha+\delta-1-\tfrac{\eta}{\alpha}, \alpha+\delta-1)\end{aligned}$$

and the last integral above only converges if $\alpha + \delta - 1 - \frac{\eta}{\alpha} > 0$, that is, $\eta < \alpha(\alpha+\delta-1)$. Furthermore,

$$(\theta-\tau)^{2\alpha+2\delta-2-\frac{\eta}{\alpha}-1} = (\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}(\theta-\tau)^{\alpha+\delta-1} \leq C(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}.$$

We then have

$$\begin{aligned}\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{\mathcal{L}(X)} &\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}+C(\theta-\tau)^{\alpha+\delta-2}(t-\theta)^{\alpha+\delta-1}\\ &\quad +C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}\\ &\leq C[(t-\theta)^{\eta}+(t-\theta)^{\alpha+\delta-1}](\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}\\ &\leq C(t-\theta)^{\eta}(\theta-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}}\end{aligned}$$

and in las inequality we used that $\eta<\alpha(\alpha+\delta-1)<\alpha+\delta-1$.

□

Corollary 3.16. *Given any* $0<\vartheta<\omega(\omega+\delta-1)$, $\tau<\theta<t$

$$\|\Phi(t,\tau)-\Phi(\theta,\tau)\|_{\mathcal{L}(Y)}\leq C(t-\theta)^{\vartheta}(\theta-\tau)^{\omega+\delta-2-\frac{\vartheta}{\omega}},$$

with $\omega+\delta-2-\frac{\vartheta}{\omega}\in(-1,0)$.

CHAPTER 4

Regularization: Abstract theory

Given $u_0 \in X$, let

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds, \quad t \in (\tau, \tau+T],$$

be the local mild solution for (P.7) obtained in Theorem 1.24, where $T > 0$ is any positive real number such that $(\tau, \tau+T] \subset (\tau, \tau_M(u_0))$. The only regularity required for this function u is continuity, that is, $u(\cdot) \in \mathcal{C}((\tau, \tau+T], X)$. When dealing with differential equations, we will usually require more properties of this function u, specially if we wish to estimate energy of the function and to establish global well-posedness for the problem. Most of the arguments used to perform those analysis on energy estimates requires to work with the equation itself.

Therefore, it is of interest to determine whether or not u satisfies (P.7) in a more convenient sense. We fix the nomenclature for a type of solution that will appear throughout the work.

Definition 4.1. *A function $u(\cdot) : (\tau, \tau+T] \to X$ is a* strong $Y-$solution *of the singularly nonautonomous semilinear equation* (P.7) *if satisfies:*

1. *$u(\cdot) \in \mathcal{C}^1((\tau, T], Y)$, $u(\tau) = u_0$ and $u(t) \in D^Y$, for all $t \in (\tau, \tau+T)$.*
2. *The equation $u'(t) = -A(t)u(t) + F(u(t))$, $\tau < t < \tau + T$, is satisfied in the usual sense.*

Remark 4.2. *Since F is a nonlinearity that has image on Y, the differential equation above takes place on Y. The nomenclature $Y-$solution goes back to Henry in [37], where the author defined $Z^\gamma-$solutions for parabolic problems where $\{Z^\xi\}$ is the scale of fractional powers spaces associated to the sectorial operator A and $F : Z^1 \to Z^\gamma$.*

The domain D^Y in the definition above stands for the domain of the operator $A^Y(t)$, that is, $A(t)$ acting in Y, as stated in (P.4). For the application in Chapter 2, D^Y was given in (2.12).

This definition of strong $Y-$solution acknowledges that we might have discontinuity at the initial time $t = \tau$ and to prove that the problem has a strong $Y-$solution, we have to understand the connection of $U(t, \tau)$ to the equation $u_t + A(t)u = 0$. This is how this chapter is structured:

1. In the first section we prove that the family $U(t, s)$ associated to $A(t), t \in \mathbb{R}$, is strongly differentiable in $\mathcal{L}(X)$ and satisfies $\partial_t U(t, s) = -A(t)U(t, s)$. In this case, $U(t, \tau)$ recovers the solution of the homogeneous equation if we define $u(t) = U(t, \tau)u_0$ and $u_t = -A(t)u(t)$.

2. We then study in the second section the nonhomogeneous (but still linear) evolution equation $u_t = A(t)u + G(t)$, $\tau < t < \tau + T$ in the space X, whose solution is given by

$$u(t) = U(t, s)u_0 + \int_\tau^t U(t, s)G(s)ds, \quad t \in (\tau, \tau + T].$$

   We prove that this function $u(\cdot)$ is actually a strong solution for the problem, under certain conditions on the perturbation $G(t)$ and on the constant α of almost sectoriality and exponent δ of Hölder continuity of the family $A(t)$, $t \in \mathbb{R}$.

3. Since our goal is to solve the semilinear equation in Y, we translate the results obtained in these two preceding sections for the equation in Y.

4. In the last section we prove that the mild solution for the semilinear case (P.7) is a strong $Y-$solution for the problem.

As we will see during this chapter, the estimates obtained in Chapter 3 are of extreme importance. The auxiliary constant η introduced in Section 3.3 plays an essential role in the differentiability of $U(t, \tau)$ in $\mathcal{L}(X)$ (in the same sense, ϑ plays an essential role in the differentiability of $U(t, \tau)$ in the space $\mathcal{L}(Y)$), whereas the auxiliary constant μ introduced in Section 3.1.2 controls the magnitude of the discontinuity at the initial time $t = \tau$ and is connected to the Hölder continuity properties of $t \mapsto F(u(t))$.

This causes in the problem the same type of behavior that we observed in Chapter 1 and Chapter 2: there is a set of conditions accountable of ensuring differentiability of the process, and on the other hand, a set of conditions that ensures the discontinuity at $t = \tau$ is under control.

Remark 4.3. *To obtain the differentiability results in this chapter we will only evaluate the right-side derivative of the functions being studied. If the right-side derivative is continuous (which will be the case) Lemma P.9 allow us to conclude the continuous differentiability.*

This results in this chapter were presented in [14].

4.1 Differentiability of $U(t, \tau)$ in $\mathcal{L}(X)$

We start this section enunciating the main theorem of this chapter, which states the differentiability of the process. We call the attention for the fact that $U(t, \tau)$ being differentiable does not depend on the nonlinearity $F : X \to Y$. Therefore, only the constants α and δ feature in the conditions established to ensure differentiability of $U(t,\tau)$ in $\mathcal{L}(X)$. In the same sense, only ω (constant of almost sectoriality in Y) and δ shows up in the conditions to ensure differentiability of $U(t,\tau)$ in $\mathcal{L}(Y)$.

We present all the results and prove them in space $\mathcal{L}(X)$. During the proofs, note that they can all be translated to $\mathcal{L}(Y)$, by simply switching α to ω and D to D^Y, since both of operators $A(t) : D \subset X \to X$ and $A^Y(t) : D^Y \subset Y \to Y$ are almost sectorial.

Theorem 4.4. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in X satisfying (P.1) - (P.3), $\alpha \in (0,1)$ the constant of almost sectoriality and $\delta \in (0,1]$ the exponent of Hölder continuity.*

1. *If $\alpha + \delta > 1$, then there exists a unique linear process of growth $1-\alpha$, $U(t,\tau)$, associated to the family $A(t), t \in \mathbb{R}$. This process satisfies $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$.*
2. *In addition, if $\alpha^2 + \alpha\delta - 1 > 0$ *, then*
   - *(a) $U(t,\tau) : X \to D$, for any $\tau < t$.*
   - *(b) $\{(t,\tau) \in \mathbb{R}^2; \tau < t\} \ni (t,\tau) \mapsto U(t,\tau) \in \mathcal{L}(X)$ is strongly differentiable, that is, for each $x \in X$, $\{(t,\tau) \in \mathbb{R}^2; \tau < t\} \ni (t,\tau) \mapsto U(t,\tau)x \in X$ is differentiable.*
   - *(c) The derivative $\partial_t U(t,\tau)$ is a bounded linear operator, strongly continuous in $\{(t,\tau) \in \mathbb{R}^2; \tau < t\}$ and satisfies:*

$$\partial_t U(t,\tau) + A(t)U(t,\tau) = 0, \quad \forall t > \tau, \tag{4.1}$$

$$\|\partial_t U(t,\tau)\|_{\mathcal{L}(X)} = \|A(t)U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-2}, \quad \forall t > \tau, \tag{4.2}$$

$$\left\|A(t)U(t,\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}, \quad \forall t > \tau. \tag{4.3}$$

Remark 4.5. *For the sectorial case ($\alpha = 1$), any $\delta > 0$ ensures that $\alpha^2 + \alpha\delta - 1 > 0$ and the differentiability of $U(t,\tau)$ holds. This agrees with the classical results on the sectorial case presented in [55, Section 1.5].*

Remark 4.6. *Sometimes the inequality $\alpha + \frac{\delta}{2} - 1 > 0$ will be necessary, but this follows readily from $\alpha^2 + \alpha\delta - 1 > 0$. Indeed $\alpha > \frac{\sqrt{\delta^2+4}}{2} - \frac{\delta}{2} > 1 - \frac{\delta}{2}$.*

The first statement has already been proved in Corollary 1.14. The second one allows us to conclude that $u(t) = U(t,\tau)u_0$ is a strong solution for the evolution equation $u_t + A(t)u = 0$, $t > 0$, $u(\tau) = u_0$ and $U(t,\tau)$ is indeed a linear process of growth $1-\alpha$ (see Remark 1.15).

To demonstrate the second statement, we use the following strategy: Given any $\gamma > 0$ and $t_0 > \tau + \gamma$, it is enough to prove the strong differentiability of $U(t,\tau)$ for $t \in [\tau + \gamma, t_0]$. From the arbitrariness of γ and t_0, the result will follow.

*This restriction appears as a consequence of the conditions required on Lemma 3.15 and Lemma 4.9.

Therefore, given $u_0 \in X$, consider

$$U(t,\tau)u_0 = T_{-A(\tau)}(t-\tau)u_0 + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0, \quad t \in [\tau+\gamma, t_0].$$

If we tried to evaluate the derivative of $U(t,\tau)u_0$ directly from the expression above, we would face a problem of convergence in the integral, since the expected value for the derivative inside the integral would be $-A(s)T_{-A(s)}(t-s)\Phi(s,\tau)u_0$ and $\|A(s)T_{-A(s)}(t-s)\Phi(s,\tau)\|_{\mathcal{L}(X)} \leq C(t-s)^{\alpha-2}(s-\tau)^{\alpha+\delta-1}$.

To overcome this problem, we consider the auxiliary family of bounded linear operators $\{U_\rho(t,\tau);\ t \in [\tau+\gamma, t_0]\}$ given by

$$U_\rho(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^{t-\rho} T_{-A(s)}(t-s)\Phi(s,\tau), \quad t \in [\tau+\gamma, t_0],$$

where $\rho > 0$ is small enough so that $t-\rho > \tau+\gamma$. This slightly retreat in the domain of integration implies that the integrand is continuously differentiable in $(\tau, t-\rho]$ and from Lemma P.6 we have $[\tau+\gamma, t_0] \ni t \mapsto U_\rho(t,\tau)u_0 \in X$ is continuously differentiable, with derivative given by

$$\begin{aligned}\frac{d}{dt}U_\rho(t,\tau)u_0 = {}&-A(\tau)T_{-A(\tau)}(t-\tau)v_0 + T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)v_0 \\ &+ \int_\tau^{t-\rho} -A(s)T_{-A(s)}(t-s)\Phi(s,\tau)v_0 ds.\end{aligned} \tag{4.4}$$

We prove in the sequel the following:

(1) $U_\rho(\cdot,\tau)u_0$ converges as $\rho \to 0$ to $U(\cdot,\tau)u_0$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

(2) $\frac{d}{dt}U_\rho(\cdot,\tau)u_0$ converges as $\rho \to 0$ to $-A(\cdot)U(\cdot,\tau)u_0$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

Then, differentiability of $t \mapsto U(t,\tau)u_0$ for $t \in [\tau+\gamma, t_0]$ follows from the fact that $\mathcal{C}^1([\tau+\gamma, t_0], X)$ is a complete metric space. Moreover, $\frac{d}{dt}U(\cdot,\tau)u_0 = -A(\cdot)U(\cdot,\tau)u_0$.

Item (1) is easily obtained: for each $t \in [\tau+\gamma, t_0]$ we have

$$\begin{aligned}\|U_\rho(t,\tau) - U(t,\tau)\|_{\mathcal{L}(X)} &= \left\|\int_{t-\rho}^t T_{-A(s)}(t-s)\Phi(s,\tau)ds\right\|_{\mathcal{L}(X)} \leq \int_{t-\rho}^t C(t-s)^{\alpha-1}(s-\tau)^{\alpha+\delta-2}ds \\ &\leq C(t-\rho-\tau)^{\alpha+\delta-2}\int_{t-\rho}^t (t-s)^{\alpha-1}ds \\ &\leq C(\gamma-\rho)^{\alpha+\delta-2}\rho^\alpha \overset{\rho\to 0}{\longrightarrow} 0.\end{aligned}$$

Item (2), on the other hand, is a more delicate matter. Ideally, we would like to rearrange the expression (4.4) for $\partial_t U_\rho(t,\tau)$ in a way that becomes visible its convergence to

$$\begin{aligned}-A(t)U(t,\tau)u_0 &= -A(t)T_{-A(\tau)}(t-\tau)u_0 - A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds \\ &= -A(t)T_{-A(\tau)}(t-\tau)u_0 - A(t)\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds \\ &\quad - A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds.\end{aligned} \tag{4.5}$$

However, the expression above might not make sense, since it is not proved yet that $U(t,\tau): X \to D$ or that the integrals on the right side belong to D. Nonetheless, we will use it as a target of what we wish to achieve when we make $\rho \to 0$ in the expression of $\partial_t U_\rho(t,\tau)$.

From first to second line in (4.5) we added and subtracted $A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)ds$ in order to obtain the difference $[\Phi(s,\tau)-\Phi(t,\tau)]$, where we can use its Hölder continuity in the first variable (Lemma 3.12) to study its convergence.

We will rearrange (4.4) in a form that it approximates the most from the expression on the right side of our idealized equality (4.5).

Lemma 4.7. *The function $[\tau+\gamma, t_0] \ni t \mapsto \partial_t U_\rho(t,\tau)$ can also be given as*

$$\begin{aligned}
\partial_t U_\rho(t,\tau)u_0 = &- A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds \\
&- \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds \\
&+ \int_{t-\rho}^{t} \varphi_1(t,s)\Phi(s,\tau)u_0 ds + [T_{-A(t-\rho)}(\rho) - I]\Phi(t,\tau)u_0 \\
&+ T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]u_0.
\end{aligned} \tag{4.6}$$

Proof. Rearranging (4.4) and taking into account the expressions (1.15) and (1.16) for $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$, respectively, we have:

$$\begin{aligned}
\partial_t U_\rho(t,\tau) &= -A(\tau)T_{-A(\tau)}(t-\tau)u_0 + \int_\tau^{t-\rho} -A(s)T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds + T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)u_0 \\
&= -A(t)T_{-A(\tau)}(t-\tau)u_0 + [A(t)-A(\tau)]T_{-A(\tau)}(t-\tau)u_0 \\
&\quad + \int_\tau^{t-\rho} [A(t)-A(s)]T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds \\
&\quad + T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)u_0 \\
&= -A(t)T_{-A(\tau)}(t-\tau)u_0 - \varphi_1(t,\tau)u_0 - \int_\tau^{t-\rho} \varphi_1(t,s)\Phi(s,\tau)u_0 ds \\
&\quad - \int_\tau^{t-\rho} A(t)T_{-A(s)}T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds + T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)u_0 \\
&= -A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)\Phi(s,\tau)u_0 ds - \varphi_1(t,\tau)u_0 \\
&\quad - \int_\tau^{t} \varphi_1(t,s)\Phi(s,\tau)u_0 ds + \int_{t-\rho}^{t} \varphi_1(t,s)\Phi(s,\tau)u_0 ds + T_{-A(t-\rho)}(\rho)\Phi(t-\rho,\tau)u_0 \\
&= -A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds \\
&\quad - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds \\
&\quad + \int_{t-\rho}^{t} \varphi_1(t,s)\Phi(s,\tau)u_0 ds + T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]u_0
\end{aligned}$$

$$+ [T_{-A(t-\rho)}(\rho) - I]\Phi(t,\tau)u_0.$$

□

Remark 4.8. *For the sectorial case, the terms in the third and fourth line of equality* (4.6) *vanish as* $\rho \to 0$ *(see [52, 55]). The same can not be said when* $A(t)$*,* $t \in \mathbb{R}$*, is almost sectorial.*

The first line of (4.6) is already suited to our purpose and converges to the first line of the right side in equality (4.5) as we can see in the next lemma.

Lemma 4.9. *Assume that the constants* α *and* δ *satisfy the inequality* $\alpha^2 + \alpha\delta - 1 > 0$*. In this case, the integral* $\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau) - \Phi(t,\tau)]u_0 ds$ *belongs to* D *and*

$$A(t) \int_\tau^{t-\rho} T_{-A(s)}(t-s)[\Phi(s,\tau) - \Phi(t,\tau)]u_0 ds \overset{\rho\to 0}{\longrightarrow} A(t) \int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau) - \Phi(t,\tau)]u_0 ds,$$

uniformly for $t \in [\tau+\gamma, t_0]$ *in the norm of* X*.*

Proof. If we prove that $\int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau) - \Phi(t,\tau)]ds$ converges, then the result follows from Corollary P.4. From Lemma 3.15, there exists $0 < \eta < \alpha(\alpha+\delta-1)$ such that

$$\begin{aligned}
&\left\| \int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau) - \Phi(t,\tau)]u_0 ds \right\|_{\mathcal{L}(X)} \\
&\leq C \int_\tau^t (t-s)^{\alpha-2}(t-s)^{\eta}(s-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} ds \\
&= C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})-1}\mathcal{B}\big(\alpha+\eta-1, \alpha+\eta-1-\tfrac{\eta}{\alpha}\big)
\end{aligned}$$

and the entries on the function $\mathcal{B}$ are positive, provided that $1-\alpha < \eta < \alpha^2 + \alpha\delta - \alpha$. The existence of a suitable η relies on the constants α and δ to satisfy $\alpha^2 + \alpha\delta - \alpha > 1 - \alpha$, that is, $\alpha^2 + \alpha\delta - 1 > 0$.

□

Remark 4.10. *Until Lemma 4.9 we only had upper bounds for* η *(see Lemma 3.15). Now we must have* $1-\alpha < \eta < \alpha(\alpha+\delta-1)$*, and the existence of such* η *happens only if* $\alpha^2 + \alpha\delta - 1 > 0$*.*

Note that this is the condition that features in Theorem 4.4 and it represents a restriction on the possible values that α *can assume. For instance, if* $\delta = \frac{3}{4}$*, this condition only holds for* $\alpha > \frac{1}{8}\left[-1+\sqrt{73}\right] \approx 0.69$*. If* $\delta = \frac{1}{4}$*, then* $\alpha > \frac{1}{8}\left[-3+\sqrt{73}\right] \approx 0.88$*. The lesser* δ *is, the harder it is to obtain differentiability for the process.*

For the remaining terms in (4.6), we will adopt a different strategy. Rather than evaluating what happens to them as $\rho \to 0$, we first study the existence of

$$A(t) \int_\tau^t T_{-A(s)}(t-s)x ds$$

for an arbitrary $x \in X$, and then we relate the outcome of this analysis to the remaining terms of (4.6).

If $T(t)$ is a C_0-semigroup with inifnitesimal generator A, an important feature of $T(t)$ is the fact that given any $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

The next results prove that $\int_\tau^t T_{-A(s)}(t-s)xds \in D$, for any $x \in X$, when $A(t)$, $t \in \mathbb{R}$, is almost sectorial, and a characterization of $A(t)\left(\int_\tau^t T_{-A(s)}(t-s)xds\right)$ that extends the one we have for C_0-semigroups is obtained.

Lemma 4.11. *Let $\alpha^2 + \alpha\delta - 1 > 0$ and consider the linear operator $\mathcal{H}(t,\tau) : D^2 \to X$, $t > \tau$, given by $\mathcal{H}(t,\tau)w = A(t)\int_\tau^t T_{-A(s)}(t-s)wds$. Then $\mathcal{H}(t,\tau)$ is a well defined operator, it is bounded in D^2, satisfies*

$$\|\mathcal{H}(t,\tau)w\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X, \quad \forall w \in D^2,$$

and admits a bounded extension to X.

Proof. The fact that $\mathcal{H}(t,\tau)$ is well defined in D^2 follows from Corollary P.4 and the estimate

$$\begin{aligned}\left\|\int_\tau^t A(t)T_{-A(s)}(t-s)wds\right\|_X &= \left\|\int_\tau^t T_{-A(s)}(t-s)A(t)wds\right\|_X \\ &\leq C\int_\tau^t (t-s)^{\alpha-1}ds\,\|A(t)w\|_X < \infty.\end{aligned}$$

We prove in the sequel that there exists a constant $C > 0$ such that, for all $w \in D^2$, $\|\mathcal{H}(t,\tau)\|_X \leq C(t-\tau)^{\alpha-1}\|w\|_X$. In Lemma 1.5 we proved that for any $y \in D^2$, the function $t \mapsto T_{-A(\tau)}(t)y$ is continuously differentiable in $[0,\infty)$ and

$$\begin{aligned}A(t)\int_\tau^t T_{-A(t)}(t-s)yds &= \int_\tau^t A(t)T_{-A(t)}(t-s)yds \\ &= \int_\tau^t \frac{d}{ds}\left[T_{-A(t)}(t-s)y\right]ds \\ &= y - T_{-A(t)}(t-\tau)y.\end{aligned} \tag{4.7}$$

Also, the function $t \mapsto T_{-A(t)}(t-s-u)T_{-A(s)}(u)w$ is continuously differentiable in $[0, t-s]$ and

$$\frac{d}{du}\left[T_{-A(t)}(t-s-u)T_{-A(s)}(u)w\right] = T_{-A(t)}(t-s-u)[A(t)-A(s)]T_{-A(s)}(u)w. \tag{4.8}$$

Therefore, (4.8), a change of variable and Fubini's Theorem ([34, Theorem 2.39]) imply

$$\mathcal{H}(t,\tau)w = A(t)\int_\tau^t T_{-A(s)}(t-s)wds$$

$$
\begin{aligned}
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t [T_{-A(s)}(t-s) - T_{-A(t)}(t-s)]wds\\
&\overset{(4.8)}{=} A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t \left[\int_0^{t-s} T_{-A(t)}(t-s-u)[A(t)-A(s)]T_{-A(s)}(u)wdu\right] ds\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t \left[\int_s^{t} T_{-A(t)}(t-\xi)[A(t)-A(s)]T_{-A(s)}(\xi-s)wd\xi\right] ds\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t \left[\int_\tau^{\xi} T_{-A(t)}(t-\xi)[A(t)-A(s)]T_{-A(s)}(\xi-s)wds\right] d\xi\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} [A(t)-A(s)]T_{-A(s)}(\xi-s)wds\right] d\xi\\
&= A(t)\int_\tau^t T_{-A(t)}(t-s)wds - A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{t} [A(s)-A(t)]T_{-A(s)}(t-s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{t} [A(s)-A(t)]T_{-A(s)}(t-s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} [A(t)-A(s)]T_{-A(s)}(\xi-s)wds\right] d\xi.
\end{aligned}
$$

Note that in the last equality for $\mathcal{H}(t,\tau)w$, the first two terms are in the form $A(t)\int_\tau^t T_{-A(t)}(t-s)yds$, where $y \in D^2$. We know how to handle these expressions using (4.7). Returning to the expression $\mathcal{H}(t,\tau)w$,

$$
\begin{aligned}
\mathcal{H}(t,\tau)w &= w - T_{-A(t)}(t-\tau)w - \int_\tau^t \varphi_1(t,s)wds + T_{-A(t)}(t-\tau)\int_\tau^t \varphi_1(t,s)wds\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{t} [A(s)-A(t)]T_{-A(s)}(t-s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} [A(t)-A(\xi)]T_{-A(s)}(\xi-s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} [A(\xi)-A(s)]T_{-A(s)}(\xi-s)wds\right] d\xi\\
&= w - T_{-A(t)}(t-\tau)w - \int_\tau^t \varphi_1(t,s)wds + T_{-A(t)}(t-\tau)\int_\tau^t \varphi_1(t,s)wds\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{t} \varphi_1(t,s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)]A(\xi)^{-1}\left[A(\xi)\int_\tau^{\xi} T_{-A(s)}(\xi-s)wds\right] d\xi\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} -\varphi(\xi,s)wds\right] d\xi\\
&= [I - T_{-A(t)}(t-\tau)]\left[w - \int_\tau^t \varphi_1(t,s)wds\right]\\
&\quad + A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^{\xi} \varphi_1(t,s)wds\right] d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+ A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\xi^t \varphi_1(t,s)wds\right]d\xi\\
&- A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left[\int_\tau^\xi \varphi_1(\xi,s)wds\right]d\xi\\
&+ A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)](A(\xi))^{-1}\mathcal{H}(\xi,\tau)wd\xi\\
=&\,[I-T_{-A(t)}(t-\tau)]\left[w-\int_\tau^t \varphi_1(t,s)wds\right]\\
&+ A(t)\int_\tau^t T_{-A(t)}(t-\xi)\left\{\int_\xi^t \varphi_1(t,s)wds+\int_\tau^\xi[\varphi_1(t,s)w-\varphi_1(\xi,s)w]ds\right\}d\xi\\
&+ A(t)\int_\tau^t T_{-A(t)}(t-\xi)[A(t)-A(\xi)](A(\xi))^{-1}\mathcal{H}(\xi,\tau)wd\xi.
\end{aligned}
$$

Using the estimates (1.6) for the semigroup, (1.2) for the Hölder continuity of $A(t)$, (1.15) for the operators $\varphi_1(\cdot,\cdot)$ and (3.5) for the Hölder continuity of $t\mapsto\varphi_1(t,\cdot)$, we have

$$
\begin{aligned}
\|\mathcal{H}(t,\tau)w\|_X \le\,& C\left(1+(t-\tau)^{\alpha-1}\right)\left(1+\int_\tau^t (t-s)^{\alpha+\delta-2}ds\right)\|w\|_X\\
&+C\int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\xi^t (t-s)^{\alpha+\delta-2}ds+\int_\tau^\xi (t-\xi)^\eta(\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}ds\right]d\xi\|w\|_X\\
&+\int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds\\
\le\,& C\left(1+(t-\tau)^{\alpha-1}\right)\left(1+(t-\tau)^{\alpha+\delta-1}\right)\|w\|_X\\
&+C\left[\int_\tau^t (t-\xi)^{(\alpha-2)+(\alpha+\delta-1)}d\xi+(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-\tfrac{\eta}{\alpha})\right]\|w\|_X\\
&+\int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds\\
\le\,& C(t-\tau)^{\alpha-1}\|w\|_X\\
&+C\left[(t-\tau)^{2\alpha+\delta-2}+(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-\tfrac{\eta}{\alpha})\right]\|w\|_X\\
&+\int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds\\
\le\,& C(t-\tau)^{\alpha-1}\|w\|_X+\int_\tau^t (t-\xi)^{\alpha+\delta-2}\|\mathcal{H}(\xi,\tau)w\|_X\,ds.
\end{aligned}
$$

The arguments used in the above estimates only hold provided that $2\alpha+\delta-2>0$ and $1-\alpha<\eta<\alpha^2+\alpha\delta-\alpha$, that is,

$$\alpha>1-\tfrac{\delta}{2}\quad\text{and}\quad\alpha^2+\alpha\delta>1.$$

The second one is more restrictive (see Remark 4.6). Finally, applying the generalized version of Gronwall inequality (see Lemma P.7) we have, for $w\in D^2$,

$$\|\mathcal{H}(t,\tau)w\|_X\le C(t-\tau)^{\alpha-1}\|w\|_X.$$

Therefore, $\mathcal{H}(t,\tau)$ can be extended to a bounded linear operator in X, which we denote the same. □

The fact that $\mathcal{H}(t,\tau)$ is bounded allows us to prove the following result.

Lemma 4.12. *Let $\alpha^2+\alpha\delta-1>0$ and $w\in X$. Then $\int_\tau^t T_{-A(s)}(t-s)wds$ belongs to D and we can obtain an expression for $A(t)\int_\tau^t T_{-A(s)}(t-s)wds$: for any $0<\rho<t-\tau$,*

$$A(t)\int_\tau^t T_{-A(s)}(t-s)wds = w-T_{-A(t-\rho)}(\rho)w-\int_{t-\rho}^t \varphi_1(t,s)wds+A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds. \quad (4.9)$$

Furthermore, $A(t)\int_\tau^t T_{-A(s)}(t-s)ds$ is a bounded linear operator satisfying

$$\left\|A(t)\int_\tau^t T_{-A(s)}(t-s)ds\right\|_{\mathcal{L}(X)} \le C(t-\tau)^{\alpha-1}. \quad (4.10)$$

Proof. Let (w_n) be a sequence in D^2 such that $w_n\to w$. Since $\int_\tau^t T_{-A(s)}(t-s)ds$ is a bounded linear operator in X, it follows that $\int_\tau^t T_{-A(s)}(t-s)w_nds\to\int_\tau^t T_{-A(s)}(t-s)wds$. The extension $\mathcal{H}(t,\tau)$ is also a bounded linear operator and

$$A(t)\int_\tau^t T_{-A(s)}(t-s)w_nds=\mathcal{H}(t,\tau)w_n\to\mathcal{H}(t,\tau)w.$$

From the closedness of $A(t)$, we derive that $\int_\tau^t T_{-A(s)}(t-s)wds\in D(A(\cdot))$ and

$$\begin{aligned}
A(t)\int_\tau^t T_{-A(s)}(t-s)wds &= \lim_{n\to\infty} A(t)\int_\tau^t T_{-A(s)}(t-s)w_nds\\
&= \lim_{n\to\infty}\left\{A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)w_nds + A(t)\int_{t-\rho}^t T_{-A(s)}(t-s)w_nds\right\}\\
&= A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds\\
&\quad+\lim_{n\to\infty}\left\{\int_{t-\rho}^t A(s)T_{-A(s)}(t-s)w_nds+\int_{t-\rho}^t [A(t)-A(s)]T_{-A(s)}(t-s)w_nds\right\}\\
&= A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds+\lim_{n\to\infty}\left\{w_n-T_{-A(t-\rho)}(\rho)w_n-\int_{t-\rho}^t\varphi_1(t,s)w_nds\right\}\\
&= w-T_{-A(t-\rho)}(\rho)w-\int_{t-\rho}^t \varphi_1(t,s)wds+A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds,
\end{aligned}$$

and in the fourth line we used (1.10). The estimate in (4.10) follows immediately from the one obtained for $\mathcal{H}(t,\tau)$ and the fact the $A(t)\int_\tau^t T_{-A(s)}(t-s)ds$ is the extension of this operator. □

Remark 4.13. *Note that, even though $\int_\tau^t T_{-A(s)}(t-s)wds\in D$ for any $w\in X$, it does not mean that $\int_\tau^t A(t)T_{-A(s)}(t-s)wds$ is defined. The second integral might not exist. We can only prove that $A(t)\left(\int_\tau^t T_{-A(s)}(t-s)wds\right)$, with the operator outside the integral, exists.*

From the results above, we can obtain all the properties enumerated in Theorem 4.4, as we see next. But prior to those conclusions, it is worth comparing such result with the existent theory for singularly nonautonomous problems with sectorial operator. At that case, to conclude the differentiability of the process, we prove that

$$A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t)xds \xrightarrow{\rho\to 0} A(t)\int_\tau^{t} T_{-A(\tau)}(t)xds,$$

and this comes as consequence of $\|A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t)ds\| \leq C$.

For the almost sectorial case, such convergence does not necessarily occur. As we can see from (4.9), it will only happen if $T_{-A(t-\rho)}(\rho)w \xrightarrow{\rho\to 0} w$, which we know is not necessarily true. Moreover, the order from which $A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t)ds$ diverges from $A(t)\int_\tau^{t} T_{-A(\tau)}(t)ds$ is the same of the semigroup of growth $1-\alpha$, $T_{-A(\tau)}(t)$, at the initial instant $t=0$. This is reinforced by the fact $\|A(t)\int_\tau^{t-\rho} T_{-A(\tau)}(t)ds\| \leq C(t-\tau)^{\alpha-1}$.We gather those considerations in the following corollary:

Corollary 4.14. *Let $\alpha^2+\alpha\delta-1>0$ and $w\in X$. Then*

$$A(t)\int_{t-\rho}^{t} T_{-A(s)}(t-s)wds = w - T_{-A(t-\rho)}(\rho)w - \int_{t-\rho}^{t}\varphi_1(t,s)wds$$

and $A(t)\int_{t-\rho}^{t} T_{-A(s)}(t-s)wds$ does not necessarily vanishes as $\rho\to 0^+$. In particular, the expression $A(t)\int_\tau^{t-\rho} T_{-A(s)}(t-s)wds$ does not necessarily converges to $A(t)\int_\tau^{t} T_{-A(s)}(t-s)wds$, as $\rho\to 0$.

We are finally in conditions to return to the derivative $\partial_t U_\rho(t,\tau)u_0$ which last characterization was given in (4.6). Note that the second and third line are exactly the right side of (4.9) for $w=\Phi(t,\tau)u_0$ (with a negative sign) and we obtain

$$\begin{aligned}\partial_t U_\rho(t,\tau)u_0 = &- A(t)T_{-A(\tau)}(t-\tau)u_0 - \int_\tau^{t-\rho} A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0 ds\\ &- A(t)\int_\tau^{t} T_{-A(s)}(t-s)\Phi(t,\tau)u_0 ds\\ &+ T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]u_0.\end{aligned}$$

Lemma 4.7 already proved the uniform (for $t\in[\tau+\gamma,t_0]$) convergence of the second term to $\int_\tau^t A(t)T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]u_0ds$. The fourth term, the last remaining, converges uniformly to zero, since

$$\left\|T_{-A(t-\rho)}(\rho)[\Phi(t-\rho,\tau)-\Phi(t,\tau)]\right\|_{\mathcal{L}(X)} \leq C\rho^{\alpha+\eta-1}(t-\rho-\tau)^{\alpha+\delta-2-\frac{\eta}{\alpha}} \xrightarrow{\rho\to 0} 0,$$

provided that $\alpha+\eta-1>0$ (which is satisfied when $1-\alpha<\eta<\alpha^2+\alpha\delta-\alpha$).

This allows us to conclude the uniform convergence of $\partial_t U_\rho(t,\tau)u_0$ to $-A(t)[T_{-A(\tau)}(t-\tau)v_0 + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)u_0ds] = -A(t)U(t,\tau)$. Hence,

$$\sup_{t\in[\tau+\gamma,t_0]}\{\|U_\rho(t,\tau)u_0-U(t,\tau)u_0\| + \|\partial_t U_\rho(t,\tau)u_0 + A(t)U(t,\tau)u_0\|\} \xrightarrow{\rho\to 0} 0$$

and items $(a)-(b)$ in Theorem 4.4 are verified, as well as (4.1). The other estimates in item (c) we prove in the sequel.

Remark 4.15. *Now that once it is proved that $U(t,\tau)$ recovers classical solutions for the equation $u_t + A(t)u = 0$, the property $U(t,\tau) = U(t,r)U(r,\tau)$, $\tau < r < t$, follows from the uniqueness of solution for the equation. Therefore, all conditions on Definition 1.11 are satisfied for the family $U(t,\tau)$ and we can address it as a linear process growth $1-\alpha$.*

4.1.1 Estimates for $A(t)U(t,\tau)$ and $A(t)U(t,\tau)A(\tau)^{-1}$

Inequality (4.2), that is, $\|\partial_t U(t,\tau)\|_{\mathcal{L}(X)} = \|A(t)U(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-2}$, is obtained from (4.10). Indeed,

$$\begin{aligned}
\|A(t)U(t,\tau)\|_{\mathcal{L}(X)} &\leq \left\|A(t)T_{-A(\tau)}(t-\tau)\right\|_{\mathcal{L}(X)} + \left\|A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds\right\|_{\mathcal{L}(X)} \\
&\leq \left\|A(t)T_{-A(\tau)}(t-\tau)\right\|_{\mathcal{L}(X)} + \left\|A(t)\int_\tau^t T_{-A(s)}(t-s)[\Phi(s,\tau)-\Phi(t,\tau)]ds\right\|_{\mathcal{L}(X)} \\
&\quad + \left\|A(t)\int_\tau^t T_{-A(s)}(t-s)\Phi(t,\tau)ds\right\|_{\mathcal{L}(X)} \\
&\leq C(t-\tau)^{\alpha-2} + C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-2-\frac{\eta}{\alpha})-1} + C(t-\tau)^{\alpha-1} \\
&\leq C(t-\tau)^{\alpha-2},
\end{aligned}$$

and for the second term at the second line, we used the estimate obtained in the proof of Lemma 4.9, while in the last inequality, we used the fact that $(\alpha+\eta-1)$ is positive and $(\alpha+\delta-2-\frac{\eta}{\alpha}) \in (-1,0)$.

To prove (4.3) in Theorem 4.4, we will provide an alternative characterization for the process when this one is restricted to D. This characterization is suitable in situations where it is necessary to use Gronwall inequality.

Proposition 4.16. *Let $\alpha^2 + \alpha\delta - 1 > 0$. The process $U(t,\tau)$ can be given as*

$$U(t,\tau)A(\tau)^{-1} = T_{-A(t)}(t-\tau)A(\tau)^{-1} - \int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds. \tag{4.11}$$

Proof. Consider the operator $[\tau,t] \ni s \mapsto w(s) = -T_{-A(t)}(t-s)U(s,\tau)A(\tau)^{-1}$. Since $A(\tau)^{-1}$ has its image in D, it follows that $[\tau,\infty) \ni s \mapsto U(s,\tau)A(\tau)^{-1}$ is continuous (see Proposition 1.16). Also, $U(s,\tau)A(\tau)^{-1}$ has its image in D and $[\tau,t] \ni s \mapsto T_{-A(t)}(t-s)U(s,\tau)A(\tau)^{-1}$ is continuous (see Lemma 1.5).

Therefore $w(\cdot)$ is continuous in $[\tau,t]$ and differentiable in (τ,t) with derivative

$$\begin{aligned}
\frac{d}{ds}w(s) &= -A(t)T_{-A(t)}(t-s)U(s,\tau)A(\tau)^{-1} + T_{-A(t)}(t-s)A(s)U(s,\tau)A(\tau)^{-1} \\
&= T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}.
\end{aligned}$$

For $0 < h < \frac{t-\tau}{2}$,

$$w(t-h) - w(\tau+h) = \int_{\tau+h}^{t-h} \frac{d}{ds}w(s)ds = \int_{\tau+h}^{t-h} T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds. \tag{4.12}$$

As $h \to 0$, from the continuity of $w(\cdot)$ in $[\tau, t]$, the left side converges to

$$w(t) - w(\tau) = -U(t,\tau)A(\tau)^{-1} + T_{-A(t)}(t-\tau)A(\tau)^{-1}.$$

The right side demands more attention. Note that,

$$\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds = \int_\tau^{t^*} T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds + \int_{t^*}^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds,$$

for any $\tau < t^* < t$. The first integral on the right side is finite, since the integrand is continuous in $[\tau, t^*]$. For the second one, from (1.2), (1.7) and (4.2), we have the following estimative

$$\begin{aligned}
&\left\| \int_{t^*}^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds \right\|_{\mathcal{L}(X)} \\
&\quad \le C\int_{t^*}^t (t-s)^{\alpha+\delta-1}\|A(s)U(s,\tau)\|_{\mathcal{L}(X)}\|A(\tau)^{-1}\|_{\mathcal{L}(X)}ds \\
&\quad \le C\int_{t^*}^t (t-s)^{\alpha+\delta-1}(s-\tau)^{\alpha-2}ds \\
&\le (t^*-\tau)^{\alpha-2}\int_{t^*}^t (t-s)^{\alpha+\delta-1}ds < \infty.
\end{aligned}$$

Since, $\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds$ exists, the right side of (4.12) converges to it and (4.11) follows. □

We can use equality (4.11) to prove (4.3). We deduce

$$\begin{aligned}
&\|A(t)U(t,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)} \\
&\quad \le \left\|A(t)T_{-A(t)}(t-\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)} + \left\|A(t)\int_\tau^t T_{-A(t)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}ds\right\|_{\mathcal{L}(X)} \\
&\quad \le C(t-\tau)^{\alpha-1} + \int_\tau^t (t-s)^{\alpha+\delta-2}\|A(s)U(s,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)}ds.
\end{aligned}$$

Applying Gronwall's inequality (Lemma P.7), we have $\|A(t)U(t,\tau)A(\tau)^{-1}\| \le C(t-\tau)^{\alpha-1}$, and the proof of Theorem 4.4 is now complete.

Before we proceed to the regularity analysis for the singularly nonautonomous and nonhomogeneous linear case (P.6), we extend the differentiability properties presented to the semigroup $T_{-A(\tau)}(t)$ in Lemma 1.5 to the family $U(t,\tau)$.

4.1.2 Further properties on the family $U(t,\tau)$

The linear process of growth $1-\alpha$, $U(t,\tau)$, obtained earlier is given by

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds.$$

Since the integral is a linear operator that usually regularizes the integrand, from the above equality, we expect that the process $U(t,\tau)$ has a similar behavior to the semigroup $T_{-A(\tau)}(t-\tau)$. In fact, the properties of continuity and differentiability stated in Chapter 1 for the semigroup extend to the process, as we see in the sequel.

Proposition 4.17. *Let $\alpha^2 + \alpha\delta - 1 > 0$ and $x \in D^2$. Then,*

$$\frac{U(\tau+h,\tau)x - x}{h} \xrightarrow{h\to 0^+} -A(\tau)x.$$

Furthermore, $U(\cdot,\tau)x : [\tau,\infty) \to X$ is continuously differentiable (including at the initial time $t=\tau$) and

$$\frac{d}{dt}U(t,\tau)x = \begin{cases} -A(t)U(t,\tau)x, & t > \tau, \\ -A(\tau)x, & t = \tau. \end{cases}$$

Proof. For $t > \tau$, Theorem 4.4 implies $\frac{d}{dt}U(t,\tau)x = -A(t)U(t,\tau)x$. It only remains to check differentiability at $t=\tau$. Consider the differential quotient

$$\frac{U(\tau+h)x - x}{h} = \frac{T_{-A(\tau)}(h)x - x}{h} + \frac{1}{h}\int_\tau^{\tau+h} U(\tau+h,s)[A(\tau)-A(s)]T_{-A(\tau)}(s-\tau)xds.$$

Lemma 1.5 implies

$$\frac{T_{-A(\tau)}(h)x - x}{h} \xrightarrow{h\to 0^+} -A(\tau)x.$$

For the second term, we have

$$\begin{aligned}
&\left\| \frac{1}{h}\int_\tau^{\tau+h} U(\tau+h,s)[A(\tau)-A(s)]T_{-A(\tau)}(s-\tau)xds \right\|_X \\
&\le h^{-1}\int_\tau^{\tau+h} \|U(\tau+h,s)\|_{\mathcal{L}(X)} \|[A(\tau)-A(s)]A(s)^{-1}\|_{\mathcal{L}(X)} \left\|A(s)T_{-A(\tau)}(s-\tau)A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \|A(\tau)x\|_X \, ds \\
&\le Ch^{-1}\int_\tau^{\tau+h} (\tau+h-s)^{\alpha-1}(s-\tau)^{\alpha+\delta-1}ds \, \|A(\tau)x\|_X \, ds \\
&= h^{2\alpha+\delta-2}\mathcal{B}(\alpha,\alpha+\delta) \xrightarrow{h\to 0} 0,
\end{aligned}$$

since $2\alpha+\delta-2 = 2\left(\alpha+\frac{\delta}{2}-1\right) > 0$ (see Remark 4.6).

Therefore,

$$\frac{d}{dt}U(t,\tau)x = \begin{cases} -A(t)U(t,\tau)x, & t > \tau, \\ -A(\tau)x, & t = \tau. \end{cases}$$

To verify the continuity at $t=\tau$,

$$\|-A(t)U(t,\tau)x - A(\tau)x\|_X = \|-A(t)U(t,\tau)A(\tau)^{-1}A(\tau)x - A(\tau)x\|_X$$

$$
\begin{aligned}
&= \|A(t)T_{-A(t)}(t-\tau)A(\tau)^{-1}A(\tau)x - A(\tau)x\|_X \\
&\quad + \left\|A(t)\int_\tau^t T_{-A(s)}(t-s)[A(s)-A(t)]U(s,\tau)A(\tau)^{-1}A(\tau)xds\right\|_X \\
&\leq \|-A(t)T_{-A(t)}(t-\tau)x + A(\tau)T_{-A(t)}(t-\tau)x\|_X \\
&\quad + \|-A(\tau)T_{-A(t)}(t-\tau)x + A(\tau)T_{-A(\tau)}(t-\tau)x\|_X \\
&\quad + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X \\
&\quad + \int_\tau^t (t-s)^{\alpha-2}\|[A(s)-A(t)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\|A(\tau)U(s,\tau)A(\tau)^{-1}\|_{\mathcal{L}(X)}\|A(\tau)x\|_X ds \\
&\leq \|[A(\tau)-A(t)]A(\tau)^{-1}\|_{\mathcal{L}(X)}\|T_{-A(t)}(t-\tau)\|_{\mathcal{L}(X)}\|A(\tau)x\|_X \\
&\quad + \|T_{-A(t)}(t-\tau) - T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)}\|A(\tau)x\|_X \\
&\quad + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X \\
&\quad + \int_\tau^t (t-s)^{\alpha+\delta-2}(s-\tau)^{\alpha-1}\|A(\tau)x\|_X ds \\
&\leq (t-\tau)^{\alpha+\delta-1}\|A(\tau)x\|_X \\
&\quad + (t-\tau)^{2\alpha+\delta-2}\|A(\tau)x\|_X \\
&\quad + \|-A(\tau)T_{-A(\tau)}(t-\tau)x - A(\tau)x\|_X \\
&\quad + C(t-\tau)^{2\alpha+\delta-2}\mathcal{B}(\alpha+\delta-1,\alpha)\|A(\tau)x\|_X,
\end{aligned}
$$

where we used (4.11) in the second line and, at the last inequality, we used (1.2), (1.6) for the first term and (1.13) for the second term. Note that all the terms above approaches zero as $t \to \tau^+$, including the third one, as a consequence of Lemma 1.5. □

In the final result on this section, before we treat the nonlinear case, we present a version of Lemma 4.12 to the linear process $U(t,\tau)$.

Lemma 4.18. *Let $\alpha^2 + \alpha\delta - 1 > 0$ and $x \in X$. Then $\int_\tau^t U(t,s)xds$ belongs to D and*

$$
\begin{aligned}
A(t)\int_\tau^t U(t,s)xds &= A(t)\int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)xd\xi\right\}ds \\
&\quad + A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right\}d\xi \\
&\quad - A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)xds\right\}d\xi.
\end{aligned}
$$

Furthermore, $\left\|A(t)\int_\tau^t U(t,s)ds\right\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\alpha-1}$.

Proof. The characterization of the linear process obtained in Corollary 1.14 and an application of Fubini's Theorem [34, Theorem 2.37] yield

$$
\int_\tau^t U(t,s)xds = \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t \left[\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)xd\xi\right]ds
$$

$$
\begin{aligned}
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(\xi,s)xds\right]d\xi \\
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi \\
&\quad + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi \Phi(t,s)xds\right]d\xi \\
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi \\
&\quad + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^t \Phi(t,s)xds\right]d\xi \\
&\quad - \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi \\
&= \int_\tau^t T_{-A(s)}(t-s)xds + \int_\tau^t T_{-A(s)}(t-s)\left[\int_\tau^t \Phi(t,\xi)xd\xi\right]d\xi \\
&\quad + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi \\
&\quad - \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi \\
&= \int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)xd\xi\right\}ds && (4.13)\\
&\quad + \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi && (4.14)\\
&\quad - \int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi. && (4.15)
\end{aligned}
$$

From Lemma 4.11 and Lemma 4.12 the expression (4.13) belongs to D and

$$
\begin{aligned}
&\left\|A(t)\left(\int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)xd\xi\right\}ds\right)\right\|_X \\
&\quad \le C(t-\tau)^{\alpha-1}\|x\|_X + C(t-\tau)^{\alpha-1}\left\|\int_\tau^t \Phi(t,\xi)xd\xi\right\|_X \\
&\quad \le C(t-\tau)^{\alpha-1}\|x\|_X + C(t-\tau)^{\alpha-1}\int_\tau^t (t-\xi)^{\alpha+\delta-2}d\xi\|x\|_X \\
&\quad \le C(t-\tau)^{\alpha-1}\|x\|_X + C(t-\tau)^{\alpha-1}(t-\tau)^{\alpha+\delta-1}\|x\|_X \\
&\quad \le C(t-\tau)^{\alpha-1}\|x\|_X.
\end{aligned} \tag{4.16}
$$

We prove that (4.14) belongs to D by proving that $\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi$ converges. The conclusion follows from Corollary P.4. In fact,

$$
\left\|\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi\right\|_X
$$

$$\begin{aligned}
&\leq \int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\tau^\xi [(t-\xi)^\eta(\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}]ds\right] d\xi\|x\|_X\\
&\leq \int_\tau^t (t-\xi)^{\alpha+\eta-2}(\xi-\tau)^{\alpha+\delta-1-\frac{\eta}{\alpha}}d\xi\|x\|_X\\
&\leq C(t-\tau)^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\mathcal{B}(\alpha+\eta-1,\alpha+\delta-\frac{\eta}{\alpha})\|x\|_X\\
&\leq C\|x\|_X,
\end{aligned}$$

since $\eta > 1-\alpha$ and $\alpha+\delta-1-\frac{\eta}{\alpha} > 0$ (conditions discussed in Lemma 4.9 and the existence of such η is guaranteed by $\alpha^2+\alpha\delta-1>0$).

Furthermore, the above estimate implies

$$\left\|A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right]d\xi\right\|_X \leq C\|x\|_X. \tag{4.17}$$

Using the same strategy, we have for (4.15) that

$$\begin{aligned}
&\left\|\int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi\right\|_X\\
&\qquad\leq \int_\tau^t (t-\xi)^{\alpha-2}\left[\int_\xi^t (t-s)^{\alpha+\delta-2}ds\right]d\xi\|x\|_X\\
&\qquad\leq \int_\tau^t (t-\xi)^{\alpha-2}(t-\xi)^{\alpha+\delta-1}d\xi\|x\|_X\\
&\qquad\leq (t-\tau)^{(\alpha-1)+(\alpha+\delta-1)}\|x\|_X\\
&\qquad\leq C\|x\|_X,
\end{aligned}$$

since $2\alpha+\delta-2>0$ (it follows from $\alpha^2+\alpha\delta-1>0$, see Remark 4.6). The above estimates imply

$$\left\|A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left[\int_\xi^t \Phi(t,s)xds\right]d\xi\right\|_X \leq C\|x\|_X. \tag{4.18}$$

Therefore, $\int_\tau^t U(t,s)xds \in D$ and the estimate for $A(t)\left(\int_\tau^t U(t,s)ds\right)$ follows from (4.16), (4.17) and (4.18).

□

Remark 4.19. *Note that the expression obtained for $A(t)\int_\tau^t U(t,s)xds$ could be replayed without any change for the case where $A(t)$ is a family of uniformly sectorial operators $\alpha=1$. Clearly, the bounds obtained for $A(t)$ applied in* (4.13), (4.14) *and* (4.15) *would improve and would hold for $\alpha=1$, implying that $\|A(t)\int_\tau^t U(t,s)ds\|_{\mathcal{L}(X)} \leq C$, in this case.*

This result for the sectorial family will be essential to prove an effect that the differential equation has on the solution called smoothing effect. *We dedicate Appendix A to discuss this topic.*

4.2 Regular solution for $u_t + A(t)u = G(t)$ in X

In the previous section we proved that the linear process $U(t,\tau)$, $t > \tau$, associated to the family $A(t), t \in \mathbb{R}$, recovers strong solutions for the evolution problem $u_t + A(t)u = 0$, $u(\tau) = u_0$. In this section we study the nonhomogeneous linear problem

$$u_t + A(t)u = G(t), \quad \tau < t < \tau + T; \quad u(\tau) = u_0 \in X. \tag{4.19}$$

It will be used in the sequel to study the semilinear equation (P.7). If $G \in L^1((\tau, \tau+T], X)$, then the function $u : (\tau, \tau+T] \to X$, given by

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)G(s)ds \tag{4.20}$$

is well defined and it is called *mild solution* of (4.19). If we impose further conditions on G, we can prove that this mild solution is actually a strong solution for the equation. As we did in the preceding section, we enunciate the main result and prove it throughout the section. The content in this section is compiled in [14].

Theorem 4.20. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in the Banach space X satisfying (P.1) - (P.3), $\alpha \in (0,1)$ the constant of almost sectoriality and $\delta \in (0,1]$ the exponent of Hölder continuity. Suppose that $\alpha^2 + \alpha\delta - 1 > 0$ and let $U(t,\tau)$ be the strongly differentiable process of growth $1-\alpha$ associated to $A(t)$, $t \in \mathbb{R}$.*

Also, assume $G : (\tau, \tau+T] \to X$ is a continuous function that satisfies

$$\|G(t) - G(s)\|_X \leq C(t-s)^\theta (s-\tau)^{-\psi}, \quad \text{for any } \tau < s < t, \tag{4.21}$$
$$\|G(t)\|_X \leq C(t-\tau)^{-\psi}, \quad \text{for any } \tau < t, \tag{4.22}$$

where θ and ψ are positive constants satisfying $\theta > 1-\alpha$, $0 < \psi < 1$.

Then, for each $u_0 \in X$, the mild solution (4.20) *is a strong solution for* (4.19)*, that is,*

1. *$u(\cdot) \in \mathcal{C}^1((\tau, \tau+T], X)$, $u(\tau) = u_0$ and $u(t) \in D$, for all $\tau < t < \tau+T$.*

2. *The equation $\frac{d}{dt}u(t) = -A(t)u(t) + G(t)$, $\tau < t < \tau+T$, is satisfied in the usual sense (in X)*

and the following expression for the derivative of $u(\cdot)$ holds

$$u_t(t) = -A(t)U(t,\tau)u_0 - A(t)\int_\tau^t U(t,s)[G(s) - G(t)]ds - A(t)\int_\tau^t U(t,s)G(t)ds + G(t).$$

Moreover, if $u_0 \in D$, then $u(\cdot)$ is continuous at $t = \tau$.

We prove this theorem in the sequel following a strategy similar to the one we adopted in Section 4.1 to treat differentiability of $U(t,\tau)$. If we tried to evaluate the derivative directly in the expression

(4.20), the first term would not pose any problem, that is, $\partial_t U(t,\tau)u_0 = -A(t)U(t,\tau)u_0$. However, the expression given by the integral would be troublesome, since the expected value inside the integral is $-A(t)U(t,s)G(s)$ and we cannot prove convergence of the integral with such integrand (recall that $\|A(t)U(t,\tau)\|_{\mathcal{L}(X)} \leq (t-\tau)^{\alpha-2}$). We denote this term as $v(t)$, that is,

$$v(t) = \int_\tau^t U(t,s)G(s)ds.$$

To overcome the problem mentioned above, we consider, for small $\rho > 0$, the approximations

$$[\tau+\gamma, t_0] \ni t \mapsto v_\rho(t) = \int_\tau^{t-\rho} U(t,s)G(s)ds,$$

where $\gamma > 0$ is arbitrary, $t_0 \in (\tau+\gamma, \tau+T]$ and ρ is small enough such that $t-\rho > \tau+\gamma$. With this slight retreat in the domain of integration, we can prove the following result for this function.

Lemma 4.21. *The function $v_\rho : [\tau+\gamma, t_0] \to X$ is continuously differentiable in X and*

$$v_\rho'(t) = U(t,t-\rho)G(t-\rho) - A(t)\int_\tau^{t-\rho} U(t,s)G(s)ds. \tag{4.23}$$

Proof. This follows readily from the fact that the integrand is continuously differentiable in $(\tau, t-\rho]$ and from an application of Lemma P.6. □

Once we know v_ρ is differentiable, we prove:

(1) $v_\rho(\cdot)$ converges as $\rho \to 0$ to $v(\cdot)$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

(2) $v_\rho'(\cdot)$ converges as $\rho \to 0$ to $-A(\cdot)v(\cdot) + G(\cdot)$ in $\mathcal{C}([\tau+\gamma, t_0], X)$.

Then, the differentiability of $t \mapsto v(t)$ for $t \in [\tau+\gamma, t_0]$ follows and $v'(t) = -A(t)v(t) + G(t)$. From the arbitrariness of $\gamma > 0$ and t_0, we have the differentiability in $(\tau, \tau+T)$.

After these two steps, Theorem 4.20 will be proved, since

$$\begin{aligned} u'(t) &= -A(t)U(t,\tau)u_0 + \frac{d}{dt}\int_\tau^t U(t,s)G(s)ds = -A(t)U(t,\tau)u_0 + v'(t) \\ &= -A(t)U(t,\tau)u_0 - A(t)v(t) + G(t) = -A(t)u(t) + G(t). \end{aligned}$$

Item (1) is easily obtained: for each $t \in [\tau+\gamma, t_0]$ we have

$$\begin{aligned} \|v_\rho(t) - v(t)\|_X &= \left\| \int_{t-\rho}^t U(t,s)G(s)ds \right\|_X \\ &\leq \int_{t-\rho}^t C(t-s)^{\alpha-1}(s-\tau)^{-\psi}ds \\ &\leq C(t-\rho-\tau)^{-\psi}\rho^\alpha \overset{\rho\to 0}{\longrightarrow} 0. \end{aligned}$$

Item (2), on the other hand, demands more attention. We first prove that $v(t) \in D$.

Lemma 4.22. *For any* $t \in [\tau+\gamma, t_0]$, $v(t) \in D$ *and*

$$\begin{aligned} -A(t)v(t) &= -A(t)\int_\tau^t U(t,s)G(s)ds \\ &= -A(t)\int_\tau^t U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^t U(t,s)G(t)ds. \end{aligned}$$

Proof. It follows from Lemma 4.18 that $\int_\tau^t U(t,s)G(t)ds \in D$. Furthermore, from (4.21) with $\theta > 1-\alpha$, we conclude that $\int_\tau^t A(t)U(t,s)[G(s)-G(t)]ds$ converges. Indeed,

$$\begin{aligned} \left\|\int_\tau^t A(t)U(t,s)[G(s)-G(t)]ds\right\|_X &\le \int_\tau^t (t-s)^{\alpha-2}(t-s)^\theta (s-\tau)^{-\psi}ds \\ &\le C(t-\tau)^{(\alpha+\theta-1)-\psi} < \infty. \end{aligned} \tag{4.24}$$

Therefore, $\int_\tau^t U(t,s)[G(s)-G(t)]ds \in D$ and

$$A(t)\int_\tau^t U(t,s)[G(s)-G(t)]ds = \int_\tau^t A(t)U(t,s)[G(s)-G(t)]ds.$$

□

To prove item (2), we must check that $v_\rho'(\cdot)$ given by (4.23) converges to $-A(\cdot)v(\cdot)+G(\cdot)$ which is also given by:

$$\begin{aligned} -A(t)v(t)+G(t) &= G(t) - A(t)\int_\tau^t U(t,s)G(s)ds \\ &= G(t) - A(t)\int_\tau^t U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^t U(t,s)G(t)ds. \end{aligned} \tag{4.25}$$

We rearrange (4.23) in a way that it approaches the most the expression (4.25) above, that is,

$$v_\rho'(t) = U(t,t-\rho)G(t-\rho) - A(t)\int_\tau^{t-\rho} U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^{t-\rho} U(t,s)G(t)ds. \tag{4.26}$$

The second term of (4.26) converges and it satisfies:

Lemma 4.23. *If* $G : (\tau, \tau+T] \to X$ *satisfies* (4.21) *with* $\theta > 1-\alpha$, *then*

$$A(t)\int_\tau^{t-\rho} U(t,s)[G(s)-G(t)]ds \overset{\rho\to 0}{\longrightarrow} A(t)\int_\tau^t U(t,s)[G(s)-G(t)]ds.$$

Proof. This follows from the existence of $\int_\tau^t A(t)U(t,s)[G(s)-G(t)]ds$ proved in Lemma 4.22, equation (4.24). Note that $\theta > 1-\alpha$ was necessary to ensure such existence. □

For the other terms in (4.26), note that the discontinuity of the process at the initial time allow situations in which

$$U(t,t-\rho)G(t-\rho) \nrightarrow G(t) \quad \text{and} \quad A(t)\int_\tau^{t-\rho} U(t,s)G(t)ds \nrightarrow A(t)\int_\tau^t U(t,s)G(t)ds,$$

as $\rho \to 0$. Therefore, we cannot work them separately and, in order to obtain the desired convergence, we have to find an alternative to overcome this situation. We will provide a way to write $A(t)\int_\tau^t U(t,s)xds$ in terms of $A(t)\int_\tau^{t-\rho} U(t,s)xds$, for a given $\rho > 0$ and $x \in X$. This is done in next lemma.

Lemma 4.24. *Let $\alpha^2 + \alpha\delta - 1 > 0$. Given any $0 < \rho < t - \tau$ and $x \in X$, the following equality holds:*

$$\begin{aligned} A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^{t-\rho} U(t,s)xds + \left\{x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^t \varphi_1(t,s)xds\right\} & \\ + A(t)\int_{t-\rho}^t T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^{\xi}[\Phi(\xi,s)-\Phi(t,s)]xds\right\}d\xi & \quad (4.27)\\ + A(t)\int_{t-\rho}^t T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^t \Phi(t,s)xds\right\}d\xi & \quad (4.28)\\ - A(t)\int_{t-\rho}^t T_{-A(\xi)}(t-\xi)\left\{\int_{\xi}^t \Phi(t,s)xds\right\}d\xi. & \quad (4.29)\end{aligned}$$

Moreover, the terms (4.27), (4.28) *and* (4.29) *vanish as $\rho \to 0^+$.*

Proof. Note first that, since $\int_\tau^t U(t,s)xds \in D$ for any $\tau, t \in \mathbb{R}$ and $x \in X$ (Lemma 4.18), we can separate the integrals as follows

$$A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^{t-\rho} U(t,s)xds + A(t)\int_{t-\rho}^t U(t,s)xds.$$

The expression available for the process $U(t,\tau)$ (Corollary 1.14) and the result on Corollary 4.14 implies that the second term in the right-side of equality above satisfies:

$$\begin{aligned} &A(t)\int_{t-\rho}^t U(t,s)xds \\ &= A(t)\int_{t-\rho}^t T_{-A(s)}(t-s)xds + A(t)\int_{t-\rho}^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)xd\xi\right\}ds \\ &= \left\{x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^t \varphi_1(t,s)xds\right\} + A(t)\int_{t-\rho}^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)xd\xi\right\}ds \end{aligned}$$

and we already obtain the first line of the desired equality:

$$\begin{aligned} A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^{t-\rho} U(t,s)xds + \left\{x - T_{-A(t-\rho)}(\rho)x - \int_{t-\rho}^t \varphi_1(t,s)xds\right\} & \\ + A(t)\int_{t-\rho}^t \left\{\int_s^t T_{-A(\xi)}(t-\xi)\Phi(\xi,s)xd\xi\right\}ds. & \quad (4.30)\end{aligned}$$

An application of Fubini's Theorem and some algebraic manipulation on (4.30) yield

$$\begin{aligned}
&A(t)\int_{t-\rho}^{t}\left\{\int_{s}^{t}T_{-A(\xi)}(t-\xi)\Phi(\xi,s)x d\xi\right\}ds = A(t)\int_{t-\rho}^{t}T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^{\xi}\Phi(\xi,s)x ds\right\}d\xi\\
&\quad = A(t)\int_{t-\rho}^{t}T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^{\xi}[\Phi(\xi,s)-\Phi(t,s)]x ds\right\}d\xi\\
&\qquad + A(t)\int_{t-\rho}^{t}T_{-A(\xi)}(t-\xi)\left\{\int_{t-\rho}^{t}\Phi(t,s)x ds\right\}d\xi\\
&\qquad - A(t)\int_{t-\rho}^{t}T_{-A(\xi)}(t-\xi)\left\{\int_{\xi}^{t}\Phi(t,s)x ds\right\}d\xi\\
&\quad = \mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)+\mathcal{I}_3(\rho).
\end{aligned}$$

The first statement of the lemma is already proved, it only remains to prove that $\mathcal{I}_1(\rho)$, $\mathcal{I}_2(\rho)$ and $\mathcal{I}_3(\rho)$ vanish as $\rho \to 0^+$. From Lemma 3.12, we obtain

$$\begin{aligned}
\|\mathcal{I}_1(\rho)\|_X &\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha-2}\left\{\int_{t-\rho}^{\xi}(t-\xi)^{\eta}(\xi-s)^{\alpha+\delta-2-\frac{\eta}{\alpha}}\|x\|_X ds\right\}d\xi\\
&\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha+\eta-2}(\xi-(t-\rho))^{\alpha+\delta-1-\frac{\eta}{\alpha}}\|x\|_X d\xi\\
&\leq C\rho^{(\alpha+\eta-1)+(\alpha+\delta-1-\frac{\eta}{\alpha})}\|x\|_X \overset{\rho\to 0}{\to} 0.
\end{aligned}$$

For $\mathcal{I}_2(\rho)$, if $w_\rho = \int_{t-\rho}^{t}\Phi(t,s)x ds$, then $\|w_\rho\|_X \overset{\rho\to 0^+}{\longrightarrow} 0$ and we have

$$\begin{aligned}
\mathcal{I}_2(\rho) &= A(t)\int_{t-\rho}^{t}T_{-A(\xi)}(t-\xi)w_\rho d\xi\\
&= A(t)\int_{\tau}^{t}T_{-A(\xi)}(t-\xi)w_\rho d\xi - A(t)A(t-\rho)^{-1}A(t-\rho)A\int_{\tau}^{t-\rho}T_{-A(\xi)}(t-\xi)w_\rho d\xi\\
&= \mathcal{H}(t,\tau)w_\rho - A(t)A(t-\rho)^{-1}\mathcal{H}(t-\rho,\tau)w_\rho.
\end{aligned}$$

Since $\mathcal{H}(\cdot,\cdot)$ is a bounded linear operator (Lemma 4.11), it follows that

$$\|\mathcal{I}_2(\rho)\|_X \leq C(t-\tau)^{\alpha-1}\|w_\rho\|_X + C(t-\rho-\tau)^{\alpha-1}\|w_\rho\|_X \overset{\rho\to 0}{\to} 0.$$

For the third term, we have

$$\begin{aligned}
\|\mathcal{I}_3(\rho)\|_X &\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha-2}\left\{\int_{\xi}^{t}(t-s)^{\alpha+\delta-2}ds\right\}d\xi\\
&\leq C\int_{t-\rho}^{t}(t-\xi)^{\alpha-2}(t-\xi)^{\alpha+\delta-1}d\xi \leq C\rho^{2\alpha+\delta-2} \overset{\rho\to 0}{\to} 0,
\end{aligned}$$

since $\alpha+\frac{\delta}{2}-1>0$ (as a consequence of $\alpha^2+\alpha\delta-1>0$). □

Equality provided in Lemma 4.24 suits well our purpose. We use the result of this lemma to rewrite equation (4.26) for v'_ρ. If $\mathcal{I}_1(\rho)$, $\mathcal{I}_2(\rho)$ and $\mathcal{I}_3(\rho)$ represent the terms (4.27), (4.28) and (4.29) with $x = G(t)$ (all of them vanishing as $\rho \to 0$), we obtain

$$\begin{aligned}
v'_\rho(t) &= U(t,t-\rho)G(t-\rho) - A(t)\int_\tau^{t-\rho} U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^{t-\rho} U(t,s)G(t)ds \\
&= T_{-A(t-\rho)}(\rho)G(t-\rho) + \int_{t-\rho}^{t} T_{-A(s)}(t-s)\Phi(s,t-\rho)G(t-\rho)ds - A(t)\int_\tau^{t-\rho} U(t,s)[G(s)-G(t)]ds \\
&\quad - A(t)\int_\tau^{t} U(t,s)G(t)ds + \left\{G(t) - T_{-A(t-\rho)}(\rho)G(t) - \int_{t-\rho}^{t}\varphi_1(t,s)G(t)ds\right\} \\
&\quad + \mathcal{I}_1(\rho) + \mathcal{I}_2(\rho) + \mathcal{I}_3(\rho) \\
&= G(t) - A(t)\int_\tau^{t-\rho} U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^{t} U(t,s)G(t)ds \qquad (4.31) \\
&\quad + T_{-A(t-\rho)}(\rho)[G(t-\rho)-G(t)] + \int_{t-\rho}^{t} T_{-A(s)}(t-s)\Phi(s,t-\rho)G(t-\rho)ds - \int_{t-\rho}^{t}\varphi_1(t,s)G(t)ds \\
&\quad + \mathcal{I}_1(\rho) + \mathcal{I}_2(\rho) + \mathcal{I}_3(\rho).
\end{aligned}$$

First line in the last equality converges to $G(t) - A(t)v(t)$, as needed (and uniformly for $t \in [\tau+\gamma, t_0]$), due to Lemma 4.23. We prove in the sequel that the remaining terms vanish as $\rho \to 0$. Note that the $\theta-$Hölder continuity of $G(\cdot)$ given in (4.21) is extremely important in the convergence analysis below, as well its controled discontinuity at initial time (given by the exponent $\psi \in (0,1)$) and the fact that $1-\alpha < \eta < \alpha(\alpha+\delta-1)$.We have

$$\|T_{-A(t-\rho)}(\rho)[G(t-\rho)-G(t)]\|_X \le C\rho^{\alpha-1}\rho^{\theta} = C\rho^{\alpha+\theta-1} \overset{\rho\to 0}{\to} 0,$$

$$\begin{aligned}
\left\|\int_{t-\rho}^{t} T_{-A(s)}(t-s)\Phi(s,t-\rho)G(t-\rho)ds\right\|_X &\le C\int_{t-\rho}^{t}(t-s)^{\alpha-1}(s-t-\rho)^{\alpha+\delta-2}(t-\rho)^{-\psi}ds \\
\le C(t-\rho)^{-\psi}\int_{t-\rho}^{t}(t-s)^{\alpha-1}(s-t-\rho)^{\alpha+\delta-2}ds &\le C(t-\rho)^{-\psi}\rho^{\alpha+\alpha+\delta-2}\mathcal{B}(\alpha,\alpha+\delta-2) \\
\le C(t-\rho)^{-\psi}\rho^{2\alpha+\delta-2} \overset{\rho\to 0}{\to} 0
\end{aligned}$$

and

$$\left\|\int_{t-\rho}^{t}\varphi_1(t,s)G(t)ds\right\|_X \le C\int_{t-\rho}^{t}(t-s)^{\alpha+\delta-2}(t-\tau)^{-\psi}ds \le C(t-\tau)^{-\psi}\rho^{\alpha+\delta-1} \overset{\rho\to 0}{\to} 0.$$

Consequently, in the expression obtained for $v'_\rho(\cdot)$ we have (4.31) converging to $G(t) - A(t)v(t)$ whereas the remaining terms converge to zero, which allow us to conclude

$$\sup_{t\in[\tau+\gamma,T]}\left\|v'_\rho(t) - \left[G(t) - A(t)\int_\tau^t U(t,s)G(s)ds\right]\right\|_X \xrightarrow{\rho\to 0^+} 0$$

and Theorem 4.20 is proved.

4.3 Regularity results stated in $\mathcal{L}(Y)$ and Y

We restate the two main theorems presented in the previous sections now in the context of the Banach space Y. The proof is exactly the same, since $A^Y(t) : D^Y \subset Y \to Y$ is also an almost sectorial operator, but with constant of sectoriality ω. We first present the differentiability of $U(t, \tau)$ in Y.

Theorem 4.25. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in Y satisfying (P.1) - (P.3), $\omega \in (0, 1)$ the constant of almost sectoriality and $\delta \in (0, 1]$ the exponent of Hölder continuity.*

1. *If $\omega + \delta > 1$, then there exists a unique linear process of growth $1 - \omega$ in $\mathcal{L}(Y)$, $U(t, \tau)$, associated to the family $A(t), t \in \mathbb{R}$. This process satisfies $\|U(t, \tau)\|_{\mathcal{L}(Y)} \leq C(t - \tau)^{\omega - 1}$.*
2. *In addition, if $\omega^2 + \omega\delta - 1 > 0$, then*
   (a) *$U(t, \tau) : Y \to D^Y$, for any $\tau < t$.*
   (b) *$\{(t, \tau) \in \mathbb{R}^2; \tau < t\} \ni (t, \tau) \mapsto U(t, \tau) \in \mathcal{L}(Y)$ is strongly differentiable, that is, for each $x \in Y$, $\{(t, \tau) \in \mathbb{R}^2; \tau < t\} \ni (t, \tau) \mapsto U(t, \tau)x \in Y$ is differentiable.*
   (c) *The derivative $\partial_t U(t, \tau)$ is a bounded linear operator, strongly continuous in $\{(t, \tau) \in \mathbb{R}^2; \tau < t\}$ and satisfies:*

$$\partial_t U(t, \tau) + A(t)U(t, \tau) = 0, \quad \forall t > \tau,$$
$$\|\partial_t U(t, \tau)\|_{\mathcal{L}(Y)} = \|A(t)U(t, \tau)\|_{\mathcal{L}(Y)} \leq C(t - \tau)^{\omega - 2}, \quad \forall t > \tau,$$
$$\left\|A(t)U(t, \tau)A(\tau)^{-1}\right\|_{\mathcal{L}(Y)} \leq C(t - \tau)^{\omega - 1}, \quad \forall t > \tau.$$

Consider the nonhomogeneous problem

$$u_t + A(t)u = G(t), \quad \tau < t < \tau + T; \quad u(\tau) = u_0 \in Y \tag{4.32}$$

now in the Banach space Y, and $G \in L^1((\tau, \tau + T), Y)$. The function $u : [\tau, \tau + T] \to Y$, given by

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)G(s)ds \tag{4.33}$$

is the *mild solution* of (4.32).

Theorem 4.26. *Let $A(t), t \in \mathbb{R}$, be a family of linear operators in the Banach space Y satisfying (P.1) - (P.3), $\omega \in (0, 1)$ the constant of almost sectoriality and $\delta \in (0, 1]$ the exponent of Hölder continuity. Suppose that $\omega^2 + \omega\delta - 1 > 0$ and let $U(t, \tau)$ be the strongly differentiable process of growth $1 - \omega$ associated to $A(t)$, $t \in \mathbb{R}$.*

Also, assume $G : (\tau, \tau + T] \to Y$ is a continuous function that satisfies

$$\|G(t) - G(s)\|_X \leq C(t - s)^\theta (s - \tau)^{-\psi}, \quad \textit{for any } \tau < s < t,$$

$$\|G(t)\|_X \le C(t-\tau)^{-\psi}, \quad \textit{for any } \tau < t,$$

where θ,ψ are positive constants satisfying $\theta > 1-\omega$ and $\psi \in (0,1)$.

Then, for each $u_0 \in Y$, the mild solution (4.33) *is a strong solution for* (4.32)*, that is,*

1. *$u(\cdot) \in \mathcal{C}^1((\tau,\tau+T],Y)$, $u(\tau)=u_0$ and $u(t) \in D^Y$, for all $\tau < t < \tau+T$.*
2. *The equation $\frac{d}{dt}u(t) = -A(t)u(t)+G(t)$ in Y, $\tau < t < \tau+T$, is satisfied in the usual sense.*

and the following expression for the derivative of $u(\cdot)$ holds

$$u_t(t) = -A(t)U(t,\tau)u_0 - A(t)\int_\tau^t U(t,s)[G(s)-G(t)]ds - A(t)\int_\tau^t U(t,s)G(t)ds + G(t).$$

Furthermore, if $u_0 \in D^Y$, then $u(\cdot)$ is continuous at $t=\tau$, that is, $u(\cdot) \in \mathcal{C}([\tau,\infty),Y) \cap \mathcal{C}^1((\tau,\infty),Y)$.

4.4 Strong $Y-$solution for the semilinear equation

We consider the semilinear evolution problem

$$u_t + A(t)u = F(u), t > \tau; \quad u(\tau) = u_0 \in X,$$

with mild solution $u : (\tau, \tau_M(u_0)) \to X$. Since the existence of this mild solution is already known, we look at the function $(\tau,\tau_M(u_0)) \ni t \mapsto F(u(t)) \in Y$. If $G(t) = F(u(t))$, then Proposition 3.7 states that for any $\tau < s < t$, given $0 < \mu < \min\{\alpha^2, \omega+\delta-1\}$, there exists $C > 0$, such that $G(\cdot) \in \mathcal{C}^\theta_\psi((\tau,T),Y)$, that is,

$$\|G(t)-G(s)\|_Y \le C(t-s)^\theta(s-\tau)^{-\psi},$$

where $0 < \theta < \min\{\mu, 1-\rho(1-\alpha)\}$, for any $0 < \mu < \min\{\alpha^2, \omega+\delta-1\}$ and

$$-\psi = \min\left\{-\tfrac{\mu}{\alpha} - \rho(1-\alpha), \delta-1-\rho(1-\alpha), \beta-\alpha-\rho(1-\alpha)\right\}.$$

In order to obtain the results claimed in Theorem 4.26, we must have

(i). $\mu > 1-\omega$ and $1-\rho(1-\alpha) > 1-\omega$.

(ii). $-\psi > -1$.

Since $0 < \mu < \min\{\alpha^2, \omega+\delta-1\}$, the interval $(1-\omega, \min\{\alpha^2,\omega+\delta-1\})$ only makes sense if

$$\alpha^2 + \omega - 1 > 0$$
$$\omega + \frac{\delta}{2} - 1 > 0.$$

It follows from $\omega^2+\delta\omega-1>$ that $\omega+\frac{\delta}{2}-1>0$ is already satisfied. The other inequality in item (i) is satisfied if

$$\rho<\frac{\omega}{1-\alpha}.$$

In order for the conditions in (ii) to hold, we have already established in Proposition 3.7 that they must satisfy

$$1\leq\rho\leq\min\left\{\frac{\beta}{1-\alpha}+1,\frac{\delta}{1-\alpha},\frac{\alpha-\mu}{\alpha(1-\alpha)}\right\}.$$

The minimum value allowed for μ such that condition (i) holds is $1-\omega$ and if we replace it in the expression above, we have

$$1\leq\rho\leq\min\left\{\frac{\beta}{1-\alpha}+1,\frac{\delta}{1-\alpha},\frac{\alpha+\omega-1}{\alpha(1-\alpha)}\right\}.$$

If all the previous conditions are satisfied, then Theorem 4.26 states that the mild solution

$$u(t)=U(t,\tau)u_0+\int_\tau^t U(t,s)G(s)ds,\quad t\in(\tau,\tau_M(u_0)),$$

for the nonautonomous linear problem

$$u_t+A(t)u=G(t),t\in(\tau,\tau_M(u_0));\quad u(\tau)=u_0,$$

is a strong solution $u(\cdot)\in\mathcal{C}^1((\tau,\tau_M(u_0)),Y)$. But since $u(\cdot):(\tau,\tau_M(u_0))\to X$ is a mild solution of

$$u_t+A(t)u=F(u),t\in(\tau,\tau_M(u_0));\quad u(\tau)=u_0,$$

and $G(t)=F(u(t))$, it follows that $u(\cdot)$ is a strong $Y-$solution for the semilinear problem and the following theorem is proved.

Theorem 4.27. *Let X,Y be Banach spaces with $X\hookrightarrow Y$. Suppose $A(t),t\in\mathbb{R}$, is $\alpha-$uniformly sectorial in X, $\omega-$uniformly sectorial in Y and $\delta-$uniformly Hölder continuous. Additionally, we assume that $A(t)$, $t\in\mathbb{R}$ satisfies the condition in (P.4) with constant $\beta\in(0,1)$ and $F:X\to Y$ is a nonlinearity satisfying (G) with growth $\rho\geq 1$. We have:*

1. *If $\alpha+\delta>1$ and $\omega+\delta>1$, then the linear process $U(t,\tau)$ exists in $\mathcal{L}(X)$, $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$, respectively.*

2. *If $\omega^2+\omega\delta-1>0$, then $U(t,\tau)$ is strongly differentiable in $\mathcal{L}(Y)$.*

3. *Furthermore, if*

$$\alpha^2+\omega-1>0$$

and

$$1\leq\rho<\min\left\{\frac{\beta}{1-\alpha},\frac{\alpha+\omega-1}{\alpha(1-\alpha)},\frac{\delta}{1-\alpha},\frac{\omega}{1-\alpha}\right\}$$

Then, for every $u_0 \in X$, the initial value problem

$$u_t + A(t)u = F(u), t > \tau, \quad u(\tau) = u_0 \in X,$$

has a unique strong $Y-$solution defined in $(\tau, \tau_M(u_0))$.

The list of necessary conditions to ensure existence of strong solutions is quite extensive. We sum them up in the next table, identifying where they are required. We also refer to each block of conditions (I, II, III or IV) as the conditions on the quadrant I, II, III or IV.

Local well-posedness	
I. Conditions on the existence of $U(t,\tau)$	II. Conditions on the existence of mild solution
$\alpha + \delta > 1$ (in $\mathcal{L}(X)$) $\omega + \delta > 1$ (in $\mathcal{L}(Y,X)$)	$1 \leq \rho < \frac{\beta}{1-\alpha}$
Regular solutions	
III. Strong differentiability $U(t,\tau)$ in $\mathcal{L}(Y)$	IV. Differentiability of the mild solution in Y
$\omega^2 + \omega\delta - 1 > 0$	$\alpha^2 + \omega - 1 > 0$ $1 \leq \rho < \min\left\{\frac{\alpha+\omega-1}{\alpha(1-\alpha)}, \frac{\delta}{1-\alpha}, \frac{\omega}{1-\alpha}\right\}$

Table 4.1: Conditions of Theorem 4.27

In Chapter 2 we started the discussion of local well-posedness for the equation in a domain with a handle and we saw how the conditions on quadrants I and II imply lower bounds for the values of q (and, consequently, upper bounds for the growth of $F_0 : U_p^0 \to U_q^0$). In the next chapter we discuss the conditions on III and IV. There will be several restriction taking place at once, but we will see that $\omega^2 + \omega\delta - 1 > 0$ is the most restrictive one for the example considered and this single condition will determine the maximal growth of F_0 that ensures existence of regular solution.

It is worth mentioning the duality in the origin of those conditions. Some of them come from the fact that we need to ensure good properties for the linear process $U(t,\tau)$. Those are the inequalities located at the left column of the table (quadrants I and III).

On the other hand, the conditions on the right column (II and IV) are those responsible to ensure that the discontinuity at initial time $t = \tau$ does not exceed a limit value that would disrupt the differentiability or existence of solution.

In the usual case where $A(t), t \in \mathbb{R}$, is a family of sectorial operators (studied for instance in [21, 37, 52, 58, 59, 60]), all those conditions are trivially satisfied and we have existence of strong solution for the problem provided F is locally Lipschitz and $A(t)$ uniformly sectorial. Indeed, the right side of the table would not pose any problem (there is no more discontinuity at the initial time $t = \tau$) and conditions on the left side are trivially satisfied (since $\alpha = \omega = 1$). This recovers the content of Theorem 7 of [55].

Moreover, if the initial condition u_0 belongs to D, the continuity obtained for the semigroup and linear process in Lemma 1.5 and Proposition 1.16, respectively, implies that the conditions on the right-column are not necessary, since there is no discontinuity at the initial time.

The nonsingular case, where $A(t) = A$ is a single almost sectorial operator can also be analyzed in view of the conditions posed in Table 4.1. In this case, δ can be chosen as 1 ($\|[A - A]A^{-1}\|_{\mathcal{L}(X)} = 0$) and the conditions one the left side are trivially satisfied.

CHAPTER 5

Domains with a handle: Strong solution

In Chapter 2 we established conditions on p and q that guaranteed existence of local mild solution for (2.1). In this chapter, we continue the analysis of the problem and study the regularity of its solution.

The initial condition (w_0, v_0) is an element chosen in the phase space $X = U_p^0$ and the mild solution obtained,

$$(w, v)(t) = U_0(t, \tau)u_0 + \int_\tau^t U_0(t, s)F_0((w, v)(s))ds,$$

is a function in $\mathcal{C}((\tau, \tau_M(u_0)), U_p^0)$. However, due to the presence of the nonlinearity $F_0 : U_p^0 \to U_q^0$, the differentiability of the process $U_0(t, \tau)$ and of the local mild solution will be studied in $\mathcal{L}(U_q^0)$ and U_q^0, respectively.

5.1 Maximal growth associated to the differentiability of the linear process $U_0(t, \tau)$

The conditions to ensure differentiability of $U_0(t, \tau)$ is the one established in Theorem 4.25

$$\omega^2 + \omega\delta - 1 > 0,$$

where ω is the constant of almost sectoriality of $A_0(t), t \in \mathbb{R}$, such that $0 < \omega < 1 - \frac{N}{2q} =: \omega^+$. Note that this is more restrictive than the condition $\omega + \delta > 1$ necessary to ensure existence of the process, since $\omega, \delta \in (0, 1]$. Therefore, we expect to obtain in this section stricter restrictions on p and ρ.

Lemma 5.1. *Let $\frac{N}{2} < q$. There exists $0 < \omega < 1 - \frac{N}{2q}$ such that $\omega^2 + \omega\delta - 1 > 0$ if and only if*

$$q > \frac{N(\sqrt{4 + \delta^2} + \delta + 2)}{4\delta}. \tag{5.1}$$

Proof. It is enough to obtain a condition on q such that $(\omega^+)^2 + \omega^+\delta - 1 > 0$, that is,

$$\left(1 - \frac{N}{2q}\right)^2 + \left(1 - \frac{N}{2q}\right)\delta - 1 > 0. \tag{5.2}$$

The left side of this inequality has only two roots for $q \in (0, \infty)$ given by the second order polynomial

$$P(q) = (4\delta)q^2 - 2N(\delta + 2)q + N^2,$$

which are

$$q_- = \tfrac{N(-\sqrt{4+\delta^2}+\delta+2)}{4\delta} \quad \text{and} \quad q_+ = \tfrac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta}.$$

Those two roots satisfy $q_- < \frac{N}{2} < q_+$ and the behavior of $\left(1 - \frac{N}{2q}\right)^2 + \left(1 - \frac{N}{2q}\right)\delta - 1$ in terms of q is given by

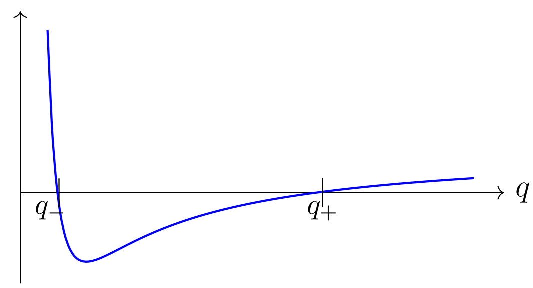


Figure 5.1: Graph of $P(q)$ when $N = 3$ and $\delta = \frac{3}{4}$

Therefore, the range of possible values of q for which (5.2) holds is given by $q > q_+$.

□

Inequality (5.1) allows us to calculate the largest growth F_0 can have so that $U_0(t, \tau)$ is strongly differentiable in $\mathcal{L}(Y)$. We will denote this growth as ρ_{III} and it is given by

$$\rho_{III} = \frac{4\delta p}{N(\sqrt{4+\delta^2} + \delta + 2)}. \tag{5.3}$$

We refer to (5.3) as *maximal growth in order to ensure differentiability of* $U_0(t, \tau)$.

5.2 Maximal growth associated to the regularity of the mild solution (w, v)

To ensure regularity of the mild solution, we must check the remaining conditions stated in Theorem 4.27, which are

$$\alpha^2 + \omega - 1 > 0,$$

$$1 \leq \rho < \min\left\{\frac{\alpha+\omega-1}{\alpha(1-\alpha)}, \frac{\delta}{1-\alpha}, \frac{\omega}{1-\alpha}\right\}.$$

Note that the first inequality provides a lower bound for the values of q:

$$\alpha^2 + w - 1 > 0 \Rightarrow q > \frac{2p^2N}{(2p-N)^2}$$

and those are obtained by replacing α^+, ω^+ in the relation and some manipulation (as it was done in Lemma 5.1, but simpler in this case since the restriction only involves q with a power 1, rather than q^2 as in the lemma just mentioned).

The second inequality provides 3 lowers bounds for q (recalling that $\rho = \frac{p}{q}$):

$$\rho < \frac{\alpha+\omega-1}{\alpha(1-\alpha)} \Rightarrow q > \frac{N(N-4p)}{2(N-2p)} \tag{5.4}$$

$$\rho < \frac{\delta}{1-\alpha} \Rightarrow q > \frac{N}{2\delta} \tag{5.5}$$

$$\rho < \frac{w}{1-\alpha} \Rightarrow q > N \tag{5.6}$$

((5.5) and (5.6) represents no further restriction beyond the ones we already had to ensure existence of local solution).

The local well-posedness of the problem is only guaranteed for $p > \max\{N, \frac{N}{2\delta}\}$, as established in Proposition 2.11. For p in this interval, the lower bounds obtained above for q are less restrictive than the one obtained in (5.1) to ensure differentiability of the process in $\mathcal{L}(Y)$. In other words

$$\max\left\{\frac{2p^2N}{(2p-N)^2}, \frac{N(N-4p)}{2(N-2p)}, \frac{N}{2\delta}, N\right\} \leq \frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta}$$

since $\delta \in (0,1]$.

This allow us to conclude that the differentiability of the mild solution follows as long as

$$\frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta} < q \leq p,$$

that is, the differentiability of the mild solution is guaranteed (for this example) once the differentiability of $U(t,\tau)$ in $\mathcal{L}(Y)$ holds.

We rewrite the conditions of Table 4.1 in terms of the restrictions obtained for q:

Local well-posedness	
I. Conditions on the existence of $U_0(t,\tau)$	II. Conditions to control the discontinuity at $t=\tau$
$\frac{N}{2\delta} < q \leq p$	$\frac{p(2N+1)}{2p+1} < q \leq p, \quad (p > N)$
Regular solutions	
III. Strong differentiability $U_0(t,\tau)$ in $\mathcal{L}(Y)$	IV. Regularity mild solution in Y
$\frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta} < q \leq p$	$\frac{2p^2N}{(2p-N)^2} < q \leq p$ $\max\left\{\frac{N(N-4p)}{2(N-2p)}, \frac{N}{2\delta}, N\right\} < q \leq p$

Table 5.1: Lower bounds for q

5.3 Strong U_q^0-solution

We already compared the conditions of quadrant IV to the condition on III. In a similar way, the lower bound for q obtained in III is more restrictive than the lower bound for q obtained in I and II, that is,

$$\max\left\{\frac{N}{2\delta}, \frac{p(N+1)}{2\delta p+1}\right\} \leq \frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta},$$

provided that $p > \max\{N, \frac{N}{2\delta}\}$, which is the case, since those are necessary conditions to ensure existence of local mild solution (quadrant I and II).

This implies that the maximal growth in order to ensure differentiability of $U_0(t,\tau)$, ρ_{III}, is smaller than ρ_I or ρ_{II}, calculated in (2.16) and (2.18). We gather those results in the following proposition:

Proposition 5.2. *Assume that $p > N$ and $\frac{N(\sqrt{4+\delta^2}+\delta+2)}{4\delta} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ satisfies (A.2) and (A.3) and $f : \mathbb{R} \to \mathbb{R}$ satisfies (A.4). Then* (2.4) *have a strong $Y-$solution $(w,v)(\cdot) : (\tau, \tau_M(u_0)) \to U_p^0$ given by*

$$(w,v)(t) = U_0(t,\tau)u_0 + \int_\tau^t U_0(t,s)F_0((w,v)(s))ds.$$

For instance, if $N = 3$ and $\delta = \frac{3}{4}$, we have the shaded region below that comprehends the possible values for p and ρ that ensures existence of strong $Y-$solution for the problem:

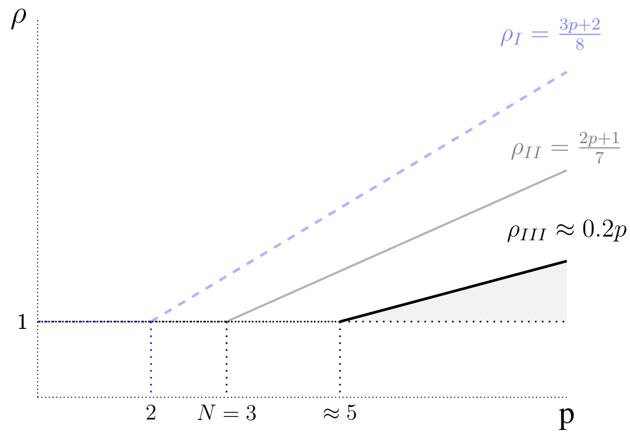


Figure 5.2: Maximal growth $N = 3$, $\delta = \frac{3}{4}$

In this case where $\delta = \frac{3}{4}$, the conditions associated to the discontinuity at $t = \tau$ are more restrictive. The translucent lines represent the conditions obtained in quadrant I and II. On the other hand, if $\delta = \frac{1}{4}$, for example, the conditions on differentiability of $U_0(t, \tau)$ (represented by the shaded triangle) become way restrictive as we see in Figure 5.3.

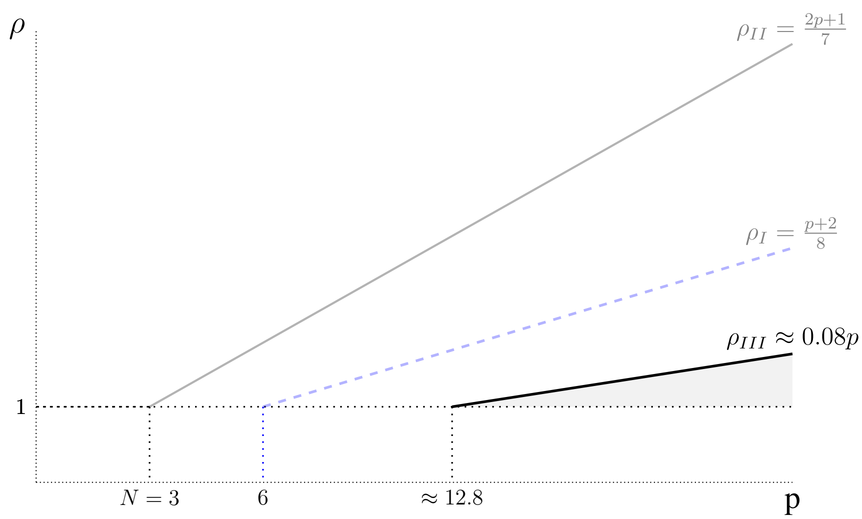


Figure 5.3: Maximal growth $N = 3$, $\delta = \frac{1}{4}$

Part III: Long-time behavior

In the last part of this work we focus our attention at the long-time behavior of the solution rather than its local properties. From Theorem 1.25, the singularly nonautonomous problem satisfies a *blow-up alternative*, that is, for each $u_0 \in X$, the solution $u : (\tau, \tau_M(u_0)) \to X$, starting at u_0 at the initial time τ and defined in its maximal interval, satisfies:

$$\tau_M(u_0) = +\infty \text{ or } \limsup_{t \to \tau_M(u_0)^-} \|u(t)\|_X = +\infty.$$

Therefore, estimates for $u(t)$ in the phase space X are useful in order to prove its global existence (also called *global well-posedness* of the problem).

We dedicate this part to present an iterative procedure that allow us to obtain estimates for the solutions in L^{2^k}, for any $k \in \mathbb{N}$, as well as identifying in the phase spaces sets with the properties of attracting the solutions to them as the evolution of the dynamics takes place. Those sets are called *pullback attracting sets* and they play an important role in describing the asymptotic dynamics of the system.

The iterative procedure and the consequently existence of pullback attractor for the equation in the domain with a handle are presented in [12].

CHAPTER 6

Global well-posedness and asymptotic dynamics

In previous chapters we mentioned that the regularity properties of the solution for the semilinear problem could be helpful if one wishes to obtain estimates of the solution directly from the differential equation. Since we know $u(\cdot) \in \mathcal{C}^1((\tau, \tau_M(u_0)), Y)$ and $u(t) \in D^Y$ for $t \in (\tau, \tau_M(u_0))$, we could perform some operations on the equation

$$u_t + A(t)u = F(u), \quad t > \tau, \tag{6.1}$$

to extract informations about $\|u\|_Y$, $\|u_t\|_Y$ and $\|A(t)u\|_Y$.

To illustrate the ideas, consider the simple case presented in Example 1.19, where $A = -\Delta$ is the Laplacian operator with Neumann boundary condition in a smooth bounded domain Ω, acting in $L^2(\Omega)$. If $u(\cdot)$ is a solution of $u_t = \Delta u + F(u)$ with regularity $u(\cdot) \in \mathcal{C}^1((\tau, \tau_M(u_0)), L^2(\Omega))$ and $u(t) \in H^2_{\mathcal{N}}$ for $t \in (\tau, \tau_M(u_0))$, then we could take the inner product in $L^2(\Omega)$ of the equation with u:

$$\int_\Omega u_t u dx = \int_\Omega (\Delta u) u dx + \int_\Omega F(u) u dx.$$

The regularity that u has allows us to integrate by parts and obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} = -\|\nabla u\|^2_{L^2} + \int_\Omega F(u) u dx.$$

This equality, added to a suitable dissipativeness assumption on F, allows us to obtain estimates of $\|u\|_{L^2}$. Those ideas will be discussed in Chapter 7 to the reaction-diffusion equation in the domain with a handle.

Moreover, if u satisfies the equation (6.1), then any estimative for two of the three terms that features in this equation, implies an estimative for the third one. For instance, if we prove that $F(u)$ and u_t are bounded in Y, then $A(t)u = F(u) - u_t$ is also bounded in Y, implying u bounded in the stronger space D^Y. This chapter presents some techniques that allows us to obtain estimates for $\|u\|$ and $\|u_t\|$.

6.1 Estimates for u_t in Y

When it comes to obtain estimates for u_t, one could try in the example above to take the inner product of the equation by u_t and to perform some calculation over it. However, it would be necessary to integrate by parts and a term $\nabla(u_t)$ would appear. With the information we have so far on the regularity of u_t ($u_t \in \mathcal{C}((\tau, \tau_M(u_0)), Y)$), this integration by parts would not be justified.

There are techniques in the literature that overcome this problem by working with regular approximations of u_t (for example, in [29, 30] the use of Steklov average to overcome the lack of regularity in the derivative, or the use of Galerkin approximation in [48, 54]). However, the time-dependence of the family $A(t)$ makes it difficult to apply those techniques.

On the other hand, the properties obtained so far for the linear process $U(t, \tau)$ associated to the family $A(t)$, $t \in \mathbb{R}$, allow us to obtain estimates for u_t by working with the mild formulation of u, (1.19). For the nonsingular ($A(t) = A$) and sectorial case, this type of result can be found in the Appendix Section of [18]. In order to reply it to the singular and almost sectorial case, we suppose that the following scenario for the problem holds:

(S). Let $A(t), t \in \mathbb{R}$, be a family of almost sectorial operators and $F : X \to Y$ a nonlinearity as in Theorem 4.27. For $u_0 \in X$, let $u = u(\cdot, \tau, u_0)$ denotes the strong $Y-$solution of the semilinear problem.

Assume that given a bounded set $B \subset X$ and $T > \tau$, there exists a constant $C = C(T, B)$ depending on T and B, such that

$$\sup_{t\in(\tau,T]} \sup_{u_0\in B} \|u(t, \tau, u_0)\|_X \leq C(T, B)(t - \tau)^{\alpha-1}, \tag{6.2}$$

where α is the constant of almost sectoriality of $A(t)$, $\alpha \in (0, 1)$.

The estimate (6.2) on bounded sets for u is transferred through the variation of constants formula to the derivative u_t, as we prove in the sequel.

Theorem 6.1. *Let X, Y be Banach spaces and assume that (S) is satisfied. Given any $T > \tau$ and B bounded set in X, there exists a constant D depending only on T and B, such that*

$$\sup_{t\in(\tau,T]} \sup_{u_0\in B} \|u_t(t, \tau, u_0)\|_Y < D(T, B)(t - \tau)^{\omega-2},$$

where ω is the almost sectoriality constant of $A(t)$ in Y.

Proof. From the conditions required in (S), the solution $u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)F(u(s))ds$ is differentiable in t and, by taking the derivative over its mild formulation, we obtain

$$u_t(t) = -A(t)U(t, \tau)u_0 - A(t)\int_\tau^t U(t, s)F(u(s))ds + F(u(t))$$

$$=-A(t)U(t,\tau)u_0 - A(t)\int_\tau^t U(t,s)[F(u(s))-F(u(t))]ds - A(t)\int_\tau^t U(t,s)F(u(t))ds + F(u(t)).$$

Since $\|u(t)\|_X \leq C(T,B)(t-\tau)^{\alpha-1}$, the growth condition on Fimplies that

$$\|F(u(t))\|_Y \leq C(1+\|u(t)\|_X^\rho) \leq D(T,B)(t-\tau)^{-\rho(1-\alpha)}. \tag{6.3}$$

We use this estimates and the results obtained in Chapter 4 to evaluate each term in the above equality for u_t. Since $X \hookrightarrow Y$, it follows from Theorem 4.25 that

$$\|A(t)U(t,\tau)u_0\|_Y \leq C(t-\tau)^{\omega-2}\|u_0\|_Y \leq C(t-\tau)^{\omega-2}\|u_0\|_X.$$

From Lemma 4.18, $A(t)\int_\tau^t U(t,s)ds$ is a bounded linear operator satisfying

$$\left\|A(t)\int_\tau^t U(t,s)ds\right\|_{\mathcal{L}(Y)} \leq C(t-\tau)^{\omega-1}.$$

Therefore, from the estimate on $\|F(u(t))\|_Y$, we obtain

$$\begin{aligned}\left\|A(t)\int_\tau^t U(t,s)F(u(t))ds\right\|_Y &\leq D(T,B)(t-\tau)^{\omega-1-\rho(1-\alpha)}\\ &\leq D(T,B)(t-\tau)^{\omega-2},\end{aligned}$$

since $\rho(1-\alpha) < \beta < 1$.

For the remaining term, recall from Proposition 3.7 (and Remark 3.10) that there exist $\theta > 0$ and $\psi \in (0,1)$ such that

$$\|F(u(t)) - F(u(s))\|_Y \leq C(t-s)^\theta(s-\tau)^{-\psi}.$$

Therefore,

$$\begin{aligned}\left\|A(t)\int_\tau^t U(t,s)[F(u(s))-F(u(t))]ds\right\|_Y &\leq C(T,B)\int_\tau^t (t-s)^{\omega-2}(t-s)^\theta(s-\tau)^{-\psi}ds\\ &\leq C(T,B)(t-\tau)^{\omega+\theta-1-\psi}\mathcal{B}(\omega+\theta-1,1-\psi)\\ &\leq D(T,B)(t-\tau)^{\omega-2}.\end{aligned}$$

The conclusion now follows from the above estimates. □

Rewriting the semilinear problem as

$$A(t)u = F(u) - u_t,$$

we note that if the right side is bounded in $(\tau,T]$ (uniformly in $B \subset X$), then $\|A(t)u\|_Y$ also is, or equivalently, $\|u(t)\|_{D^Y}$.

Corollary 6.2. *Under the conditions of Theorem 6.1,*

$$\sup_{t\in(\tau,T]}\sup_{u_0\in B}\|u\|_{D^Y} < D(T,B)(t-\tau)^{\omega-2}.$$

The same result holds if $A(t)$ is sectorial ($\alpha = \omega = 1$). The estimate in this case is slightly improved, since $A(t) \int_\tau^t U(t,s)ds$ is a bounded linear operator with norm no larger than $C(t-\tau)^{-1}$ (it is enough to replay the almost sectorial case with constant of sectoriality equals 1). Recall that for the sectorial case, all conditions on Table 4.1 are trivially satisfied. Moreover, in the sectorial case there is no discontinuity at the initial time $t = \tau$ and the estimate can be taken in $[\tau, T]$.

This justifies the following corollary that complements the theory for nonsingular problems for the sectorial case.

Corollary 6.3. *Suppose that $A(t)$, $t \in \mathbb{R}$, is a uniformly sectorial family of linear operators, $T > \tau$ and $B \subset X$ is a bounded set. If $F : X \to Y$ is a nonlinearity with growth condition (G) and the compatibility condition (P.4) between $A(t)$ holds, then, whenever a solution $u(t)$ of the semilinear problem*

$$u_t + A(t)u = F(u), t > \tau; \quad u(\tau) = u_0 \in X,$$

is bounded in $[\tau, T]$,

$$\sup_{t\in[\tau,T]} \sup_{u_0\in B} \|u_t(t,\tau,u_0)\|_X < C(T,B),$$

the derivative u_t is also bounded in the same interval

$$\sup_{t\in[\tau,T]} \sup_{u_0\in B} \|u_t(t,\tau,u_0)\|_Y < D(T,B)(t-\tau)^{-1},$$

where $C(T,B)$ and $D(T,B)$ are positive constants that depend only on $T > \tau$ and $B \subset X$.

6.2 Fractional powers of sectorial operators and smoothing effect of the differential equation

Theorem 6.1 on its on is not enough to deal with the application we are investigating in this work: the reaction-diffusion equation in Ω_0. As we will see in the next chapter, it will be necessary to estimate the solution (w, v) in more regular spaces than U_p^0 in order to obtain the existence of compact attracting sets. To tackle this problem we will use one of its features: the fact that the system is weakly coupled.

By decoupling the problem, we will obtain two equations, one in Ω and other in R_0, and each one of them, separately, will be associated to a family of sectorial operators ($\alpha = \omega = 1$).

In order to take advantage of this fact, we introduce in this section the definition of fractional powers of sectorial operators and we study the smoothing effect that the differential equation with sectorial operators has.

Therefore, unlike the rest of this work, this section is focused exclusively in the case where the family $A(t), t \in \mathbb{R}$, is sectorial.

Remark 6.4. *The fractional powers of almost sectorial operators can also be defined (as it is done, for instance, in [19]). However, due to the resolvent deficiency, some restrictions on the values of the powers* *available for the almost sectorial operator A emerges. For instance, A^{θ}, for a real number θ, is only defined if $1-\alpha<\theta<1$, where α is the constant of almost sectoriality.*

The restrictions increases if we wish to use a momentum type inequality. For this case, $1-\alpha<\theta<\alpha$ (see [19, (8)]). Those conditions difficult the analysis of smoothing effect in which features almost sectorial operators.

We assume that $A : D(A) \subset X \to X$ is a sectorial operator in a Banach space X such that $Re(\sigma(A)) > 0$. We refer to it as being *sectorial and positive*.

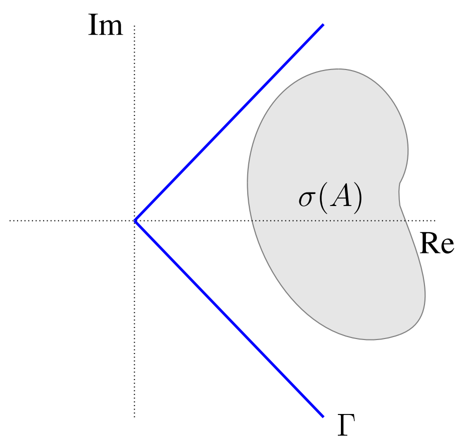


Figure 6.1: Sectorial and positive operator

For an operator A with those properties, if Γ is a curve as above, orientated from $\infty e^{-i\psi}$ to $\infty e^{i\psi}$ ($\psi \in (0, \frac{\pi}{2})$) in a way that $\sigma(A)$ is contained inside the region determined by it, then, for $\theta > 0$, $A^{-\theta}$ given by

$$A^{-\theta} = \frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\theta}(\lambda + A)^{-1}d\lambda. \tag{6.4}$$

For a deeper discussion on fractional powers of unbounded operators, we recommend [3, 49]. There are expressions equivalent to (6.4) that also defines the fractional power of A, as one can see in [11, 46, 47], even for more general operators A. We focus on the case A is sectorial and positive and we gather some properties of this operator in the sequel.

Proposition 6.5. *[52, Section 2.6] Let $A : D(A) \subset X \to$ be a sectorial and positive operator. Then, for any $\theta > 0$, the operator $A^{-\theta}$ given in* (6.4) *is well defined in X and it is a bounded linear operator. Moreover,*

1. *$A^{-\theta} : X \to X$ is one-to-one.*

2. *For $\xi, \theta > 0$, $A^{-\theta}A^{-\xi} = A^{-\theta-\xi}$.*

The fact that $A^{-\theta}$ is one-to-one allows the following definition:

Definition 6.6. *For every $\theta > 0$, $A^\theta := (A^{-\theta})^{-1}$. We denote by X^θ the domain of A^θ equipped with the norm*

$$\|x\|_{X^\theta} = \|A^\theta x\|_X, \quad \forall x \in D(A^\theta),$$

and we refer to X^θ as a fractional power space. *For $\theta = 0$, $X^0 = X$.*

Those fractional power spaces $\{X^\theta\}_{\theta \geq 0}$ are Banach and they establish a scale of spaces, as we see in the next proposition.

Proposition 6.7. *[25, Section 1.3.3] For each $\theta > 0$, X^θ is a Banach space. Moreover, if $0 \leq \theta < \xi$, then X^ξ is continuously embedded in X^θ, which we denote by $X^\xi \hookrightarrow X^\theta$. If A has compact resolvent, then X^ξ is compactly embedded in X^θ, denoted by $X^\xi \overset{c}{\hookrightarrow} X^\theta$.*

We return to the family $A(t)$, $t \in \mathbb{R}$, of uniformly sectorial ($\alpha = 1$) and uniformly $\delta-$Hölder continuous operators. For each $t \in \mathbb{R}$, $A(t)$ has its fractional powers well-defined.

As a consequence of the Hölder continuity (P.3), given any arbitrarily large compact set in $\mathbb{R}^2$, there exists a constant $C > 0$ such that $\|A(t)A^{-1}(\tau)\|_{\mathcal{L}(X)} \leq C$, for all (t, τ) in this compact set. In this case, for $t, \tau \in [-M, M]$, the norms $\|\cdot\|_{D(A(t))} = \|A(t)\cdot\|_X$ and $\|\cdot\|_{D(A(\tau))} = \|A(\tau)\cdot\|_X$ defined by the operators $A(t)$ and $A(\tau)$, respectively, are equivalent. In the same way, $\|\cdot\|_{D(A(t)^\theta)} = \|A(t)^\theta\cdot\|_X$ and $\|\cdot\|_{D(A(\tau)^\theta)} = \|A(\tau)^\theta\cdot\|_X$ are equivalent.

We will fix one operator $A_0 = A(t_0)$ as a reference and we shall refer to this norm as $\|\cdot\|_{X^1}$. We notate by X^θ the domain of $A_0^\theta = A(t_0)^\theta$ endowed with the norm $\|\cdot\|_{X^\theta} = \|A_0^\theta\cdot\|_X$. From the equivalence obtained above, we can refer to X^θ as domain of any operator $A(t)^\theta$.

The scale of fractional powers associated to $A(t)$, $t \in \mathbb{R}$, allows us to work with semilinear problems in the following setting:

$$\begin{aligned} &u_t + A(t)u = F(t, u), \quad t > \tau; \\ &u(\tau) = u_0 \in X^\theta, \end{aligned} \tag{6.5}$$

where F is a nonlinearity such that $F : \mathbb{R} \times X^\theta \to X$, $0 \leq \theta < 1$. In [55, Theorem 7]) local well-posedness of (6.5) was proved, which we state next:

Theorem 6.8. *[55, Theorem 7] Let $A(t)$, $t \in \mathbb{R}$, be a family of uniformly sectorial operators ($\alpha = 1$) and uniformly $\delta-$Hölder continuous, and $F : \mathbb{R} \times X^\theta \to X$ a locally Hölder continuous function in the first variable and locally Lipschitz in the second variable, $0 \leq \theta < 1$. Then, given any $u_0 \in X^\theta$,*

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)F(s, u(s))ds$$

is a strong solution for (6.5)*, that is,*

1. *$u(\cdot) \in \mathcal{C}([\tau, T), X) \cap \mathcal{C}^1((\tau, T), X)$ and $u(t) \in D$, for $\tau < t < T$;*

2. *u satisfies the equation in the usual sense $\frac{d}{dt}u(t,x) = -A(t)u(t,x) + F(u(t,x))$, for all $t \in (\tau, T)$.*

Moreover, if $\|u(t)\|_{X^\theta}$ is bounded in any bounded set $[\tau, t^*]$, then the solution is globally defined in time.

Note that this type of result is similar to Theorem 4.27 proved in Chapter 4. The parabolic structure of the problem allows us to obtain regularity for the solution $u(t)$, that is $u(t) \in X^\xi$ for $t \in (\tau, T)$ and any $0 \leq \xi \leq 1$. As before, we call this property *regularization*.

Another important feature of parabolic equation is the one called *smoothing effect* which refers to the increase of regularity of $u_t(t)$. From Theorem 6.8 above, $u_t(t) \in X$, but we can prove that the derivative belongs to better spaces X^ξ, with $0 \leq \xi < \delta$, where δ is the Hölder exponent of the family $A(t)$.

The theorem stated in the sequel extends Theorem 3.5.2 in [37], where the author proved the smoothing effect to the nonsingular ($A(t) = A$) and sectorial case. For the singular case with sectorial operators, an indication of such result is presented in [55]. We provide a proof of such fact in Appendix A, based on the theory developed in the previous chapter. This result was presented in [13] alongside its consequence in the study of the asymptotic dynamics of singularly nonautonomous problems.

Theorem 6.9. *Let $A(t)$, $t \in \mathbb{R}$, be a family of uniformly sectorial operators ($\alpha = 1$) and uniformly $\delta-$Hölder continuous, and $F : X^\theta \to X$ a locally Lipschitz function, $0 \leq \theta < 1$. If $u : [\tau, T) \to X$ is the solution of*

$$u_t(t) + A(t)u = F(u), t \in (\tau, T); \quad u(\tau) = u_0 \in X^\theta,$$

then, for any $0 \leq \xi < \delta$, $u_t(t) \in X^\xi$ and satisfies the estimate

$$\|u_t(t)\|_{X^\xi} \leq C(t-\tau)^{-1-\xi+\theta}\|u_0\|_{X^\theta}.$$

The illustration below describes the regularization effect (blue) and the smoothing effect (red) of the parabolic equation. In the vertical axis we represent the scale of fractional powers X^ξ, starting at X^0 and increasing upwards. In the horizontal axis, we have two points representing the solution u and its derivative u_t. The vertical lines denote the spaces u/u_t belongs.

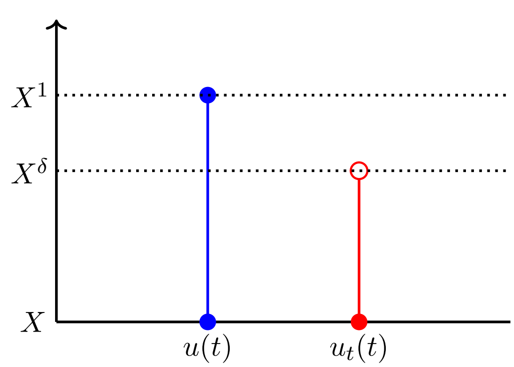


Figure 6.2: Regularization and smoothing effect

Remark 6.10. *The smoothing effect will play an essential role in the application when we transfer the information of the dynamics in Ω to the channel R_0. Estimates of w_t in stronger norms will be required at this point.*

To help us obtain estimate of the solutions in more regular spaces, we have the next results that provide estimates of the linear operators $T_{-A(t)}(s)$ and $U(t,\tau)$ in the space $\mathcal{L}(X, X^\theta)$.

Proposition 6.11. *[37, Theorem 1.4.3] If $A(\tau)$ is sectorial and $Re(\sigma(A(\tau))) > 0$, for any $\theta, \gamma \geq 0$ such that $\gamma < \theta$, there exists a constant $C(\theta,\gamma) > 0$ such that $\|T_{-A(\tau)}(t)\|_{\mathcal{L}(X^\gamma, X^\theta)} \leq Ct^{-\beta+\gamma}$, for all $\tau > 0$.*

Proposition 6.12. *[21, Theorem 2.2] Let $\tau < t$ and $0 \leq \gamma \leq \theta < 1 + \delta$. Then*

$$\|A(t)^\beta U(t,\tau) A(\tau)^{-\gamma}\|_{\mathcal{L}(X)} \leq C(\theta,\beta)(t-\tau)^{\gamma-\beta}.$$

6.3 Pullback attractor

In this last section we provide a brief review in the theory of pullback attractors. For more details, we recommend [17, 20, 24]. Let X be a Banach space and $\{S(t,\tau) : X \to X; t \geq \tau\}$ a family of operators satisfying:

1. $S(t,t) = I_X$, for all $t \in \mathbb{R}$.
2. $S(t,s) = S(t,\tau)S(\tau,s)$, for all $t \geq \tau \geq s$, $s \in \mathbb{R}$.
3. $(s,\infty) \ni t \mapsto S(t,s)x$ is continuous for all $x \in X$.

Such family is called a *process in* X and we also denote it by $S(\cdot,\cdot)$. We will usually call it *nonlinear process* to distinguish from the family $U(t,\tau)$ obtained in Definition 1.14.

Given any $x \in X$, there are two distinct manners of studying the asymptotic dynamics of such evolution process: One called the *pullback dynamics* that basically fixes the final time t and evaluate what happens to $S(t,s)x$ when $s \to -\infty$ and the other called *forward dynamics*, which consider $S(t,s)x$ when s is fixed and $t \to \infty$. The pullback dynamics can be described by an object in the phase space called *pullback attractor*. We recall in the sequel some basic concepts and results of the theory of pullback attractor.

To compare the distance between two sets in the phase space X, we use the Hausdorff semidistance: given $A, B \subset X$, the *Hausdorff semidistance* between A and B is

$$dist(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b).$$

This semidistance measures how much the set A is located outside the set B. If A and B are closed, then $dist(A,B) = 0 \Rightarrow A \subset B$.

Definition 6.13. *Let $S(\cdot,\cdot)$ be a process. A family $A(\cdot) = \{A(t) \subset X; t \in \mathbb{R}\}$* pullback attracts *$B \subset X$ if, for each $t \in \mathbb{R}$,*

$$dist(S(t,s)B, A(t)) \xrightarrow{s\to-\infty} 0.$$

Definition 6.14. *The* pullback attractor *of $S(\cdot,\cdot)$ is a family $\mathcal{A}(\cdot) = \{\mathcal{A}(t) \subset X;\ t \in \mathbb{R}\}$ that satisfies:*

1. *$\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$.*
2. *$\mathcal{A}(\cdot)$ is* invariant *by $S(\cdot,\cdot)$, that is, $S(t,s)\mathcal{A}(s) = \mathcal{A}(t)$, for all $t \geq s$, $s \in \mathbb{R}$.*
3. *$\mathcal{A}(\cdot)$ pullback attracts bounded sets of X.*
4. *$\mathcal{A}(\cdot)$ is the minimal closed family that satisfies (3).*

The existence of such object in the phase space is guaranteed whenever we find a family of compact pullback attracting sets.

Theorem 6.15. *[20, Theorem 2.12] Let $S(\cdot,\cdot)$ be a process. The statements below are equivalent:*

1. *$S(\cdot,\cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$.*
2. *There exists a family of compact sets $K(\cdot)$ that pullback attracts bounded sets of X.*

Corollary 6.16. *If there exists a fixed compact set $K \subset X$ such that, for any bounded set $B \subset X$*

$$dist(S(t,s)B, K) \to 0 \text{ when } s \to -\infty,$$

then $S(\cdot,\cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$ such that $\cup_{t\in\mathbb{R}}\mathcal{A}(t) \subset K$.

The description of the pullback attractor can be given in terms of global bounded solutions, which we define in the sequel.

Definition 6.17. *A continuous function $\xi : \mathbb{R} \to X$ is a* global solution *of $S(\cdot,\cdot)$ if for any $t, s \in \mathbb{R}$, $t \geq s$,*

$$S(t,s)\xi(s) = \xi(t).$$

Moreover, we say that a global solution $\xi(\cdot) : \mathbb{R} \to X$ of $S(\cdot,\cdot)$ is bounded in the past *if there exists $\tau \in \mathbb{R}$ such that $\{\xi(t) :\ t \leq \tau\}$ is bounded in X.*

From the pullback attraction property and the invariance of the global solution, we readily obtain:

Proposition 6.18. *Let $S(\cdot,\cdot)$ be a process with pullback attractor $\mathcal{A}(\cdot)$. If $\xi(\cdot)$ is a global solution bounded in the past, then $\xi(t) \in \mathcal{A}(t)$, $\forall t \in \mathbb{R}$.*

Therefore, if $\mathcal{U}(t) = \{\xi(t) : \ \xi(\cdot) : \mathbb{R} \to X$ is a global solution bounded in the past$\}$ and $\{\mathcal{A}(t); t \in \mathbb{R}\}$ is the pullback attractor for the process $S(\cdot,\cdot)$, then

$$\mathcal{U}(t) \subset \mathcal{A}(t), \ \forall t \in \mathbb{R}.$$

To achieve a characterization for the pullback attractor in terms of global bounded solutions, as we have for the global attractor, it is necessary to require a certain boundedness for the pullback attractor.

Proposition 6.19. *[20, Theorem 1.17] Suppose that the pullback attractor $\mathcal{A}(\cdot)$ for the process $S(\cdot,\cdot)$ is bounded in the past. Then*

$$\mathcal{A}(t) = \mathcal{U}(t) = \{\xi(t) : \ \xi(\cdot) : \mathbb{R} \to X \textit{ is a global solution bounded in the past}\}.$$

CHAPTER 7

Domains with a handle: Attractors

In this chapter we study the asymptotic dynamics of the singularly nonautonomous reaction-diffusion equation (2.1). The majority of the content developed here was compiled and presented in [12]. Note that the iteration technique developed in this chapter, Section 7.1, is quite general and can be be applied to other parabolic problems in which the linear operator is a second order regular elliptic boundary value problem.

In order to study the asymptotic dynamics of the problem, we will take advantage of the fact that the system is weakly coupled and the equation in Ω is independent of the dynamics in the channel. In Ω the problem is

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \partial_n w = 0, & x \in \partial\Omega, \\ w(\tau) = w_0 \in L^p(\Omega). \end{cases} \tag{7.1}$$

If we denote $B(t)w = -div(a(t,x)\nabla w) + w$ and $D(B(t)) = \{w \in W^{2,p}(\Omega); \partial_n w = 0\}$ then, proceeding as it is done in Lemma 2.4, we obtain the following properties for this family:

Lemma 7.1. *Let $B(t)$, $t \in \mathbb{R}$, be the family of linear operators $B(t)w = -div(a(t,x)\nabla w) + w$, defined in $D(B(t)) = D(B) = \{W^{2,q}(\Omega); \partial_n w = 0\}$. This family satisfies:*

(1) $B(t)$, $t \in \mathbb{R}$, is uniformly sectorial and uniformly $\delta-$Hölder continuous

$$\|[B(t) - B(s)]B(\tau)^{-1}\|_{\mathcal{L}(L^q(\Omega))} \leq C|t-s|^\delta, \quad \textit{for all } \tau, s, t \in \mathbb{R}.$$

(2) Each operator $B(t)$ is positive (in the sense that $Re(\sigma(B(t))) > 0$) and their fractional powers $B(t)^\theta$, $\theta \in \mathbb{R}$, are well-defined. We denote $Y_q^\theta = D(B(t)^\theta)$, which is independent of t.

(3) Those spaces define a scale of fractional power spaces Y_q^θ, $\theta > 0$, such that the following embeddings hold

$$
\begin{aligned}
&Y_q^\theta \hookrightarrow C^{1,\eta}(\Omega) \quad && \textit{for some } \eta > 0, \textit{ if } \theta > \tfrac{1}{2} + \tfrac{N}{2q},\\
&Y_q^\theta \hookrightarrow C^{\nu}(\Omega) \quad && \textit{for some } \nu > 0, \textit{ if } \theta > \tfrac{N}{2q},\\
&Y_q^\theta \hookrightarrow L^{r}(\Omega) \quad && \textit{when } -\tfrac{N}{r} < 2\theta - \tfrac{N}{q}, \textit{ for } r \geq q.
\end{aligned}
$$

In particular, $Y_q^\theta \hookrightarrow L^p(\Omega)$ *if* $\theta > \frac{N}{2q}\left(\frac{\rho-1}{\rho}\right)$, *where* $\rho = \frac{p}{q}$.

(4) If $0 \leq \theta < \xi \leq 1$, *then* Y_q^ξ *is compactly embedded in* Y_q^θ.

(5) The spectrum of $B(t)$ *consists entirely of isolated eigenvalues, all of them positive and real. To be precise,*

$$\sigma(B(t)) = \{\mu_i(t) : 1 = \mu_1(t) \leq \mu_2(t) \leq ... \leq \mu_n(t) \leq ...\}.$$

The embedding results can be found in [37, Theorem 1.6.1]. Statement (4) is a consequence of the compactness of the resolvent of $B(t)$ (see Proposition 6.7). We drop the dependence of Y_q^θ on t due to the fact that $\|B(t)B(s)^{-1}\|_{\mathcal{L}(L^q(\Omega))} \leq C$ for all $t, s \in \mathbb{R}$. The last statement follows from Proposition 2.5.

We illustrate embeddings of item (3) above in Figure 7.1. Recall that, in order for the problem to be well defined, $q > \frac{N}{2}$ (see (2.10)) and $\frac{1}{2} + \frac{N}{2q} < 1$.

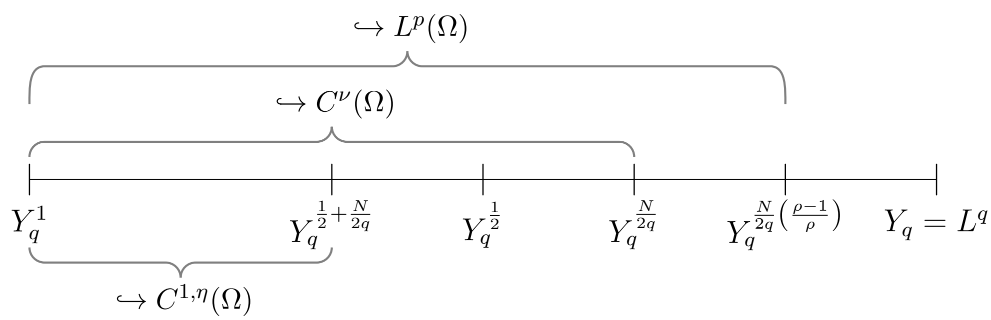


Figure 7.1: Embeddings of Y_q^θ.

Remark 7.2. *It is usual when we are dealing with parabolic problems to consider an equation like*

$$u_t + Au = F(u),\ t > \tau; \quad u(\tau) = u_0 \in X^\alpha,$$

where $-A$ *generates an analytic semigroup in the Banach space* X *and* $F : X^\alpha \to X$, *for some* $\alpha \in [0, 1)$. *Even though the above equation is in* X *and the linear semigroup* e^{-At} *belongs to* $\mathcal{L}(X)$, *we can consider initial conditions in the more regular space* X^α, $0 \leq \alpha < 1$, *and search for a nonlinear semigroup (solution of the problem)* $S(t) : X^\alpha \to X^\alpha$. *To obtain global well-posedness and existence of attractor for* $S(t)$ *in* X^α, *we must obtain estimates of the* $X^\alpha-$*norm of the solution, that is,* $\|\cdot\|_{X^\alpha} = \|A^\alpha \cdot\|_X$.

In the same spirit, the reaction-diffusion equation (2) *studied in this work is such that the equality is satisfied in the Banach space* $U_q^0 = L^q(\Omega) \times L^q(0, 1)$. *However, the initial conditions are given in* U_p^0

and we search for nonlinear process (obtained by the solutions of the problem) and attractor both in U_p^0. In other words U_p^0 plays the role that the space X^α plays in usual parabolic problems whereas U_q^0 plays the role of the space X.

For this reason, we will focus next on obtaining estimates in $L^p(\Omega) \times L^p(0,1)$, as well as compact attracting sets embedded in this space.

In the same way that we did for the equation in Ω_0, let $F : L^p(\Omega) \to L^q(\Omega)$ be the Nemytskii operator associated to f. From (2.8) and (2.9), we obtain:

Lemma 7.3. *For any $\theta > \frac{N}{2q}\left(\frac{\rho-1}{\rho}\right)$, the nonlinearity $F : Y_q^\theta \subset L^p(\Omega) \to L^q(\Omega)$ is locally Lipschitz and satisfies*

$$\begin{aligned} \|F(w) - F(\tilde{w})\|_{L^q(\Omega)} &\leq C\|w - \tilde{w}\|_{Y_q^\theta}(1 + \|w\|_{Y_q^\theta}^{\rho-1} + \|\tilde{w}\|_{Y_q^\theta}^{\rho-1}), \\ \|F(w)\|_{L^q(\Omega)} &\leq C(1 + \|w\|_{L^p(\Omega)}^\rho) \leq C(1 + \|w\|_{Y_q^\theta}^\rho). \end{aligned} \tag{7.2}$$

Proof. Proceeding as in Lemma 2.7 (but only for Ω), we obtain

$$\begin{aligned} \|F(w) - F(\tilde{w})\|_{L^q(\Omega)} &\leq C\|w - \tilde{w}\|_{L^p(\Omega)}(1 + \|w\|_{L^p(\Omega)}^{\rho-1} + \|\tilde{w}\|_{L^p(\Omega)}^{\rho-1}), \\ \|F(w)\|_{L^q(\Omega)} &\leq C(1 + \|w\|_{L^p(\Omega)}^\rho). \end{aligned}$$

If for a given $\theta \in (0,1)$, $Y_q^\theta \hookrightarrow L^p(\Omega)$, then $\|\cdot\|_{L^p(\Omega)} \leq C\|\cdot\|_{Y_q^\theta}$ and the desired estimates follow. According to Lemma 7.1, and recalling that $\rho = \frac{p}{q}$, this embedding occurs for $\theta > \frac{N(p-q)}{2qp} = \frac{N}{2q}\left(\frac{\rho-1}{\rho}\right)$. ☐

Therefore the linear family $B(t), t \in \mathbb{R}$, and the nonlinearity $F : Y_q^\theta \to L^q(\Omega)$, for a fixed $\theta > \frac{N}{2q}\left(\frac{\rho-1}{\rho}\right)$, satisfy the conditions of Theorem 6.8. We will denote by $\mathcal{P}_\Omega(t,\tau) : L^q(\Omega) \to L^q(\Omega)$ the linear process associated to the family $B(t), t \in \mathbb{R}$, and, from the variation of constants formula, w can be given as

$$w(t,\tau,w_0) = \mathcal{P}_\Omega(t,\tau)w_0 + \int_\tau^t \mathcal{P}_\Omega(t,s)F(w(s))ds.$$

Before we proceed to the calculus of estimates for w, we make a last observation concerning the solution of the equations in Ω_0 and Ω: note that the decoupling of the equations is justified, that is, the solution obtained in Ω when we consider the coupled problem is the same as the solution obtained in Ω for the decoupled problem. Indeed, suppose $w_0 \in L^p(\Omega)$ is a initial condition for (7.1) and $w(\cdot,\tau,w_0) : [\tau,\tau+T] \to L^p(\Omega)$ the solution starting at w_0. Given any $v_0 \in L^p(0,1)$, the problem (2) has a solution $(w^*,v^*)(\cdot,\tau,(w_0,v_0)) : (\tau,\tau+T^*] \to U_p^0$. Since $w^* : (\tau,\tau+T^*] \to L^p(\Omega)$ is also a solution for (7.1) satisfying the same initial condition, it follows from uniqueness that

$$w^*(t) = w(t), \quad \text{for } t \in (\tau, \tau + \min\{T, T^*\}].$$

The outline we adopt in the sequel to obtain estimates for $(w,v)(\cdot)$ is: we first work with the solution in Ω in order to evaluate $\|w(t)\|_{L^p(\Omega)}$. Then, we translate this information to the channel via the junction points p_0 and p_1. The equation in R_0 is a reaction-diffusion equation in $1-$dimension with nonhomogeneous time-dependent boundary conditions (conditions provided by $w(p_0,t)$ and $w(p_1,t)$) and we can estimate $\|v(t)\|_{L^p(0,1)}$ in terms of the values of $w(p_0,t)$ and $w(p_1,t)$.

Once $\|(w,v)(t)\|_{U_p^0}$ is controlled, the results on Chapter 6 allows us to obtain existence of pullback attractor for the equation in $\Omega_0 = \Omega \cup R_0$.

To obtain global well-posedness, we assume that f satisfies an appropriate dissipativeness condition:

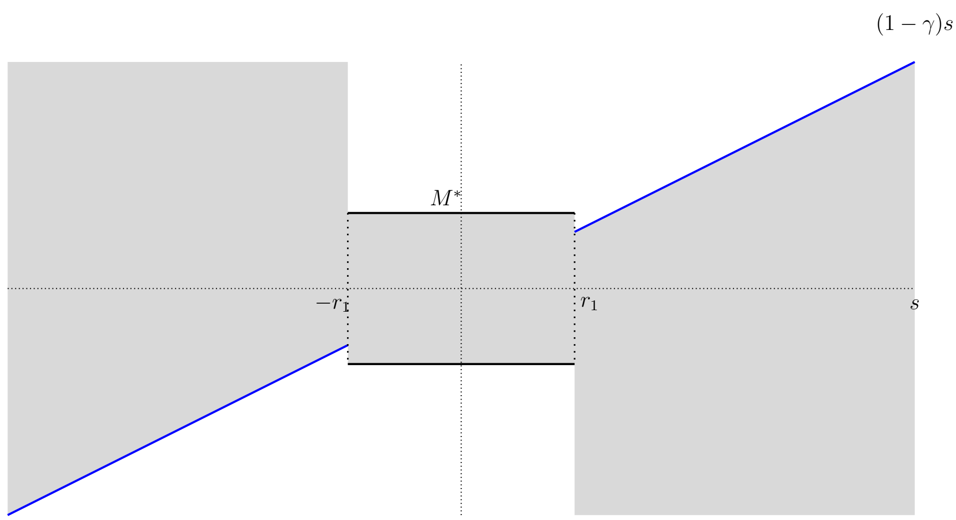
$$\textbf{(D).}\ \limsup_{|s|\to\infty} \frac{f(s)}{s} < 1.$$

Remark 7.4. *The value 1 comes from the fact that the first eigenvalue of $A_0(t)$ is $\lambda_1(t) = 1$, for all $t \in \mathbb{R}$ (see Proposition 2.5). From the definition of* Limsup*, we have $\inf_{r>0}\sup_{|s|>r} \frac{f(s)}{s} < 1$. Therefore, there exists $r_1 > 0$ such that $S := \sup_{|s|>r_1} \frac{f(s)}{s} < 1$. We denote $\gamma_1 := 1 - S$ and $S = 1 - \gamma_1$. We can assume $\gamma_1 < 1$. If that is not the case, it is enough to increase S.*

Hence, for $|s| > r_1$ and $\gamma \in (0,\gamma_1)$, $\frac{f(s)}{s} \le S = 1 - \gamma_1 < 1 - \gamma$. From the continuity of f, it follows that there exists $M > 0$ such that $f(s) \le M$ for all $s \in [-r_1, r_1]$. Gathering all those results, we can state that there exists $\gamma_1 \in (0,1)$ such that, for every $\gamma \in (0,\gamma_1)$, f satisfies

$$\begin{cases} f(s) < (1-\gamma)s, & s > r_1, \\ f(s) > (1-\gamma)s, & s < -r_1, \\ |f(s)| \le M^*, & s \in [-r_1, r_1]. \end{cases}$$

Grafically, this means that f has its image in the shaded region:



Multiplying for s, we have:

$$\begin{cases} sf(s) < (1-\gamma)s^2, & s > r_1, \\ sf(s) < (1-\gamma)s^2, & s < -r_1, \\ |sf(s)| \le M^* r_1 = M, & s \in [-r_1, r_1], \end{cases}$$

and we conclude that

$$f(s)s \leq (1-\gamma)s^2 + M, \quad \forall s \in \mathbb{R}. \tag{7.3}$$

Remark 7.5. *In terms of physical interpretation for the dissipativeness condition, what happens is that the term w at the left side of the equation*

$$w_t - div(a(t,x)\nabla w) + w = f(w)$$

works as a term that draws energy from the system (as we will see in equality (7.4)*). The nonlinearity f, on the other hand, can increase the system's energy. The dissipative condition states that the increase provided by f will never exceed the decrease of energy that w at the left side produces.*

7.1 The equation in Ω

7.1.1 Estimates for w

We first estimate the L^2-norm for the solution $w(t,\tau,w_0)$ (recall that $w(t) \in L^q(\Omega) \hookrightarrow L^2(\Omega)$). We do it to take advantage of the Hilbert structure that $L^2(\Omega)$ possesses. Then we develop an iteration procedure (inspired in the method proposed by Moser-Alikakos - see [2, 25, 28, 29]) to obtain L^{2^k} estimates for the solution w, where $k \in \mathbb{N}$.

L^2 estimate for w

By taking the inner product in $L^2(\Omega)$ of the equation (7.1) with w, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} = -\int_\Omega a(t,x)|\nabla w|^2 dx - \|w\|^2_{L^2(\Omega)} + \int_\Omega f(w)w dx \tag{7.4}$$

and using (A.2),

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leq -\|w\|^2_{L^2(\Omega)} + \int_\Omega f(w)w dx. \tag{7.5}$$

Proposition 7.6. *The solution $w(\cdot,\tau,w_0)$ for* (7.1) *satisfies*

$$\|w(t,\tau,w_0)\|_{L^2(\Omega)} \leq 2^{\frac{1}{2}}\left[e^{-\gamma(t-\tau)}\|w_0\|_{L^2(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right],$$

as long as it exists. The constants γ and M come from the dissipativeness condition.

Proof. Inequality (7.3) can be applied in (7.5) in order to obtain

$$\|w\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leq \int_\Omega \left[(1-\gamma)w^2 + M\right] dx$$

$$2\gamma\|w\|^2_{L^2(\Omega)} + \frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leq 2M|\Omega|$$

$$2\gamma e^{2\gamma(t-\tau)}\|w\|^2_{L^2(\Omega)} + e^{2\gamma(t-\tau)}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leq e^{2\gamma(t-\tau)}2M|\Omega|$$
$$\frac{d}{dt}\left[e^{2\gamma(t-\tau)}\|w\|^2_{L^2(\Omega)}\right] \leq e^{2\gamma(t-\tau)}2M|\Omega|. \tag{7.6}$$

Integrating from τ to t we derive

$$e^{2\gamma(t-\tau)}\|w(t)\|_{L^2(\Omega)} - \|w_0\|_{L^2(\Omega)} \leq [e^{2\gamma(t-\tau)} - 1]\frac{M}{\gamma}|\Omega|$$
$$\|w(t)\|^2_{L^2(\Omega)} \leq e^{-2\gamma(t-\tau)}\|w_0\|^2_{L^2(\Omega)} + \frac{M}{\gamma}|\Omega|.$$

Taking the square roots on both sides and using the inequality $|a+b|^r \leq 2^r(|a|^r + |b|^r)$ for any $r > 0$, we obtain the desired inequality.

□

$L^{2^k}-$*estimate for* w

The iteration technique consists in obtaining $L^{2^k}-$estimates of w by using the estimate in the $L^{2^{k-1}}-$norm. In other words, it is an inductive procedure. Therefore, from the L^2-norm obtained above, we derive L^4-estimate, than L^8 and so on. This is the procedure applied at the following lemma, but before we enunciate it, we discuss a convention that we adopt in the next results.

Remark 7.7. *Given a bounded set $B \subset L^p(\Omega)$ such that $\|w_0\|_{L^p(\Omega)} \leq L$ for $w_0 \in B$, after any evolution in time, the solutions starting with initial conditions in B become bounded in stronger norms. Indeed, from the continuity of the solution, for any $\tau < t^*$, with t^* arbitrarily close to τ, $\|w(t,\tau,w_0)\|_{L^p(\Omega)} \leq CL$, for all $\tau \leq t \leq t^*$, and from the variation of constants formula, for $\theta \in [0,1)$,*

$$\begin{aligned}\|w(t^*,\tau,w_0)\|_{Y^\theta_q} &\leq \|\mathcal{P}_\Omega(t^*,\tau)w_0\|_{Y^\theta_q} + \int_\tau^{t^*} \|\mathcal{P}_\Omega(t^*,s)F(w(s,\tau,w_0))\|_{Y^\theta_q} ds \\ &\leq \|\mathcal{P}_\Omega(t^*,\tau)\|_{\mathcal{L}(L^q(\Omega),Y^\theta_q)}\|w_0\|_{L^q(\Omega)} + \int_\tau^{t^*} \|\mathcal{P}_\Omega(t^*,s)\|_{\mathcal{L}(L^q(\Omega),Y^\theta_q)}\|F(w(s,\tau,w_0))\|_{L^q(\Omega)} ds \\ &\leq C(\theta)(t^*-\tau)^{-\theta}L + \int_\tau^{t^*} C(t^*-s)^{-\theta}(1+\|w(s,\tau,w_0)\|^\rho_{L^p(\Omega)})ds \leq C(\theta,\rho,\|w_0\|_{L^p(\Omega)}),\end{aligned}$$

where we used Theorem 6.12 to estimate $\|\mathcal{P}_\Omega(t^*,\tau)\|_{\mathcal{L}(L^q(\Omega),Y^\theta_q)}$ and $C(\theta,\rho,\|w_0\|_{L^p(\Omega)})$ denotes a constant that depends on θ, ρ, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ and $\|w_0\|_{L^p(\Omega)}$.

In particular, for $\theta > \frac{N}{2q}$ (see Figure 7.1) we obtain $\|w(t^*,\tau,w_0)\|_{L^\infty(\Omega)} \leq C(\rho,\|w_0\|_{L^p(\Omega)})$. In conclusion, given any bounded set B in $L^p(\Omega)$, after an arbitrarily small evolution takes place, this set B becomes bounded in $L^\infty(\Omega)$. Since we are interested in the asymptotic dynamics of the problem, whenever we wish to estimate the solution, we will assume that given any bounded set of initial condition B in $L^p(\Omega)$, this set will also be bounded in $L^\infty(\Omega)$. If that is not the case, we evolve the system any arbitrary time and restart the evolution from this point.

For this reason, we de not lose generality in the next results by assuming that $w_0 \in L^\infty(\Omega)$ (consequently, $w_0 \in L^{2^k}(\Omega)$).

Lemma 7.8. *Let $w(\cdot,\tau,w_0)$ be the solution of* (7.1) *and assume that $w_0 \in L^\infty(\Omega)$. Given any $k \in \mathbb{N}$, there exists constant $c > 0$ independent of k such that, for $t > \tau$,*

$$\|w(t)\|^{2^k}_{L^{2^k}(\Omega)} \leq e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|w(s)\|^{2^k}_{L^{2^{k-1}}(\Omega)}ds + \left[\tfrac{M}{\gamma}|\Omega|\right], \quad (7.7)$$

as long as the solution exists.

Proof. Multiplying the equation in (7.1) by w^{2^k-1} and integrating in Ω, we obtain

$$\int_\Omega w_t w^{2^k-1}dx = \int_\Omega div(a(t,x)\nabla w)w^{2^k-1}dx - \int_\Omega w^{2^k}dx + \int_\Omega f(w)ww^{2^k-2}dx.$$

The term on the left side can be replaced by

$$\frac{1}{2^k}\frac{d}{dt}\int_\Omega w^{2^k}dx = \int_\Omega w_t w^{2^k-1}dx,$$

whereas from the dissipativeness condition, we obtain

$$\int_\Omega f(w)ww^{2^k-2}dx \leq \int_\Omega (1-\gamma)w^{2^k}dx + M\int_\Omega w^{2^k-2}dx \leq \int_\Omega (1-\gamma)w^{2^k}dx + M\left[\int_\Omega w^{2^k}+1dx\right].$$

In the last inequality we used the fact that $a^{2^k-2} < a^{2^k}+1$ for any positive a. This inequality holds since $a^{2^k-2} < 1$, if $a < 1$, and $a^{2^k-2} = \frac{a^{2^k}}{a^2} \leq a^{2^k}$, if $a > 1$. Thus,

$$\frac{1}{2^k}\frac{d}{dt}\int_\Omega w^{2^k}dx \leq \int_\Omega div(a(t,x)\nabla w)w^{2^k-1}dx + [M-\gamma]\int_\Omega w^{2^k}dx + M|\Omega|.$$

Integration by parts leads to

$$-\int_\Omega [a(t,x)\nabla w](2^k-1)w^{2^k-2}\nabla w dx = -(2^k-1)\int_\Omega a(t,x)(\nabla w)^2 w^{2^k-2}dx.$$

Note that

$$\nabla\left(w^{2^{k-1}}\right) = 2^{k-1}w^{2^{(k-1)}-1}\nabla w \quad \text{and} \quad \left[\nabla\left(w^{2^{(k-1)}}\right)\right]^2 = 2^{2(k-1)}w^{2^k-2}(\nabla w)^2,$$

so the term $w^{2^k-2}(\nabla w)^2$ can be replaced by $\frac{1}{2^{2(k-1)}}\left[\nabla\left(w^{2^{(k-1)}}\right)\right]^2$. Therefore,

$$-(2^k-1)\int_\Omega a(t,x)(\nabla w)^2 w^{2^k-2}dx \leq -a_0(2^k-1)(2^{2-2k})\int_\Omega \left[\nabla\left(w^{2^{(k-1)}}\right)\right]^2 dx$$

and the inequality studied becomes (after multiplying both sides by 2^k),

$$\frac{d}{dt}\int_\Omega w^{2^k}dx \leq -a_0(2^k-1)(2^{2-k})\int_\Omega \left[\nabla\left(w^{2^{(k-1)}}\right)\right]^2 dx + 2^k[M-\gamma]\int_\Omega w^{2^k}dx + 2^k M|\Omega|. \quad (7.8)$$

If for a certain $u \in W^{1,2}(\Omega) \cap L^1(\Omega)$ we apply Nirenberg-Gagliardo's inequality (Lemma P.12) with $j = 0$, $p = 2$, $m = 1$, $r = 2$, $q = 1$ and $\theta = \frac{N}{N+2}$, we obtain

$$\|u\|_{L^2(\Omega)} \leq C(N,\Omega)\|\nabla u\|^{\frac{N}{N+2}}_{L^2(\Omega)}\|u\|^{\frac{2}{N+2}}_{L^1(\Omega)}.$$

If we also use the Young generalized inequality (Lemma P.13) with conjugated exponents $\xi = \frac{1}{\theta} = \frac{N+2}{N}$ and $\xi' = \frac{N+2}{2}$, we obtain

$$\|u\|_{L^2(\Omega)} \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} + \frac{1}{\varepsilon^{\frac{N}{2}}} \|u\|_{L^1(\Omega)}.$$

Taking the square power on both sides (and rearranging ε^2 for ε),

$$(1-\varepsilon)\|u\|^2_{L^2(\Omega)} \leq \|u\|^2_{L^2(\Omega)} \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^{\frac{N}{2}}} \|u\|^2_{L^1(\Omega)}.$$

We apply the above inequality for $u = w^{2^{(k-1)}}$

$$\begin{aligned}
\|w^{2^{(k-1)}}\|^2_{L^2(\Omega)} &= \int_\Omega w^{2\left(2^{(k-1)}\right)} dx = \int_\Omega w^{2^k} dx, \\
\left\| \nabla \left[w^{2^{(k-1)}} \right] \right\|^2_{L^2(\Omega)} &= \int_\Omega \left| \nabla \left[w^{2^{(k-1)}} \right] \right|^2 dx, \\
\|w^{2^{(k-1)}}\|^2_{L^1(\Omega)} &= \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2,
\end{aligned}$$

and it becomes

$$-\int_\Omega \left| \nabla \left[w^{2^{(k-1)}} \right] \right|^2 dx \leq \frac{1}{\varepsilon^{\frac{N}{2}+1}} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 - \frac{(1-\varepsilon)}{\varepsilon} \int_\Omega w^{2^k} dx.$$

We use this inequality with a proper choice of ε and apply it at (7.8) in order to obtain a negative term multiplying $\int_\Omega w^{2^k} dx$.

$$\begin{aligned}
\frac{d}{dt} \int_\Omega w^{2^k} dx \leq & - a_0 \frac{2^k - 1}{(2^{k-2})} \frac{(1-\varepsilon)}{\varepsilon} \int_\Omega w^{2^k} dx + a_0 \frac{2^k - 1}{(2^{k-2})} \frac{1}{\varepsilon^{\frac{N}{2}+1}} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 \\
& + 2^k [M - \gamma] \int_\Omega w^{2^k} dx + 2^k M |\Omega|.
\end{aligned}$$

Note that $2 \leq \frac{2^k - 1}{(2^{k-2})} \leq 4$, and we can readjust the preceding inequality to obtain

$$\frac{d}{dt} \int_\Omega w^{2^k} dx \leq \left[-2a_0 \frac{(1-\varepsilon)}{\varepsilon} + 2^k [M - \gamma] \right] \int_\Omega w^{2^k} dx + 4a_0 \frac{1}{\varepsilon^{\frac{N}{2}+1}} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 + 2^k M |\Omega|.$$

Choosing $\varepsilon = c 2^{-k}$, with c small enough to ensure that

$$2a_0 \frac{(1-\varepsilon)}{\varepsilon} = 2a_0 \frac{1 - c2^{-k}}{c2^{-k}} > \frac{2a_0}{c} 2^k > 2^k ([M-\gamma]+1),$$

(for example, $c = \frac{a_0}{([M-\gamma]+1)}$) we obtain

$$\frac{d}{dt} \int_\Omega w^{2^k} dx \leq - 2^k \int_\Omega w^{2^k} dx + 4a_0 \frac{1}{c^{\frac{N}{2}+1} (2^{-k})^{\frac{N}{2}+1}} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 + 2^k M |\Omega|$$

$$= -2^k \int_\Omega w^{2^k} dx + c(2^k)^{\frac{N}{2}+1} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 + 2^k M|\Omega|.$$

We have then achieved the desired differential inequality

$$2^k \int_\Omega w^{2^k} dx + \frac{d}{dt} \int_\Omega w^{2^k} dx \leq c(2^k)^{\frac{N}{2}+1} \left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 + 2^k M|\Omega|. \tag{7.9}$$

Note that

$$\left(\int_\Omega w^{2^{(k-1)}} dx \right)^2 = \left[\left(\int_\Omega w^{2^{(k-1)}} dx \right)^{\frac{1}{2^{(k-1)}}} \right]^{2^k} = \left[\|w\|_{L^{2^{(k-1)}}(\Omega)} \right]^{2^k}.$$

Inequality (7.9) becomes

$$2^k \|w(t)\|^{2^k}_{L^{2^k}(\Omega)} + \frac{d}{dt} \|w(t)\|^{2^k}_{L^{2^k}(\Omega)} \leq c(2^k)^{\frac{N}{2}+1} \|w(t)\|^{2^k}_{L^{2^{k-1}}(\Omega)} + 2^k M|\Omega|$$

and then

$$\frac{d}{dt} \left[e^{2^k(t-\tau)} \|w(t)\|^{2^k}_{L^{2^k}(\Omega)} \right] \leq e^{2^k(t-\tau)} c(2^k)^{\frac{N}{2}+1} \|w(t)\|^{2^k}_{L^{2^{k-1}}(\Omega)} + e^{2^k(t-\tau)} 2^k M|\Omega|. \tag{7.10}$$

Integrating from τ to t, we obtain

$$e^{2^k(t-\tau)} \|w\|^{2^k}_{L^{2^k}(\Omega)} \leq \|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1} \int_\tau^t e^{2^k(s-\tau)} \|w(s)\|^{2^k}_{L^{2^{k-1}}(\Omega)} ds + [e^{2^k(t-\tau)} - 1] M|\Omega|$$

$$\|w\|^{2^k}_{L^{2^k}(\Omega)} \leq e^{-2^k(t-\tau)} \|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1} e^{-2^k(t-\tau)} \int_\tau^t e^{2^k(s-\tau)} \|w(s)\|^{2^k}_{L^{2^{k-1}}(\Omega)} ds + M|\Omega|$$

and the statement of the lemma follows by noting that we can consider $\gamma \in (0,1)$ and $M|\Omega| \leq \left[\frac{M}{\gamma} |\Omega| \right]$ (this adjustment of constant is just to facilitate future calculus and notations).

□

From the recurrence formula obtained in Lemma 7.8, we derive the next proposition:

Proposition 7.9. *Let M, γ be the constants obtained from the dissipativeness condition and assume $w_0 \in L^\infty(\Omega)$. Given any $k \in \mathbb{N}$, there exists a constant $D = D(N,k,\gamma)$ that depends on k, N and γ, such that, for $t > \tau$,*

$$\|w(t)\|_{L^{2^k}(\Omega)} \leq D(N,k,\gamma) \left[e^{-\gamma(t-\tau)} \|w_0\|_{L^{2^k}(\Omega)} + \left[\frac{M}{\gamma} |\Omega| \right]^{\frac{1}{2}} \right],$$

as long as the solution exists.

Proof. We prove for $k = 2$ and $k = 3$ to see the pattern. The result follows from induction. We will use the inequality $|a+b|^r \leq 2^r(|a|^r + |b|^r)$, for any $r > 0$, whenever we need to estimate the power of a sum of two terms.

Let $k = 2$. Rather then replacing the value of k in the inequalities, we will keep it to help us obtain a generalization for any $k \in \mathbb{N}$. We first estimate the integral that appears in (7.7) using the L^2-bound obtained for the solution in Proposition 7.6:

$$\begin{aligned}
\int_\tau^t e^{2^k(s-\tau)}\|w(s)\|^{2^k}_{L^2(\Omega)}ds &\leq \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}\left\{e^{-\gamma(s-\tau)}\|w_0\|_{L^2(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\}^{2^k} ds \\
&\leq \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left\{e^{-2^k\gamma(s-\tau)}\|w_0\|^{2^k}_{L^2(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\} ds \\
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\int_\tau^t e^{2^k(1-\gamma)(s-\tau)}\|w_0\|^{2^k}_{L^2(\Omega)}ds + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\frac{1}{2^k}e^{2^k(t-\tau)} \\
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|w_0\|^{2^k}_{L^2(\Omega)} + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^k}e^{2^k(t-\tau)}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}.
\end{aligned}$$

Therefore, replacing it in (7.7), we have

$$\begin{aligned}
\|w(t)\|^{2^k}_{L^{2^k}(\Omega)} &\leq e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^k(1-\gamma)}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^2(\Omega)} \\
&\quad + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^k}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} + \left[\tfrac{M}{\gamma}|\Omega|\right] \\
&\leq e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^k(1-\gamma)}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^2(\Omega)} \\
&\quad + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} \\
&\leq e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^2(\Omega)} \\
&\quad + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}
\end{aligned}$$

and we assumed that $c(2^k)^{\frac{N}{2}+1}$ and $\frac{M}{\gamma}|\Omega|$ are larger or equal than 1 (we can increase c, M if that is not the case). Moreover, let $d_{k-1,k}$ denotes the embedding constant of $L^{2^k}(\Omega) \hookrightarrow L^{2^{k-1}}(\Omega)$. Then $\|w_0\|_{L^2} \leq d_{k-1,k}\|w_0\|_{L^{2^k}}$ and inequality above can be written as

$$\begin{aligned}
\|w(t)\|^{2^k}_{L^{2^k}(\Omega)} &\leq e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + c(2^k)^{\frac{N}{2}+1}\frac{d^{2^k}_{k-1,k}}{1-\gamma}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} \\
&\quad + c(2^k)^{\frac{N}{2}+1}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} \\
&\leq c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d^{2^k}_{k-1,k})(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left\{e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.
\end{aligned}$$

Extracting the 2^k root and denoting $C(k,N,\gamma) = \left\{c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d^{2^k}_{k-1,k})\right\}^{\frac{1}{2^k}}$, we obtain

$$\|w(t)\|_{L^{2^k}(\Omega)} \leq C(k,N,\gamma)2(2^{\frac{1}{2}})2^{\frac{1}{2^k}}\left\{e^{-\gamma(t-\tau)}\|w_0\|_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\}$$

$$= D(2,N,\gamma)\left\{e^{-\gamma(t-\tau)}\|w_0\|_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right\},$$

where $D(k,N,\gamma) = 2^{k-1}\left(\prod_{i=1}^k 2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^k C(i,N,\gamma)\right)$.

For $k=3$ the calculation is analogous. The previous information about $\|w(t)\|_{L^4(\Omega)}$ allows us to obtain now estimates on $\|w(t)\|_{L^{2^3}(\Omega)}$. First, the integral on (7.7) satisfies

$$\int_\tau^t e^{2^k(s-\tau)}\|w(s)\|^{2^k}_{L^4(\Omega)} \le D(2,N,\gamma)^{2^k}(2)^{2^k}\int_\tau^t e^{2^k(s-\tau)}\left\{e^{-2^k\gamma(s-\tau)}\|w_0\|^{2^k}_{L^4(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}ds$$
$$\le D(2,N,\gamma)^{2^k}(2)^{2^k}\left\{\frac{1}{2^k(1-\gamma)}e^{2^k(1-\gamma)(t-\tau)}\|w_0\|^{2^k}_{L^4(\Omega)} + \frac{1}{2^k}e^{2^k(t-\tau)}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.$$

Replacing this expression in (7.7), we obtain

$$\|w(t)\|^{2^k}_{L^{2^k}(\Omega)} \le e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{2^k}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^4(\Omega)}$$
$$+ D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{2^k}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}} + \left[\tfrac{M}{\gamma}|\Omega|\right]$$
$$\le e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^4(\Omega)}$$
$$+ D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}$$
$$\le e^{-2^k(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}e^{-2^k\gamma(t-\tau)}d^{2^k}_{k-1,k}\|w_0\|^{2^k}_{L^{2^k}(\Omega)}$$
$$+ D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}$$
$$\le D(2,N,\gamma)^{2^k}(2)^{2^k}c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d^{2^k}_{k-1,k})\left\{e^{-2^k\gamma(t-\tau)}\|w_0\|^{2^k}_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{2^k}{2}}\right\}.$$

Extracting the 2^k–root and using $\left\{c(2^k)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d^{2^k}_{k-1,k})\right\}^{\frac{1}{2^k}} = C(k,N,\gamma)$, we obtain

$$\|w(t)\|_{L^{2^k}(\Omega)} \le D(2,N,\gamma)C(3,N,\gamma)(2)(2^{\frac{1}{2^k}})\left[e^{-\gamma(t-\tau)}\|w_0\|_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right]$$
$$= D(3,N,\gamma)\left[e^{-\gamma(t-\tau)}\|w_0\|_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right].$$

In general,

$$\|w(t)\|_{L^{2^k}(\Omega)} \le D(k,N,\gamma)\left[e^{-\gamma(t-\tau)}\|w_0\|_{L^{2^k}(\Omega)} + \left[\tfrac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}\right],$$

where $C(i,N,\gamma) = \left\{c(2^i)^{\frac{N}{2}+1}\frac{1}{1-\gamma}(1+d^{2^i}_{i-1,i})\right\}^{\frac{1}{2^i}}$ and

$$D(k,N,\gamma) = 2^{k-1}\left(\prod_{i=1}^k 2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^k C(i,N,\gamma)\right). \quad (7.11)$$

□

Y_q^θ estimates for w

The estimates of the solution obtained so far allow us to derive estimates in better norms. To be precise, we can obtain estimates in each Y_q^θ space for $0 \leq \theta < 1$. In (7.2) we proved that

$$\|F(w(t))\|_{L^q(\Omega)} \leq C(1 + \|w(t)\|^\rho_{L^p(\Omega)}).$$

For $p > N$ fixed, let $k_0 \in \mathbb{N}$ such that

$$k_0 \geq \log_2 p. \tag{7.12}$$

In this case $2^{k_0} \geq p$ and from classical L^r-embedding, we can rewrite inequality above as

$$\|F(w(t))\|_{L^q(\Omega)} \leq C(1 + \|w(t)\|^\rho_{L^{2^{k_0}}(\Omega)}),$$

adjusting the constant C. This fact and the results obtained earlier imply:

Proposition 7.10. *Let $0 \leq \theta < 1$, $w_0 \in L^\infty(\Omega)$ and $k_0 \in \mathbb{N}$ such that $k_0 \geq \log_2 p$. There exist constants E_1 and E_2 depending on Ω, θ, ρ, N, k_0, M and γ, such that, for $\tau < t - 1 < t$,*

$$\|w(t, \tau, w_0)\|_{Y_q^\theta} \leq E_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} \right) + E_2,$$

as long as the solution exists.

Proof. From the variation of constants formula, Proposition 7.9, the embeddings $L^{r_1}(\Omega) \hookrightarrow L^{r_2}(\Omega)$ whenever $r_2 \leq r_1$ (used for $q \leq p \leq 2^{k_0}$) and estimate for $\mathcal{P}_\Omega(\cdot, \cdot)$ derived in Theorem 6.9, we obtain (adjusting constants when needed)

$$\begin{aligned}
&\|w(t, \tau, w_0)\|_{Y_q^\theta} = \|w(t, t-1, w(t-1, \tau, w_0))\|_{Y_q^\theta} \\
&\leq \|\mathcal{P}_\Omega(t, t-1) w(t-1, \tau, w_0)\|_{Y_q^\theta} + \int_{t-1}^t \|\mathcal{P}_\Omega(t, s) F(w(s, \tau, w_0))\|_{Y_q^\theta} ds \\
&\leq C(\theta)(1)^{-\theta} \|w(t-1, \tau, w_0)\|_{L^q(\Omega)} + \int_{t-1}^t C(\theta)(t-s)^{-\theta} \|F(w(s, \tau, w_0))\|_{L^q(\Omega)} ds \\
&\leq C(\theta)|\Omega|^{\frac{1}{q} - \frac{1}{2^{k_0}}} \|w(t-1, \tau, w_0)\|_{L^{2^{k_0}}(\Omega)} + C(\theta) \int_{t-1}^t (t-s)^{-\theta} \left(1 + \|w(t)\|^\rho_{L^{2^{k_0}}(\Omega)} \right) ds \\
&\leq C(\theta)|\Omega|^{\frac{1}{q} - \frac{1}{2^{k_0}}} D(k_0, N, \gamma) \left\{ e^{-\gamma(t-1-\tau)} \|w_0\|_{L^{2^{k_0}}(\Omega)} + \left[\tfrac{M}{\gamma} |\Omega| \right]^{\frac{1}{2}} \right\} \\
&\quad + C(\theta) \int_{t-1}^t (t-s)^{-\theta} \left(1 + D(k_0, N, \gamma)^\rho \left\{ e^{-\gamma(s-\tau)} \|w_0\|_{L^{2^{k_0}}(\Omega)} + \left[\tfrac{M}{\gamma} |\Omega| \right]^{\frac{1}{2}} \right\}^\rho \right) ds \\
&\leq C(\theta)|\Omega|^{\frac{1}{q} - \frac{1}{2^{k_0}}} D(k_0, N, \gamma) \left\{ e^{-\gamma(t-\tau)} \|w_0\|_{L^{2^{k_0}}(\Omega)} + \left[\tfrac{M}{\gamma} |\Omega| \right]^{\frac{1}{2}} \right\} \\
&\quad + C(\theta) \int_{t-1}^t (t-s)^{-\theta} \left(1 + D(k_0, N, \gamma)^\rho (2^\rho) \left\{ e^{-\rho\gamma(s-\tau)} \|w_0\|^\rho_{L^{2^{k_0}}(\Omega)} + \left[\tfrac{M}{\gamma} |\Omega| \right]^{\frac{\rho}{2}} \right\} \right) ds
\end{aligned}$$

$$\leq C(\theta)|\Omega|^{\frac{1}{q}-\frac{1}{2^{k_0}}} D(k_0,N,\gamma)e^{-\gamma(t-\tau)}\|w_0\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2^{k_0}}} + C(\theta)|\Omega|^{\frac{1}{q}-\frac{1}{2^{k_0}}} D(k_0,N,\gamma)\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}}$$
$$+ C(\theta) + C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)e^{-\rho\gamma(t-1-\tau)}\|w_0\|^\rho_{L^{2^{k_0}}(\Omega)} + C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}$$
$$\leq C(\theta)|\Omega|^{\frac{1}{q}} D(k_0,N,\gamma)e^{-\gamma(t-\tau)}\|w_0\|_{L^\infty(\Omega)} + C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)e^{-\rho\gamma(t-\tau)}\|w_0\|^\rho_{L^\infty(\Omega)}|\Omega|^{\frac{\rho}{2^{k_0}}}$$
$$+ C(\theta) + C(\theta)D(k_0,N,\gamma)|\Omega|^{\frac{1}{q}-\frac{1}{2^{k_0}}}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{1}{2}} + C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}$$
$$\leq E_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + E_2,$$

where $D(k_0,N,\gamma)$ is the constant given in (7.11) and we assumed that $D(k_0,N)$, $\left[\frac{M}{\gamma}|\Omega|\right] > 1$ to group the terms above as

$$\begin{aligned} E_1 &= C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)\max\{|\Omega|^{\frac{1}{q}}, |\Omega|^{\frac{\rho}{2^{k_0}}}\}, \\ E_2 &= 3C(\theta)D(k_0,N,\gamma)^\rho(2^\rho)\max\{1, |\Omega|^{\frac{1}{q}-\frac{1}{2^{k_0}}}\}\left[\frac{M}{\gamma}|\Omega|\right]^{\frac{\rho}{2}}. \end{aligned} \tag{7.13}$$

□

7.1.2 Global well-posedness and Pullback attractor

The $Y_q^\theta-$estimate obtained in Proposition 7.10 for $\frac{N}{2q}\left(\frac{\rho-1}{\rho}\right) < \theta < 1$ (see Figure 7.1) implies, in particular, that $\|w(t,\tau,w_0)\|_{L^p(\Omega)}$ is finite in each bounded interval $[\tau,T]$. Then, $w(t,\tau,w_0)$ is globally defined in time (Theorem 6.8) and we can obtain a nonlinear process $S_\Omega(t,\tau) : L^p(\Omega) \to L^p(\Omega)$ given by the solution $w(t,\tau,w_0)$, that is, $S_\Omega(t,\tau)w_0 = w(t,\tau,u_0)$. If $\mathcal{P}_\Omega(t,\tau)$ is the linear process associated to $B(t)$, $t \in \mathbb{R}$, then

$$S_\Omega(t,\tau)w_0 = \mathcal{P}_\Omega(t,\tau)w_0 + \int_\tau^t \mathcal{P}_\Omega(t,s)F(w(s))ds.$$

We prove this result and the existence of pullback attractor in next two theorems.

Theorem 7.11. *Let $w(\cdot,\tau,w_0)$ be the solution of* (7.1) *and M,γ the constants obtained from the dissipativeness condition* (7.3)*. Then $w(\cdot,\tau,w_0)$ is globally defined and associated to it there is a nonlinear process $S_\Omega(t,\tau)$ in $L^p(\Omega)$ given by $S_\Omega(t,\tau)w_0 = w(t,\tau,w_0)$, for all $t \geq \tau$.*

Moreover, the closed ball in Y_q^θ centered in zero and with radius E_2, $B_{Y_q^\theta}[0,E_2]$, is a pullback attracting set for the process $S_\Omega(t,\tau)$ in the topology of Y_q^θ, where E_2 is given in (7.13) *and depends on θ,ρ,N,k_0,M and γ.*

Proof. Let $w_0 \in L^p(\Omega)$. It follows from Remark 7.7 that after any arbitrarily small evolution in time, $w^* = w(t^*,\tau,w_0) \in L^\infty(\Omega)$. Therefore, if we start the evolution at instant t^* and at the point w^*, Proposition 7.10 implies that $\|w(t,t^*,w^*)\|_{Y_q^\theta}$ is finite in any bounded interval $[t^*,T]$. For $\theta > \frac{N}{2q}\left(\frac{\rho-1}{\rho}\right)$, this boundedness implies global existence of the solution.

Moreover, let $B \subset L^p(\Omega)$ be a bounded set such that $\|w_0\|_{L^p(\Omega)} \leq L$ for any $w_0 \in B$. It also follows from Remark 7.7 that after an arbitrarily small evolution in time, $t^* > \tau$, the elements $w^* = w(t^*, \tau, w_0)$ are bounded in $L^\infty(\Omega)$, that is $\|w^*\|_{L^\infty(\Omega)} \leq \tilde{L}$. Therefore,

$$\|w(t, t^*, w^*)\|_{Y_q^\theta} \leq E_1 e^{-\gamma(t-t^*)}(\tilde{L} + \tilde{L}^\rho) + E_2,$$

and $dist(S_\Omega(t, t^*)S_\Omega(t^*, \tau)w_0, B_{Y_q^\theta}[0, E_2]) = dist(w(t, t^*, w^*), B_{Y_q^\theta}[0, E_2]) \xrightarrow{t-t^*\to\infty} 0$, uniformly for $w_0 \in B$.

□

The existence of pullback attractor for $S_\Omega(t, \tau)$ is now a consequence of the previous result.

Theorem 7.12. *Assume that* $p > N$ *and* $\max\left\{\frac{N}{2\delta}, \frac{p(2N+1)}{2p+1}\right\} < q \leq p$, $X = U_p^0$, $Y = U_q^0$, $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ *satisfies (A.2) and (A.3) and* $f : \mathbb{R} \to \mathbb{R}$ *satisfies (A.4) and (D).*

The solution $w(t)$ *for the equation* (7.1) *in* Ω *defines a nonlinear process* $S_\Omega(t, \tau)$ *in* $L^p(\Omega)$ *which has a pullback attractor* $\mathcal{A}_\Omega(t)$ *in* $L^p(\Omega)$. *Moreover,* $\cup_{t\in\mathbb{R}}\mathcal{A}_\Omega(t) \subset \mathcal{C}^{1,\eta}(\Omega)$, *for some* $\eta > 0$, *and pullback attracts bounded sets of* $L^p(\Omega)$ *in the topology of* $\mathcal{C}^{1,\eta}(\Omega)$.

Proof. The conditions required ensure existence of local mild solution for the problem in $\Omega_0 = \Omega \cup R_0$ (see Proposition 2.11). In particular, they ensure the existence of solution in Ω.

Moreover, in Theorem 7.11 we proved the existence of a pullback attracting bounded set in Y_q^θ for any $0 \leq \theta < 1$. Since Y_q^θ is compactly embedded in $L^p(\Omega)$ for $\theta > \frac{N}{2q}(\frac{\rho-1}{\rho})$ (as seen in Lemma 7.1), we conclude that $B_{Y_q^\theta}[0, E_2]$ is a compact pullback attracting set for the process $S_\Omega(t, \tau)$ in $L^p(\Omega)$.

Therefore, from Corollary 6.16, there exists a pullback attractor

$$\mathcal{A}_\Omega(t) \subset B_{Y_q^\theta}[0, E_2] \overset{c}{\hookrightarrow} L^p(\Omega), \quad \forall t \in \mathbb{R}.$$

that attracts bounded sets of $L^p(\Omega)$ in the topology of Y_q^θ. Moreover, if $\theta > \frac{1}{2} + \frac{N}{2q}$, then $Y_q^\theta \hookrightarrow \mathcal{C}^{1,\eta}(\Omega)$ and the last statement follows.

□

Remark 7.13. *Note we only required the conditions posed in Proposition 2.11 rather then the more restrictive conditions of Proposition 5.2. We do that because once the problem is decoupled, the fact that* $B(t)$ *is sectorial ensures differentiability of the mild solution* w.

7.1.3 Properties of w_t

Before we proceed to the analysis in the channel R_0, some properties of w_t, the derivative of w in Ω, are required. As we will see in the next section, the nonhomogeneous condition at the junction points (given by $w(t, p_0)$ and $w(t, p_1)$) will be incorporated at the equation in the channel via an appropriate change of variable. This will cause the appearance of a term depending on w_t in the equation and the properties presented in this section will be necessary.

Let us first recall that one of the conditions required to obtain local solution for the equation in $\Omega_0 = \Omega \cup R_0$ is that $q > \frac{N}{2\delta}$ (see Table 2.1), that is $\delta > \frac{N}{2q}$.

This condition, as we see in the next proposition, ensures that $w_t(t,x)$ belongs to $Y_q^\theta \hookrightarrow C^\nu(\Omega)$, for $\theta \in (\frac{N}{2\delta}, \delta)$, and estimate for the derivative in the $C^\nu(\Omega)-$norm is available (see Theorem 6.9). Moreover, after a certain time, those derivatives are enclosed in a compact set of the phase space and the variation of the solution in the long-time is somehow controlled.

We say in the next proposition that a given set K pullback absorbs $w_t(t,\tau,w_0)$ in the sense that there exists $T^* > 0$ such that $t - \tau > T^*$ implies that $w_t(t,\tau,w_0) \in K$.

Proposition 7.14. *Let $0 \le \theta < \delta$ and $w_0 \in L^\infty(\Omega)$. Then there exists constants F_1 and F_2 depending on Ω, θ, ρ, k_0, N, M and γ, such that, for $\tau < t-1 < t$,*

$$\|w_t(t,\tau,w_0)\|_{Y_q^\beta} \le F_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + F_2,$$

and, for any $\varepsilon > 0$, $B_{Y_q^\theta}[0, F_2+\varepsilon]$ is a pullback absorbing bounded set for $w_t(t,\tau,w_0)$ in the topology of Y_q^θ.

Moreover, for $\frac{N}{2q} < \theta < \delta$, there exists $\nu > 0$ such that $B_{C^\nu(\Omega)}[0, F_2+\varepsilon] \overset{c}{\hookrightarrow} L^p(\Omega)$ is a compact pullback absorbing set for $w_t(t,\tau,w_0)$ in the topology of $C^\nu(\Omega)$.

Proof. From Theorem 6.9, $w_t(t) \in Y_q^\theta$ for any $0 \le \theta < \delta$ and $t > \tau$. Together with Proposition 7.10, we obtain

$$\begin{aligned}\|w_t(t,\tau,w_0)\|_{Y_q^\beta} &\le C(\theta)(t-(t-1))^{-1-\theta}\|w(t-1,\tau,w_0)\|_{L^q(\Omega)}\\ &\le C(\theta)E_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + C(\theta)E_2.\end{aligned}$$

Taking $F_i = C(\theta)E_i$, $i = 1,2$, we derive the desired inequality. The other statements follows from Lemma 7.1. $\square$

From the proposition above we conclude that $t \mapsto |w_t(t,x)| \in \mathbb{R}$ is bounded for each $x \in \overline{\Omega}$. In this case, the solution $t \mapsto |w(t,x)| \in \mathbb{R}$ can increase/decrease in the long-time dynamics, but those variations are somehow controlled and limited.

7.2 The equation on the line segment R_0

Now that we have estimates for both w and w_t in Ω, we turn our attention to the reaction-diffusion equation that takes place at the line segment R_0.

Unlike the dynamics in Ω that is indifferent to what happens at the line segment, the evolution in R_0 is subordinated to the values w assume at the points, p_0 and p_1, at the time t. This works as a boundary condition for the equation in R_0 which can be seen as a heat equation in $[0,1]$ with nonhomogeneous and time-dependent boundary conditions, that is,

$$\begin{cases} v_t(t,r) - \partial_r[a(t,r)\partial_r v(t,r)] + v(t,r) = f(v), & t > \tau, r \in (0,1), \\ v(t,0) = w(t,p_0) \text{ and } v(t,1) = w(t,p_1), & t > \tau, \\ v(0,r) = v_0(r) \in L^p(0,1). \end{cases} \tag{7.14}$$

The boundary conditions $v(t,0) = w(t,p_0)$ and $v(t,1) = w(t,p_1)$ only make sense because $p \geq q > \frac{N}{2}$ and $w(t,\cdot) \in W^{2,q}(\Omega) \hookrightarrow C(\overline{\Omega})$. Moreover, we also have that $w_t(t,\cdot) \in \mathcal{C}^\nu(\overline{\Omega})$ (Proposition 7.14) and there are estimates for both $w(t,\cdot)$ and $w_t(t,\cdot)$ in the space of continuous functions.

Given each $w_0 \in L^p(\Omega)$, the evolution equation in Ω produces a solution $w(\cdot,\tau,w_0)$ defined in $[\tau,\infty)$ that dictates the values of v at the junction points. In other words, each initial condition w_0 determines a different evolution equation (7.14) in R_0.

In this section we first study global existence and asymptotic dynamics for the problem (7.14) when a given function $w(t,\tau,w_0)$ is the solution for the problem in Ω. Then we extend those concepts for the coupled equations, obtaining the existence of pullback attractor for the problem in $U_p^0 = L^p(\Omega) \times L^p(0,1)$.

7.2.1 Associated problem with autonomous Dirichlet boundary condition

To treat the equation in R_0, we can perform a change of variables in a manner that the conditions on the boundary are incorporated to the equation.

Given $w_0 \in L^p(\Omega)$, let $w(t)$ be the solution for the equation in Ω with initial condition $w(\tau) = w_0$. We establish the following notations:

1. $w(t,\tau,w_0)$ denotes the function $w(t) \in L^p(\Omega)$ that assumes the value w_0 at the instant τ.
2. The value of $w(t)$ at the junction points p_0 and p_1 are denoted by $w(t)(p_0)$ and $w(t)(p_1)$.
3. If we need to emphasize both aspects (the initial condition and the value at one of those points), we denote $w(t,\tau,w_0)(p_i)$, $i = 0,1$.

Consider the (time-dependent) boundary problem associated to the instant τ and initial condition w_0:

$$\begin{cases} \partial_r(a(t,r)\partial_r \xi) = 0, \quad r \in (0,1), \\ \xi(t,0) = w(t,\tau,w_0)(p_0); \ \xi(t,1) = w(t,\tau,w_0)(p_1). \end{cases} \tag{7.15}$$

Integrating in r the equation one time, we obtain $\partial_r \xi(t) = \frac{c}{a(t,r)}$ and integrating one more time from 0 to r, we obtain

$$\xi(t,r) = \xi(t,0) + c\int_0^r \frac{1}{a(t,\theta)} d\theta = w(t,\tau,w_0)(p_0) + c\int_0^r \frac{1}{a(t,\theta)} d\theta. \tag{7.16}$$

For $r = 1$,

$$w(t,\tau,w_0)(p_1) = w(t,\tau,w_0)(p_0) + c\int_0^1 \frac{1}{a(t,\theta)}d\theta \Rightarrow c = \frac{w(t,\tau,w_0)(p_1) - w(t,\tau,w_0)(p_0)}{\int_0^1 \frac{1}{a(t,\theta)}d\theta}$$

and replacing this value of c in (7.16), we obtain

$$\xi(t,r) = w(t,\tau,w_0)(p_0)\left[\frac{\displaystyle\int_r^1 \frac{1}{a(t,\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right] + w(t,\tau,w_0)(p_1)\left[\frac{\displaystyle\int_0^r \frac{1}{a(t,\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right],$$

or

$$\xi(t,r) = w(t,\tau,w_0)(p_0)\mathcal{X}_0(t,r) + w(t,\tau,w_0)(p_1)\mathcal{X}_1(t,r),$$

where

$$\mathcal{X}_0(t,r) = \left[\frac{\displaystyle\int_r^1 \frac{1}{a(t,\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right], \quad \mathcal{X}_1(t,r) = \left[\frac{\displaystyle\int_0^r \frac{1}{a(t,\theta)}d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right]. \tag{7.17}$$

This function ξ is the solution of (7.15). To emphasize the dependence on τ and w_0, we denote this solution by $\xi(t,r;(\tau,w_0))$.

Definition 7.15. *Given $\tau \in \mathbb{R}$ and $w_0 \in L^p(\Omega)$, for $t \geq \tau$ and $r \in [0,1]$,*

$$\xi(t,r;(\tau,w_0)) := \xi(t,r) = w(t,\tau,w_0)(p_0)\mathcal{X}_0(t,r) + w(t,\tau,w_0)(p_1)\mathcal{X}_1(t,r) \tag{7.18}$$

is the solution of the equation (7.15) *associated to $(\tau,w_0) \in \mathbb{R}\times L^p(\Omega)$. We denote by $\xi_t(t,r;(\tau,w_0))$ its derivative in time.*

If we consider $z(t,r) = v(t,r)-\xi(t,r)$, then $z(t,0) = v(t,0)-w(t)(p_0) = 0 = v(t,1)-w(t)(p_1) = z(t,1)$ and the differential equation becomes:

$$\begin{aligned}
z_t = v_t - \xi_t &= [\partial_r(a(t,r)\partial_r v) - v + f(v)] - \xi_t \\
&= [\partial_r(a(t,s)\partial_r v) - \partial_r(a(t,r)\partial_r \xi) - v + \xi] + \{-\xi - \xi_t + f(v)\} \\
&= [\partial_r(a(t,s)\partial_r z) - z] + \{-\xi - \xi_t + f(v)\} \\
&= [\partial_r(a(t,s)\partial_r z) - z] + \{-\xi - \xi_t + f(z+\xi)\} \\
&= \partial_r(a(t,s)\partial_r z) - z + \psi(t,z)
\end{aligned}$$

where $\psi(t,z) = -\xi - \xi_t + f(z,\xi)$ is a nonlinearity depending on ξ (given in (7.18)). Therefore, $\psi(t,z)$ also depends on $(\tau,w_0) \in \mathbb{R}\times L^p(\Omega)$ and we can emphasize this dependence by denoting

$$\psi(t,z;(\tau,w_0)) = -\xi(t,r;(\tau,w_0)) - \xi_t(t,r;(\tau,w_0)) + f\left(\xi(t,r;(\tau,w_0)) + z\right). \tag{7.19}$$

After the change of variables $z = v - \xi$, problem (7.14) can be written as

$$\begin{cases} z_t - \partial_r(a(t,r)\partial_r z) + z = \psi(t,z), & t > \tau, r \in (0,1), \\ z(t,0) = 0 \text{ and } z(t,1) = 0, & t > \tau, \\ z(\tau,r) = v_0(r) - \xi(\tau,r) =: z_0 \in L^p(0,1), \end{cases} \tag{7.20}$$

and we have homogeneous Dirichlet boundary conditions for the problem, which allows us to define the linear operator $\mathcal{L}(t)z = -\partial_r(a(t,r)\partial_r z) + z$ with domain $D(\mathcal{L}(t)) = D = W^{2,q}(0,1) \cap W_0^{1,q}(0,1)$.

Remark 7.16. *We will refer to* (7.20) *as the* associated problem with homogeneous (Dirichlet) boundary condition. *Note that the initial condition $z_0 \in L^p(0,1)$,*

$$z_0(r) = v_0(r) - \xi(\tau,r),$$

is given in terms of the initial condition $v_0 \in L^p(0,1)$ and in terms of the function $\xi(\tau,r)$ determined by the pair (τ, w_0) and the evolution in Ω.

Proceeding in the same as it was done in Lemma 7.1, we obtain similar properties for the linear operator $\mathcal{L}(t)$ that features in (7.20).

Lemma 7.17. *Let $\mathcal{L}(t)$, $t \in \mathbb{R}$, be the family of linear operators $\mathcal{L}(t)z = -\partial_r(a(t,r)\partial_r z) + z$, $D(\mathcal{L}(t)) = D(\mathcal{L}) = W^{2,q}(0,1) \cap W_0^{1,q}(0,1)$. This family satisfies:*

1. *$\mathcal{L}(t)$, $t \in \mathbb{R}$, is uniformly sectorial and uniformly $\delta-$Hölder continuous*

$$\|[\mathcal{L}(t) - \mathcal{L}(s)]\mathcal{L}(\tau)^{-1}\|_{\mathcal{L}(L^q(0,1))} \leq C|t-s|^\delta, \quad \textit{for all } \tau, s, t \in \mathbb{R}.$$

2. *Each operator $\mathcal{L}(t)$ is positive (in the sense that $Re(\sigma(\mathcal{L}(t))) > 0$) and their fractional powers $\mathcal{L}(t)^\theta$, $\theta \in \mathbb{R}$, are well-defined. We denote $V_q^\theta = D(\mathcal{L})^\theta$.*

3. *Those spaces define a scale of fractional power spaces V_q^θ, $\theta > 0$, such that the following embeddings hold*

$$\begin{array}{ll} V_q^\theta \hookrightarrow \mathcal{C}^{1,\eta}(0,1) & \textit{for some } 0 < \eta, \textit{ if } \theta > \frac{1}{2} + \frac{1}{2q}, \\ V_q^\theta \hookrightarrow C^\nu(0,1) & \textit{for some } 0 < \nu, \textit{ if } \theta > \frac{1}{2q}, \\ V_q^\theta \hookrightarrow L^r(0,1) & \textit{when } -\frac{N}{r} < 2\theta - \frac{1}{q}, \end{array}$$

in particular, $V_q^\theta \hookrightarrow L^p(0,1)$, if $\theta > \frac{1}{2q}\left(\frac{\rho-1}{\rho}\right)$.

4. *If $0 \leq \theta < \xi \leq 1$, then V_q^ξ is compactly embedded in V_q^θ.*

5. *The spectrum of $\mathcal{L}(t)$ consists entirely of isolated eigenvalues, all of them positive and real. To be more precise,*

$$\sigma(\mathcal{L}(t)) = \{\tau_i(t) : 1 < \tau_1(t) \leq \tau_2(t) \leq ... \leq \tau_n(t) \leq ...\}.$$

The embeddings above are represented in the following figure:

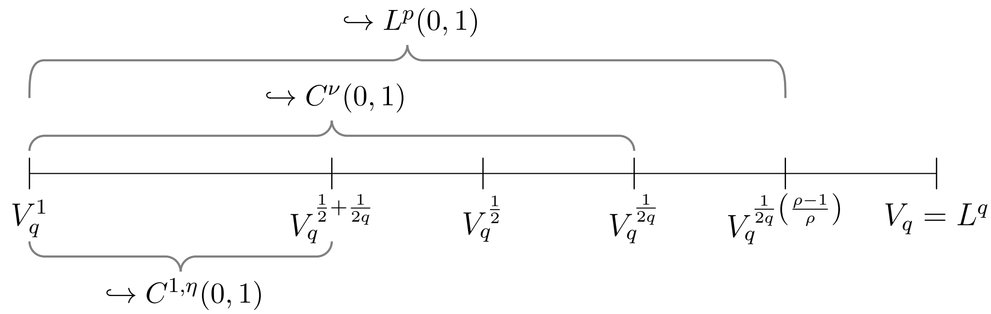


Figure 7.2: Embeddings of V_q^θ

In the same way that was done for the equation in Ω, we wish to obtain estimates for z in $L^2(0,1)$, $L^{2^k}(0,1)$ and V_q^θ, ensuring global well-posedness and existence of pullback attractor. To perform such analysis, we must figure out whether or not $\psi(t,z)$ satisfies some type of dissipation.

7.2.2 Dissipation property of $\psi(t,z)$

The next two lemmas will help studying the properties of ψ.

Lemma 7.18. *If $\mathcal{X}_0(t,r)$ and $\mathcal{X}_1(t,r)$ are given by* (7.17)*, then they are differentiable in t and there exist constants A_1, A_2, A_3 and A_4 such that, for all $(t,r) \in \mathbb{R} \times [0,1]$,*

$$|\mathcal{X}_0(t,r)| \leq A_1, \quad |\mathcal{X}_1(t,r)| \leq A_2, \quad \left|\frac{d}{dt}\mathcal{X}_0(t,r)\right| \leq A_3 \quad \textit{and} \quad \left|\frac{d}{dt}\mathcal{X}_1(t,r)\right| \leq A_4.$$

Proof. It follows from condition (A.2) required for the function $a(\cdot,\cdot)$ and a directly differentiation of the functions $\mathcal{X}_0$ and $\mathcal{X}_1$ in time. $\square$

Taking into account Remark 7.8, we will assume in the next results that $w_0 \in L^\infty(\Omega)$. If that is not the case, we evolve the system an arbitrarily small time and restart the evolution from this point.

Lemma 7.19. *Let $p_i \in \{p_0, p_1\}$ and $\tau < t-1 < t$. Assuming that $w_0 \in L^\infty(\Omega)$, we have*

i. If E_1 and E_2 are the constants given in (7.13) *(depending on Ω, ρ, k_0 N, M and γ), then*

$$|w(t,\tau,w_0)(p_i)| \leq E_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + E_2,$$

ii. If F_1 and F_2 are the constants given in Proposition 7.14 (depending on Ω, ρ, k_0 N, M and γ), then

$$|w_t(t,\tau,w_0)(p_i)| \leq F_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + F_2,$$

Proof. The statements follow from Propositions 7.10 and 7.14, respectively. $\square$

As a consequence of the two properties above, we have global estimate for $\xi(t,r)$ and $\xi_t(t,r)$:

Proposition 7.20. *Let $(\tau, w_0) \in \mathbb{R} \times L^\infty(\Omega)$. There exists constants G_1, G_2, depending on Ω, ρ, k_0, N, M and γ (but not on (τ, w_0)), such that, for $\tau < t-1 < t$,*

$$|\xi(t,r;(\tau,w_0))| + |\xi_t(t,r;(\tau,w_0))| \le G_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} \right) + G_2.$$

Proof. From (7.18), we have

$$|\xi(t,r)| \le |w(t,\tau,w_0)(p_0)|A_1 + |w(t,\tau,w_0)(p_1)|A_2$$

and

$$\begin{aligned} |\xi_t(t,r)| &\le \left| w_t(t)(p_0)\mathcal{X}_0(t,r) + w(t)(p_0)\frac{d}{dt}\mathcal{X}_0(t,r) + w_t(t)(p_1)\mathcal{X}_1(t,r) + w(t)(p_1)\frac{d}{dt}\mathcal{X}_1(t,r) \right| \\ &\le |w_t(t)(p_0)|A_1 + |w(t)(p_0)|A_3 + |w_t(t)(p_1)|A_2 + |w(t)(p_1)|A_4. \end{aligned}$$

Therefore, from Lemma 7.19 we obtain

$$\begin{aligned} |\xi(t,r)| + |\xi_t(t,r)| &\le C(|w(t)(p_0)| + |w(t)(p_1)| + |w_t(t)(p_0)| + |w_t(t)(p_1)|) \\ &\le C(2E_1 + 2F_1)e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} \right) + C(2E_2 + 2F_2), \end{aligned}$$

where E_1, E_2 are constants given in 7.13, F_1, F_2 constants given in Proposition 7.14. The result follows by considering

$$G_1 = C(2E_1 + 2F_1), \quad \text{and} \quad G_2 = C(2E_2 + 2F_2). \tag{7.21}$$

□

Since both ξ and ξ_t are bounded (uniformly in t and r), we can prove that ψ defined in (7.19) satisfies an appropriate dissipative condition for the problem (7.20).

Proposition 7.21. *Given any $(\tau, w_0) \in \mathbb{R} \times L^\infty(\Omega)$ and $\xi(t) = \xi(t,r;(\tau,w_0))$, the nonlinearity*

$$\psi(t,z;(\tau,w_0)) = \psi(t,z) = -\xi(t) - \xi_t(t) + f(\xi(t) + z)$$

satisfies

$$\limsup_{|z|\to\infty} \frac{\psi(t,z)}{z} < 1.$$

Moreover, there exist constants $\gamma > 0$ and H_1, H_2 depending only on Ω, ρ, k_0, N, and γ (but not on t, τ or w_0), such that

$$z\psi(t,z;(\tau,w_0)) \le (1-\gamma)z^2 + H_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)} \right) + H_2. \tag{7.22}$$

Proof. From the dissipativeness condition (D) on f and the fact that ξ and ξ_t are bounded, we obtain

$$\limsup_{|z|\to\infty} \frac{\psi(t,z)}{z} = \limsup_{|z|\to\infty} \frac{-\xi(t)-\xi_t(t)}{z} + \limsup_{|z|\to\infty} \frac{f(\xi(t)+z)}{z} < 0+1 = 1.$$

Proceeding as in Remark 7.4, there exists $\gamma_1 \in (0,1)$ such that, for every $\gamma \in (0,\gamma_1)$, ψ satisfies

$$\begin{cases} \psi(t,z) < (1-\gamma)z, & z > r_1, \\ \psi(t,z) > (1-\gamma)z, & z < -r_1, \\ |\psi(t,z)| \le M^{**}, & z \in [-r_1, r_1], \end{cases} \tag{7.23}$$

but in this case M^{**} is a constant that depends on t, τ, w_0. Indeed, it follows from inequality (2.9), the estimate obtained in Proposition 7.20 and the fact that

$$\begin{aligned} &|a+b|^\theta \le 2^\theta(|a|^\theta + |b|^\theta), \\ &|a+b+c|^\theta \le 3^\theta(|a|^\theta + |b|^\theta + |c|^\theta), \end{aligned}$$

for $\theta > 0$, that

$$\begin{aligned} |f(\xi(t)+z)| &\le C(1+|\xi(t)+z|^\rho) \le C2^\rho(1+|z|^\rho + |\xi(t)|^\rho) \\ &\le C2^\rho \left(1 + r_1^\rho + \left[G_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + G_2\right]^\rho\right) \\ &\le C2^\rho(1+r_1^\rho) + C2^\rho 3^\rho \left[G_1^\rho e^{-\rho\gamma(t-\tau)}\left(\|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right) + G_2^\rho\right] \\ &\le C2^\rho(1+r_1^\rho) + C2^\rho 3^\rho G_1^\rho e^{-\rho\gamma(t-\tau)}\left(\|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right) + C2^\rho 3^\rho G_2^\rho. \end{aligned}$$

Since $\Psi(t,z) = -\xi(t) - \xi_t(t) + f(\xi(t)+z)$, we deduce, for $z \in [-r_1, r_1]$,

$$\begin{aligned} |\psi(t,z)| &\le |f(\xi(t)+z)| + |\xi(t)| + |\xi_t(t)| \\ &\le C2^\rho(1+r_1^\rho) + C2^\rho 3^\rho G_1^\rho e^{-\rho\gamma(t-\tau)}\left(\|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right) + C2^\rho 3^\rho G_2^\rho \\ &\quad + G_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + G_2 \\ &\le 2C2^\rho 3^\rho G_1^\rho e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right) + (C2^\rho(1+r_1^\rho) + 2C2^\rho 3^\rho G_2^\rho) \\ &= M^{**}, \end{aligned}$$

where we assumed $G_1, G_2 > 1$ and we used the fact that $e^{-\rho\gamma(t-\tau)} \le e^{-\gamma(t-\tau)}$.

Multiplying (7.23) for z, we obtain:

$$\begin{cases} z\psi(t,z) < (1-\gamma)z^2, & z > r_1, \\ z\psi(t,z) < (1-\gamma)z^2, & z < -r_1, \\ |z\psi(t,z)| < M^{**} r_1, & z \in [-r_1, r_1], \end{cases}$$

and we conclude that

$$z\psi(t,z) \le (1-\gamma)z^2 + H_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^{\rho}_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)} \right) + H_2,$$

where

$$H_1 = r_1 2C2^\rho 3^\rho G_1^\rho, \quad H_2 = r_1(C2^\rho(1+r_1^\rho) + 2C2^\rho 3^\rho G_2^\rho) \tag{7.24}$$

and G_1, G_2 are constants given in (7.21).

□

To simplify the notation in next sections, we denote by

$$\begin{aligned} m_2(w_0) &= H_1 \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^{\rho}_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)} \right), \\ M_2(t,\tau,w_0) &= e^{-\gamma(t-\tau)} m_2(w_0). \end{aligned} \tag{7.25}$$

Remark 7.22. *Note that*

(1) the function $M_2(t,\tau,w_0)$ is decreasing in t and $M_2(t,\tau,w_0) \xrightarrow{t-\tau\to\infty} 0$, uniformly for w_0 in bounded sets of $L^\infty(\Omega)$.

(2) for any $t > t-1 > \tau$,

$$M_2(t,\tau,w_0) = e^{-\gamma(t-\tau)} m_2(w_0) = e^{-\gamma} e^{-\gamma(t-1-\tau)} m_2(w_0) = e^{-\gamma} M_2(t-1,\tau,w_0).$$

(3) the dissipativeness condition is restated as

$$s\psi(t,s;(\tau,w_0)) \le (1-\gamma)s^2 + M_2(t,\tau,w_0) + H_2.$$

7.2.3 Estimates in the channel R_0

The ideas for this section are essentially the same of the ones used to obtain estimates for the function w in Ω. However, rather than dealing with v, we will estimate the function z, solution of the associated problem with homogeneous boundary condition:

$$z_t - \mathcal{L}(t)z = \psi(t,z),\ t > \tau; \quad z(\tau,r) = z_0(r) = v_0(r) - \xi(0,r) \in L^p(0,1),$$

where $\mathcal{L}(t)$ is the family of linear operators given in Lemma 7.17.

Once we obtain estimates for z, those are transferred to v, since we know that ξ is bounded (in $\mathcal{C}^\nu(\Omega)$) and $z = v - \xi$. The outline to obtain the estimates is the following:

1. We first evaluate $\|z(t)\|_{L^2(0,1)}$ by performing formal calculus over the differential equation.

2. Then, via the iterative procedure we evaluate $\|z(t)\|_{L^{2^k}(0,1)}$.

3. From the previous items we can obtain estimates for $z(t)$ in the spaces V_q^θ.

Moreover, we will denote by $\mathcal{P}_{R_0}(t,\tau) : L^q(0,1) \to L^q(0,1)$ the linear process associated to the family $\mathcal{L}(t), t \in \mathbb{R}$, and, from the variation of constants formula, z can be given as

$$z(t,\tau,z_0) = \mathcal{P}_{R_0}(t,\tau)z_0 + \int_\tau^t \mathcal{P}_{R_0}(t,s)\psi(t,z(s))ds. \tag{7.26}$$

Due to the similarity of the calculations involved in determining those estimates and the ones done in Section 7.1, we simply point out the differences in the proofs. Those differences come from the dissipativeness condition (7.22) that slightly differs from the one used for the problem in Ω. Note that in this case, the dissipation depends on (τ, w_0).

Remark 7.23. *The parabolic structure of the associated problem* (7.20) *implies in similar regularization properties as the ones discussed in Remark 7.7, this time applied to the initial condition $z_0 = v_0 - \xi(\tau, w_0)$. Therefore, if $z_0 \in L^p(0,1)$, after any arbitrarily small evolution $\tau < t^*$ in time, $z(t^*,\tau,z_0)$ will be an element of $L^\infty(0,1)$ (and consequently, $v(t^*) = z(t^*) - \xi(t^*) \in L^\infty(0,1)$). Moreover, bounded sets of $L^p(0,1)$ are taken in bounded sets of $L^\infty(0,1)$ through the variation of constants formula* (7.26)*.*

For this reason, we do not lose generality in the next results by assuming that $(w_0, v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$ and, consequently, $z_0 \in L^\infty(0,1)$.

L^2-estimate for z

Proposition 7.24. *Let $(w_0, v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$ and $z_0 = v_0 - \xi(\tau, w_0) \in L^\infty(0,1)$. If H_1, H_2 are the constants obtained in Proposition 7.21 depending only on Ω, ρ, k_0, N, and γ (but not on t,τ or w_0) and $M_2(t,\tau,w_0)$ is given in* (7.25)*, then the solution $z(\cdot,\tau,z_0)$ to* (7.20) *satisfies, as long as it exists,*

$$\|z(t,\tau,z_0)\|_{L^2(0,1)} \le 2^{\frac{1}{2}} \left[e^{-\gamma(t-\tau)}\|z_0\|_{L^2(0,1)} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0) + \tfrac{2}{\gamma}H_2\right]^{\frac{1}{2}}\right].$$

Proof. We take the inner product in $L^2(0,1)$ of the equation (7.20) with z,

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(0,1)} = -\int_0^1 a(t,r)|\partial_r z|^2 dr - \|z\|^2_{L^2(0,1)} + \int_0^1 \psi(t,z)zdr$$

and item (3) of Remark 7.22 leads to inequality

$$\|z\|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(0,1)} \le (1-\gamma)\|z\|^2_{L^2(0,1)} + M_2(t,\tau,w_0) + H_2.$$

Adjusting the terms above and using $M(t,\tau,w_0) = e^{-\gamma(t-\tau)}m_2(w_0)$, we obtain

$$2\gamma\|z\|^2_{L^2(0,1)} + \frac{d}{dt}\|z\|^2_{L^2(0,1)} \le 2\left[M_2(t,\tau,w_0) + H_2\right] = e^{-\gamma(t-\tau)}\left[2m_2(w_0) + 2e^{\gamma(t-\tau)}H_2\right].$$

The steps here differ from Proposition 7.6 at (7.6) due to the time dependence of the right side of inequality. This time dependence is important and must not be overlooked since it provides the exponential decay for the terms in which $\|w_0\|_{L^2(\Omega)}$ features. After multiplying by $e^{2\gamma(t-\tau)}$ we obtain

$$\frac{d}{dt}\left[e^{2\gamma(t-\tau)}\|z\|^2_{L^2(0,1)}\right] \leq e^{\gamma(t-\tau)}\left[2m_2(w_0) + e^{\gamma(t-\tau)}2H_2\right].$$

Integrating from τ to t, and using that $m_2(w_0) + e^{\gamma(t-\tau)}H_2$ is increasing in t, we obtain

$$\begin{aligned} e^{2\gamma(t-\tau)}\|z(t)\|^2_{L^2(0,1)} - \|z_0\|^2_{L^2(0,1)} &\leq \left[2m_2(w_0) + e^{\gamma(t-\tau)}2H_2\right]\int_\tau^t e^{\gamma(s-\tau)}ds \\ &\leq e^{\gamma(t-\tau)}\frac{1}{\gamma}\left[2m_2(w_0) + e^{\gamma(t-\tau)}2H_2\right]. \end{aligned}$$

Therefore,

$$\|z(t)\|^2_{L^2(0,1)} \leq e^{-2\gamma(t-\tau)}\|z_0\|^2_{L^2(0,1)} + \left[\frac{2}{\gamma}M_2(t,\tau,w_0) + \frac{2}{\gamma}H_2\right].$$

Taking the square roots on both sides and using the inequality $|a+b|^r \leq 2^r(|a|^r + |b|^r)$, for $r > 0$, we obtain the desired inequality. □

$L^{2^k}-$ *estimate for* z

With the same iterative procedure used for the solution in Ω, we estimate $L^{2^k}-$norm of the solution in R_0.

Lemma 7.25. *Let* $(w_0, v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$*,* $z_0 = v_0 - \xi(\tau, w_0) \in L^\infty(0,1)$ *and* $z(\cdot,\tau,z_0)$ *the solution of* (7.20)*. Given any* $k \in \mathbb{N}$*, there exists constant* $c > 0$ *independent of* k *such that, for* $t > \tau$*,*

$$\begin{aligned} \|z(t)\|^{2^k}_{L^{2^k}(0,1)} \leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|z(s)\|^{2^k}_{L^{2^{k-1}}(0,1)}ds \\ + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0) + \tfrac{2}{\gamma}H_2\right], \end{aligned} \tag{7.27}$$

as long as the solution exists. The constant H_2 *is given in* (7.24) *and the function* $M_2(t,\tau,w_0)$ *in* (7.25).

Proof. The proof follow the exactly same steps as the proof of Lemma 7.8 up the inequality (7.10). From this point forward, some differences appear due to the time dependence of the term $M_2(t,\tau,w_0)$. Notice that, in this case, we replace $|\Omega|$ by the measure of $(0,1)$, which is 1, and the dimension of the space is $N = 1$. Therefore, we have

$$\frac{d}{dt}\left[e^{2^k(t-\tau)}\|z(t)\|^{2^k}_{L^{2^k}(0,1)}\right] \leq e^{2^k(t-\tau)}c(2^k)^{\frac{3}{2}}\|z(t)\|^{2^k}_{L^{2^{k-1}}(0,1)} + e^{2^k(t-\tau)}2^k\left[M_2(t,\tau,w_0) + H_2\right]. \tag{7.28}$$

We will integrate the above inequality from τ to t, but in order to understand the behavior of the term in which $M_2(t,\tau,w_0)$ appears, we evaluate it separately:

$$\int_\tau^t e^{2^k(s-\tau)}2^k[M_2(s,\tau,w_0) + H_2]ds = \int_\tau^t e^{2^k(s-\tau)}2^k e^{-\gamma(s-\tau)}[m_2(w_0) + e^{\gamma(s-\tau)}H_2]ds$$

$$\leq 2^k[m_2(w_0) + e^{\gamma(t-\tau)}H_2]\int_\tau^t e^{(2^k-\gamma)(s-\tau)}ds$$
$$\leq \frac{2^k}{2^k-\gamma}e^{(2^k-\gamma)(t-\tau)}[m_2(w_0) + e^{\gamma(t-\tau)}H_2]$$
$$\leq \frac{2^k}{2^k-\gamma}e^{2^k(t-\tau)}\left[M_2(t,\tau,w_0) + H_2\right].$$

Therefore, integrating (7.28) from τ to t and some manipulation imply

$$\|z(t)\|^{2^k}_{L^{2^k}(0,1)} \leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|z(s)\|^{2^k}_{L^{2^{k-1}}(0,1)}ds$$
$$+ \frac{2^{k-1}}{2^k-1}\Big[\tfrac{2}{\gamma}M_2(t,\tau,w_0) + \tfrac{2}{\gamma}H_2\Big].$$

The statement of the lemma follows by noting that $\gamma \in (0,1)$ and $\dfrac{2^{k-1}}{2^k-1} < 1$ (these adjustment of constant is just to facilitate future calculus and notations).

□

From the recurrence formula obtained in Lemma 7.25, we derive the next proposition:

Proposition 7.26. *Let* $(w_0, v_0) \in L^\infty(\Omega)\times L^\infty(0,1)$, $z_0 = v_0 - \xi(\tau, w_0) \in L^\infty(0,1)$ *and* $z(\cdot,\tau,z_0)$ *the solution of* (7.20). *Given any* $k \in \mathbb{N}$, *there exists a constant* $\tilde{D}(k,\gamma)$ *that depends on* k *and* γ *such that, for* $t > \tau$,

$$\|z(t)\|_{L^{2^k}(0,1)} \leq \tilde{D}(k,\gamma)\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \Big[\tfrac{2}{\gamma}M_2(t,\tau,w_0) + \tfrac{2}{\gamma}H_2\Big]^{\frac{1}{2}}\right\},$$

as long as the solution exists. The constant H_2 *is given in* (7.24) *and the function* $M_2(t,\tau,w_0)$ *in* (7.25).

Proof. We verify for $k = 2$ to see the pattern. The result follows from induction. We will use the inequality $|a+b|^r \leq 2^r(|a|^r + |b|^r)$, for $r > 0$, whenever we need to estimate the power of a sum of two terms.

Let $k = 2$. Rather then replacing the value of k in the inequalities, we keep it to help us obtain a generalization for any $k \in \mathbb{N}$. We first estimate the integral that appears in (7.27) using the L^2-bound obtained for the solution in Proposition 7.24. We have

$$\int_\tau^t e^{2^k(s-\tau)}\|z(s)\|^{2^k}_{L^2(0,1)}ds$$
$$\leq \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}\left\{e^{-\gamma(s-\tau)}\|z_0\|_{L^2(0,1)} + \Big[\tfrac{2}{\gamma}M_2(s,\tau,w_0) + \tfrac{2}{\gamma}H_2\Big]^{\frac{1}{2}}\right\}^{2^k}ds$$
$$\leq \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\left\{e^{-2^k\gamma(s-\tau)}\|z_0\|^{2^k}_{L^2(0,1)}\right\}ds$$
$$+ \int_\tau^t e^{2^k(s-\tau)}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}e^{-2^{k-1}\gamma(s-\tau)}\Big[\tfrac{2}{\gamma}m_2(w_0) + \tfrac{2}{\gamma}H_2e^{\gamma(s-\tau)}\Big]^{\frac{2^k}{2}}ds$$

$$
\begin{aligned}
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)} + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\int_\tau^t e^{2^k(2-\gamma)(s-\tau)}\left[\tfrac{2}{\gamma}m_2(w_0)+\tfrac{2}{\gamma}e^{\gamma(s-\tau)}H_2\right]^{\frac{2^k}{2}}ds\\
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)} + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^{k-1}(2-\gamma)(t-\tau)}}{2^{k-1}(2-\gamma)}\left[\tfrac{2}{\gamma}m_2(w_0)+\tfrac{2}{\gamma}e^{\gamma(t-\tau)}H_2\right]^{\frac{2^k}{2}}\\
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)} + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(t-\tau)}}{2^{k-1}(2-\gamma)}(e^{-\gamma(t-\tau)})^{\frac{2^k}{2}}\left[\tfrac{2}{\gamma}m_2(w_0)+\tfrac{2}{\gamma}e^{\gamma(t-\tau)}H_2\right]^{\frac{2^k}{2}}\\
&\leq (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)} + (2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{2^k(t-\tau)}}{2^{k-1}(2-\gamma)}\left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{2^k}{2}}.
\end{aligned}
$$

Therefore, replacing it in (7.27) and noticing that in the set $(0,1)$, the embedding constants of $L^{2^k}(0,1)\hookrightarrow L^{2^{k-1}}(0,1)$ are less than 1, that is, $\|\cdot\|_{L^{2^{k-1}}(0,1)}\leq\|\cdot\|_{L^{2^k}(0,1)}$, we obtain

$$
\begin{aligned}
\|z(t)\|^{2^k}_{L^{2^k}(0,1)} &\leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}e^{-2^k(t-\tau)}\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)}\\
&\quad + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^{k-1}(2-\gamma)}\left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{2^k}{2}} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]\\
&\leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{-2^k\gamma(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^2(0,1)}\\
&\quad + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{2^{k-1}(1-\gamma)}\left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{2^k}{2}} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]\\
&\leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{e^{-2^k\gamma(t-\tau)}}{1-\gamma}\|z_0\|^{2^k}_{L^{2^k}(0,1)}\\
&\quad + c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{1}{1-\gamma}\left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{2^k}{2}}\\
&\leq c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{2}{1-\gamma}\left\{e^{-2^k\gamma(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{2^k}{2}}\right\}
\end{aligned}
$$

and we assumed that $c(2^k)^{\frac{3}{2}}(2^{\frac{1}{2}})^{2^k}(2)^{2^k}\frac{2}{1-\gamma}$ and $\frac{2}{\gamma}H_2$ are larger or equal than 1. Extracting the 2^k−root on both sides and denoting $\tilde{C}(k,\gamma)=\left\{c(2^k)^{\frac{3}{2}}\frac{2}{1-\gamma}\right\}^{\frac{1}{2^k}}$, we obtain

$$
\begin{aligned}
\|z(t)\|_{L^{2^k}(0,1)} &\leq \tilde{C}(k,\gamma)(2)(2^{\frac{1}{2}})(2^{\frac{1}{2^k}})\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \left[\frac{2}{\gamma}M_2(t,\tau,w_0)+\frac{2}{\gamma}H_2\right]^{\frac{1}{2}}\right\}\\
&= \tilde{D}(2,\gamma)\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{1}{2}}\right\},
\end{aligned}
$$

where $\tilde{D}(k,\gamma)=2^{k-1}\left(\prod_{i=1}^k 2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^k\tilde{C}(i,\gamma)\right)$.

For $k=3$, we obtain, in the same way as we did in Lemma 7.8,

$$
\|z(t)\|_{L^{2^k}(0,1)} \leq \tilde{D}(3,\gamma)\left[e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{1}{2}}\right].
$$

In general,

$$\|z(t)\|_{L^{2^k}(0,1)} \leq \tilde{D}(k,\gamma)\left[e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \left[\tfrac{2}{\gamma}M_2(w_0) + \tfrac{2}{\gamma}H_2\right]^{\frac{1}{2}}\right],$$

where $\tilde{C}(i,\gamma) = \left\{c(2^i)^{\frac{3}{2}}\frac{2}{1-\gamma}\right\}^{\frac{1}{2^i}}$ and

$$\tilde{D}(k,\gamma) = 2^{k-1}\left(\prod_{i=1}^{k} 2^{\frac{1}{2^i}}\right)\left(\prod_{i=2}^{k}\tilde{C}(i,\gamma)\right).$$

□

V_q^θ estimate for z

The estimates of the solution obtained so far allow us to derive estimates in better norms. To be precise, we can obtain estimates in each $V_q^\theta = D(\mathcal{L}(t)^\theta)$, for $0 \leq \theta < 1$, but in order to do that, we first need to evaluate $\|\psi(t,z(t))\|_{L^q(0,1)}$.

Remark 7.27. *From this point forward we will group all the constants depending on $\theta, \Omega, \rho, p, q, N, M$ and γ in a single constants $C^* = C^*(\theta, \Omega, \rho, p, q, N, M, \gamma) > 0$, enlarging it whenever necessary. We will lose precision on the estimates by doing this, but on the other hand, it will be easier to identify the main ideas in the sequel.*

Lemma 7.28. *Let $(w_0, v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$, $z_0 = v_0 - \xi(\tau, w_0) \in L^\infty(0,1)$ and $z(\cdot,\tau,z_0)$ the solution of* (7.20)*, then there exists $C^* = C^*(\Omega, \rho, p, q, N, M, \gamma) > 0$ depending on Ω, ρ, p, q, N, M and γ (but not on τ or w_0) such that*

$$\|\psi(t,z)\|_{L^q(0,1)} \leq C^*\left[e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^\infty(0,1)} + [M_2(t,\tau,w_0)+1]^\rho\right].$$

Proof. The nonlinearity $\psi(t,z)$ is given by $\psi(t,z) = -\xi(t) - \xi_t(t) + f(\xi(t)+z)$. From the growth of f and the fact that the embedding constants of $L^{r_1}(0,1) \hookrightarrow L^{r_2}(0,1)$ whenever $r_1 > r_2$ are all less then 1, we obtain

$$\begin{aligned}
\|\psi(t,z)\|_{L^q(0,1)} &= \|-\xi(t) - \xi_t(t) + f(\xi(t)+z)\|_{L^q(0,1)} \\
&\leq \|\xi(t)+\xi_t(t)\|_{L^q(0,1)} + C(1 + \|\xi(t)+z(t)\|^\rho_{L^p(0,1)}) \\
&\leq \|\xi(t)+\xi_t(t)\|_{L^\infty(0,1)} + C\|\xi(t)\|^\rho_{L^\infty(0,1)} + C\|z(t)\|^\rho_{L^{2^{k_0}}(0,1)},
\end{aligned}$$

where k_0 is the positive integer such that $k_0 \geq \log_2 p$ and $L^{2^{k_0}}(0,1) \hookrightarrow L^p(0,1)$.

It follows from Proposition 7.20 that

$$\begin{aligned}
\|\xi(t)+\xi_t(t)\|_{L^\infty(0,1)} &\leq \left[\frac{G_1}{H_1}M_2(t,\tau,w_0) + G_2\right] \\
C\|\xi(t)\|^\rho_{L^\infty(0,1)} &\leq \left[C^{\frac{1}{\rho}}\frac{G_1}{H_1}M_2(t,\tau,w_0) + C^{\frac{1}{\rho}}G_2\right]^\rho.
\end{aligned}$$

Enlarging the constants such that the terms inside the brackets is larger than 1, if necessary, we have

$$\|\xi(t)+\xi_t(t)\|_{L^\infty(0,1)}+C\|\xi(t)\|^\rho_{L^\infty(0,1)}\leq C^*\left[M_2(t,\tau,w_0)+1\right]^\rho, \tag{7.29}$$

with $C^*=C^*(\theta,\Omega,\rho,p,q,N,M,\gamma)>0$.

Moreover, it follows from Proposition 7.26 and the inequality $|a+b|^r\leq 2^r(|a|^r+|b|^r)$, for $r>0$, that

$$\begin{aligned}
C\|z(t)\|^\rho_{L^p(0,1)} &\leq C\|z(t)\|^\rho_{L^{2^{k_0}}(0,1)}\\
&\leq C\tilde{D}(k,\gamma)^\rho(2^\rho)e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^{2^{k_0}}(0,1)}\\
&\quad + C\tilde{D}(k,\gamma)^\rho(2^\rho)\left[\tfrac{2}{\gamma}M_2(t,\tau,w_0)+\tfrac{2}{\gamma}H_2\right]^{\frac{\rho}{2}}\\
&\leq C\tilde{D}(k,\gamma)^\rho e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^\infty(0,1)}+C\tilde{D}(k,\gamma)^\rho\left(\frac{2}{\gamma}\right)^\rho\left[M_2(t,\tau,w_0)+H_2\right]^{\frac{\rho}{2}},\\
&\leq C^*\left[e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^\infty(0,1)}+\left[M_2(t,\tau,w_0)+1\right]^{\frac{\rho}{2}}\right].
\end{aligned} \tag{7.30}$$

Therefore, the desired result follows from (7.29) and (7.30), adjusting the constant C^*. □

The results obtained earlier imply:

Proposition 7.29. *Let* $(w_0,v_0)\in L^\infty(\Omega)\times L^\infty(0,1)$*,* $z_0=v_0-\xi(\tau,w_0)\in L^\infty(0,1)$ *and* $z(\cdot,\tau,z_0)$ *the solution of* (7.20)*. Then, for any* $0\leq\theta<1$*, there exists* $C^*=C^*(\theta,\Omega,\rho,p,q,N,M,\gamma)>0$ *depending on* $\theta,\Omega,\rho,p,q,N,M$ *and* γ *(but not on* τ *or* w_0*) such that*

$$\|z(t)\|_{V^\theta_q}\leq C^*\left[e^{-\gamma(t-\tau)}(\|z_0\|_{L^\infty(0,1)}+\|z_0\|^\rho_{L^\infty(0,1)})+\left[M_2(t,\tau,w_0)+1\right]^\rho\right],$$

as long as the solution exists.

Proof. From the variation of constants formula and Proposition 6.12, we obtain

$$\begin{aligned}
&\|z(t,\tau,z_0)\|_{V^\theta_q}=\|z(t,t-1,z(t-1,\tau,z_0))\|_{V^\theta_q}\\
&\leq\|\mathcal{P}_{R_0}(t,t-1)z(t-1,\tau,z_0)\|_{V^\theta_q}+\int_{t-1}^t\|\mathcal{P}_{R_0}(t,s)\psi(s,z(s)))\|_{V^\theta_q}ds\\
&\leq C(\theta)(t-(t-1))^{-\theta}\|z(t-1,\tau,z_0)\|_{L^q(0,1)}\\
&\quad+\int_{t-1}^t C(\theta)(t-s)^{-\theta}\|\psi(s,z(s))\|_{L^q(0,1)}ds.
\end{aligned}$$

Note that $\|z(t-1,\tau,z_0)\|_{L^q(0,1)}$ is estimated in the same way as in (7.30):

$$\|z(t)\|^\rho_{L^q(0,1)}\leq C^*\left[e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^\infty(0,1)}+\left[M_2(t,\tau,w_0)+1\right]^{\frac{\rho}{2}}\right],$$

whereas $\|\psi(s,\tau)\|_{L^q(0,1)}$ was estimated in Lemma 7.28. From those results, we obtain

$$\|z(t,\tau,z_0)\|_{V^\theta_q}\leq C(\theta)C^*(2^{\frac{1}{\rho}})\left[e^{-\gamma(t-1-\tau)}\|z_0\|_{L^\infty(0,1)}+\left[M_2(t-1,\tau,w_0)+1\right]^{\frac{1}{2}}\right]$$

$$
\begin{aligned}
&+\int_{t-1}^{t} C(\theta)(t-s)^{-\theta}C^*\left[e^{-\rho\gamma(s-\tau)}\|z_0\|^{\rho}_{L^\infty(0,1)}+[M_2(s,\tau,w_0)+1]^{\rho}\right]ds\\
&\leq C^*(2^{\frac{1}{\rho}})\left[e^{-\gamma(t-\tau)}\|z_0\|_{L^\infty(0,1)}+[M_2(t,\tau,w_0)+1]^{\frac{1}{2}}\right]\\
&\quad+C^*\left[e^{-\rho(t-1-\tau)}\|z_0\|^{\rho}_{L^\infty(0,1)}+[M_2(t-1,\tau,w_0)+1]^{\rho}\right]\\
&\leq C^*\left[e^{-\gamma(t-\tau)}(\|z_0\|_{L^\infty(0,1)}+\|z_0\|^{\rho}_{L^\infty(0,1)})+[M_2(t,\tau,w_0)+1]^{\rho}\right],
\end{aligned}
$$

where we used the relation presented in Remark 7.22, item (2), to replace $M_2(t-1,\tau,w_0)$ by $M_2(t,\tau,w_0)$ and we adjusted C^* to incorporate the other constants.

□

7.2.4 Global well-posedness and Pullback attracting set for $z(t)$

The results on the previous section allow us to obtain global well-posedness and the existence of attracting set for the solution $z(t,\tau,z_0)$ of the problem

$$z_t+\mathcal{L}(t)z=\psi(s,z),t>\tau;\quad z(\tau)=z_0,$$

which can be transferred to $v(t)$ trough the relation $v(t)=z(t)+\xi(t)$. Before we state the results, we recall some essential points:

1. The constant $M_2(t,\tau,w_0)$ given in (7.25) satisfies

$$M_2(t,\tau,w_0)\xrightarrow{t-\tau\to\infty}0, \text{ uniformly for } w_0 \text{ in bounded sets of } L^\infty(\Omega).$$

2. $\xi(t,r;(\tau,w_0))$, estimated in Proposition 7.20, satisfies

$$|\xi(t,r;(\tau,w_0))|\leq G_1e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)}+\|w_0\|^{\rho}_{L^\infty(\Omega)}\right)+G_2,$$

and

$$|\xi(t,r;(\tau,w_0))|\xrightarrow{t-\tau\to\infty}G_2 \text{ uniformly for } w_0 \text{ in bounded sets of } L^\infty(\Omega).$$

The $L^{2^k}-$estimate obtained in Proposition 7.26 implies that $\|z(t,\tau,z_0)\|_{L^p(0,1)}$ is bounded in each bounded interval $[\tau,T]$. Consequently, $v(t)$ is bounded in any interval $[\tau,T]$. Therefore, $z(t,\tau,z_0)$ and $v(t,\tau)$ are globally defined in time (Theorem 6.8). We prove with details this statement in the next proposition.

Proposition 7.30. *Let* $z(\cdot,\tau,z_0)$ *be the solution of* (7.20) *associated to the initial conditions* $(w_0,v_0)\in L^\infty(\Omega)\times L^\infty(0,1)$ *and* $z_0=v_0-\xi(\tau,w_0)\in L^\infty(0,1)$. *Then* $z(\cdot,\tau,z_0)$ *is globally defined. Moreover,* *there exists a closed ball in V_q^θ centered in zero and with a radius $C^* = C^*(\theta, \Omega, \rho, p, q, N, M, \gamma) > 0$ (independent of τ, w_0, or z_0),*

$$B_{V_q^\theta}[0, C^*],$$

such that $z(t, \tau, z_0)$ is pullback attracted by this set in the topology of V_q^θ, uniformly for z_0 in bounded sets of $L^p(0,1)$ and for w_0 in bounded sets of $L^\infty(\Omega)$.

Proof. Let $z_0 \in L^p(0,1)$. It follows from Remark 7.23 that after any arbitrarily small evolution in time, $z^* = z(t^*, \tau, z_0) \in L^\infty(0,1)$. Therefore, if we start the evolution at instant t^* and at the point z^*, Proposition 7.29 implies that $\|z(t, t^*, z^*)\|_{V_q^\theta}$ is finite in any bounded interval $[t^*, T]$. For $\theta > \frac{1}{2q}\left(\frac{\rho-1}{\rho}\right)$, this boundedness implies $L^p(0,1)$ boundedness and global existence of the solution (see Figure 7.2).

Moreover, let $B \subset L^p(0,1)$ be a bounded set such that $\|z_0\|_{L^p(0,1)} \leq L$ for any $z_0 \in B$. It also follows from Remark 7.23 that after an arbitrarily small evolution in time, $t^* > \tau$, the elements $z^* = z(t^*, \tau, z_0)$ are bounded in $L^\infty(0,1)$, that is $\|z^*\|_{L^\infty(0,1)} \leq \tilde{L}$. We denote by $B^* = \{z^* = z(t^*, \tau, z_0); z_0 \in B\}$. Therefore, Proposition 7.29 states that there exists a constant $C^* = C^*(\theta, \Omega, \rho, p, q, N, M, \gamma) > 0$ such that

$$\|z(t)\|_{V_q^\theta} \leq C^* \left[e^{-\gamma(t-\tau)}(\|z_0\|_{L^\infty(0,1)} + \|z_0\|^\rho_{L^\infty(0,1)}) + [M_2(t, \tau, w_0) + 1]^\rho \right].$$

In particular, for $z^* \in B^*$, we obtain

$$\|z(t, t^*, z^*)\|_{V_q^\theta} \leq C^* \left[e^{-\gamma(t-t^*)}(\tilde{L} + \tilde{L}^\rho) + [M_2(t, \tau, w_0) + 1]^\rho \right]$$

and since $M_2(t, \tau, w_0) \xrightarrow{t-\tau\to\infty} 0$, uniformly for w_0 in bounded sets of $L^\infty(\Omega)$, we have

$$dist(z(t, t^*, z^*), B_{V_q^\theta}[0, C^*]) \xrightarrow{t-t^*\to\infty} 0,$$

uniformly for $z^* \in B^*$ and w_0 in bounded sets of $L^\infty(\Omega)$. □

In order to transfer the results to $v(t)$, we must estimate $\xi(t, r)$ in the $\|\cdot\|_{V_q^\theta}$-norm, since $\|v(t,\cdot)\|_{V_q^\theta} \leq \|z(t,\cdot)\|_{V_q^\theta} + \|\xi(t,\cdot)\|_{V_q^\theta}$. This is done in the next lemma.

Lemma 7.31. *Let $w_0 \in L^\infty(\Omega)$, $\tau < t - 1 < t$, E_1, E_2 the constants given in* (7.13) *and H_1 given in* (7.24) *(they only depend on Ω, ρ, N, M, γ), then*

$$\|\xi(t, r; (\tau, w_0))\|_{V_q^\theta} \leq C\frac{E_1}{H_1} M_2(t, \tau, w_0) + CE_2.$$

Proof. Recall that

$$\xi(t, r) = w(t, \tau, w_0)(p_0)\mathcal{X}_0(t, r) + w(t, \tau, w_0)(p_1)\mathcal{X}_1(t, r),$$

where

$$\mathcal{X}_0(t, r) = \left[\frac{\displaystyle\int_r^1 \frac{1}{a(t,\theta)} d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)} d\theta} \right], \quad \mathcal{X}_1(t, r) = \left[\frac{\displaystyle\int_0^r \frac{1}{a(t,\theta)} d\theta}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)} d\theta} \right].$$

Differentiating $\mathcal{X}_0(t,r)$ in r two times, we obtain

$$\partial_r^2 \mathcal{X}_0(t,r) = \frac{\dfrac{\partial_r a(t,r)}{(a(t,r)^2)}}{\displaystyle\int_0^1 \frac{1}{a(t,\theta)} d\theta} \quad \text{and} \quad |\partial_r^2 \mathcal{X}_0(t,r)| \le C,$$

which is bounded due to Assumption (A.3) (this estimate does not dependent of t, τ, w_0 or v_0). The same holds for $\partial_r^2 \mathcal{X}_1(t,r)$. Therefore, $\|\mathcal{X}_0(t,r)\|_{V_q^\theta}$ and $\|\mathcal{X}_1(t,r)\|_{V_q^\theta}$ are bounded by a constant C for any $0 \le \theta < 1$. From Lemma 7.19, we obtain

$$\begin{aligned}
\|\xi(t,r;(\tau,w_0))\|_{V_q^\theta} &\le |w(t,\tau,w_0)(p_0)| \|\mathcal{X}_0(t,r)\|_{V_q^\theta} + |w(t,\tau,w_0)(p_1)| \|\mathcal{X}_1(t,r)\|_{V_q^\theta} \\
&\le C(|w(t,\tau,w_0)(p_0)| + |w(t,\tau,w_0)(p_1)|) \\
&\le CE_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} \right) + CE_2. \\
&= C\frac{E_1}{H_1} M_2(t,\tau,w_0) + CE_2.
\end{aligned}$$

□

As a consequence of Lemma 7.31, Proposition 7.29 and fact that $v(t,r) = \xi(t,r) + z(t,r)$, we obtain

Proposition 7.32. *Let* $(w_0,v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$, $z_0 = v_0 - \xi(\tau,w_0) \in L^\infty(0,1)$, $z(\cdot,\tau,z_0)$ *the solution of* (7.20) *and* $v(t) = \xi(t) + z(t)$. *Then, for* $0 \le \theta < 1$, *there exists* $C^* = C^*(\theta,\Omega,\rho,p,q,N,M,\gamma) > 0$ *depending on* $\theta,\Omega,\rho,p,q,N,M$ *and* γ *(but not on* τ, w_0 *or* z_0*) such that*

$$\|v(t)\|_{V_q^\theta} \le C^* \left[e^{-\gamma(t-\tau)} (\|z_0\|_{L^\infty(0,1)} + \|z_0\|^\rho_{L^\infty(0,1)}) + [M_2(t,\tau,w_0)+1]^\rho \right].$$

Proof. From Proposition 7.29 and Lemma 7.31

$$\begin{aligned}
\|v(t)\|_{V_q^\theta} \le \|z(t)\|_{V_q^\theta} + \|\xi(t)\|_{V_q^\theta} &\le C^* \left[e^{-\gamma(t-\tau)} (\|z_0\|_{L^\infty(0,1)} + \|z_0\|^\rho_{L^\infty(0,1)}) + [M_2(t,\tau,w_0)+1]^\rho \right] \\
&\quad + C\frac{E_1}{H_1} M_2(t,\tau,w_0) + CE_2.
\end{aligned}$$

By adjusting the constant C^* the result follows. □

7.3 Pullback attractor in $\Omega_0 = \Omega \cup R_0$

We finally return to the equation

$$(w,v)_t(t) = -A_0(t)(w,v)(t) + F_0((w,v)(t)), \ t > \tau; \quad (w,v)(\tau) = (w_0,v_0) \in U_p^0,$$

whose solution $(w,v)(t) : (\tau, \tau_M(w_0,v_0)) \to U_p^0$ is given by

$$(w,v)(t) = U_0(t,\tau)(w_0,v_0) + \int_\tau^t U_0(t,s) F_0(t,s) ds.$$

It follows from the estimate obtained for $w(t)$ in $\|\cdot\|_{Y_q^\theta}$ in Proposition 7.10 and the estimate for $v(t)$ in $\|\cdot\|_{V_q^\theta}$ in Proposition 7.32 the global existence of the solution as well as the existence of a compact pullback absorbing set, summarized in the next proposition.

Proposition 7.33. *Let E_2 be the constant obtained in* (7.13) *and $C^* > 0$ the constant obtained in Proposition 7.32 (depending only on Ω, ρ, p, q, N, M and γ, but independent of t, τ, w_0 or v_0). Given $(w_0, v_0) \in U_p^0$, we have:*

1. *The solution $(w, v)(t, \tau, (w_0, v_0))$ exists for all $t > \tau$ and defines a nonlinear process*
$$S(t,\tau)(w_0,v_0) = (w,v)(t) = U_0(t,\tau)(w_0,v_0) + \int_\tau^t U_0(t,s)F_0(t,s)ds.$$
2. *For any $0 \le \theta < 1$,*
$$K_{\theta,q} = B_{Y_q^\theta}[0, E_2] \times B_{V_q^\theta}[0, C^*]$$
is a pullback attracting set for the process $S(t,\tau)$ in the $Y_q^\theta \times V_q^\theta-$topology.

The existence of such pullback attracting set implies that the nonlinear process has pullback attractor.

Theorem 7.34. *Assume that $p > N$ and $\max\left\{\frac{N}{2\delta}, \frac{p(2N+1)}{2p+1}\right\} < q \le p$, $X = U_p^0$, $Y = U_q^0$, $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ satisfies (A.2) and (A.3) and $f : \mathbb{R} \to \mathbb{R}$ satisfies (A.4) and (D). The solution $(w, v)(t)$ for the problem*
$$\begin{cases} (w,v)_t + A_0(t)(w,v) = F_0(w,v), & t > \tau, \\ (w,v)(\tau) = (w_0, v_0) \in U_p^0, \end{cases}$$
defines a nonlinear process $S(t,\tau)$ in $U_p^0 = L^p(\Omega) \times L^p(0,1)$ which has a pullback attractor $\mathcal{A}(t)$ in U_p^0. Moreover, $\cup_{t\in\mathbb{R}}\mathcal{A}(t) \subset \mathcal{C}^{1,\eta}(\Omega) \times \mathcal{C}^{1,\eta}(0,1)$, for some $\eta > 0$, and pullback attracts bounded sets of U_p^0 in the topology of $\mathcal{C}^{1,\eta}(\Omega)$.

Proof. In Proposition 7.33 we proved the existence of a pullback attracting bounded set in $Y_q^\theta \times V_q^\theta$ for any $0 \le \theta < 1$. Since

1. Y_q^θ is compactly embedded in $L^p(\Omega)$ for $\theta > \frac{N}{2q}(\frac{\rho-1}{\rho})$ (Lemma 7.1),
2. V_q^θ is compactly embedded in $L^p(0,1)$ for $\theta > \frac{1}{2q}\left(\frac{\rho-1}{\rho}\right)$ (Lemma 7.17),

we conclude that
$$K_{\theta,q} = B_{Y_q^\theta}[0, E_2] \times B_{V_q^\theta}[0, C^*] \overset{c}{\hookrightarrow} U_p^0$$
and $K_{\theta,q}$ is a compact pullback attracting set for the process $S(t,\tau)$ in U_p^0. It follows from Corollary 6.16 that there exists a pullback attractor
$$\mathcal{A}(t) \subset K_{\theta,q} \overset{c}{\hookrightarrow} U_p^0, \quad \forall t \in \mathbb{R},$$
that attracts bounded sets of U_p^0 in the topology of $Y_q^\theta \times V_q^\theta$. Moreover, if $\theta > \frac{1}{2} + \frac{N}{2q}$, then $Y_q^\theta \times V_q^\theta \hookrightarrow \mathcal{C}^{1,\eta}(\Omega) \times \mathcal{C}^{1,\eta}(0,1)$ and the last statement follows.

□

We can derive some conclusions and observations from the steps taken to obtain the existence of pullback attractor:

1. For this particular problem, it was not necessary all the restrictive conditions required in Proposition 5.2 to treat the asymptotic dynamics. This is a consequence of the fact that the equations are weakly coupled and allowed us to work with them separately, enjoying the parabolic structure of each one, with sectorial operators rather then almost sectorial.

2. The pullback attractor $\mathcal{A}(t)$ obtained has two components, one acting in Ω and the other in the channel R_0. Since the dynamics in Ω is independent of R_0, the pullback attractor in Ω, $\mathcal{A}_\Omega(t)$, obtained in Theorem 7.12, and the part of $\mathcal{A}(t)$ in Ω must be the same. In other words, if $\Pi_1 : U_p^0 \to L^p(\Omega)$ is the projection in the first coordinate, then

$$\Pi_1(\mathcal{A}(t)) = \mathcal{A}_\Omega(t).$$

3. Since

$$\begin{aligned} &\cup_{t\in\mathbb{R}} \mathcal{A}_\Omega(t) \subset B_{Y_q^\theta}\left[0, E_2\right], \\ &\cup_{t\in\mathbb{R}} \mathcal{A}(t) \subset K_{\theta,q}, \end{aligned}$$

both attractors are bounded in the past and they are given by the union of all bounded global solutions (Proposition 6.19), that is

$$\begin{aligned} &\mathcal{A}_\Omega(t) = \{\phi(t); \phi : \mathbb{R} \to L^p(\Omega) \text{ is a bounded global solution}\} \\ &\mathcal{A}(t) = \{(\phi,\zeta)(t); (\phi,\zeta) : \mathbb{R} \to L^p(\Omega) \times L^p(0,1) \text{ is a bounded global solution}\}. \end{aligned}$$

Moreover, if $w_0 \in L^p(\Omega)$ is such that there exists a bounded global solution $\phi : \mathbb{R} \to L^p(0,1)$ with $\phi(\tau) = w_0$, then ϕ (or (τ, w_0)) originates one problem in the channel R_0 that, after the change of variables, is given by

$$\begin{cases} z_t - \partial_r(a(t,r)\partial_r z) + z = -\xi(t) - \xi_t(t) + f\left(\xi + z\right), & t > \tau, \in (0,1), \\ z(t,0) = 0 \text{ and } z(t,1) = 0, & t > \tau, \\ z(0,r) = z_0(r) = v_0(r) - \xi(0,r) \in L^p(0,1). \end{cases}$$

The solution $v(t)$ in the channel is given by $v(t,r) = \xi(t,r;(\tau,w_0)) + z(t,r)$ and in Proposition 7.30 we proved the existence of a constant $C^* > 0$ (depending on $\theta, \Omega, \rho, p, q, N, M$ and γ) such that $B_{V_q^\theta}\left[0, C^*\right]$ is a compact set that pullback attracts $z(t,\tau,z_0)$ uniformly in bounded sets in the $V_q^\theta-$topology.

Since $V_q^\theta \overset{c}{\hookrightarrow} L^p(0,1)$, this implies that the dynamical system originated by $z(t,\tau,z_0)$ has a pullback attractor $\mathcal{A}_{(\tau,w_0)}(t)$. We keep the subindex (τ, w_0) to indicate that this attractor depends on the initial conditions chosen for the evolution in Ω.

In this case,

$$\xi(t) + \mathcal{A}_{(\tau,w_0)}$$

pullback attracts the solution $v(t)$ in the channel and since $\xi(t, r; (\tau, w_0))$ is bounded, we have

$$(\phi(t), \xi(t) + \mathcal{A}_{(\tau,w_0)}) \subset \mathcal{A}(t).$$

In other words, for a given bounded global solution ϕ in Ω such that $\phi(\tau) = w_0$, the set $(\phi(t), \xi(t) + \mathcal{A}_{(\tau,w_0)})$ is a "piece" of the pullback attractor. This illustrates how the dynamics in the channel (given by the second coordinate) can collaborate to form the pullback attractor.

CHAPTER 8

Remarks and discussion

We dedicate this chapter to enlarge the discussion of some points made during the text and also to pose some problems related to the topic studied. We begin by connecting the theory developed in the previous chapters with the one in [19], where the authors used a fractional power approach to treat the semilinear case.

8.1 Fractional power spaces

Suppose $A(t) : D(A(t)) \subset Z \to Z$ is a family of uniformly almost sectorial operators (with constant $\phi \in (0,1)$) and uniformly Hölder continuous (with exponent δ) in the Banach space Z. In [53, 57] a functional calculus and fractional powers for almost sectorial operators were established.

Denoting $A_0^{\xi} = [A(t_0)]^{\xi}$ the fractional power for a fixed operator $A(t_0)$, we can obtain an associated scale of fractional power spaces $Z^{\xi} = D(A_0^{\xi})$ in the same sense that we do for sectorial operators.

However, the deficiency in the resolvent allows us to define those powers only on the interval $1-\phi < \xi < 1$ and the momentum inequality only holds for $1 - \phi < \xi < \phi$ (see [19, p. 24]). Those restrictions reflect on the semilinear problem, as we see next.

Let $F : Z^{\gamma} \to Z^{\theta}$, with $1 - \phi < \theta < \gamma < 1$ and assume $1 - \phi < \gamma - \theta < \phi^2$. Suppose also that F has a growth given by $\rho \geq 1$. Under those conditions, Theorem 3.1 in [19] proves the existence of mild solutions for

$$u_t + A(t)u = F(u),\ t > \tau; \quad u(\tau) = u_0 \in Z^{\gamma}.$$

Comparing with the terminology used in this work, we set $X = Z^{\gamma}$ and $Y = Z^{\theta}$. Using the almost sectoriality of $A(t)$, the characterization of the resolvent $(\lambda + A(t))^{-1}$ as the Laplace transform of the semigroup $T_{-A(t)}$, which also holds for almost sectorial operators (see [51, Lemma 3.1]), and the Momentum inequality [19, Proposition 2.1], we have

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Y,X)} = \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Z^{\theta},Z^{\gamma})} = \|A_0^{\gamma}(\lambda + A(t))^{-1}A_0^{-\theta}\|_{\mathcal{L}(Z)}$$

$$\leq \left\| A_0^{\gamma-\theta} \int_0^\infty e^{-\lambda s} T_{-A(t)}(s) ds \right\|_{\mathcal{L}(Z)} \leq C \int_0^\infty e^{-\lambda s} s^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} ds$$
$$\leq C \int_0^\infty e^{-u} u^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} \left(\frac{1}{\lambda}\right)^{-1+\phi-\frac{(\gamma-\theta)}{\phi}} \frac{1}{\lambda} du = C \frac{1}{\lambda^{\phi-\frac{(\gamma-\theta)}{\phi}}} \Gamma\left(\phi - \tfrac{(\gamma-\theta)}{\phi}\right),$$

since $\gamma - \theta < \phi^2$.

Therefore, the constant β in (P.4) (Chapter 1), in this case, would be $\beta = \phi - \frac{(\gamma-\theta)}{\phi}$. Furthermore, if $\gamma = \theta$ then we can see that

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} = \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Y)} = \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Z)} \leq \frac{1}{\lambda^\phi},$$

that is, the constant of sectoriality in X, Y and Z are all the same: $\alpha = \omega = \phi$.

Theorem 1.24 in Chapter 1 states that local solvability for the problem is guaranteed if

$$\rho < \frac{\beta}{1-\alpha} = \frac{\phi - \frac{(\gamma-\theta)}{\phi}}{1-\phi} \quad \text{and} \quad \phi + \delta > 1.$$

The first inequality above is exactly the maximal growth condition established in Theorem 3.1 of [19] and $\phi + \delta > 1$ appears in Proposition 2.3 in order to estimate the norm $\|A(t)^\gamma U(t,\tau) A(t)^{-\theta}\|$, which is used in the proof of Theorem 3.1.

This allows us to conclude that the strategy adopted here to treat the problem and the one developed in [19] communicates quite well. However, we understand that the approach developed in the text can incorporate cases that the fractional power approach fails to, especially examples where the domains of the fractional powers are unknown (which is the case for the reaction-diffusion equation in the domain with handle). Even if we can describe $D(A(t)^\xi)$ we usually need to know sharp embeddings of this space in the Bessel potential spaces $H^s_q(\Omega)$, which, in general, are not known (see Section 2.1 of [10] for further details on Bessel spaces and their embeddings on Sobolev spaces).

For instance, in [19, Section 4], in order to solve the semilinear reaction-diffusion equation in a domain with handle, they had to assume that F is Lipschitz continuous and has no growth ($\rho = 1$), possibly due to the absence of a good description of the domain of the fractional powers of the linear operator.

On the downside, working with the abstract setting involving the spaces X and Y prevent us to obtain smoothing effects as the one described in Theorem 6.9 for the sectorial case. If we do not know intermediate scales between Y and X, we cannot improve the regularity of the derivative u_t.

8.2 The nonsingular reaction-diffusion equation in Ω

It is worth mentioning that in order to perform the study in Section 7.1 that culminates with existence of pullback attractor $\mathcal{A}_\Omega(t)$, no additional condition concerning monotonicity, decay ($a'(t) \leq 0$) or asymptotic behavior for the function $a(t,x)$ was necessary, which differs this part from some studies existent in the literature.

For example, in [61], in order to study the asymptotic dynamics of a singularly nonautonomous linear equations

$$u_t + A(t)u = f(t),$$

the author assumed that $A(t)$ approaches an operator $A(\infty)$ as $t \to \infty$, in the following sense: There exists a closed linear operator $A(\infty) : D \subset X \to X$ such that

$$\lim_{t\to\infty} \|(A(t) - A(\infty))A(0)^{-1}\| = 0.$$

In this case, the author was able to prove an exponential decay for the linear process associated to $A(t)$ and study the asymptotic dynamics for the equation.

Other works, like [16, 27, 32, 33], treated singularly nonautonomous damped wave equations in $\mathbb{R}^N$ of the type

$$u_{tt} - a(t)\Delta u + b(t)u_t = f(u).$$

They refer to this class of equations as *wave equations with time-dependent speed and damping*. By assuming conditions on the derivative of a, it is possible to obtain an energy function (or Lyapunov function) for the system and derive global existence of solution.

We searched for a way in which neither asymptotic conditions or monotonicity/decay of $a(t, x)$ were necessary. The techniques employed in this work to treat the singularly nonautonomous reaction-diffusion equation - the iteration technique to obtain $L^{2^k}-$estimates and the smoothing effect that the differential equation has on w_t and v_t - enable us to study long-time behavior of the solutions without any restriction on the sign of $a'(t)$.

8.3 Asymptotic dynamics in Ω

For a nonsingular reaction-diffusion equation ($A(t) = A$) in Ω, for example,

$$\begin{aligned} &u_t - div(a(x)\nabla u) + u = f(u), t > 0 \\ &\partial_n u = 0, \end{aligned}$$

the construction of a Lyapunov function for the system is usually available. This provides further information on the long-time behavior of the solution. For instance, if f is time-independent, the equation is autonomous and, under suitable conditions, has an associated semigroup $T(\cdot)$. If

$$\mathcal{E} = \{y : T(t)y = y \text{ for all } t \geq 0\}$$

denotes the set of equilibrium point for $T(\cdot)$ (which we assumed discrete), then all solutions converges to an equilibrium point. In other words, all the solutions converges to a constant function in the long-time dynamics, the derivative in time will approach zero and the solution will be close to a solution of the associated elliptic equation $Au = f(u)$. This allows a better description of the attractor in terms of equilibria and heteroclinic orbits connecting them (see [20, Chapter 12] or [54, Chapter 10] for a deeper discussion on the structure of the attractor and Lyapunov functions).

For the singularly nonautonomous case, this situation changes, especially due to the fact that the "elliptic" operator itself changes with time and the associated equation is $A(t)u = f(u)$. There are no reasons to say that the solution approaches a constant value (an equilibrium) as the dynamics evolves. The derivative in time for the solution does not vanish in the long-time. However, we were able to prove in Section 7.1.3 that, after a certain time, those derivatives are enclosed in a compact set of the phase space and the variation of the solution in the long-time is somehow controlled.

CHAPTER 9

Conclusions/Conclusiones/Conclusões

Conclusions

In this work we were interested in developing an abstract theory suitable to treat singularly nonautonomous problem in which the time-dependent family of linear operator is uniformly almost sectorial, that is, problems of the form

$$\begin{aligned} &u_t + A(t)u = F(u),\ t > \tau; \\ &u(\tau) = u_0 \in X. \end{aligned} \tag{9.1}$$

Those uniformly almost sectorial operators $A(t)$ usually emerge in applications when we consider elliptic operators defined in more regular phase spaces, as the space of Hölder continuous functions that vanish in the boundary of a bounded domain, $X = \mathcal{C}_0^\mu(\Omega)$, see [19, Section 4.1], or when we are dealing with parabolic equations in certain singular domains, as the domain with a handle Ω_0 introduced in Chapter 2. The second case was explored throughout this work and illustrated the abstract theory developed.

In terms of the results achieved, we were able to provide conditions that ensure local well-posedness for the semilinear problem (9.1), regularity of this local solution and conditions to study the long-time dynamics of the problem. The work was structured in three different parts, each dedicated to attend one of those topics.

In the first part we introduced the concepts of semigroups and linear process of growth $1-\alpha$ and we gather a series of properties for these two families necessary for the development of the theory. The main result in this part is the existence of local mild solution, obtained via the variation of constants formula

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds. \tag{9.2}$$

This expression for the mild solution allowed us to obtain regularity results for $u(t)$. There were no results so far in the literature concerning regularity of the mild solution for the singularly nonautonomous case (9.1). Our contribution on this topic (presented in Chapter 4) were gathered in the paper [14]. Actually, the literature on almost sectorial operators lacked results in this direction even when $A(t) = A$ does not depend on time. In [31] the author explored some results in the direction of obtaining strong solutions for semilinear problems of the form

$$\begin{aligned} &u_t + Au = F(u),\ t > 0; \\ &u(0) = u_0 \in X, \end{aligned} \tag{9.3}$$

with almost sectorial operators, but the conditions required to obtain the strong solutions were very restrictive. With the results developed in Part II of this work, we were also able to extend the theory of regularity for problems of the form (9.3) without requiring the conditions adopted in [31]. The new results obtained for the autonomous case were presented in the paper [15].

As far as the long-time dynamics of the problem, the fact that $A(t)$ is time-dependent prevented us to use the classical approaches to obtain global estimates for parabolic problems, which consists in the construction of a Lyapunov function or the use of comparison result and monotonicity of solutions. As a matter of fact, there were no results so far in the literature that allowed us to deal with the singularly nonautonomous case even when $A(t)$ is sectorial, unless some monotonicity condition or decay in time were required for this family of linear operators.

Therefore, in order to treat the equation

$$\begin{aligned} &u_t - div(a(t,x)\nabla u) + u = f(u),\ x \in \Omega,\ t > \tau; \\ &\partial_n u = 0, \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega, \end{aligned} \tag{9.4}$$

on the bounded smooth domain Ω, to which we can associate the sectorial family $B(t)u = -div(a(t,x)u) + u$, we developed the iterative method presented in Chapter 7, Section 7.1. This method combines an iterative procedure and regularization properties of the differential equation in order to obtain estimates for the solution in stronger norms.

This iterative procedure can be extended to more general second order parabolic equations in the divergent form. For example, it can be applied to obtain global estimates for the solution of

$$u_t = \sum_{i,j=1}^n \partial_j\left(a_{ij}(t,x)\partial_i u\right) + \sum_{i=1}^n b_i(t,x)\partial_i u + f(t,x,u), \quad t > \tau,\ x \in \Omega, \tag{9.5}$$

where u satisfies homogeneous Dirichlet or Neumann boundary conditions, that is,

$$u|_{\partial\Omega} = 0 \quad \text{or} \quad \left(\sum_{i,j=1}^n a_{ij}(t,x)\partial_i u \cos(N(x), x_j)\right)\Bigg|_{\partial\Omega} = 0,$$

(the expression $\cos(N(x), x_j)$ denotes the cosine between the normal vector $N(x)$ at the point $x \in \partial\Omega$ and the coordinate axis x_j). If Ω is a bounded smooth domain, $\partial\Omega$ is C^2, $a_{ij}, b_i \in C^1(\mathbb{R} \times \overline{\Omega})$, $f \in C(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ and an appropriate ellipticity condition on a_{ij}

$$\exists_{\alpha_0>0} \, \forall_{x\in\Omega, t\in\mathbb{R}} \, \forall_{\xi\in\mathbb{R}^n} \quad \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq a_0|\xi|^2, \tag{9.6}$$

is satisfied, as well as an appropriate growth condition for f

$$\exists_{C,D>0} \, \forall_{x\in\Omega, t\in\mathbb{R}} \, \forall_{v\in\mathbb{R}} \quad vf(t,x,v) \leq Cv^2 + D, \tag{9.7}$$

we can obtain a result similar to Lemma 7.8. The details of this method as well as the smoothing effect of differential equations with sectorial operators were gathered in the paper [13].

Finally, to treat the reaction-diffusion equation in $\Omega_0 = \Omega \cup R_0$, we take advantage of the fact that the equations in Ω (the open domain in $\mathbb{R}^n$) and R_0 (the line segment) are weakly coupled, and when we separate the problems we obtain sectorial operators for each one of them. The iterative method was then applied to treat the problems separately and estimates of the solutions in each component became available, as well as the existence of pullback attracting sets.

One interesting feature in this decoupling strategy was the analysis required on the nonlinearity when we consider the equation in the line segment R_0. It was necessary to prove that the nonlinearity, after the decoupling, would satisfy and appropriate dissipation condition. The results concerning the long-time dynamics of the equation in Ω_0 were presented in the paper [12].

Future works

Parabolic problems of the form (9.1) (or more generally for $F = F(t, u, \nabla u)$) still have several open problems to be studied, rather we are dealing with a sectorial family of linear operators or a almost sectorial family.

One of the difficulties of working with this type of problem comes from the fact that it is essentially nonautonomous. To make it precise, consider the case where $A(t) = A$ is time independent. The solution for the semilinear problem in this case is given in terms of the semigroup generated by $-A$, that is

$$u(t) = T_{-A}(t-\tau) + \int_{\tau}^{t} T_{-A}(t-s)F(s, u(s), \nabla u(s))ds \tag{9.8}$$

and $u(t)$ can be seen as a nonautonomous perturbation of the linear semigroup $T_{-A}(\cdot)$. The situation in (9.2) is different since the family of linear operator $U(t,\tau)$ is itself nonautonomous, making it difficult to study some properties of the solution $u(t)$. The implications of this fact are enumerated in the sequel and are interesting problems to be studied.

1. The concept of "equilibrium solutions" that plays an essential role in the description of the structure of the attractor (for autonomous problems) is not available and we do not expect the solutions to

converge to stationary states. This lack of information on the structure of the pullback attractor makes it difficult the analysis of lower semicontinuity of pullback attractor, when we consider perturbations of the problem.

2. For the nonsingular case (9.3), if we consider perturbations of the problem of the form $u_t + A_\varepsilon u = F_\varepsilon(u)$, one can prove convergence of the linear semigroup $T_{-A_\varepsilon}(t)$ to $T_{-A}(t)$, provided that we have convergence of the resolvent of A_ε to the resolvent of A (see [4, 6, 22]). This convergence is then transferred to the solution of the semilinear problem via the variation of constants formula. For the singular case, if $U_\varepsilon(t,\tau)$ is a linear process associated to $A_\varepsilon(t)$, an interesting analysis would be to understand how the resolvent convergence of $A_\varepsilon(t)$ to $A(t)$ can be transferred to the linear process and, consequently, to the solution of the semilinear problem.

Conclusiones

En este trabajo nos interesaba desarrollar una teoría abstracta adecuada para tratar un problema singularmente no autónomo en el que la familia de operadores lineales dependientes del tiempo es uniformemente casi sectorial, es decir, problemas de la forma (9.1).

Esos operadores uniformemente casi sectoriales $A(t)$ generalmente surgen en aplicaciones cuando consideramos operadores elípticos definidos en espacios de fase más regulares, como el espacio de funciones Hölder continuas que se anulan en la frontera de un dominio acotado, $X = \mathcal{C}_0^{\mu}(\Omega)$, consulte [19, Sección 4.1], o cuando se trata de ecuaciones parabólicas en ciertos dominios singulares, como el dominio con una "asa" Ω_0 introducido en el Capítulo 2. El segundo caso fue explorado a lo largo de este trabajo e ilustró la teoría abstracta desarrollada.

En términos de los resultados obtenidos, pudimos proporcionar condiciones que aseguran el buen planteamento local para los problemas semilineales (9.1), la regularidad de esta solución local y las condiciones para estudiar la dinámica asintótica del problema. El trabajo se estructuró en tres partes, cada una dedicada a atender uno de esos temas.

En la primera parte introducimos los conceptos de semigrupos y proceso lineales de crecimiento $1-\alpha$ y reunimos una serie de propiedades para estas dos familias necesarias para el desarrollo de la teoría. El principal resultado en esta parte es la existencia de una solución "mild" local, obtenida mediante la fórmula de variación de constantes (9.2).

Esta expresión para la solución "mild" nos permitió obtener resultados de regularidad para $u(t)$. Hasta ahora no hay resultados en la literatura sobre la regularidad de la solución "mild" para el caso singularmente no autónomo (9.1). Nuestra contribución sobre este tema (presentada en el Capítulo 4) fue reunida en el artículo [14]. En realidad, la literatura sobre operadores casi sectoriales carecía de resultados en esta dirección incluso cuando $A(t) = A$ no depende del tiempo. En [31] el autor exploró algunos resultados en la dirección de obtener soluciones fuertes para problemas semilineales de la forma (9.3) con operadores casi sectoriales, pero las condiciones requeridas para obtener las soluciones fuertes eran muy restrictivas. Con los resultados desarrollados en la Parte II de este trabajo, también pudimos extender la teoría de la regularidad para problemas de la forma (9.3) sin requerir las condiciones adoptadas en [31]. Los nuevos resultados obtenidos para el caso autónomo fueron presentados en el artículo [15].

En cuanto a la dinámica asintótica del problema, el hecho de que $A(t)$ sea dependiente del tiempo nos impidió utilizar los enfoques clásicos para obtener estimaciones globales de problemas parabólicos, que consisten en la construcción de una función de Lyapunov o el uso de resultados de comparación y monotonicidad de las soluciones. De hecho, hasta ahora no ha habido resultados en la literatura que nos permitan abordar el caso singularmente no autónomo incluso cuando $A(t)$ es sectorial, a menos que se requiera alguna condición de monotonicidad o decrecimiento en el tiempo para esta familia de operadores lineales.

Por tanto, para tratar la ecuación (9.4) en el dominio suave acotado Ω, al que podemos asociar la familia sectorial $B(t)u = -div(a(t,x)u)+u$, desarrollamos el método iterativo presentado en el Capítulo 7, Sección 7.1. Este método combina un procedimiento iterativo y propiedades de regularización de la ecuación diferencial con el fin de obtener estimaciones para la solución en normas más fuertes.

Este procedimiento iterativo puede extenderse a ecuaciones parabólicas de segunda orden más generales en la forma divergente del operador. Por ejemplo, se puede aplicar el método para obtener estimaciones globales para la solución de (9.5) donde u satisface condiciones de frontera homogéneas de Dirichlet o Neumann, es decir,

$$u|_{\partial\Omega} = 0 \quad \text{o} \quad \left(\sum_{i,j=1}^{n} a_{ij}(t,x)\partial_i u \cos(N(x), x_j)\right)\Bigg|_{\partial\Omega} = 0,$$

(la expresión $\cos(N(x), x_j)$ denota el coseno entre el vector normal $N(x)$ en el punto $x \in \partial\Omega$ y el eje de coordenadas x_j). Si Ω es un dominio acotado, $\partial\Omega$ es C^2, $a_{ij}, b_i \in C^1(\mathbb{R}\times\overline{\Omega})$, $f \in C(\mathbb{R}\times\overline{\Omega}\times\mathbb{R})$ y la condición de elipticidad (9.6) para a_{ij} está satisfecha, así como la condición de crecimiento (9.7) para f, podemos obtener un resultado similar al Lema 7.8. Los detalles de este método, así como el efecto suavizante de la ecuación diferencial con operador sectorial, fueron recopilados en el artículo [13].

Finalmente, para tratar la ecuación de reacción-difusión en $\Omega_0 = \Omega \cup R_0$, aprovechamos el hecho de que las ecuaciones en Ω (el dominio abierto en $\mathbb{R}^n$) y R_0 (el segmento de línea) están débilmente acoplados, y cuando separamos los problemas obtenemos operadores sectoriales para cada uno de ellos. Luego se aplicó el método iterativo para tratar los problemas por separado y se dispuso de estimaciones de las soluciones en cada componente, así como de la existencia de conjuntos que atraen *pullback*.

Una característica interesante en esta estrategia de desacoplamiento fue el análisis requerido sobre la no linealidad cuando consideramos la ecuación en el segmento de línea R_0. Era necesario demostrar que la no linealidad, después del desacoplamiento, satisfaría una condición de disipación adecuada. Los resultados relacionados con la dinámica asintótica de la ecuación en Ω_0 se presentaron en el artículo [12].

Trabajos futuros

Los problemas parabólicos de la forma (9.1) (o más generalmente cuando $F = F(t, u, \nabla u)$) todavía tienen varios problemas abiertos por estudiar, tanto en el caso sectorial como en el caso casi sectorial.

Una de las dificultades de trabajar con este tipo de problema proviene del hecho de que es esencialmente no autónomo. Para hacerlo más preciso, considere el caso en el que $A(t) = A$ es independiente del tiempo. La solución para el problema semilineal en este caso se da en términos del semigrupo generado por $-A$, es decir, $u(t)$ es dado por (9.8) y se puede verlo como una perturbación no autónoma del semigrupo lineal $T_{-A}(\cdot)$. La situación en (9.2) es diferente ya que la familia de operadores lineales $U(t,\tau)$ no es autónoma, lo que dificulta el estudio de algunas propiedades de la solución $u(t)$. Las implicaciones de este hecho se enumeran a continuación y son problemas interesantes para estudiar.

1. El concepto de "soluciones de equilibrio" que tiene un papel esencial en la descripción de la estructura del atractor (para problemas autónomos) no está disponible y no esperamos que las soluciones converjan a estados estacionarios. Esta falta de información sobre la estructura del atractor *pullback* dificulta el análisis de la semicontinuidad inferior de este cuando consideramos perturbaciones del problema.

2. Para el caso no singular (9.3), si consideramos las perturbaciones del problema de la forma $u_t + A_\varepsilon u = F_\varepsilon(u)$, se puede demostrar la convergencia del semigrupo lineal $T_{-A_\varepsilon}(t)$ a $T_{-A}(t)$, una vez que tengamos convergencia de la resolvente de A_ε a la resolvente de A (ver [4, 6, 22]). Esta convergencia se transfiere luego a la solución del problema semilineal mediante la fórmula de variación de constantes. Para el caso singular, si $U_\varepsilon(t,\tau)$ es un proceso lineal asociado a $A_\varepsilon(t)$, un análisis interesante sería entender cómo la convergencia resolutiva de $A_\varepsilon(t)$ a $A(t)$ se puede transferir al proceso lineal y, en consecuencia, a la solución del problema semilineal.

Conclusões

Neste trabalho, estávamos interessados em desenvolver uma teoria abstrata adequada para lidar com problemas singularmente não autônomos nos quais a família de operadores lineares dependentes do tempo é uniformemente quase setorial, ou seja, problemas da forma (9.1).

Esses operadores uniformemente quase setoriais $A(t)$ geralmente surgem em aplicações quando consideramos operadores elípticos definidos em espaços de fase mais regulares, como o espaço de funções Hölder contínuas que se anulam na fronteira de um domínio limitado, $X = \mathcal{C}_0^{\mu}(\Omega)$, veja [19, Seção 4.1], ou quando se trata de equações parabólicas em certos domínios singulares, como o domínio com uma "alça" Ω_0 introduzido no Capítulo 2. O segundo caso foi explorado ao longo deste trabalho e ilustrou a teoria abstrata desenvolvida.

Em termos de resultados obtidos, foi possível fornecer condições que garantem a boa postura local dos problemas semilineares (9.1), a regularidade desta solução local e condições para estudar a dinâmica assintótica do problema. O trabalho foi estruturado em três partes, cada uma dedicada a abordar uma dessas questões.

Na primeira parte, introduzimos os conceitos de semigrupos e processos lineares de crescimento $1-\alpha$ e reunimos uma série de propriedades para essas duas famílias necessárias para o desenvolvimento da teoria. O principal resultado nesta parte é a existência de uma solução local "mild", obtida através da fórmula da variação das constantes (9.2).

Esta expressão para a solução "mild" permitiu obter resultados de regularidade para $u(t)$. Até então, não havia resultados na literatura sobre a regularidade da solução "mild" para o caso singularmente não autônomo (9.1). Nossa contribuição neste tópico (apresentada no Capítulo 4) foi reunida no artigo [14]. Na realidade, a literatura sobre operadores quase setoriais carecia de resultados nessa direção mesmo quando $A(t) = A$ não depende do tempo. Em [31] o autor explorou alguns resultados no sentido de obter soluções fortes para problemas semilineares da forma (9.3) com operadores quase setoriais, mas as condições necessárias para obter soluções fortes eram muito restritivas. Com os resultados desenvolvidos na Parte II deste trabalho, também fomos capazes de estender a teoria de regularidade para problemas da forma (9.3) sem requerer as condições adotadas em [31]. Os novos resultados obtidos para o caso autônomo foram apresentados no artigo [15].

Em relação à dinâmica assintótica do problema, o fato de $A(t)$ ser dependente do tempo nos impediu de usar as abordagens clássicas para obter estimativas globais de soluções de problemas parabólicos, que consistem na construção de uma função de Lyapunov ou na utilização de resultados de comparação e monotonicidade das soluções. Não havia resultados na literatura que nos permitiam abordar o caso singularmente não autônomo, mesmo quando $A(t)$ é setorial, a menos que alguma condição de monotonicidade ou decrescimento no tempo fosse requisitada para esta família de operadores lineares.

Portanto, para tratar a equação (9.4) no domínio limitado Ω, ao qual podemos associar a família de operadores setoriais $B(t)u = -div(a(t,x)u) + u$, desenvolvemos o método iterativo apresentado no Capítulo 7, Seção 7.1. Este método combina um procedimento iterativo e propriedades de regularização da equação diferencial para obter estimativas para a solução em normas mais fortes.

Este procedimento iterativo pode ser estendido para equações parabólicas de segunda ordem mais gerais na forma divergente. Por exemplo, pode-se aplicar o método para obter estimativas globais para a solução de(9.5) onde u satisfaz condições de contorno homogêneas de Dirichlet ou Neumann, ou seja,

$$u|_{\partial\Omega} = 0 \quad \text{ou} \quad \left(\sum_{i,j=1}^{n} a_{ij}(t,x)\partial_i u \cos(N(x), x_j) \right) \Bigg|_{\partial\Omega} = 0,$$

(a expressão $\cos(N(x), x_j)$ denota o cosseno entre o vetor normal $N(x)$ no ponto $x \in \partial\Omega$ e o eixo coordenado x_j). Se Ω é um domínio limitado, $\partial\Omega$ é C^2, $a_{ij}, b_i \in C^1(\mathbb{R} \times \overline{\Omega})$, $f \in C(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ e a condição de elipticidade (9.6) para a_{ij} está satisfeita, assim como a condição de crescimento (9.7) para f, podemos obter um resultado semelhante ao Lemma 7.8. Os detalhes deste método, bem como o efeito de suavização da equação diferencial com operador setorial, foram compilados no artigo [13].

Finalmente, para lidar com a equação de reação-difusão em $\Omega_0 = \Omega \cup R_0$, usamos o fato de que as equações em Ω (o domínio aberto em $\mathbb{R}^n$) e R_0 (o segmento de linha) são fracamente acopladas e, quando separamos os problemas, obtemos operadores setoriais para cada um deles. Em seguida, o método iterativo foi aplicado para tratar os problemas separadamente e foram obtidas estimativas das soluções em cada componente, bem como a existência de conjuntos que atraem *pullback*.

Uma característica interessante dessa estratégia de desacoplamento foi a análise necessária sobre a não linearidade quando consideramos a equação no segmento R_0. Era necessário mostrar que a não linearidade, após o desacoplamento, satisfaria uma condição de dissipação adequada. Os resultados relacionados à dinâmica assintótica da equação em Ω_0 foram apresentados no artigo [12].

Trabalhos futuros

Problemas parabólicos da forma (9.1) (ou mais geralmente quando $F = F(t, u, \nabla u)$) ainda possuem vários problemas em aberto, tanto no caso setorial quanto no caso quase setorial.

Uma das dificuldades de se trabalhar com esse tipo de problema vem do fato deste ser essencialmente não autônomo. Para tornar mais preciso, considere o caso em que $A(t) = A$ é independente do tempo. A solução para o problema semilinear neste caso é dada em termos do semigrupo gerado por $-A$, ou seja, $u(t)$ é dado por (9.8) e pode-se ver tal solução como uma perturbação não autônoma do semigrupo linear $T_{-A}(\cdot)$. A situação em (9.2) é diferente, pois a família do operador linear $U(t,\tau)$ não é autônoma, o que dificulta o estudo de algumas propriedades da solução $u(t)$. As implicações desse fato estão listadas abaixo e são problemas interessantes para se estudar.

1. O conceito de "soluções de equilíbrio", o qual tem um papel essencial na descrição da estrutura do atrator para problemas autônomos, não está disponível e não esperamos que as soluções convirjam para estados estacionários. Esta falta de informação sobre a estrutura do atrator *pullback* dificulta a análise da semicontinuidade inferior quando consideramos perturbações no problema.

2. Para o caso não singular (9.3), se consideramos perturbações do problema da forma $u_t + A_\varepsilon u = F_\varepsilon(u)$, podemos provar a convergência do semigrupo linear $T_{-A_\varepsilon}(t)$ para $T_{-A}(t)$, uma vez que temos convergência do resolvente A_ε para o resolvente de A (ver [4, 6, 22]). Essa convergência é então transferida para a solução do problema semilinear usando a fórmula da variação das constantes. Para o caso singular, se $U_\epsilon(t,\tau)$ é um processo linear associado a $A_\epsilon(t)$, uma análise interessante seria entender como a convergência do resolvente de $A_\varepsilon(t)$ ao resolvente de $A(t)$ pode ser transferida para o processo linear e, consequentemente, para a solução do problema semilinear.

APPENDIX A

Smoothing effect of the differential equation for the sectorial case

At this point we focus our attention to the case where the family $A(t)$, $t \in \mathbb{R}$, is uniformly sectorial ($\alpha = 1$) and uniformly Hölder continuous. Associated to this family there is a scale of fractional power spaces that we denote by $\{X^\gamma\}_{\gamma \geq 0}$ and our goal is to prove the smoothing effect that the differential equation has on the solution of the problem, presented in Theorem 6.9, which we reproduce in the sequel.

Theorem. *Let $A(t)$, $t \in \mathbb{R}$, be uniformly sectorial ($\alpha = 1$) and uniformly $\delta-$Hölder continuous, and $F : X^\gamma \mapsto X$ a locally Lipschitz function, $0 \leq \gamma < 1$. If $u : [\tau, T) \to X$ is the solution of*

$$u_t(t) + A(t)u = F(u),\ t \in (\tau, T); \quad u(\tau) = u_0 \in X^\gamma,$$

then, for any $0 \leq \beta < \delta$, $u_t(t) \in X^\beta$ and satisfies the estimate

$$\|u_t(t)\|_{X^\beta} \leq C(t-\tau)^{-1-\xi+\gamma}\|u_0\|_{X^\gamma}.$$

In this appendix section, β does not designate the constant that appears in the compatibility condition (P.4) posed in Chapter (1), neither γ designates the constant of the dissipativeness condition of f. They will be arbitrarily positive constants.

Several of the necessary steps to prove the result above has already appear throughout this work, in the context of almost sectorial operators. We will briefly mention them for the sake of completeness, but we will omit most of the calculus on estimate. This Appendix chapter and the consequences of the results here established are presented in [13].

A.1 Estimates in the fractional power spaces

The major difference for the results with almost sectorial operators performed so far is, rather then working with the norms in the phase spaces $\mathcal{L}(X)$, $\mathcal{L}(Y, X)$ or $\mathcal{L}(Y)$, our family of linear operators

($T_{-A(\tau)}(t-\tau)$, $U(t,\tau)$, $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$) will be estimated in $\mathcal{L}(X, X^\beta)$

Estimates for the semigroup $T_{-A(\tau)}(s)$:

The most basic estimate for the semigroup $T_{-A(\tau)}(s)$ generated by a positive sectorial operator is $\|T_{-A(\tau)}(s)\|_{\mathcal{L}(X)} \leq C$, for all $s \geq 0, \tau \in \mathbb{R}$.

Proposition A.1. *[21, Proposition 7] There exists constant $C > 0$, independent of β and t such that*

$$\|A(t)^\beta T_{-A(t)}(\tau)\|_{\mathcal{L}(X)} \leq C\tau^{-\beta}, \quad \forall\, \beta \geq 0, \tau > 0,$$
$$\|[T_{-A(t)}(\tau) - I]A(t)^{-\beta}\|_{\mathcal{L}(X)} \leq C\tau^{\beta}, \quad \forall\, 0 \leq \beta \leq 1, \tau > 0.$$

Proposition A.2. *[21, Proposition 8] For any $\xi \in \mathbb{R}$, $t \leq r$, $\tau > 0$ and $0 \leq \beta \leq 1$*

$$\|A(\xi)^\beta [T_{-A(r)}(\tau) - T_{-A(t)}(\tau)]\|_{\mathcal{L}(X)} \leq C\tau^{-\beta}(r-t)^\delta,$$
$$\|A(\xi)^\beta [A(r)T_{-A(r)}(\tau) - A(t)T_{-A(t)}(\tau)]\|_{\mathcal{L}(X)} \leq C\tau^{-\beta-1}(r-t)^{\delta(1-\beta)}.$$

Estimates for the families $\varphi_1(t,\tau)$, $\Phi(t,\tau)$:

Lemma A.3. *[52, Section 5.6] The families $\varphi_1(t,\tau)$ and $\Phi(t,\tau)$ satisfy*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\delta-1} \quad \textit{and} \quad \|\Phi(t,\tau)\|_{\mathcal{L}(X)} \leq C(t-\tau)^{\delta-1}.$$

Proposition A.4. *Let $0 \leq \beta < \delta$. There exists a constant $C > 0$ depending only on β such that, for any $t > \tau$,*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(X,X^\beta)} \leq C(t-\tau)^{\delta-\beta-1} \quad \textit{and} \quad \|\Phi(t,\tau)\|_{\mathcal{L}(X,X^\beta)} \leq C(t-\tau)^{\delta-\beta-1}.$$

Proof. The statement for the family $\varphi_1(t,\tau)$ follows from the Hölder continuity of the family $A(t)$ and the estimates for the semigroup:

$$\|A(\xi)^\beta \varphi_1(t,\tau)\|_{\mathcal{L}(X)} = \|[A(\tau) - A(t)]A(\xi)^{-1}A(\xi)^{1+\beta}T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X)} < C(t-\tau)^{\delta-\beta-1},$$

whereas the estimate for $\Phi(t,\tau)$ follows from

$$\begin{aligned}\|\Phi(t,\tau)\|_{\mathcal{L}(X,X^\beta)} &\leq \|\varphi_1(t,\tau)\|_{\mathcal{L}(X,X^\beta)} + \int_\tau^t \|\varphi_1(t,s)\|_{\mathcal{L}(X,X^\beta)}\|\Phi(s,\tau)\|_{\mathcal{L}(X)}ds \\ &\leq C(t-\tau)^{\delta-\beta-1} + C(t-\tau)^{2\delta-\beta-1} \leq C(t-\tau)^{\delta-\beta-1}.\end{aligned}$$

□

Proceeding in the same way as it is done in [21, Propositions 3 and 4], we have:

Proposition A.5. *Let $\tau < \theta < t$. Given any $\beta < \delta$ and $0 \leq \eta < \delta - \beta$,*

$$\|\varphi_1(t,\tau) - \varphi_1(\theta,\tau)\|_{\mathcal{L}(X,X^\beta)} \leq C(t-\theta)^\eta(\theta-\tau)^{(\delta-\eta)-\beta-1}, \tag{A.1}$$
$$\|\Phi(t,\tau) - \Phi(\theta,\tau)\|_{\mathcal{L}(X,X^\beta)} \leq C(t-\theta)^\eta(\theta-\tau)^{(\delta-\eta)-\beta-1}. \tag{A.2}$$

In the same lines of the preceding result, we also need an estimate for the families φ_1 and Φ when both initial and final instant evolves a quantity $h > 0$. The proof of the next result is similar to the proof of Proposition 3.6 but $\omega = 1$ in this case.

Proposition A.6. *Let $\tau < t$ and $h > 0$. Then, given any $0 \leq \eta < \delta$, we have*

$$\|\varphi_1(t+h,\tau+h) - \varphi_1(t,\tau)\|_{\mathcal{L}(X)} \leq Ch^\eta (t-\tau)^{(\delta-\eta)-1},$$
$$\|\Phi(t+h,\tau+h) - \Phi(t,\tau)\|_{\mathcal{L}(X)} \leq Ch^\eta (t-\tau)^{(\delta-\eta)-1}.$$

Estimates for the linear process $U(t,\tau)$:

Besides the estimate $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq C$ for the linear process, we also need the following results:

Proposition A.7. *[21, Theorem 2.2] Let $\tau < t$ and $0 \leq \gamma \leq \beta < 1+\delta$. Then*

$$\|A(t)^\beta U(t,\tau) A(\tau)^{-\gamma}\|_{\mathcal{L}(X)} \leq C(\gamma,\beta)(t-\tau)^{\gamma-\beta}.$$

Proposition A.8. *If $\gamma > \beta$ and $0 < \gamma - \beta < 1$, then*

$$\|A(t)^\beta [U(t,\tau) - I] A(\tau)^{-\gamma}\|_{\mathcal{L}(X)} \leq C(\gamma,\beta)(t-\tau)^{\gamma-\beta}.$$

A.2 The operators $A\int_\tau^t T_{-A(\tau)}(s)ds$ and $A\int_\tau^t U(t,s)(s)ds$

We can state Lemmas 4.11 and 4.12 for the sectorial case. Their proof is identical, being only necessary to consider $\alpha = 1$ when it shows up.

Lemma A.9. *For any $x \in X$, $\int_\tau^t T_{-A(s)}(t-s)wds$ belongs to D and $A(t)\int_\tau^t T_{-A(s)}(t-s)ds$ is a bounded linear operator satisfying $\left\|A(t)\int_\tau^t T_{-A(s)}(t-s)ds\right\|_{\mathcal{L}(X)} \leq C$.*

Lemma A.10. *For any $x \in X$, $\int_\tau^t U(t,s)xds$ belongs to D and the following equality for $A(t)\int_\tau^t U(t,s)xds$ holds*

$$\begin{aligned} A(t)\int_\tau^t U(t,s)xds = A(t)\int_\tau^t T_{-A(s)}(t-s)\left\{x + \int_\tau^t \Phi(t,\xi)xd\xi\right\}ds \\ + A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]xds\right\}d\xi \\ - A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)xds\right\}d\xi. \end{aligned}$$

Furthermore, $A(t)\int_\tau^t U(t,s)ds$ satisfies $\left\|A(t)\int_\tau^t U(t,s)ds\right\|_{\mathcal{L}(X)} \leq C$.

A.3 Smoothing effect of the differential equation

A.3.1 Linear problem

Rather than considering the semilinear problem directly, we first deal with the nonautonomous linear case

$$\begin{aligned} &x_t(t) + A(t)x = g(t), \quad t \in (\tau, T), \\ &x(\tau) = x_0 \in X, \end{aligned} \tag{A.3}$$

whose solution is given by $x(t) = U(t,\tau)x_0 + \int_\tau^t U(t,s)g(s)ds$.

The characterization obtained in Lemma A.10 for $A(t)\int_\tau^t U(t,s)ds$ is applied in the expression for $x_t(t)$, calculated next, resulting

$$\begin{aligned} x_t(t) &= -A(t)U(t,\tau)x_0 - A(t)\int_\tau^t U(t,s)[g(s)-g(t)]ds - A(t)\int_\tau^t U(t,s)g(t)ds + g(t) \\ &= -A(t)U(t,\tau)x_0 - A(t)\int_\tau^t U(t,s)[g(s)-g(t)]ds - \left\{\int_\tau^t \Phi(t,\xi)g(t)d\xi\right\} \\ &\quad + T_{-A(\tau)}(t-\tau)\left\{g(t) + \int_\tau^t \Phi(t,\xi)g(t)d\xi\right\} \\ &\quad - \int_\tau^t [A(t)-A(s)]T_{-A(s)}(t-s)\left\{g(t) + \int_\tau^t \Phi(t,\xi)g(t)d\xi\right\} ds \\ &\quad - A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]g(t)ds\right\} d\xi \\ &\quad + A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)g(t)ds\right\} d\xi \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6 + \mathcal{I}_7. \end{aligned}$$

It might seem that equality above would only complicate the analysis. However, the nonlinear term $g(t) \in X$ no longer features in the expression for x_t and all terms (from $\mathcal{I}_1$ to $\mathcal{I}_7$) belong to a space X^ξ, $\xi > 0$, with more regularity, as we see in next lemma.

Lemma A.11. *Let*

1. *$A(t) : D \subset X \to X$ be uniformly sectorial and uniformly $\delta-$Hölder continuous, with $\delta \in (0,1]$.*
2. *$g : (\tau,T) \to X$ a continuous function such that there exists $0 < \lambda \leq 1$, $0 \leq \theta < 1$ and $C > 0$, for which $\|g(t)-g(s)\|_X \leq C(t-s)^\lambda (s-\tau)^{-\theta}$, $\tau < s < t < T$.*

Given any $0 \leq \beta < \min\{\lambda,\delta\}$, the terms $\mathcal{I}_1$ to $\mathcal{I}_7$ of the above equality belong to X^β and satisfy

$$\begin{array}{lll} \|\mathcal{I}_1\|_{X^\beta} \leq C(t-\tau)^{-1-\beta} & \|\mathcal{I}_2\|_{X^\beta} \leq C(t-\tau)^{(\lambda-\beta)-\theta} & \|\mathcal{I}_3\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta} \\ \|\mathcal{I}_4\|_{X^\beta} \leq C(t-\tau)^{-\beta-\theta} & \|\mathcal{I}_5\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta} & \|\mathcal{I}_6\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta} \\ \|\mathcal{I}_7\|_{X^\beta} \leq C(t-\tau)^{(\delta-\beta)-\theta}, & & \end{array}$$

where the constant C depends on β.

Proof. For each term we estimate its norm in X^β, $0 \leq \beta < \min\{\lambda, \delta\}$, proving that it belongs to X^β. From the results enumerated in the beginning of this chapter, we obtain:

$$\|\mathcal{I}_1\|_{X^\beta} = \|A(t)U(t,\tau)x_0\|_{X^\beta} = \|A(\xi)^\beta A(t)U(t,\tau)x_0\|_X \leq C(\beta)(t-\tau)^{-1-\beta}\|x_0\|_X.$$

$$\begin{aligned}\|\mathcal{I}_2\|_{X^\beta} &= \left\|A(\xi)^\beta A(t)\int_\tau^t U(t,s)[g(s)-g(t)]ds\right\|_X \leq \int_\tau^t (t-s)^{-1+(\lambda-\beta)}(s-\tau)^{-\theta}ds \\ &\overset{\beta<\lambda}{\leq} C(\beta)(t-\tau)^{(\lambda-\beta)-\theta}.\end{aligned}$$

From Proposition A.4, we obtain

$$\|\mathcal{I}_3\|_{X^\beta} = \left\|A(\xi)^\beta \int_\tau^t \Phi(t,\xi)g(t)\right\|_X \leq C\int_\tau^t (t-\xi)^{\delta-1-\beta}(t-\tau)^{-\theta}d\xi \leq C(\beta)(t-\tau)^{(\delta-\beta)-\theta}.$$

Let $H(t) = g(t) + \int_\tau^t \Phi(t,\xi)g(t)d\xi$. From the properties of g and $\Phi(t,\tau)$, we obtain $\|H(t)\|_X \leq C(t-\tau)^{-\theta}$ and

$$\|\mathcal{I}_4\|_{X^\beta} = \left\|A(\xi)^\beta T_{-A(\tau)}(t-\tau)H(t)\right\|_X \leq C(t-\tau)^{-\beta-\theta}.$$

$$\begin{aligned}\|\mathcal{I}_5\|_{X^\beta} &= \left\|A(\xi)^\beta \int_\tau^t [A(t)-A(s)]T_{-A(s)}(t-s)H(t)ds\right\|_X \\ &\leq C(\beta)\int_\tau^t (t-s)^{\delta-\beta-1}ds(t-\tau)^{-\theta} \leq C(\beta)(t-\tau)^{(\delta-\beta)-\theta}.\end{aligned}$$

Applying (A.2), with $\eta \in (\beta,\delta)$, we obtain

$$\begin{aligned}\|\mathcal{I}_6\|_{X^\beta} &= \left\|A(\xi)^\beta \int_\tau^t A(t)T_{-A(\xi)}(t-\xi)\left\{\int_\tau^\xi [\Phi(\xi,s)-\Phi(t,s)]g(t)ds\right\}d\xi\right\|_X \\ &\leq C\int_\tau^t (t-\xi)^{-1+(\eta-\beta)}(\xi-\tau)^{(\delta-\eta)}d\xi(t-\tau)^{-\theta} \leq C(t-\tau)^{(\delta-\beta)-\theta}.\end{aligned}$$

The last term follows from Proposition A.1 and the estimate (1.17) for $\Phi(t,\tau)$

$$\begin{aligned}\|\mathcal{I}_7\|_{X^\beta} &= \left\|A(\xi)^\beta A(t)\int_\tau^t T_{-A(\xi)}(t-\xi)\left\{\int_\xi^t \Phi(t,s)g(t)ds\right\}d\xi\right\|_X \\ &\leq \int_\tau^t (t-\xi)^{-1-\beta}\left\{\int_\xi^t (t-s)^{\delta-1}ds\right\}d\xi(t-\tau)^{-\theta} \leq C(t-\tau)^{(\delta-\beta)-\theta}\end{aligned}$$

□

The previous Lemma is a major part in the proof of next theorem.

Theorem A.12. *Let*

1. *$A(t) : D \subset X \to X$ be uniformly sectorial and uniformly $\delta-$Hölder continuous, with $\delta \in (0,1]$.*
2. *$g : (\tau, T) \to X$ a continuous function such that there exists $0 < \lambda \leq 1$, $0 \leq \theta < 1$ and $C > 0$, for which $\|g(t) - g(s)\|_X \leq C(t-s)^{\lambda}(s-\tau)^{-\theta}$, $\tau < s < t < T$.*

If $x : [\tau, T) \to X$ is the solution of

$$x_t(t) + A(t)x = g(t), t \in (\tau, T); \quad x(\tau) = x_0 \in X,$$

then, for any $0 \leq \beta < \min\{\lambda, \delta\}$, $x_t(t)$ is in X^{β} for $t \in (\tau, T)$ and satisfies the estimate $\|x_t(t)\|_{X^{\beta}} \leq C(\beta)(t-\tau)^{-1-\beta}$.

Moreover, if $x_0 \in X^{\gamma}$, the estimate on $x_t(t)$ can be improved to $\|x_t(t)\|_{X^{\beta}} \leq C(t-\tau)^{-1-\beta+\gamma}\|x_0\|_{X^{\gamma}}$.

Proof. The result follows from Lemma A.11, with the exception of the last assertion. This one follows from the fact that $\|\mathcal{I}_1\|_{X^{\beta}}$ in the previous lemma can be improved if $x_0 \in X^{\gamma}$ by using Theorem A.7 (and note that this term is the one that restrict the most the estimate of $\|x_t(t)\|_{X^{\beta}}$):

$$\|\mathcal{I}_1\|_{X^{\beta}} = \|A(\xi)^{\beta}A(t)U(t,\tau)A(\xi)^{-\gamma}A(\xi)^{\gamma}x_0\|_X \leq C(t-\tau)^{-1-\beta+\gamma}\|x_0\|_{X^{\gamma}}.$$

□

A.3.2 Semilinear problem

The linear nonautonomous case (A.3) works as an intermediate step in the proof of the smoothing effect for u_t. That is, consider now the semilinear case

$$\begin{aligned} &u_t(t) + A(t)u = F(u(t)), \quad t \in (\tau, T), \\ &u(\tau) = u_0 \in X^{\gamma}, \end{aligned}$$

for $0 \leq \gamma < 1$. Under the properties required for $A(t)$ and $F : X^{\gamma} \to X$ in Theorem 6.9, this problem has a local solution given by $u(t) = U(t,\tau)u_0 + \int_{\tau}^{t} U(t,s)F(u(s))ds$.

If we define $g : (\tau, T) \to X$ as $g(t) := F(u(t))$, then u also satisfies

$$u_t + A(t)u = g(t), t \in (\tau, T); \quad u(\tau) = u_0 \in X^{\gamma}.$$

By proving that this $g(\cdot)$ satisfies $\|g(t) - g(s)\|_X \leq C(t-s)^{\lambda}(s-\tau)^{-\theta}$, $\tau < s < t < T$, for some $\lambda \in (0,1]$ and $\theta \in [0,1)$, then the results on Theorem A.12 can be translated to the semilinear case.

The proof is exactly the same of Proposition 3.7, simplified by the fact that $\alpha = 1$ and there is no discontinuity at the initial time.

Lemma A.13. *Let*

1. *$A(t) : D \subset X \to X$ be uniformly sectorial and uniformly $\delta-$Hölder continuous, with $\delta \in (0,1]$.*

2. *$F : X^\gamma \to X$ a locally Lipschitz function, $0 \le \gamma < 1$.*

If $u : [\tau, T) \to X$ is the solution of

$$u_t(t) + A(t)u = f(t, u(t)), t \in (\tau, T); \quad u(\tau) = u_0 \in X^\gamma,$$

then, $g(\cdot)$ satisfies $\|g(t) - g(s)\|_X \le C(t-s)^\eta (s-\tau)^{-\max\{\gamma,\eta\}}$, $\tau < s < t < T$, for any $\eta \in [0, \delta)$.

Proof. The fact that f is locally Lipschitz implies that, for $t > \tau$ and $h > 0$ small,

$$\|g(t+h) - g(t)\|_X = \|f(u(t+h)) - f(u(t))\|_X \le C\|u(t+h) - u(t)\|_{X^\gamma}$$

and

$$\begin{aligned} u(t+h) - u(t) &= [U(t+h,t) - I]U(t,\tau)u_0 + \int_\tau^{\tau+h} U(t+h,s)g(s)ds \\ &\quad + \int_\tau^t [U(t+h,s+h)g(s+h) - U(t,s)g(s)]ds \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

From Propositions A.7 and A.8, given any $\eta \in [0,1)$, we obtain

$$\|[U(t+h,t) - I]U(t,\tau)u_0\|_{X^\gamma} = \|[U(t+h,t) - I]A(\xi)^{-\eta}A(\xi)^{\eta+\gamma}U(t,\tau)u_0\|_X \le Ch^\eta (t-\tau)^{-\eta}\|u_0\|_{X^\gamma}$$

and

$$\left\| \int_\tau^{\tau+h} U(t+h,s)g(s)ds \right\|_{X^\gamma} \le C \int_\tau^{\tau+h} (t+h-s)^{-\gamma}\|g(s)\|_X ds \le Ch(t-\tau)^{-\gamma}.$$

The last term can be separated into 5 terms: $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$, as in the proof of Proposition 3.7, which we estimate next. From Proposition A.2,

$$\|\mathcal{S}_1\|_{X^\gamma} \le C \int_\tau^t h^\delta (t-s)^{-\gamma}\|g(s+h)\|_X ds \le Ch^\delta (t-\tau)^{1-\gamma},$$

$$\|\mathcal{S}_2\|_{X^\gamma} \le \int_\tau^t \|A(\xi)^\gamma T_{-A(s)}(t-s)\|\|g(s+h) - g(s)\|_X ds \le C \int_\tau^t (t-s)^{-\gamma}\|g(s+h) - g(s)\|_X ds.$$

Term $\mathcal{S}_3$ also follows from Proposition A.2 and the estimate for the family $\Phi(\cdot,\cdot)$

$$\|\mathcal{S}_3\|_{X^\gamma} \le C \int_\tau^t \left\{ \int_s^t h^\delta (t-\xi)^{-\gamma}(\xi-s)^{\delta-1} d\xi \right\} ds \le Ch^\delta (t-\tau)^{\delta-\gamma+1}.$$

From Proposition A.6, given any $0 \le \nu < \delta$,

$$\|\mathcal{S}_4\|_{X^\gamma} \le C \int_\tau^t \left\{ \int_s^t (t-\xi)^{-\gamma} h^\nu (\xi-s)^{(\delta-\nu)-1} d\xi \right\} ds \le Ch^\nu (t-\tau)^{1-\gamma+\delta-\nu}$$

and

$$\|\mathcal{S}_5\|_{X^\gamma} \leq C\int_\tau^t (t-s)^{\delta-\gamma}\|g(s+h)-g(s)\|_X ds.$$

Using the estimates above, we obtain

$$\|\mathcal{I}_3\|_{X^\gamma} \leq Ch^\nu + C\int_\tau^t (t-s)^{-\gamma}\|g(s+h)-g(s)\|_X ds.$$

From $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, we conclude that, for any $\eta \in [0,1)$ and $\nu \in [0,\delta)$,

$$\|u(t+h)-u(t)\|_{X^\gamma} \leq Ch^\nu[(t-\tau)^{-\nu}+(t-\tau)^{-\gamma}] + C\int_\tau^t (t-s)^{-\gamma}\|g(s+h)-g(s)\|_X ds$$

and

$$\|g(t+h)-g(t)\|_X \leq Ch^\nu[(t-\tau)^{-\nu}C(t-\tau)^{-\gamma}] + C\int_\tau^t (t-s)^{-\gamma}\|g(s+h)-g(s)\|_X ds. \qquad \text{(A.4)}$$

An application of Gronwall's inequality yields $\|g(t+h)-g(t)\|_X \leq Ch^\nu(t-\tau)^{-\max\{\gamma,\nu\}}$.

□

Proof of Theorem 6.9:

If $u:(\tau,T)\to X^\gamma$, $\gamma\in[0,1)$, is the solution of

$$u_t(t)+A(t)u = F(u(t)), t\in(\tau,T); \quad u(\tau)=u_0\in X^\gamma$$

then $g(t)=F(t,u(t))$ is η locally Hölder continuous for any $\eta\in[0,\delta)$. In this case, Theorem A.12 states that the solution $x:[\tau,T)\to X$ of

$$x_t(t)+A(t)x = g(t), t\in(\tau,T); \quad x(\tau)=u_0\in X^\gamma$$

satisfies, for $0\leq\beta<\delta$, $x_t(t)\in X^\beta$ and $\|x_t(t)\|_{X^\beta}\leq C(t-\tau)^{-1-\beta+\gamma}\|x_0\|_{X^\gamma}$. From the variation of constants formula,

$$x(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)g(s)ds = U(t,\tau)u_0 + \int_\tau^t U(t,s)f(u(s))ds = u(t)$$

and we obtain the desired properties for $u_t(t)$, which proves Theorem 6.9.

Remark A.14. *In [37, Theorem 3.5.2], the author proved the smoothing effect on $u_t(t)$ when $A(t)=A$ and $f(u)$ is locally Lipschitz. In the notation of Theorem 6.9, this case corresponds to $\delta=1$. Therefore, for any $0\leq\beta<1$, $u_t(t)\in X^\beta$ and*

$$\|u_t(t)\|_{X^\beta} \leq C(t-\tau)^{-1-\beta+\gamma},$$

matching the result found in [37].

Remark A.15. *The fact that $u_t(t) \in X^\beta$ for $\beta \in [0, \delta)$ is independent of γ, as long as $\gamma \in [0, 1)$.*

Remark A.16. *There are some works in the literature that deals with the case $\gamma = 1$ in $F : X^\gamma \to X$, which is called* critical case *(see [5, 10] for the nonsingular case and [21] for the singular case). For a class of functions called $\varepsilon-$*regular maps*, the existence of local mild solution can be proved in this situation (uniqueness is a more delicate matter, as discussed in Section 4 of [5]).*

The estimates of items $\mathcal{S}_1 - \mathcal{S}_5$ on Lemma A.13, which results in the estimate (A.4) *for the difference $u(t+h) - u(t)$, would all be impaired if $\gamma = 1$, preventing us to extend those results of regularization and smoothing effect to the $\varepsilon-$regular solution constructed for the problem in the papers just mentioned.*

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