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NONTRIVIAL COMPACT BLOW-UP SETS OF LOWER DIMENSION IN A HALF-SPACE

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Abstract. In this paper we provide examples of blowing-up solutions to parabolic problems in a half space, $\mathbb{R}^N_+ \times \mathbb{R}^M = \{x_N > 0\} \times \mathbb{R}^M$, with nontrivial blow-up sets of dimension strictly smaller than the space dimension. To this end we prove existence of a nontrivial compactly supported solution to $\nabla(|\nabla\varphi|^{p-2}\nabla\varphi) = \varphi$ in the half space $\mathbb{R}^N_+ = \{x_N > 0\}$ with the nonlinear boundary condition $-|\nabla\varphi|^{p-2}\frac{\partial\varphi}{\partial x_N} = \varphi^{p-1}$ on $\partial\mathbb{R}^N_+ = \{x_N = 0\}$.

1. Introduction and main results

Our main concern in this paper is to find examples of blowing-up solutions to parabolic problems which exhibit a nontrivial compact blow-up set of dimension strictly less than the space dimension.

For many parabolic equations it is well known that solutions may develop singularities in finite time. In particular, it may happen that the L^∞-norm of the solution goes to infinity in finite time; that is, there exists T such that $\lim_{t\to T}\|u(\cdot,t)\|_\infty = +\infty$ (see [16]). This phenomenon is called *blow-up* and has been the object of active research in recent years; see the surveys [1] and [10], the book [16] and references therein.

An important aspect of blow-up problems is the spatial structure of the set where the solution becomes unbounded, that is, the *blow-up set*. More precisely, the blow-up set of a solution u that blows up at time T is defined as

$$B(u) = \{x : \text{ there exist } x_n \to x,\, t_n \nearrow T, \text{ with } u(x_n,t_n) \to \infty\}.$$

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A problem which has attracted some attention in the literature is the identification of possible blow-up sets. There are several situations where the blow-up set is a single point (single-point blow-up); for instance for $u_t = \Delta u + u^p$ with $p > 1$, [2]; a proper subset of the spatial domain of the same dimension (regional blow-up), for example for $u_t = \Delta u^m + u^m$, with $m > 1$, [4] and [5]; or the whole space (global blow-up), as happens for $u_t = \Delta(u^m) + u^p$ with $(1 < p < m)$, [16]. Moreover, considering radial solutions to $u_t = \Delta u + u^p$ it is easy to construct an example with blow-up set a sphere, $B(u) = \{|x| = r\}$. In addition, from the results of [18] and [19], some regularity of the blow-up set is known for solutions to the above-mentioned equation $u_t = \Delta u + u^p$. So, up to now we have as possible blow-up sets isolated points, the whole space, balls and spheres.

As we will show in this paper, there exist many other examples of blow-up sets. For instance, we can find a solution whose blow-up set is a segment, $B(u) = [0, c] \times \{0\}$, in $\mathbb{R}_+ \times \mathbb{R} = \{x_1 > 0\} \times \mathbb{R}$. In general, given any dimensions N and M, we can construct a compact subset, $K \subset \mathbb{R}^N_+$, of dimension N and a solution of a parabolic problem in $\mathbb{R}^N_+ \times \mathbb{R}^M$ with blow-up set $B(u) = K \times \{0\}$, or, more generally, $B(u) = K \times \{y_1, \dots, y_k\}$ for any given set of points $\{y_1, \dots, y_k\} \in \mathbb{R}^M$.

To give those examples we propose to study self-similar solutions to the following parabolic problem combining the doubly nonlinear operator in $\mathbb{R}^N_+$ (with $p > 2$ and $m > 0$ as parameters) and the Laplacian operator in $\mathbb{R}^M$, on the product space $\mathbb{R}^N_+ \times \mathbb{R}^M = \mathbb{R}^N_+ \times \mathbb{R}^M = \{x_N > 0\} \times \mathbb{R}^M$, with a nonlinear boundary reaction that produces blow-up, namely,

$$\begin{cases} (u^m)_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u^m, & \text{in } \mathbb{R}^N_+ \times \mathbb{R}^M \times (0, T), \\ -|\nabla_x u|^{p-2}\dfrac{\partial u}{\partial x_N} = u^{p-1}, & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}^M \times (0, T), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N_+$, $y \in \mathbb{R}^M$ and $\nabla_x(|\nabla_x u|^{p-2}\nabla_x u)$ denotes the p-Laplacian in the x variable and $\Delta_y u$ is the usual Laplacian in the y variable.

As particular cases of (1.1) we have that if $m = 1$ the equation becomes $u_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \triangle_y u$, that is, a combination of the p-Laplacian and the Laplacian operators. If we do not consider the y variable in (1.1), we face the description of the asymptotic behavior of the blowing-up solutions to the parabolic problem

$$\begin{cases} (u^m)_t = \nabla(|\nabla u|^{p-2}\nabla u), & \text{in } \mathbb{R}^N_+ \times (0, T), \\ -|\nabla u|^{p-2}\dfrac{\partial u}{\partial x_N} = u^{p-1}, & \text{on } \partial\mathbb{R}^N_+ \times (0, T). \end{cases} \tag{1.2}$$

In the analysis of blow-up problems, self-similar profiles are used to study the fine asymptotic behavior of a solution of the parabolic equation near its blow-up time; see, for instance, [13] and [14]. It often happens that the spatial shape of the solution near blow-up is close to a self-similar profile; see [4], [5], [11] and [14]. If we consider a solution to (1.2) of the form $u(x,t) = (T-t)^{-1/(p-2)}v(x,t)$, the rescaled solution $v(x,t)$ is expected to converge to a stationary profile as $t \nearrow T$. For the case $N = 1$ ((1.2) in an interval) we refer to [9]. There, the authors show that the phenomenon of regional blow-up appears due to the existence of compactly supported self-similar profiles. Note that the previous rescaling preserves the original spatial variable. This fact means that the blow-up set of the solution is directly related to the support of the profile.

When dealing with problem (1.1) we will consider self-similar solutions of the form

$$u(x,y,t) = \varphi(x)\psi(y,t). \tag{1.3}$$

If u is a solution to (1.1) of the form (1.3) we obtain that φ and ψ must solve the following elliptic and parabolic problems, respectively:

$$\begin{cases} \varphi^m = \nabla(|\nabla\varphi|^{p-2}\nabla\varphi), & \text{in } \mathbb{R}^N_+, \\ -|\nabla\varphi|^{p-2}\dfrac{\partial\varphi}{\partial x_N} = \varphi^{p-1}, & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.4}$$

and

$$(\psi^m)_t = \Delta\psi^m + \psi^{p-1}, \qquad \text{in } \mathbb{R}^M \times (0,T). \tag{1.5}$$

Observe that the former equation written for $\widetilde{\psi} = \psi^m$ is the heat equation with a source given by the term $\widetilde{\psi}^{(p-1)/m}$, whose solutions are global if $p-1 \le m$. Since our interest is to identify the blow-up set of u, we will consider $p-1 > m$ in the sequel.

Note that the blow-up set of a solution $u(x,y,t)$ of the form (1.3) is given by

$$B(u) = \operatorname{supp}(\varphi) \times B(\psi),$$

where $B(\psi)$ is the blow-up set of ψ. The set $B(\psi)$ is known to be a finite set of points and generically a single point; recall that we are assuming $p-1 > m$. Hence, to find the desired blow-up set $B(u)$ we need to determine the support of φ. As we have mentioned, in one space dimension the support of φ is explicit, since (1.1) reduces to an ODE problem; see [9]. So we need to extend the existence result of a compactly supported profile φ to several space dimensions. In doing this extension some new difficulties arise. If $N \ge 2$ the boundary condition makes it impossible to choose φ as a radial function, and hence we cannot reduce (1.4) to an ODE. Nevertheless, we can look for solutions being radial in the tangential variables; that is, if we denote $x \in \mathbb{R}^N_+$ by $x = (x^{'}, x_N)$, $x^{'} \in \mathbb{R}^{N-1}$, then u satisfies

$$u(x^{'}, x_N) = u(|x^{'}|, x_N). \tag{1.6}$$

Now let us state our main result.

Theorem 1.1. *There exists a nontrivial nonnegative compactly supported solution to* (1.4)*, satisfying* (1.6)

We observe that the problem of uniqueness up to translations in the tangential variables $(x_1, \dots, x_{N-1})$ of compactly supported solutions to (1.4) remains open.

Let us briefly explain our strategy to prove Theorem 1.1. First, for $R > 0$ large enough, we consider the problem

$$\begin{cases} \nabla(|\nabla u_R|^{p-2}\nabla u_R) = (u_R)^m, & \text{in } B_R^+, \\ -|\nabla u_R|^{p-2}\dfrac{\partial u_R}{\partial x_N} = (u_R)^{p-1}, & \text{on } \Gamma_1, \\ u_R = 0, & \text{on } \Gamma_2, \end{cases} \tag{1.7}$$

where B_R^+ denotes $B(0,R)_+ = \{x : \|x\| < R,\, x_N > 0\}$ and Γ_i, $i = 1, 2$, the boundaries $\partial B_R^+ \cup \{x_N = 0\}$ and $\partial B_R^+ \cup \{x_N > 0\}$ respectively.

We will show that for R sufficiently large there exists a nonnegative nontrivial solution of (1.7) such that

$$\max_{x \in \operatorname{supp}(u_R)} |x| < R.$$

Thus, u_R is a compactly supported solution to (1.1).

At this stage let us recall that by well-known results, it is possible to have compactly supported solutions to $\nabla(|\nabla u|^{p-2}\nabla u) = u^\alpha$ in the whole of $\mathbb{R}^N$ if and only if $\alpha + 1 < p$; see [15]. Therefore our assumption $m + 1 < p$ is natural in this sense.

Once this analysis is performed we deduce some corollaries concerning problem (1.1).

Corollary 1.2. *Every nonnegative nontrivial solution to* (1.1) *blows up in finite time for* $1 < (p-1)/m < 1 + 2/M$*.*

This fact follows by contradiction. Assume that v is a global nontrivial solution. Since v is a solution to (1.1) its support in x expands (being the whole space in y) and eventually covers the support of a self-similar solution $u(x, y, t) = \varphi(x)\psi(y, t)$. The proof ends with the use of a comparison argument using that every solution to (1.5) blows up when $(p-1)/m$ is below the critical Fujita exponent; that is, $1 < (p-1)/m < 1 + 2/M$.

Corollary 1.3. *There exists a solution to* (1.1) *with a blow-up set composed of an arbitrary number of connected components of dimension N.*

In fact, we may consider a solution of the form (1.3) with a profile whose blow-up set $B(\psi)$ is composed by k points, $\{y_1, \dots, y_k\}$. Therefore the blow-up set of u consists of k disjoint copies of the compactly supported solution provided by Theorem 1.1; that is, $B(u) = \cup_{i=1}^{k} K \times \{y_i\}$.

Moreover, we conjecture that these self-similar solutions introduced above give the asymptotic behavior of any solution to (1.1) near its blow-up time.

Let us remark that our study can also be performed in more general situations, for example, imposing boundary conditions also in the y variable. In fact, we can consider the problem for $x \in \mathbb{R}_+^{\mathbb{N}}$ and $y \in \mathbb{R}_+^{\mathbb{M}}$ and deal with

$$\begin{cases} (u^m)_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u^m, & \text{in } \mathbb{R}_+^N \times \mathbb{R}_+^M \times (0,T), \\ -|\nabla_x u|^{p-2}\dfrac{\partial u}{\partial x_N} = u^{p-1}, & \text{on } \partial\mathbb{R}_+^N \times \mathbb{R}_+^M \times (0,T), \\ -\dfrac{\partial u^m}{\partial y_M} = u^m \text{ or } \dfrac{\partial u^m}{\partial y_M} = 0 \text{ or } u = 0, & \text{on } \mathbb{R}_+^N \times \partial\mathbb{R}_+^M \times (0,T). \end{cases} \tag{1.8}$$

Also for these problems there are solutions of the form

$$u(x,y,t) = \varphi(x)\psi(y,t),$$

with φ a solution to (1.4) and ψ a solution to

$$\begin{cases} (\psi^m)_t = \Delta\psi^m + \psi^{p-1}, & \text{in } \mathbb{R}_+^M \times (0,T), \\ -\dfrac{\partial \psi^m}{\partial y_M} = \psi^m \text{ or } \dfrac{\partial \psi^m}{\partial y_M} = 0 \text{ or } \psi = 0, & \text{on } \partial\mathbb{R}_+^M \times (0,T). \end{cases}$$

Again, the blow-up set of u is given by $B(u) = \operatorname{supp}(\varphi) \times B(\psi)$. Note that ψ solves a parabolic problem more easily than the original one (1.8).

Moreover, the same ideas can be applied to deal with porous-medium-type equations as well:

$$\begin{cases} u_t = \Delta_x u^m + \Delta_y u, & \text{in } \mathbb{R}_+^N \times \mathbb{R}^M \times (0,T), \\ -\dfrac{\partial u^m}{\partial x_N} = u^m, & \text{on } \partial\mathbb{R}_+^N \times \mathbb{R}^M \times (0,T), \end{cases}$$

considering ψ a solution to $\psi_t = \Delta\psi + \psi^m$ and φ a compactly supported solution (constructed in [3] and [7]) to the problem

$$\begin{cases} \Delta\varphi^m = \varphi, & \text{in } \mathbb{R}^N_+, \\ -\dfrac{\partial\varphi^m}{\partial x_N} = \varphi^m, & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Organization of the paper: In Section 2 we study the auxiliary problem (1.7) in a half-ball and give a comparison principle, some symmetry and growth properties of these solutions. In Section 3 we prove that the support of such solutions is strictly included in the half-ball, providing a solution to our original problem.

2. Existence and properties of the solutions of the auxiliary problem

In this section we deal with problem (1.7). To find nontrivial solutions we look for a natural variational setting. Let us consider the space $W = \{u \in W^{1,p}(B_R^+)$ satisfying $u|_{\Gamma_2} = 0\}$ with the norm

$$\|u\|_W^p = \int_{B_R^+} |\nabla u|^p.$$

Note that Poincaré's inequality is also applicable to functions vanishing on a nontrivial part of the boundary of the domain. Hence, $\| \ \|_W$ is equivalent to the usual $W^{1,p}$-norm in W.

Minimizing the functional

$$J_R(u) = \frac{\frac{m+1}{p}\left(\int_{B_R^+} |\nabla u|^p - \int_{\Gamma_1} u^p\right)}{\left(\int_{B_R^+} u^{m+1}\right)^{p/(m+1)}}$$

over W, we will find a nontrivial solution to (1.7). See [8] for related arguments.

Lemma 2.1. *For every R sufficiently large J_R attains a minimum in W. Moreover, there exists a nontrivial minimizer that is a weak solution to* (1.7)

Proof. Let us show that for every R large there exists a constant $K(R)$ such that

$$\inf_{u\in W,\, u\neq 0} J_R(u) \geq -K > -\infty;$$

that is,

$$\frac{m+1}{p}\int_{B_R^+} |\nabla u|^p + K\Big(\int_{B_R^+} u^{m+1}\Big)^{p/(m+1)} \geq \frac{m+1}{p}\int_{\Gamma_1} u^p, \quad \forall u \in W.$$

If not, there exists a sequence $u_n \in W$ such that $\int_{\Gamma_1} u_n^p = 1$ satisfying

$$\frac{m+1}{p}\int_{B_R^+} |\nabla u_n|^p + n\Big(\int_{B_R^+} u_n^{m+1}\Big)^{p/(m+1)} \le \frac{m+1}{p}, \quad \forall n \ge 1. \tag{2.1}$$

From (2.1) it follows that, up to a subsequence, we have

$$\begin{array}{ll} u_n \rightharpoonup u & \text{weakly in } W, \\ u_n \to u & \text{strongly in } L^p(B_R^+), \\ u_n|_{\Gamma_1} \to u & \text{strongly in } L^p(\Gamma_1). \end{array} \tag{2.2}$$

By the last convergence we get $\int_{\Gamma_1} u^p = 1$. On the other hand (2.1) also implies that $u = 0$, which is a contradiction.

The next step consists of showing that

$$\inf_{u\in W,\, u\neq 0} J_R(u) < 0,$$

which ensures that the minimizer is not trivial.

We apply J_R to $\varphi_{1,R}$, the eigenfunction associated with the first eigenvalue $\lambda_1(R)$ of the following problem:

$$\begin{cases} \nabla(|\nabla\varphi|^{p-2}\nabla\varphi) = 0, & \text{in } B_R^+, \\ -|\nabla\varphi|^{p-2}\dfrac{\partial\varphi}{\partial x_N} = \lambda\varphi^{p-1}, & \text{on } \Gamma_1, \\ \varphi = 0, & \text{on } \Gamma_2. \end{cases} \tag{2.3}$$

Using the same ideas from [8] we get existence of such an eigenvalue and a positive corresponding eigenfunction. We claim that, for R large,

$$J_R(\varphi_{1,R}) = \Big(\frac{m+1}{p}\Big)\frac{(\lambda_1(R)-1)\int_{\Gamma_1}\varphi_{1,R}^p}{\Big(\int_{B_R^+}\varphi_{1,R}^{m+1}\Big)^{p/(m+1)}} < 0. \tag{2.4}$$

The value of $\lambda_1(R)$ depends on R in the following way:

$$\lambda_1(R) = \min_{\varphi\in W\setminus\{0\}} \frac{\int_{B_R^+}|\nabla\varphi|^p}{\int_{\Gamma_1}\varphi^p} = \frac{1}{R}\frac{\int_{B_1^+}|\nabla\widetilde{\varphi}|^p}{\int_{\Gamma_1(B_1^+)}\widetilde{\varphi}^p} = \frac{\lambda_1(1)}{R},$$

where we have performed the change of variables $\widetilde{\varphi}(x) = \varphi(Rx)$. Taking R large enough we obtain $\lambda_1(R) < 1$ and the claim (2.4) follows.

Let us see that the minimum of J_R in W is attained. Consider u_n, a minimizing sequence for J_R, such that $\int_{\Gamma_1} u_n^p = 1$. For n large $J_R(u_n) < 0$ by the previous step, which implies that $\int_{B_R^+}|\nabla u_n|^p < 1$. This assures the convergences in (2.2) and that u is not trivial. We can assume that $u \ge 0$.

From the lower semicontinuity of the norm and the weak convergence in W it holds that

$$\inf J_R \le J_R(u) \le \liminf_{n\to\infty} J_R(u_n) = \inf J_R.$$

Finally, by homogeneity, multiplying a minimizer by the appropriate constant, we get a solution of (1.7). We find this constant minimizing J_R among the functions in W with $\int_{B_R^+} u^{m+1} = 1$. This gives us the existence of a Lagrange multiplier

$$\int_{B_R^+} |\nabla u|^{p-2}\nabla u \nabla v - \int_{\Gamma_1} u^{p-1} v = \lambda \int_{B_R^+} u^m v, \quad \forall v \in W.$$

Taking $v = u$ in the expression above we see that $\lambda < 0$, since it has the same sign as $J_R(u)$. It is not difficult to check that $(-\lambda)^{-1/(p-m-1)}u$ is a minimizer solving (1.7). $\square$

The following estimates will be used in the course of next section. As we will see later, they are also valid for the solution to the original problem.

Lemma 2.2. *Let R be sufficiently large. Then, if u_R is a nonnegative minimizer of J_R, there exists a constant C independent of R such that*

$$\|u_R\|_{L^{m+1}(B_R^+)} \le C, \quad \|u_R\|_{L^\infty(B_R^+)} \le C \quad \textit{and} \quad \|\nabla u_R\|_{L^\infty(B_{R/2}^+)} \le C.$$

Proof. Let us take R_0 such that the first eigenvalue for problem (2.3) satisfies

$$\lambda_1(R_0) - 1 < 0.$$

We extend by zero the first eigenfunction associated with $\lambda_1(R_0)$. Then, for $R > R_0$

$$\inf J_R \le J_R(\varphi_{1,R_0}) = \Big(\frac{m+1}{p}\Big)\frac{(\lambda_1(R_0)-1)\int_{\Gamma_1}\varphi_{1,R_0}^p}{\Big(\int_{B_R^+}\varphi_{1,R_0}^{m+1}\Big)^{p/(m+1)}} = -C_0. \tag{2.5}$$

On the other hand, multiplying (1.7) by u_R and integrating by parts we get

$$-\int_{B_R^+} |\nabla u_R|^p + \int_{\Gamma_1} u_R^p = \int_{B_R^+} u_R^{m+1}.$$

From this identity and (2.5) the first estimate easily follows. The remaining estimates can be obtained from general regularity theory; see [17]. $\square$

We enclose in the following lemma some useful inequalities related to the p-Laplacian operator.

Lemma 2.3. *For all $\eta, \eta' \in \mathbb{R}^N$, there exist positive constants such that*

i) *if* $p \geq 1$ *and* $|\eta| + |\eta'| > 0$ *it holds that* $\big||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'\big| \leq C_1|\eta - \eta'|(|\eta| + |\eta'|)^{p-2}$,

ii) *if* $p \geq 2$ *we have* $|\eta|^p \geq |\eta'|^p + p|\eta|^{p-2}\langle \eta, \eta - \eta'\rangle + C(p)|\eta - \eta'|^p$.

From the previous estimates we show that it is possible to compare two solutions to (1.7) despite the Neumann boundary condition, whenever the measure of the region of this boundary is sufficiently small. This comparison principle plays a fundamental role in several parts of the paper.

Definition 2.1. *We say that* $\omega \in W^{1,p}(\Omega)$ *is a supersolution if it satisfies*

$$\begin{cases} \omega^m \geq \nabla(|\nabla\omega|^{p-2}\nabla\omega), & \text{in } \Omega, \\ \omega \geq 0, & \text{on } \partial\Omega \cap \{x_N > 0\}, \\ -|\nabla\omega|^{p-2}\dfrac{\partial\omega}{\partial x_N} \geq \omega^{p-1}, & \text{on } \partial\Omega \cap \{x_N = 0\}. \end{cases} \tag{2.6}$$

Analogously we say that ω *is a subsolution if it satisfies* (2.6) *with the reverse inequalities.*

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^N_+$ *be an open bounded domain, with a Lipschitz boundary. Suppose that* $\omega_i \in W^{1,p}(\Omega)$, $i = 1, 2$ *are bounded super- and subsolutions to the problem*

$$\begin{cases} \nabla(|\nabla\omega|^{p-2}\nabla\omega) - \omega^m = 0, & \text{in } \Omega, \\ \omega = 0, & \text{on } \partial\Omega \cap \{x_N > 0\}, \\ -|\nabla\omega|^{p-2}\dfrac{\partial\omega}{\partial x_N} = \omega^{p-1}, & \text{on } \partial\Omega \cap \{x_N = 0\}, \end{cases}$$

respectively in the sense of Definition 2.1. *If the* $N-1$*-dimensional measure of the set* $\partial\Omega \cap \{x_N = 0\}$ *satisfies* $\mu(\partial\Omega \cap \{x_N = 0\}) < \delta$ *for some* $\delta > 0$ *small, then* $(\omega_1 - \omega_2) \geq 0$ *in* Ω.

Proof. We multiply the inequalities satisfied by ω_i, $i = 1, 2$ by $h(\omega_2 - \omega_1)$ and integrate in Ω, with $h(x) = -\min\{0, x\}$. This gives

$$\begin{aligned} &\int_\Omega \Big(\nabla(|\nabla\omega_2|^{p-2}\nabla\omega_2) - \nabla(|\nabla\omega_1|^{p-2}\nabla\omega_1)\Big)h(\omega_2 - \omega_1) \\ &\leq \int_\Omega (\omega_2^m - \omega_1^m)h(\omega_2 - \omega_1). \end{aligned}$$

After integration by parts using the boundary condition it holds that

$$\int_{\Omega\cap\{x_N=0\}\cap A} \Big(|\nabla\omega_2|^{p-2}\nabla\omega_2 - |\nabla\omega_1|^{p-2}\nabla\omega_1\Big)\nabla(\omega_2 - \omega_1)$$

$$+\int_{\Omega\cap A}(\omega_2^m-\omega_1^m)(\omega_2-\omega_1)\le\int_{\partial\Omega\cap A}(\omega_2^{p-2}-\omega_1^{p-2})(\omega_2-\omega_1),$$

where $A=\{x\in\Omega$ such that $\omega_2(x)\le\omega_1(x)\}$. Applying *ii*) of the previous lemma to the first and the second of the above integrals and *i*) to the last one, we get, using Holder's inequality on the right-hand side (recall that $p>2$),

$$\begin{aligned}C_1(p)&\int_\Omega|\nabla h(\omega_2-\omega_1)|^p+\int_{\Omega\cap A}(\omega_2^m-\omega_1^m)(\omega_2-\omega_1)\\&\le C_2(\|\omega_i\|_\infty,p)\int_{\partial\Omega}\Big(h(\omega_2-\omega_1)\Big)^2\\&\le C_2\Big(\int_{\partial\Omega\cap\{x_N=0\}}h(\omega_2-\omega_1)^p\Big)^{2/p}\Big(\mu(\partial\Omega\cap\{x_N=0\})\Big)^{1-2/p}.\end{aligned}\tag{2.7}$$

From here it is easy to see that, if $\mu(\partial\Omega\cap\{x_N=0\})$ is sufficiently small, the last integral in (2.7) can be absorbed into the first one. This implies that $h(\omega_2-\omega_1)\equiv0$ in Ω, and the comparison principle holds. $\square$

In the sequel, by u_R we denote a nontrivial nonnegative solution of (1.7) satisfying (1.6), which is a minimizer of J_R on the space of the functions belonging to W that satisfy (1.6). The existence of such a minimizer can be proved as in Lemma 2.1.

We conclude this section by showing a tangential radial growth property, which will be also satisfied by a solution to (1.4).

To carry out this task, we establish some useful notation in order to apply the moving-plane method, which was introduced in [12]. We define $S^\lambda=\{x\in\mathbb{R}^N:x_1>\lambda\}$, Π^λ the hyperplane ∂S^λ and $x^\lambda=2(\lambda-x_1)e_1+x$, that is, the reflection across Π^λ. Finally, $u_R^\lambda(x)=u_R(x^\lambda)$ and $v^\lambda=u_R^\lambda-u_R$. Assume that $D=\operatorname{supp}(u_R)$ is connected.

Lemma 2.5. *Let u_R be a solution to* (1.4)*. Then it satisfies* (1.6)*. Moreover, $u_R(|x'|,x_N)$ is decreasing in $|x'|$ and x_N.*

Proof. This result is proved in several steps.

First step. Let us show that if $v^\lambda(x_0)=0$ for any $x_0\in S^\lambda$, then $v^\lambda\equiv0$ in S^λ. A similar proof is performed in Lemma 2.1 in [6] for any point $x_0\in S^\lambda\cap\mathbb{R}^N_+$. On $\{x_N=0\}$ we apply the Hopf boundary lemma to conclude that $v^\lambda\equiv0$.

Second step. Let us define

$$\lambda_0=\inf\{\lambda\in\mathbb{R}:v^\lambda(x)\ge0,\ \text{for all } x\in S^\lambda\}.$$

Since u_R is compactly supported λ_0 is well defined, and we have that $-\infty < \lambda_0 < \infty$. Let us see that $S^{\lambda_0} \cap D \neq \emptyset$.

If λ is large then $S^\lambda \cap D = \emptyset$ and consequently $v^\lambda \geq 0$, whereas if $-\lambda$ is large then $(\mathbb{R}^N_+ \setminus S^\lambda) \cap D = \emptyset$, which implies that $v^\lambda < 0$. Note that it is possible to take a $\widetilde{\lambda}$ such that $\mu(S^{\widetilde{\lambda}} \cap D \cap \{x_N = 0\})$ is small enough, so as to apply Lemma 2.4 to $u^{\widetilde{\lambda}}$ and u in $S^{\widetilde{\lambda}} \cap D \cap \mathbb{R}^N_+$, getting $v^{\widetilde{\lambda}} \geq 0$.

Third step. v^{λ_0} vanishes in $S^{\lambda_0} \cap \mathbb{R}^N_+$.

Arguing by contradiction, we suppose that $v^{\lambda_0} \not\equiv 0$; thus, by the first step $v^{\lambda_0} > 0$ in $S^{\lambda_0} \cap D$. The second step assures that $S^{\lambda_0} \cap D \cap \{x_N = 0\} \neq \emptyset$. Let us take a compact K, $K \subset S^{\lambda_0} \cap D \cap \{x_N = 0\}$ and λ sufficiently close to λ_0 such that $\mu((S^\lambda \cap D \setminus K) \cap \{x_N = 0\}) \leq \delta/2$, for which $v^\lambda > 0$ in K, since $v^{\lambda_0} > 0$ in K. Let us denote by $D^- = \{x \in S_\lambda \text{ such that } v^\lambda < 0\}$. The definition of λ_0 implies that $D^- \neq \emptyset$ if $\lambda < \lambda_0$. Since $D^- \subset S^\lambda \cap D \setminus K$ the previous considerations ensure that $\mu(D^- \cap \{x_N = 0\})$ is small. Applying the comparison principle of Lemma 2.4 to u_R^λ and u_R in D^- it follows that $v^\lambda \geq 0$ in D^-, which is a contradiction.

Fourth step. We show now that u_R is decreasing in x_N. In the sequel of the proof let us denote by $S^\lambda = \{x \in \mathbb{R}^N_+ \text{ such that } x_N > \lambda\}$ and keep the remaining notation.

Take $\lambda \in (R/2, R)$. It is easy to see that

$$\begin{cases} \nabla(|\nabla u_R^\lambda|^{p-2}\nabla u_R^\lambda) - \nabla(|\nabla u_R|^{p-2}\nabla u_R) = (u_R^\lambda)^m - u_R^m, & \text{in } S^\lambda \cap B_R, \\ v^\lambda = 0, & \text{on } \Pi^\lambda \cap B_R, \\ v^\lambda \geq 0, & \text{on } S^\lambda \cap \partial B_R. \end{cases} \tag{2.8}$$

Multiplying the equation by $h(v^\lambda)$ and integrating in $S^\lambda \cap B_R$ we get

$$\begin{aligned} &\int_{S^\lambda \cap B_R} (|\nabla u_R^\lambda|^{p-2}\nabla u_R^\lambda - |\nabla u_R|^{p-2}\nabla u_R)\nabla h(v^\lambda) + \int_{S^\lambda \cap B_R} ((u_R^\lambda)^m - u_R^m) h(v^\lambda) \\ &\leq \int_{\partial(S^\lambda \cap B_R)} \left(|\nabla u_R^\lambda|^{p-2} \frac{\partial u_R^\lambda}{\partial x_N} - |\nabla u_R|^{p-2} \frac{\partial u_R}{\partial x_N} \right) h(v^\lambda). \end{aligned} \tag{2.9}$$

The boundary integral vanishes and consequently $v^\lambda \geq 0$ in $S^\lambda \cap B_R$ for every $\lambda \in (R/2, R)$.

We consider now $\lambda \in (R/4, R/2)$. Then v^λ satisfies $(2.8)_1$ in $B_R \cap \{\lambda < x_N < 2\lambda\}$. To argue as in (2.9) we show that the boundary integral corresponding to these values of λ is nonpositive. It is immediate to see that $v^\lambda \geq 0$ on $\partial B_R \cap \{\lambda < x_N < 2\lambda\}$ and vanishes over $\{x_N = \lambda\}$. Finally, we show that the integrand is nonpositive over $B_R \cap \{x_N = 2\lambda\}$. If $x_N > R/2$ we have already shown that u_R is decreasing with respect to x_N. Thus, $\frac{\partial u_R}{\partial x_N} \le 0$. Conversely, $\frac{\partial u_R^\lambda}{\partial x_N} \ge 0$, getting the desired result. Repeating this argument using (2.9) we can conclude that $v^\lambda \ge 0$ for every $\lambda > 0$. □

3. Existence of a compactly supported solution

The following task is to show that, for R sufficiently large, u_R indeed solves (1.4).

Proposition 3.1. *Let u_R be a solution to* (1.4) *satisfying* (1.6). *For R large enough*

$$\max_{x \in supp(u_R)} |x| < R.$$

Proof. We show first that u_R is compactly supported in the variable x_N. Note that by Lemma 2.5 we have, for every $|y'| \le |x'|$ and every $0 < y_N \le x_N$,

$$\int_A u_R^{m+1}(x', x_N)dy \le \int_A u_R^{m+1}(y', y_N)dy,$$

where $A = \{(y', y_N) : |y'| \le |x'|,\ 0 < y_N \le x_N\}$. Taking now into account the first estimate of Lemma 2.2, we can deduce

$$u_R(x', x_N) \le \frac{C}{|x'|^{(N-1)/(m+1)}|x_N|^{1/(m+1)}} \le \frac{C(2L)^{(N-1)/(m+1)}}{R_1^{1/(m+1)}} \le \frac{1}{2}, \quad (3.1)$$

for any $|x'| \ge \frac{1}{2L}$ and $x_N \ge R_1$ for $R_1 \le R$ large enough, where by L we denote the uniform bound $\|\nabla u\|_{L^\infty(B^+_{R/2})} \le L$ of Lemma 2.2. If $y' \in \mathbb{R}^{N-1}_+$ is such that $|x' - y'| \le \frac{1}{2L}$, then

$$|u_R(x', R_1) - u_R(y', R_1)| \le L|x' - y'| \le \frac{1}{2},$$

which together with (3.1) gives

$$u_R(x', R_1) \le 1, \qquad \forall x'. \quad (3.2)$$

Let us consider a supersolution with compact support in x_N to the problem

$$\begin{cases} \nabla(|\nabla\omega|^{p-2}\nabla\omega) = \omega^m, & \text{in } \{x_N > R_1\} \cap B_R^+, \\ \omega(R_1) = 1, & \text{in } \{x_N = R_1\} \cap B_R^+. \end{cases} \quad (3.3)$$

An integration by parts of the equations that ω and u_R satisfy, multiplied by $h(\omega - u)$, gives

$$\begin{aligned}\int_{\Omega\cap\{\omega\le u_R\}} &\Big(|\nabla\omega|^{p-2}\nabla\omega - |\nabla u_R|^{p-2}\nabla u_R\Big)\nabla(\omega - u_R)\\ &+ \int_{\Omega\cap\{\omega\le u_R\}} (\omega^m - u_R^m)(\omega - u_R)\\ \le -&\int_{\partial\Omega}\Big(|\nabla\omega|^{p-2}\frac{\partial\omega}{\partial\eta} - |\nabla u|^{p-2}\frac{\partial u_R}{\partial\eta}\Big)h(\omega - u_R),\end{aligned}$$

where $\Omega = \{x_N > R_1\}\cap B_R^+$ and η is the outward unit normal vector. Note that $\omega \ge u_R$ in $\partial\Omega$, due to (3.2) and to the fact that $u_R \equiv 0$ on $\{x_N > R_1\}\cap\partial B_R^+$. Thus the boundary integral vanishes and we get that $\omega \ge u_R$ in $\{x_N > R_1\}\cap B_R^+$. Therefore, if $x_N \ge R_2$ then $u(x', x_N) = 0$.

As such a supersolution, we can take for instance

$$\omega = \beta\big((\gamma - x_N)^+\big)^\alpha, \tag{3.4}$$

where $f^+ = \max\{f, 0\}$ and $\alpha,\ \beta\ ,\gamma$ being such that

$$\begin{aligned}&\alpha = \frac{p}{p-(m+1)},\\ &2\beta^{p-(m+1)}p^{p-1}(m+1)(p-1) = \big(p-(m+1)\big)^p,\\ &\beta(\gamma - R_1)^{p/(p-(m+1))} = 1.\end{aligned} \tag{3.5}$$

We prove now that the support of u_R is bounded in the direction of x'. Let us take $x_0 \in \{x_N = 0\}$. We show that for $|x_0|$ and R sufficiently large, $u_R \le \psi$, with ψ a vanishing function in a small neighbourhood of x_0. We construct such a function ψ satisfying as well the following:

$$\nabla(|\nabla\psi|^{p-2}\nabla\psi) \le \psi^m, \qquad \text{in }\ \Omega\cap\{x_N > 0\}, \tag{3.6}$$

$$-|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial x_N} \ge \psi^{p-1}, \qquad \text{on }\ \partial\Omega\cap\{x_N = 0\}, \tag{3.7}$$

$$\varepsilon := \inf_{\partial\Omega\cap\{x_N>0\}}\psi > 0, \tag{3.8}$$

where $\Omega = B(x_0, r_0)$, with $0 < r_0 < 1$ chosen sufficiently small.

From the estimate (3.1) we can find R_3 such that

$$u_R(x', x_N) \le \frac{\varepsilon}{2}, \qquad \forall |x'| \ge R_3, \quad x_N \ge \frac{\varepsilon}{2L},$$

denoting by L the uniform bound for $|\nabla u_R|$ of Lemma 2.2. Arguing as in (3.2) it follows that

$$u_R(x', x_N) \le \varepsilon, \qquad \forall x' \text{ such that } |x'| = R_3, \ \forall x_N > 0.$$

Choosing $x_0 \in \mathbb{R}^N_+$ such that $|x_0| = R_3 + r_0$, the comparison provided by Lemma 2.4 holds and $\psi \ge u_R$ in $\Omega \cap \mathbb{R}^N_+$, with $R \ge R_3 + 2r_0$. Note that x_0 was any of the points in $\partial B_{R_3+r_0} \cap \{x_N = 0\}$; thus, u_R vanishes in a neighbourhood of this set. The monotonicity of u_R in $|x'|$ and x_N concludes the proof.

It remains to construct the function ψ with the desired properties. We denote $x_0 = (x_0', 0)$. For simplicity, let us assume that ψ is radial around the point (x_0', d); that is, $\psi = \psi(r^2)$, where $r^2 = r_1^2 + (x_N - d)^2$ and $r_1^2 = |x' - x_0'|^2$. More precisely, we look for a function of the form

$$\psi = a\big((r^2 - b)^+\big)^\alpha, \quad \text{with } \alpha = \frac{p}{p-1-m}, \tag{3.9}$$

where $a, b, d > 0$ will be fixed conveniently depending only on r_0, m and p. Since

$$-|\psi'|^{p-2}\frac{\partial \psi}{\partial x_N} = (2a\alpha)^{p-1} r^{p-2} \big((r^2-b)^+\big)^{(p-2)(\alpha-1)} (x_N - d),$$

condition (3.7) reads as

$$(2\alpha)^{p-1} r^{p-2} d \ge \big((r^2-b)^+\big)^{p-1}.$$

Taking $d \ge r_0^p$ and

$$d^2 - b < 0, \tag{3.10}$$

we get that (3.7) holds. Note that (3.10) implies also that ψ vanishes in a neighbourhood of $r_1 = 0$ and $x_N = 0$. To satisfy (3.8) we need the following inequality,

$$r_1^2 + (x_N - d)^2 - b = r_1^2 + x_N^2 + d^2 - 2x_N d - b \ge r_0(r_0 - 2d) + d^2 - b, \tag{3.11}$$

since $r_1^2 + x_N^2 = r_0^2$ and thus $x_N \le r_0$. Then, if $d < r_0/2$ we can choose an appropriate b such that (3.11) holds. Finally, we deal with (3.6), which written in radial variables reads

$$(p-1)|\psi'|^{p-2}\psi'' + (N-1)\frac{|\psi'|^{p-2}\psi'}{r} \ge c\psi^m.$$

This holds if a is small enough, more precisely, if

$$a^{p-1-m}(2\alpha)^{p-1} r^{p-2}\big(2r^2(\alpha-1)(p-1) + (p+N-2)(r^2-b)\big) \le 1.$$

This ends the proof. $\square$

Lemma 3.1. *If $u \in W^{1,p}(\mathbb{R}^N_+)$ solves* (1.4), *then*

$$\|u\|_{L^{m+1}(\mathbb{R}^N_+)} \leq C, \;\; \|u\|_{L^\infty(\mathbb{R}^N_+)} \leq C \;\; \text{and} \;\; \|\nabla u\|_{L^\infty(\mathbb{R}^N_+)} \leq C.$$

Proof. Multiplying the equation (1.4) and integrating by parts,

$$\int_{\mathbb{R}^N_+} |\nabla u|^p + \int_{\mathbb{R}^N_+} u^{m+1} = \int_{\{x_N=0\}} u^p,$$

the first bound follows, since $u \in W^{1,p}(\mathbb{R}^N_+)$. The remaining estimates are derived from the first one as follows.

First we deal with the uniform bound for u. Note that if u solves (1.4), by the results in Lemma 2.2 we know that $u \in L^\infty(B^+_R)$, and $\nabla u \in L^\infty(B^+_{R/2})$ for $R > 0$ large. Let us argue by contradiction. Suppose that there exists a sequence $\{x_n\} \in \mathbb{R}^N_+$ such that $|x_n| \to \infty$ and $|u(x_n)| \to \infty$. Taking any $y \in B(x_n, R/2)^+$ one can see that

$$u(y) \geq u(x_n) - LR/2, \tag{3.12}$$

where L is the uniform bound for the gradient of u in the ball $B(x_n, R/2)^+$. On the other hand, by the previous step we have $\|u\|_{L^{m+1}(\mathbb{R}^+)} \leq C$, which plugged into (3.12) gives

$$C \geq \int_{\mathbb{R}^N_+} u^{m+1} \geq \int_{B^+_R} u^{m+1} \geq \mu(B^+_R)(u(x_n) - LR/2)^{m+1},$$

which is not bounded and gives the desired contradiction.

To prove the remaining estimate we argue similarly, but using the Hölder continuity of the gradient of u (see [17]) since we lack the uniform bound for the second derivative. We get the analog to (3.12)

$$|\nabla u(y)| \geq |\nabla u(x_n)| - C(R/2)^\alpha,$$

for any $y \in B(x_n, R/2)$. But this contradicts that $u \in W^{1,p}(\mathbb{R}^N_+)$ since

$$C \geq \int_{\mathbb{R}^N_+} |\nabla u|^p \geq \int_{B(x_n,R/2)^+} |\nabla u|^p \geq \mu(B(x_n, R/2))(\|\nabla u(x_n)\| - C(R/2)^\alpha)^p,$$

which is not bounded, and the lemma is proved. $\square$

Proof of Theorem 1.1. We start by showing that u becomes small outside B^+_R; that is,

$$\lim_{R\to\infty} \sup_{\mathbb{R}^N_+ \setminus B^+_R} u = 0. \tag{3.13}$$

If not, there exist $\varepsilon_0 > 0$ and $x_n \in \overline{\mathbb{R}^N_+}$ such that $u(x_n) \ge \varepsilon_0$ as $|x_n| \to \infty$. Let L be the uniform bound in $\mathbb{R}^N_+$ for the gradient of u provided by the previous lemma and $r = \frac{\varepsilon_0}{2L}$. For every $y_n \in B(x_n, r)$ it holds that

$$|u(x_n) - u(y_n)| \le \|\nabla u\|_{L_\infty(\mathbb{R}^N_+)}|x_n - y_n| \le \frac{\varepsilon_0}{2}.$$

Thus, $u \ge \varepsilon_0/2$ in $B(x_n, r) \cap \mathbb{R}^N_+$, $\forall n$. Then, for a subsequence of disjoint balls $B(x_n, r)$, we have

$$\int_{\mathbb{R}^N_+} u^{m+1} \ge \sum_n \int_{B(x_n,r)\cap\mathbb{R}^N_+} u^{m+1} = \infty,$$

which contradicts Lemma 3.1.

By (3.13) we can find $R_1 > 0$ such that $u(x', R_1) \le 1$, $\forall x' \in \mathbb{R}^{N-1}$. Considering the same function ω of Proposition 3.1 for this R_1 it follows that

$$\omega \ge u \ \text{ on } \ \{x_N = R_1\} \tag{3.14}$$

and

$$\liminf_{|x|\to\infty} (\omega - u) \ge 0. \tag{3.15}$$

Note that (3.14) and (3.15) imply that $u \le \omega$ in $\{x_N > R_1\}$. This fact can be easily seen by multiplying the equation that u and ω satisfy by $h(\omega - u)$ once more and integrating by parts. This gives

$$\begin{aligned}\int_{\Omega\cap\{\omega\le u\}} \big(|\nabla\omega|^{p-2}\nabla\omega - |\nabla u|^{p-2}\nabla u\big)\nabla(\omega - u) + \int_{\Omega\cap\{\omega\le u\}} (\omega^m - u^m)(\omega - u) \\ \le -\int_{\partial\Omega} \big(|\nabla\omega|^{p-2}\frac{\partial\omega}{\partial\eta} - |\nabla u|^{p-2}\frac{\partial u}{\partial\eta}\big)h(\omega - u),\end{aligned}$$

where $\Omega = \{x_N > R_1\}$. Since the boundary integral vanishes, $h(\omega - u) \equiv 0$ in Ω and we obtain the desired result. Thus there exists R_2 such that $u(x', x_N) = 0$ for all $x' \in \mathbb{R}^{N-1}$, and $x_N > R_2$.

Next we show that $u(x', 0) = 0$ for any $x' \in \mathbb{R}^{N-1}$ with $|x'|$ large. We apply the comparison principle of Lemma 2.4 in $B(x_0, r_0) \cap \mathbb{R}^N_+$ with $r_0 > 0$ small enough and $x_0 \in \{x_N = 0\}$, to the function ψ constructed in Proposition 3.1 in (3.9) and u. It follows that

$$\inf_{\partial B(x_0,r_0)\cap\{x_N>0\}} \psi = \varepsilon;$$

see (3.8). Since (3.13) holds, it is possible to find $R_3 > 0$ large such that if $|x_0| > R_3$ then $u \le \varepsilon$ in $B(x_0, r_0) \cap \mathbb{R}^N_+$. Therefore, $u \le \psi$ in this domain, and this fact implies that u vanishes in a neighbourhood of x_0, since ψ did.

To conclude the proof of the compactness of the support of u we consider a function ω with the same expression (3.4) but as a function of x_k for any direction $k = 1, \dots, N-1$, that is,

$$\omega = \beta\big((\gamma - x_k)^+\big)^\alpha,$$

with α, β and γ taken as in (3.5). Analogously, take R_1 large enough ensuring that $u(x) \le 1$ if $x_k \ge R_1$ and $x_N > 0$. As before, considering problem (3.3) in $\{x_k > R_1\} \cap \mathbb{R}^N_+$ it follows that $u \le \omega$ in this region. Thus $u = 0$ for x_k large and $x_N > 0$. Since x_k was arbitrary we have deduced that u is compactly supported.

Once we have the compactness of the support of u, we easily deduce from steps 1–3 in Lemma 2.5 the symmetry property (1.6). □

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