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New examples of subsets of c with the FPP and stability of the FPP in hyperconvex spaces

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Abstract. The purpose of this work is two-fold. On the one side, we focus on the space of real convergent sequences c where we study non-weakly compact sets with the fixed point property. Our approach brings a positive answer to a recent question raised by Gallagher et al. in (J Math Anal Appl 431(1):471–481, 2015). On the other side, we introduce a new metric structure closely related to the notion of relative uniform normal structure, for which we show that it implies the fixed point property under adequate conditions. This will provide some stability fixed point results in the context of hyperconvex metric spaces. As a particular case, we will prove that the set $M = [-1, 1]^{\mathbb{N}}$ has the fixed point property for d-nonexpansive mappings where $d(\cdot, \cdot)$ is a metric verifying certain restrictions. Applications to some Nakano-type norms are also given.

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1. Introduction

The main topic of this work is the study of existence of fixed points for nonexpansive selfmappings. It is well-known that c_0, the Banach space of all null-convergent sequences, and c, the Banach space of all convergent sequences, satisfy that their convex weakly compact sets have the fixed point property (FPP for short) for $\|\cdot\|_\infty$-nonexpansive mappings [5] (for definitions see Sect. 2). In regards to $(c_0, \|\cdot\|_\infty)$, it was shown in [8,10,11] that convex weakly compact sets are the only convex closed subsets of c_0 with the FPP, that is, if a closed and convex subset of c_0 has the FPP then it has to be

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weakly compact. This raised the question of whether the FPP for $\|\cdot\|_\infty$-nonexpansive mappings could lead to a similar fixed point characterization of weak compactness in $(c, \|\cdot\|_\infty)$. This question was answered negatively in [14], although weak compactness in c has actually been characterized in terms of fixed point properties for some larger family of mappings. In fact, it was shown in [16] that a closed and convex subset of c is weakly compact if, and only if, every (so-called) cascading nonexpansive selfmapping defined on it has a fixed point.

In [14] the authors gave several examples of non-weakly compact, closed and convex subsets of c with the fixed point property for $\|\cdot\|_\infty$-nonexpansive mappings. The fact is that those examples were all hyperconvex and so a new question was posed: is it possible to find such a non-weakly compact set lacking the property of being hyperconvex? This question motivates the first part of the present work. We will show that the actual answer is yes using the notion of uniform normal structure in metric spaces. In the second part we will explore new generalizations of the concept of relative uniform normal structure in metric spaces in order to bring new facts on fixed points for nonexpansive mappings.

This work is organized as follows: In Sect. 2 we recall some notions and results that will be needed afterwards. In Sect. 3 we build two examples (family of examples, actually) of bounded closed and convex subsets of c which enjoy the fixed point property for nonexpansive mappings but fail to be both weakly compact and hyperconvex. This will answer the question raised in [14] in the positive and will spark new natural questions. In particular, whether it is possible to characterize bounded closed and convex subsets of $(c, \|\cdot\|_\infty)$ with FPP for nonexpansive mappings in terms of topological or metric properties remains open. In Sect. 4, we present an extension of the notion of relative uniform normal structure for a metric space and show that, under certain assumptions, this extension, likewise uniform normal structure and relative uniform normal structure, implies the FPP. As a consequence, we will achieve a fixed point stability result for hyperconvex metric spaces. This stability property will be applied to several situations and examples, including some kinds of Nakano sequence spaces. As a byproduct we offer a new approach to the long standing problem of the fixed point property for uniformly Lipschitzian mappings defined on hyperconvex spaces (see, for instance, [26]). We finish this work with a last section where we bring forward some remarks and we set some questions related to the content of the article.

2. Definitions and basic results

In this section we recall some needed definitions and results. We begin with the definition of nonexpansive mapping and fixed point property.

Definition 2.1. Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be nonexpansive if $d(Tx, Ty) \le d(x, y)$ for every $x, y \in M$. M is said to have the fixed point property for nonexpansive mappings (FPP for short) if every nonexpansive mapping $T : M \to M$ has a fixed point, that is, there exists $x \in M$ such that $Tx = x$.

The interested reader may consult [20] for a thorough exposition of fixed point theory, in this connection, see also the early papers by S. Reich [23] and [24]. In this work, the notation $B_d(x, r)$ will always refer to the closed ball with center $x \in M$ and radius $r \geq 0$ with respect to the metric d. We now introduce a family of subsets of a metric space which will be heavily used in the next sections.

Definition 2.2. Let (M, d) be a metric space, $A \subset M$ is said to be admissible if it is an intersection of closed balls of M. We will denote the set of admissible subsets of M by $\mathcal{A}_d(M)$.

When there is no danger of confusion we will omit the subscript d in notations like $\mathcal{A}_d(M)$, $B_d(x, r)$ and others that might come up.

A very important property for a family of subsets of a given set is that of compactness.

Definition 2.3. Let M be a set. A family $\{X_\lambda\}_{\lambda\in\Gamma}$ of subsets of M is said to have the finite intersection property if $\bigcap_{\lambda\in\Gamma_f} X_\lambda \neq \emptyset$ for all finite $\Gamma_f \subset \Gamma$.

Now compactness may be defined as follows.

Definition 2.4. A family $\mathcal{L}$ of subsets of a set M is said to be compact if every subset of it which satisfies the finite intersection property has nonempty intersection.

Some interesting properties can be deduced from the compactness of a family of sets when combined with other topological structures as, for instance, the one stated in the next proposition.

Proposition 2.5. [12, Proposition 3.2] *Let (M, d) be a metric space such that $\mathcal{A}(M)$ is compact. Then (M, d) is complete.*

We give next the definition of hyperconvex metric spaces. The interested reader may find more information on these spaces in [12].

Definition 2.6. A metric space (M, d) is said to be hyperconvex if, for every subset $X \subset M$ and for every family of radii $\{r_x \geq 0 : x \in X\}$, we have that

$$\bigcap_{x\in X} B(x, r_x) \neq \emptyset$$

whenever $d(x, y) \leq r_x + r_y$ for all x and $y \in X$.

Hyperconvex metric spaces have been extensively studied in fixed point theory as they have been shown to be a very relevant class of metric spaces and provide a particular example of metric structure for which abstract results work (see [12]). In particular, bounded hyperconvex metric spaces, as it is very well-known, enjoy the FPP for nonexpansive mappings (see [2,3,12,19]).

This work is also involved with uniform normal structure and relative uniform normal structure in metric spaces. We give next these notions and recall some relevant results.

Definition 2.7. Given a nonempty subset X of a metric space (M,d) we define

$$\begin{aligned} r_a(X) &= \sup\{d(a,x) : x \in X\}\, for any a \in M, \\ r(X) &= \inf\{r_x(X) : x \in X\}\, and \\ \operatorname{diam}(X) &= \sup\{d(x,y) : x,y \in X\}, \end{aligned}$$

where $r_a(X)$ and $r(X)$ stand for the Chebyshev radius of X with respect to a and the Chebyshev radius of X in M respectively, and $\operatorname{diam}(X)$ is the diameter of X.

Definition 2.8. A metric space (M,d) is said to have uniform normal structure (UNS) if there exists some $c \in (0,1)$ such that $r(A) \le c \cdot \operatorname{diam}(A)$ for every $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A) < +\infty$. For short, we say that M has UNS or c-UNS if c is needed.

Spaces with UNS have been deeply studied in the literature and uniformly convex Banach spaces are known to have UNS (see, for instance, [17]). Finite dimensional normed spaces have it too as the following proposition shows:

Proposition 2.9. *[1, Proposition 2.12] If X is a normed space of dimension n, then it has UNS with constant $\frac{n}{n+1}$.*

The following theorem shows how the concepts of UNS and FPP are related.

Theorem 2.10. [18, Theorem 10] *Let (M,d) be a bounded complete metric space. If (M,d) has uniform normal structure then it has the fixed point property.*

Initially, the previous theorem (see [21, Theorem I]) was stated with the additional requirement that the family $\mathcal{A}(M)$ was compact or at least countably compact (condition (R) in [18]). It turns out that this compactness condition is superfluous as stated in the following theorem.

Theorem 2.11. [17, Theorem 5.4] *Suppose (M,d) is a bounded complete metric space which has UNS. Then $\mathcal{A}(M)$ is compact.*

It can be shown that if (M,d) is a hyperconvex metric space then it has $\frac{1}{2}$-UNS (see [12, Lemma 3.3]). Thus, in particular, since hyperconvex metric spaces are complete, it follows from Theorem 2.11 that the family $\mathcal{A}(M)$ is compact.

The following definition was introduced by Soardi in [25].

Definition 2.12. A metric space (M,d) is said to have relative uniform normal structure (RUNS or c-RUNS) if there exists some $c \in (0,1)$ such that, for every nonempty $A \in \mathcal{A}(M)$ with $0 < \operatorname{diam}(A) < +\infty$, the following conditions are satisfied:

(i) There exists $z_A \in M$ with

$$r_{z_A}(A) \leq c \cdot \operatorname{diam}(A).$$

(ii) For z_A as above and $x \in M$ with $r_x(A) \leq c \cdot \operatorname{diam}(A)$, we have that

$$d(x, z_A) \leq c \cdot \operatorname{diam}(A).$$

There is a subtle but important difference between Definition 2.8 and Definition 2.12 since the point z_A given in Definition 2.12 does not necessarily belong to the set A. Because of that, the extra condition ii) is needed to assure the existence of a fixed point as the following theorem states.

Theorem 2.13. [17, Theorem 5.6] *Let (M, d) be a bounded metric space such that $\mathcal{A}(M)$ is compact and (M, d) has relative uniform normal structure. Then M has the FPP.*

Note that if (M, d) has uniform normal structure then it has relative uniform normal structure. Besides, it is not known whether RUNS implies compactness of $\mathcal{A}(M)$ as it is the case for UNS (Theorem 2.11), in this connection see [6] and [7].

3. On subsets of $(c, \|\cdot\|_\infty)$ with the FPP

Consider the following subset of c:

$$W = \{x = (x_n) \in \ell_\infty : 1 \geq x_1 \geq x_2 \geq \ldots \geq 0\}.$$

In [14] the authors introduce the subset W as a first example of a bounded closed and convex subset of c with the FPP but failing to be weakly compact. This closed the door to a possible characterization result for weakly compactness in c as the one provided for c_0 in [10] (although it opened the scenario to considering fixed point theorems for a larger family of mappings, see [16]).

The set W was not the only example of a convex closed bounded and non-weakly compact subset of c with the FPP provided in [14]. For $q = (q_n)$ a positive convergent sequence with positive limit, subsets of c, defined out of such sequences q as follows

$$W_q = \{u = (q_n y_n) \in \ell_\infty : 1 \geq y_1 \geq y_2 \geq \ldots \geq 0\},$$

were proved to enjoy the same properties as those described for W.

The key fact why all these sets, W as well as the family of sets W_q, enjoy the FPP is that they all happen to be hyperconvex. Therefore, authors of [14] wondered to what extent hyperconvexity was essential to determine the FPP for non-weakly compact sets in c. The goal of this section is to provide a deeper understanding of this situation. We give a first example showing that hyperconvexity can actually be avoided.

Example. Let $Q = \left\{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 + x_3 = 1\right\}$, let W be as previously described. If we set

$$M = \left(Q, \|\cdot\|_{\infty,3}\right) \oplus_\infty (W, \|\cdot\|_\infty), where \, \|\cdot\|_{\infty,3} \, is the sup norm in \mathbb{R}^3,$$

then M is a closed, non-weakly compact, bounded and convex subset of c which is not hyperconvex but still has the FPP.

Proof. Since $W \subset c$, it is obvious that $M \subset c$. It is also obvious that M is closed, bounded and convex. We claim that W has UNS. Indeed, observe that $\left(Q, \|\cdot\|_{\infty,3}\right)$ is isometric to $\left(P, \|\cdot\|_{\infty,3}\right)$ where

$$P = \left\{(x_1, x_2, x_3) \in [0,1]^2 \times [-1,0] : x_1 + x_2 + x_3 = 0\right\}.$$

Also, P is contained in $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\right\}$ which is a 2-dimensional subspace of $\mathbb{R}^3$. Thus, it follows from Proposition 2.9 that P with the $\|\cdot\|_{\infty,3}$ norm has $\frac{2}{3}$-UNS and therefore, by isometry, $\left(Q, \|\cdot\|_{\infty,3}\right)$ has $\frac{2}{3}$-UNS too.

Since $(W, \|\cdot\|_\infty)$ is hyperconvex, it follows that $(W, \|\cdot\|_\infty)$ has $\frac{1}{2}$-UNS and therefore, given that direct maximum sums of spaces with UNS have UNS ([18, Proposition 12]), we have that $(M, \|\cdot\|_\infty)$ has $\frac{2}{3}$-UNS. This implies that $(M, \|\cdot\|_\infty)$ has the FPP.

To show that M is not hyperconvex we consider the family of closed balls in M given by:

$$\left\{B\left(e_i, \frac{1}{2}\right) : 1 \leq i \leq 3\right\},$$

where e_i stands for the i-th standard basis element. Since $\|e_i - e_j\|_\infty = 1$ whenever $i \neq j$, this collection of balls should have nonempty intersection with M if M were hyperconvex. However, it follows that, for any $x = (x_n)$ in these three balls, it must be the case that $|x_i - 1| \leq \frac{1}{2}$ and $|x_i| \leq \frac{1}{2}$ for all $1 \leq i \leq 3$. Thus, $x_1 = x_2 = x_3 = \frac{1}{2}$ and therefore, $x \notin M$. Hence $(M, \|\cdot\|_\infty)$ is not hyperconvex.

Finally, since M is a direct sum with a nonweakly compact subset of c, it follows that M is not weakly compact either. □

The previous example can be modified by replacing the first term of the direct sum by any non hyperconvex, bounded, closed and convex subset of any finite dimensional normed space since such sets, as stated in Proposition 2.9, have UNS.

A direct sum, as the previous example, may be seen as a too straightforward construction and one can wonder if there is any such example which cannot be obtained in that way. This will be the case for the next one.

Example. Let

$$D = \left\{x = (x_n) \in [0,1]^{\mathbb{N}} : x_1 + x_2 + x_3 = 1, x_3 \geq x_4 \geq \ldots \geq 0\right\}.$$

Then D is a bounded, closed, convex and non-weakly compact subset of c which is not hyperconvex and has the FPP.

Proof. It is obvious that $D \subset c$ and D is closed, bounded and convex. Similar arguments to those used in the previous example show that D is neither weakly compact nor hyperconvex.

We will apply Theorem 2.10 to prove that D has the FPP. It suffices to show that D has $\frac{2}{3}$-UNS. Let $A \in \mathcal{A}(D)$ such that $\delta = \operatorname{diam}(A) > 0$. For each $n \in \mathbb{N}$, define

$$a_n = \inf_{x \in A} x_n, b_n = \sup_{x \in A} x_n and z_n = \frac{a_n + b_n}{2}.$$

Observe that $|z_n - x_n| \leq \frac{1}{2}\,\delta$ for all $x \in A$ and $n \in \mathbb{N}$. This implies that $A \subset B(z, \frac{1}{2}\,\delta)$ where $z = (z_n)$. However, we cannot assure that $z \in D$ although it is still true that $1 \geq z_3 \geq z_4 \geq z_5 \geq \ldots \geq 0$.

Consider the projection mapping:

$$\begin{aligned} \pi : (\ell_\infty, \tau) &\to \left(\mathbb{R}^3, \|\cdot\|_\infty\right) \\ x = (x_n) &\mapsto (x_1, x_2, x_3) \end{aligned}$$

Let τ be the coordinatewise convergence topology on ℓ_∞. We have that D is τ-compact, which gives us that any $A \in \mathcal{A}(D)$ is τ-compact. Since A is convex too, $\pi(A)$ is a compact and convex subset of $\left(\mathbb{R}^3, \|\cdot\|_\infty\right)$. From the proof of [1, Proposition 2.12] together with the argument we used in the previous example, we know that there is $w = (w_n) \in A$ such that

$$\|\pi(w) - \pi(x)\|_\infty \leq \frac{2}{3} \operatorname{diam}(\pi(A)) \leq \frac{2}{3}\,\delta \text{ for all } x \in A.$$

This implies that, for $x \in A$, $|w_n - x_n| \leq \frac{2}{3}\,\delta$ for $1 \leq n \leq 3$. For $n \geq 4$ we still know that $w_n \in [a_n, b_n]$.

Now, we are going to show that $A \cap \bigcap_{y \in A} B\left(y, \frac{2}{3}\,\delta\right) \neq \emptyset$. To this end, let us define

$$\tilde{w}_n = \begin{cases} w_n & \text{if } 1 \leq n \leq 3, \\ \min\{w_3, z_n\} & \text{if } n \geq 4. \end{cases}$$

Observe that $\tilde{w}_1 + \tilde{w}_2 + \tilde{w}_3 = 1$ and $|\tilde{w}_n - x_n| \leq \frac{2}{3}\,\delta$ for $1 \leq n \leq 3$ and $x \in A$. Also, since $0 \leq a_n \leq w_n \leq w_3$ and $a_n \leq z_n \leq b_n$ for all $n \geq 4$, it follows that $\tilde{w}_n \in [a_n, b_n]$ for $n \in \mathbb{N}$.

Since $A \in \mathcal{A}(D)$ we have that

$$A = \bigcap_{i \in I} B\left(p^i, r_i\right) \cap D$$

for a certain collection of centers $p^i = \left(p^i_n\right) \in D$ for $i \in I$. Therefore, given $x = (x_n) \in A$, we have that $x \in B\left(p^i, r_i\right)$ for all $i \in I$. In particular, we have that $\left|x_n - p^i_n\right| \leq r_i$ for all $n \in \mathbb{N}$ and $i \in I$. Thus, given $n \in \mathbb{N}$,

$$p^i_n - r_i \leq x_n \leq p^i_n + r_i$$

for all $x \in A$ and $i \in I$. Therefore, $p_n^i - r_i \leq a_n \leq b_n \leq p_n^i + r_i$ and we can conclude that

$$[a_n, b_n] \subset \bigcap_{i \in I} \left[p_n^i - r_i, p_n^i + r_i\right].$$

Hence,

$$\tilde{w} \in \bigcap_{i \in I} B\left(p^i, r_i\right)$$

and therefore, since $\tilde{w} \in D$, $\tilde{w} \in A$.

Let $x = (x_n) \in A$. We already know that $|\tilde{w}_n - x_n| \leq \frac{2}{3}\,\delta$ for $1 \leq n \leq 3$. Let $n \geq 4$. We have the following three cases to study:

- Case $z_n \leq w_3$. In this case we have that $\tilde{w}_n = z_n$ which gives that
$$|\tilde{w}_n - x_n| = |z_n - x_n| \leq \frac{1}{2}\,\delta \leq \frac{2}{3}\,\delta.$$
- Case $w_3 < z_n$ and $x_n \leq z_n$. In this case, since $\tilde{w}_n,\ x_n \in [a_n, b_n]$ and z_n is the middle point of $[a_n, b_n]$, we have that
$$|x_n - \tilde{w}_n| \leq \frac{1}{2}\,\delta \leq \frac{2}{3}\,\delta.$$
- Case $w_3 < z_n$ and $z_n < x_n$. In this case we have that
$$|x_n - \tilde{w}_n| = |x_n - w_3| = x_n - w_3 \leq x_3 - w_3 \leq \frac{2}{3}\,\delta.$$

Since these are the only possible cases, we can conclude that

$$|x_n - \tilde{w}_n| \leq \frac{2}{3}\,\delta \text{ for all } n \geq 4.$$

Hence, $|\tilde{w}_n - x_n| \leq \frac{2}{3}\,\delta$ for all $n \in \mathbb{N}$ and $x \in A$. Therefore, $\tilde{w} \in \bigcap_{y \in A} B\left(y, \frac{2}{3}\,\delta\right)$ and, finally,

$$A \cap \bigcap_{y \in A} B\left(y, \frac{2}{3}\,\delta\right) \neq \emptyset.$$

Now, since A was an arbitrary admissible subset of D with positive diameter, we have that D has $\frac{2}{3}$-UNS and, by Theorem 2.10, D has the FPP. □

4. The (p, q)-relative uniform normal structure

In this section we introduce a concept related to the relative uniform normal structure that will be applied to obtain some new fixed point results as well as stability of the FPP in metric spaces. Under the scope of a metric given through a norm, a stability result for the fixed point property for nonexpansive mappings is stated as follows: Assume that C is a subset of a Banach space $(X, \|\cdot\|)$ such that $(C, \|\cdot\|)$ satisfies the FPP. Let $|\cdot|$ be an equivalent norm on X. What can be said regarding the fulfillment of the FPP for the metric space $(C, |\cdot|)$?

With the backdrop of a suitable geometry for the Banach space $(X, \|\cdot\|)$ itself, an extensive literature has been published leading to the fulfillment of the FPP for the new metric space $(C, |\cdot|)$ when the Banach-Mazur distance between the spaces $(X, \|\cdot\|)$ and $(X, |\cdot|)$ is close enough (see for instance [20, Chapter 7] and references therein). It was shown in [4] that it is still possible to obtain stability results for some specific closed convex bounded subsets of dual Banach spaces, being this property local, in the sense that it is not shared by all the closed convex bounded subsets of the Banach space. This shows that the concept of the stability of the FPP depends on geometric and topological properties of the domain C rather than on the Banach space in which it lies. Furthermore, stability can be easily translated to the more general scope of metric spaces: Assume that (M, d) is a metric space with the FPP and that $d'(\cdot, \cdot)$ is an equivalent metric defined on M. Is it possible to determine whether (M, d') also fulfills the FPP in terms of how "close" distances d and d' are?

It must be stressed that, in particular, the Banach space $(\ell_\infty, \|\cdot\|_\infty)$ may be considered the worst case scenario when it comes to obtaining general fixed point results for nonexpansive mappings. This is due to the fact that the FPP is separably determined [15, Chapter 3] and that every separable Banach space can be isometrically embedded in $(\ell_\infty, \|\cdot\|_\infty)$. In spite of that, the metric condition of hyperconvexity has turned out to be an essential tool to ensure that, for some bounded subsets of $(\ell_\infty, \|\cdot\|_\infty)$, the FPP is still possible. Thus, a question is raised in a natural way: Assume that C is a bounded hyperconvex subset of $(\ell_\infty, \|\cdot\|_\infty)$, and therefore with the FPP. Does $(C, \|\cdot\|_\infty)$ have stability of the FPP?

At this stage two facts stand out:

(i) The first one is that hyperconvexity is an extremely tight property, which shows itself very unstable by renormings (it strongly depends on the shape of the closed balls). Indeed, fix some $n \in \mathbb{N}$, $n > 1$ and, for $1 < p < \infty$, we define the following equivalent norm on ℓ_∞:

$$\|x\|_{p,\infty} = \max\{\|P_n(x)\|_p, \|Q_n(x)\|_\infty\}$$

where $P_n(x) = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $Q_n(x) = (x_{n+1}, x_{n+2}, \ldots)$ for all $x = (x_n) \in \ell_\infty$. It can be easily checked that $\|x\|_\infty \leq \|x\|_{p,\infty} \leq n^{\frac{1}{p}} \|x\|_\infty$ for every $x \in \ell_\infty$. It is well known that $(\mathbb{R}^n, \|\cdot\|_p)$ is not hyperconvex (for $1 < p < \infty$) so neither is $(\ell_\infty, \|\cdot\|_{p,\infty})$. Thus, for every $\epsilon > 0$ we can find a norm $\|\cdot\|_{p,\infty}$ with $\|x\|_\infty \leq \|x\|_{p,\infty} \leq (1+\epsilon)\|x\|_\infty$ for all $x \in \ell_\infty$ and such that $(\ell_\infty, \|\cdot\|_{p,\infty})$ fails to be hyperconvex.

(ii) The second fact worth mentioning is that the stability of the FPP is not given for free, that is, some conditions over the equivalent metric must be imposed. For instance, suppose that we consider the subset W as in Sect. 3, that is

$$W = \{x = (x_n) \in \ell_\infty : 1 \geq x_1 \geq x_2 \geq \ldots \geq 0\},$$

for which we know that $(W, \|\cdot\|_\infty)$ satisfies the FPP. Given $x = (x_n) \in W$, we denote by $x_0 := \lim_n x_n$. Thus the mapping $T : W \to W$ given by

$$T(x_1, x_2, x_3, \ldots) = (1, x_1 - x_0, x_2 - x_0, x_3 - x_0, \ldots)$$

is well-defined and fixed point free. It turns out that $\|Tx - Ty\|_\infty \leq 2\| x - y\|_\infty$ for all $x, y \in W$ and the constant 2 cannot be relaxed. Let $w \in \ell_\infty^*$ be a Banach limit and define the equivalent norm on ℓ_∞ given by

$$|x|_w = \|x\|_\infty + |w(x)|$$

which yields $\|x\|_\infty \leq |x|_w \leq 2\|x\|_\infty$ for every $x \in \ell_\infty$. Taking into account that $Tx \in c_0$ for all $x \in W$ and that $w(x) = 0$ for $x \in c_0$, it is easy to check that $|Tx - Ty|_w \leq |x - y|_w$ for all $x, y \in W$ and, therefore, the FPP is not preserved for all equivalent norms on ℓ_∞ as it may be expected.

We next introduce a generalization of the relative uniform normal structure given in Definition 2.12 which will lead us to the fulfillment of the FPP and to obtain stability results for bounded hyperconvex metric spaces.

We will make use of the following notation. Given a metric space (M, d), $X \subset M$ and $r > 0$, we define

$$B\left[X, r\right] = \bigcap_{x \in X} B\left(x, r\right) = \left\{y \in M : X \subset B\left(y, r\right)\right\}.$$

Note that $y \in B[X, r]$ if and only if $X \subset B(y, r)$, $\operatorname{diam}(X \cap B[X, r]) \leq r$ for all $r > 0$ and $B\left[X, r\right] \in \mathcal{A}\left(M\right)$ for all $X \subseteq M$. With this notation in mind, UNS and RUNS can be rewritten as the following proposition states:

Proposition 4.1. 1. *(M, d) has c-UNS if for every $A \in \mathcal{A}\left(M\right)$ with finite and positive diameter we have*

$$A \cap B\left[A, c \cdot \operatorname{diam}\left(A\right)\right] \neq \emptyset.$$

2. *(M, d) has c-RUNS if for every $A \in \mathcal{A}\left(M\right)$ with finite and positive diameter we have*

$$B\left[A, c \cdot \operatorname{diam}\left(A\right)\right] \cap B\left[B\left[A, c \cdot \operatorname{diam}\left(A\right)\right], c \cdot \operatorname{diam}\left(A\right)\right] \neq \emptyset.$$

Definition 4.2. A metric space (M, d) is said to have (p, q)-RUNS for some $p > 0$ and $q \in (0, 1)$ if

$$B\left[A, p \cdot \operatorname{diam}\left(A\right)\right] \cap B\left[B\left[A, p \cdot \operatorname{diam}\left(A\right)\right], q \cdot \operatorname{diam}\left(A\right)\right] \neq \emptyset$$

for every $A \in \mathcal{A}\left(M\right)$ with finite and positive diameter.

Observe that when $p = q = c \in (0, 1)$ we have the c-RUNS and so (p, q)-RUNS provides a formal extension of both UNS and c-UNS.

To prove our fixed point results for metric spaces with (p, q)-RUNS, we will proceed with some technical lemmas first. Although some of our arguments follow standard procedures in the theory, see for instance [25] or [18, Section 5.4], we have included them on behalf of completeness. In the following lemmas, M is a metric space and $T : M \to M$ is a nonexpansive mapping. We will also make use of the next notation.

Definition 4.3. Let (M,d) be a metric space and let $T : M \to M$ be a mapping. Define

$$\mathcal{A}_T(M) = \{A \in \mathcal{A}(M),\ A \neq \emptyset,\ T(A) \subset A\}.$$

Given $X \subset M$, we define

$$\mathrm{co}\,(X) = \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}(M)}} A and \mathrm{co}_T\,(X) = \bigcap_{\substack{X \subset A \\ A \in \mathcal{A}_T(M)}} A,$$

as, respectively, the admissible and T-admissible hulls of X in M.

Since $\mathcal{A}\,(M)$ is closed under intersections, both hulls in the previous definition are elements of $\mathcal{A}\,(M)$. In fact, $\mathrm{co}_T(X) \in \mathcal{A}_T(M)$. It is also easy to check that $X \subset \mathrm{co}\,(X) \subset \mathrm{co}_T\,(X)$ and $B\,[X,r] = B\,[\mathrm{co}\,(X)\,,r]$.

Lemma 4.4. *Let $A \in \mathcal{A}_T\,(M)$ and let $s > 0$ such that $A \cap B\,[A,s] \neq \emptyset$. If we set*

$$\tilde{A} := co_T\,(A \cap B\,[A,s])\,,$$

it holds that $\mathrm{diam}(\tilde{A}) \leq s$.

Proof. The main purpose here is to prove $\tilde{A} \subset \tilde{A} \cap B[\tilde{A},s]$, which automatically implies $\mathrm{diam}(\tilde{A}) \leq s$. We will check that

(i) $A \cap B[A,s] \subset \tilde{A} \cap B[\tilde{A},s]$ and
(ii) $\tilde{A} \cap B[\tilde{A},s]$ is T-invariant.

Since $\tilde{A} \cap B[\tilde{A},s] \in \mathcal{A}\,(M)$, the conclusion follows.

(i) From the definition, $A \cap B\,[A,s] \subset \tilde{A}$. Since A is T-invariant and $A \cap B\,[A,s] \subset A$, we have $\tilde{A} \subset A$ and so $B\,[A,s] \subset B[\tilde{A},s]$. Thus, $A \cap B\,[A,s] \subset B[\tilde{A},s]$ and so

$$A \cap B\,[A,s] \subset \tilde{A} \cap B[\tilde{A},s].$$

(ii) Let $z \in \tilde{A} \cap B[\tilde{A},s]$. Since $\tilde{A}$ is T-invariant, we have that $Tz \in \tilde{A}$. It remains to show that $Tz \in B[\tilde{A},s]$ which is equivalent to the fact that $\tilde{A} \subset B\,(Tz,s)$. Therefore, it suffices to show that

$$\tilde{A} \cap B\,(Tz,s) \in \{L \in \mathcal{A}_T\,(M) : L \supset A \cap B\,[A,s]\}\,.$$

It is obvious that $\tilde{A} \cap B\,(Tz,s) \in \mathcal{A}\,(M)$. Since $B\,[A,s] \subset B[\tilde{A},s]$ and $Tz \in \tilde{A}$, given $x \in A \cap B\,[A,s]$, it follows that $d\,(x,Tz) \leq s$, which implies $A \cap B\,[A,s] \subset B\,(Tz,s)$ and

$$A \cap B\,[A,s] \subset \tilde{A} \cap B\,(Tz,s)\,.$$

Now, take $y \in \tilde{A} \cap B\,(Tz,s)$. Since $\tilde{A}$ is T-invariant, it follows that $Ty \in \tilde{A}$. Since T is nonexpansive, $y \in \tilde{A}$ and $z \in B[\tilde{A},s]$ we have

$$d\,(Ty,Tz) \leq d\,(y,z) \leq s,$$

which implies $Ty \in B\,(Tz,s)$. Thus, $\tilde{A} \cap B\,(Tz,s)$ is T-invariant.

From the above, we conclude that $\tilde{A} \cap B[\tilde{A}, s] \in \mathcal{A}_T(M)$ and contains $A \cap B[A, s]$. Thus, $\tilde{A} \subset \tilde{A} \cap B[\tilde{A}, s]$ and $\operatorname{diam}(\tilde{A}) \leq s$.

□

Lemma 4.5. *Let $\mathcal{A}(M)$ be compact and let $A \in \mathcal{A}_T(M)$. Then there exists $A_0 \subset A$ such that:*

(i) *$A_0 \in \mathcal{A}_T(M)$ and*
(ii) *$B[A_0, r] \in \mathcal{A}_T(M)$ whenever $B[A_0, r]$ is nonempty.*

Proof. Let $\mathcal{L}_A := \{L \in \mathcal{A}_T(M) : \ L \subset A\}$. Since $A \in \mathcal{A}_T(M)$ we have that $\mathcal{L}_A \neq \emptyset$. Using the compactness of $\mathcal{A}(M)$ and Zorn's lemma, we can find a minimal element A_0 of $\mathcal{L}_A$.

Since $A_0 \in \mathcal{A}_T(M)$ we have that $\operatorname{co}(T(A_0)) \subset A_0$, and so

$$T(\operatorname{co}(T(A_0))) \subset T(A_0) \subset \operatorname{co}(T(A_0)).$$

Therefore, $\operatorname{co}(T(A_0)) \in \mathcal{L}_A$. The minimality of A_0 implies that

$$A_0 = \operatorname{co}(T(A_0)).$$

Finally, given $r > 0$, the nonexpansivity of T leads us to

$$T(B[A_0, r]) \subset B[T(A_0), r] = B[\operatorname{co}(T(A_0)), r] = B[A_0, r],$$

which completes the proof of the lemma. □

Lemma 4.6. *Let $\mathcal{A}(M)$ be compact and $A \in \mathcal{A}_T(M)$. Let $A_0 \subset A$ as in Lemma 4.5. If $B[A_0, r] \cap B[B[A_0, r], s] \neq \emptyset$, then the set*

$$\tilde{A}_0 := co_T(B[A_0, r] \cap B[B[A_0, r], s])$$

satisfies:

(i) *$\tilde{A}_0$ is T-invariant.*
(ii) *$\tilde{A}_0 \subset B[A_0, r]$.*
(iii) *$\operatorname{diam}(\tilde{A}_0) \leq s$.*

Proof. Assertions i) and iii) easily follow from the definition of $\tilde{A}_0$ and Lemma 4.4. Assertion ii) follows from the definition of $\tilde{A}_0$ and from the T-invariance of $B[A_0, r]$ shown in Lemma 4.5. □

With the previous lemmas in hand we can now show the following theorem.

Theorem 4.7. *Let (M, d) be a bounded metric space with (p, q)-RUNS for some $0 < q < 1$ and such that $\mathcal{A}(M)$ is compact. Then (M, d) has the FPP.*

Proof. Assume that we construct a sequence of subsets (A_n) in $\mathcal{A}_T(M)$ fulfilling the following properties:

(1) $A_n \subset B[A_{n-1}, p \operatorname{diam}(A_{n-1})]$.
(2) $\operatorname{diam}(A_n) \leq q \operatorname{diam}(A_{n-1})$.

Under the above assumptions, take $x_n \in A_n$ for all $n \in \mathbb{N}$. Condition (2) and the fact that every A_n is a T-invariant set imply that (x_n) is an approximate fixed point sequence, that is, $\lim_n d(x_n, Tx_n) = 0$. Furthermore, we can deduce that $\text{diam}(A_n) \leq q^n \cdot \text{diam}(M)$ from (2). Condition (1) ensures that $d(x_n, x_{n-1}) \leq p \cdot \text{diam}(A_{n-1}) \leq p \cdot q^{n-1}\text{diam}(M)$, which implies that (x_n) is a Cauchy sequence. By completeness and continuity, the limit point of (x_n) is a fixed point for T.

Now we proceed by induction to construct a sequence $(A_n) \subset \mathcal{A}_T(M)$ satisfying (1) and (2). We will denote the $\text{diam}\,(A_n)$ and $\text{diam}\,(\tilde{A}_n)$ by δ_n and $\tilde{\delta}_n$ respectively.

Take $A_0 \in \mathcal{A}_T\,(M)$ resulting from making $A = M$ in Lemma 4.5. Since M has (p, q)-RUNS, we have that

$$B\,[A_0, p\;\delta_0] \cap B\,[B\,[A_0, p\;\delta_0]\,, q\;\delta_0] \neq \emptyset.$$

Thus, defining $\tilde{A}_0 = \text{co}_T\,(B\,[A_0, p\delta_0] \cap B\,[B\,[A_0, p\delta_0]\,, q\delta_0])$ as in Lemma 4.6, we have that

$$\tilde{A}_0 \in \mathcal{A}_T\,(M)\,, \tilde{A}_0 \subset B\,[A_0, p\;\delta_0]\, and \tilde{\delta}_0 \leq q\;\delta_0.$$

Now, taking $A = \tilde{A}_0$ and applying Lemma 4.5 again, there exists $A_1 \in \mathcal{A}_T\,(M)$ with $A_1 \subset \tilde{A}_0$. Hence

$$\delta_1 \leq q\;\delta_0 and A_1 \subset B\,[A_0, p\;\delta_0]\,.$$

Thus, defining $\tilde{A}_1 = \text{co}_T\,(B\,[A_1, p\;\delta_1] \cap B\,[B\,[A_1, p\;\delta_1]\,, q\;\delta_1])$ as above, we have that

$$\tilde{A}_1 \in \mathcal{A}_T\,(M)\,, \tilde{A}_1 \subset B\,[A_1, p\;\delta_1]\, and \tilde{\delta}_1 \leq q\;\delta_1.$$

Proceeding inductively in this way, we construct a sequence of sets (A_n) satisfying properties (1) and (2). This completes the proof. □

Next, we prove that, given a hyperconvex metric space (M, d) endowed with a further equivalent metric d_1, the new metric space (M, d_1) satisfies the (p, q)-RUNS when d and d_1 are close enough.

Proposition 4.8. *Let (M, d) be a hyperconvex metric space. Let d_1 be an equivalent metric such that*

$$a \cdot d\,(x, y) \leq d_1\,(x, y) \leq b \cdot d\,(x, y)\, for all x, y \in M.$$

Then, for any $L \subset M$ with $\delta_1 := \text{diam}_{d_1}(L) > 0$ we have

$$B_{d_1}\left[L, \frac{b}{2a}\;\delta_1\right] \cap B_{d_1}\left[B_{d_1}\left[L, \frac{b}{2a}\delta_1\right], \frac{b^2}{2a^2}\delta_1\right] \neq \emptyset.$$

Proof. Assume that we can ensure that the following intersection of balls for (M, d)

$$A := B_d\left[L, \frac{1}{2a}\delta_1\right] \cap B_d\left[B_{d_1}\left[L, \frac{b}{2a}\delta_1\right], \frac{b}{2a^2}\delta_1\right]$$

is nonempty. Taking into account that $B_{d_1}\,(x, a\,r) \subset B_d\,(x, r) \subset B_{d_1}\,(x, b\,r)$ for all $x \in M$ and $r > 0$ it follows that $B_{d_1}[P, ar] \subset B_d[P, r] \subset B_{d_1}[P, br]$ for all $P \subset M$. Thus, when $A \neq \emptyset$, the claim asserted in the statement of the proposition holds.

Since A is in fact an intersection of d-balls in M and (M, d) is hyperconvex, it is enough to check the hyperconvexity condition for the centers of all the d-balls involved. To do that, it suffices to consider the following three cases:

- Case $x, y \in L$. In this case we have that

$$d(x,y) \leq \frac{1}{a}\, d_1(x,y) \leq \frac{1}{a}\, \delta_1 = \frac{1}{2a}\, \delta_1 + \frac{1}{2a}\, \delta_1.$$

- Case $x \in L$ and $y \in B_{d_1}\left[L, \frac{b}{2a}\delta_1\right]$. In this case we have that

$$d(x,y) \leq \frac{1}{a}\, d_1(x,y) \leq \frac{b}{2a^2}\, \delta_1 \leq \frac{b}{2a^2}\, \delta_1 + \frac{1}{2a}\, \delta_1.$$

- Case $x, y \in B_{d_1}\left[L, \frac{b}{2a}\delta_1\right]$. In this case we have that for any $z \in L$,

$$d(x,y) \leq \frac{1}{a}\, d_1(x,y) \leq \frac{1}{a}\, d_1(x,z) + \frac{1}{a}\, d_1(z,y) \leq \frac{b}{2a^2}\, \delta_1 + \frac{b}{2a^2}\, \delta_1.$$

Thus, from the hyperconvexity of (M, d), it follows that $A \neq \emptyset$ and the proof is finished. □

Theorem 4.9. *Let (M, d) be a hyperconvex metric space. Let d_1 be an equivalent metric such that*

$$a \cdot d(x,y) \leq d_1(x,y) \leq b \cdot d(x,y)\, forall x, y \in M. \tag{1}$$

The metric space (M, d_1) has $\left(\frac{b}{2a}, \frac{b^2}{2a^2}\right)$-RUNS when $\frac{b}{a} < \sqrt{2}$.

Proof. If follows just by taking $L \in \mathcal{A}_{d_1}(M)$ in Proposition 4.8 □

As a corollary, we obtain the following stability result for the FPP.

Corollary 4.10. *Let (M, d) be a bounded hyperconvex metric space. Let d_1 be an equivalent metric such that $a \cdot d(x,y) \leq d_1(x,y) \leq b \cdot d(x,y)$ for all $x, y \in M$ and $\frac{b}{a} < \sqrt{2}$. If the family $\mathcal{A}_{d_1}(M)$ is compact then the metric space (M, d_1) has the FPP.*

Proof. It is a direct consequence of Theorem 4.7 and Theorem 4.9. □

Although the result in Corollary 4.10 requires the assumption of $\mathcal{A}_{d_1}(M)$ being compact, we next provide a plethora of examples where this condition comes for free.

Consider the Banach space $X = (\ell_\infty(I), \|\cdot\|_\infty)$ where I is any nonempty index set (in fact every metric space can be embedded isometrically in $\ell_\infty(I)$ for some I using the Kuratowski embedding). Let τ be the product topology on $\ell_\infty(I)$, which is the topology of the coordinatewise convergence. From Tychonoff's theorem, we know that every τ-closed bounded set is τ-compact.

Assume that M is a τ-closed bounded subset of $\ell_\infty(I)$ and let d_1 be any equivalent metric on M such that its closed balls are τ-closed. Then, the family $\mathcal{A}_{d_1}(M)$ is compact. Thus, we can assure that $M = \prod_{i\in I}[a_i, b_i]$ for $a_i \le b_i$, being $(a_i)_{i\in I}$, $(b_i)_{i\in I}$ bounded sets in $\mathbb{R}$, have the FPP for all metrics on M such that the balls are coordinatewise closed and verifying inequalities (1) for $\frac{b}{a} < \sqrt{2}$.

If we particularize to the case $I = \mathbb{N}$, the coordinatewise topology on ℓ_∞ coincides with the $\sigma(\ell_\infty, \ell_1)$-topology for bounded sets of ℓ_∞. In case that a metric on ℓ_∞ comes from an equivalent norm $|\cdot|$, it is also worth noting that the the closed balls for the $|\cdot|$ norm are $\sigma(\ell_\infty, \ell_1)$-closed if and only if $|\cdot|$ is a dual norm, that is, there exists an equivalent norm $|\cdot|_1$ on ℓ_1 for which the dual $(\ell_1, |\cdot|_1)^*$ is isometric to $(\ell_\infty, |\cdot|)$ (see for instance [13, Lemma 8.8]). Thus we can state:

Corollary 4.11. *Let M be a bounded subset of ℓ_∞. Assume that M is coordinatewise closed and $(M, \|\cdot\|_\infty)$ is hyperconvex. Let $\|\cdot\|$ be a dual norm on ℓ_∞ such that*

$$\|x\|_\infty \le \|x\| \le b\,\|x\|_\infty \, forall x \in M.$$

Then $(M, \|\cdot\|)$ has the FPP if $b < \sqrt{2}$.

We next apply the above stability result to deduce the fulfillment of the FPP for some closed convex bounded subsets of ℓ_∞ endowed with some Nakano-type norm.

We recall that if (p_n) is a sequence in $[1, +\infty)$, the modular function $\rho(x) := \sum_{n=1}^{\infty} |x_n|^{p_n}$ for $x = (x_n)$ gives place to the sequence Nakano space, which is defined by $\ell_{p_n} = \left\{ x = (x_n) : \rho\left(\frac{x}{\lambda}\right) < +\infty, \text{for some } \lambda > 0 \right\}$. This space is endowed with Luxemburg norm defined by

$$\|x\|_{p_n} = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1 \right\}.$$

Observe that $(\ell_{p_n}, \|\cdot\|_{p_n})$ becomes the sequence Banach space $(\ell_p, \|\cdot\|_p)$ if (p_n) is the constant sequence $p_n = p$ for every $n \in \mathbb{N}$ (see for instance [22]).

In case that the sequence (p_n) is unbounded, $\left(\ell_{p_n}, \|\cdot\|_{p_n}\right)$ contains an isometric copy of ℓ_∞ so, once more, every failure of the FPP has its own isometric reflection in $\left(\ell_{p_n}, \|\cdot\|_{p_n}\right)$ (see for instance [9, Theorem 2.3]). We will next prove that, under some additional restriction over the divergence of the sequence (p_n), we can still prove that there are some closed convex bounded sets with the FPP for the Luxemburg norm.

Example. Let (p_n) be a sequence in $[1, +\infty)$ such that there exists $b \in \left(1, \sqrt{2}\right)$ for which $\sum_{n=1}^{\infty} \left(\frac{1}{b}\right)^{p_n} \le 1$. Let M be a bounded subset of ℓ_∞ with $(M, \|\cdot\|_\infty)$ hyperconvex. Then $\left(M, \|\cdot\|_{p_n}\right)$ has the FPP.

Proof. Let $x=(x_n)\in\ell_\infty$. Under the previous assumptions

$$\sum_{n=1}^{\infty}\left|\frac{x_n}{b\left\|x\right\|_\infty}\right|^{p_n}\leq\sum_{n=1}^{\infty}\left(\frac{1}{b}\right)^{p_n}\leq 1,$$

and the identity map is an isomorphism between ℓ_∞ and ℓ_{p_n}, for which $\left\|x\right\|_\infty\leq\left\|x\right\|_{p_n}\leq r\left\|x\right\|_\infty$ for all $x\in\ell_\infty$. Furthermore, the closed unit balls for the norm $\left\|\cdot\right\|_{p_n}$ are $\sigma\left(\ell_\infty,\ell_1\right)$-closed, since $\left\|\cdot\right\|_{p_n}$ is the dual norm of the corresponding Nakano space ℓ_{q_n} with $\dfrac{1}{p_n}+\dfrac{1}{q_n}=1$ endowed with the Orlicz norm. Applying Corollary 4.11 we obtain the result. □

From Corollary 4.11 we can go further in the study of the FPP for bounded non-weakly compact subsets of c. For example, if we consider the original example from [14]

$$W=\left\{x=(x_n)\in\ell_\infty:1\geq x_1\geq x_2\geq\ldots\geq 0\right\},$$

we now know that, given a metric d equivalent to $\left\|\cdot\right\|_\infty$ such that $\delta\left(\left\|\cdot\right\|_\infty,d\right)<\sqrt{2}$ and for which the closed d-balls are coordinatewise closed in ℓ_∞, then (W,d) has the FPP. In particular, $\left(W,\left\|\cdot\right\|_{p_n}\right)$ has the FPP where (p_n) satisfies the assumptions of Example 4.

5. Some comments and remarks

We finish the paper by bringing forward some questions raised in the course of the elaboration of this article:

1. As we already mentioned, it was proved in [14] that, unlike the c_0-case, weak compactness does not determine those closed convex subsets of the Banach space c satisfying the FPP for nonexpansive mappings (see also [16]). We wonder whether there is either a topological or geometric property that could lead to a characterization of those closed convex subsets of c with the FPP for nonexpansive mappings.
2. Related to the above question, we wonder whether there is a convex closed subset C of c with the FPP for nonexpansive mappings, which additionally fails to be closed for the coordinatewise topology. In the particular case that C is a subset of c_0 with the FPP, we know that C is weakly compact and therefore closed coordinatewise. If C is a convex weakly compact subset of c, it has the FPP but it is also compact for the coordinatewise topology. The examples of non-weakly compact closed convex subsets of c with the FPP given in [14] and in Sect. 3 of this article are all coordinatewise compact.
3. The problem of existence of fixed points for k-uniformly Lipschitzian mappings (that is, $d(T^nx,T^ny)\leq kd(x,y)$ for any x and y in M and $n\in\mathbb{N}$) has been studied by many authors and, in particular, it remains open if such mappings have a fixed point when defined from a bounded hyperconvex space into itself. The conjecture claims that the most plausible highest value for k to obtain a fixed point is $\sqrt{2}$. The

interested reader may check, for instance, [26]. Corollary 4.10 provides a new approach to this problem. Indeed, consider a bounded hyperconvex metric space (M, d) and a k-uniform Lipschitzian mapping $T : M \to M$. Define

$$d_1 (x, y) = \sup \{d (T^n x, T^n y) : n \geq 0\}.$$

Then d_1 is a metric on M that satisfies

$$d (x, y) \leq d_1 (x, y) \leq kd (x, y) forall x, y \in M.$$

Therefore, if $k < \sqrt{2}$, Corollary 4.10 implies that (M, d_1) has the FPP as long as the family of d_1-admissible subsets of M is compact. Now, since T is nonexpansive for d_1, it would have a fixed point.

4. It is not clear for the authors whether the hypothesis of the compactness of $\mathcal{A}_{d_1}(M)$ in Corollary 4.10 can be dropped. Likewise, whether or not the thesis of Corollary 4.11 would hold without the assumption of $\|\cdot\|$ being an equivalent dual norm. It is worth noting that this observation is strictly related to the above remark. Indeed, if for every $b > 1$ we could find a dual norm $\|\cdot\|$ on ℓ_∞ with $\|x\|_\infty \leq \|x\| \leq b\|x\|_\infty$ for every $x \in \ell_\infty$ and for which the set $[-1, 1]^{\mathbb{N}}$ fails to have the FPP for $\|\cdot\|$-nonexpansive mappings, we would automatically have a fixed point free mapping $T : [-1, 1]^{\mathbb{N}} \to [-1, 1]^{\mathbb{N}}$ which is b-uniformly Lipschitzian for the $\|\cdot\|_\infty$ norm, showing a counterexample to the conjecture that every hyperconvex bounded metric space has the FPP for k-uniformly Lipschitzian mappings with $k < \sqrt{2}$ [26].

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References

[1] Amir, D.: On Jung's constant and related constants in normed linear spaces. Pac. J. Math. **118**(1), 1–15 (1985)

[2] Aronszajn, N., Panitchpakdi, P.: Extensions of uniformly continuous transformations and hyperconvex metric spaces. Pac. J. Math. **6**, 405–439 (1956)

[3] Baillon, J.B.: Nonexpansive mappings and hyperconvex spaces. Contemp. Math. **72**, 11–19 (1988)

[4] Betiuk, A., Domínguez-Benavides, T., Japón, M.: Existence of fixed points in a class of convex sets. Z. Anal. Anwed. **38**(3), 351–374 (2019)

[5] Borwein, J., Sims, B.: Nonexpansive mappings on Banach lattices and related topics. Houston J. Math **10**(3), 339–356 (1984)

[6] Budzyńska, Monika, Kuczumow, Tadeusz, Reich, Simeon: Uniform asymptotic normal structure, the uniform semi-Opial property and fixed points of asymptotically regular uniformly Lipschitzian semigroups. I. Abstr. Appl. Anal. **3**(1–2), 133–151 (1998)

[7] Budzyńska, Monika, Kuczumow, Tadeusz, Reich, Simeon: Uniform asymptotic normal structure, the uniform semi-Opial property, and fixed points of asymptotically regular uniformly Lipschitzian semigroups. II. Abstr. Appl. Anal. **3**(3–4), 247–263 (1998)

[8] Domínguez-Benavides, T.: The failure of the fixed point property for unbounded sets in c_0. Proc. Am. Math. Soc. **140**(2), 645–650 (2012)

[9] Domínguez-Benavides, T., Japón, M.: Fixed point properties and reflexivity in variable Lebesgue spaces. J. Funct. Anal. **280**(6), 1–22 (2021)

[10] Dowling, P.N., Lennard, C.J., Turett, B.: Characterizations of weakly compact sets and new fixed point free maps in c_0. Studia Math. **154**(3), 277–293 (2003)

[11] Dowling, P.N., Lennard, C.J., Turett, B.: Weak compactness is equivalent to the fixed point property in c_0. Proc. Am. Math. Soc. **132**(6), 1659–1666 (2004)

[12] Espínola, R., Khamsi, M.A.: Introduction to hyperconvex spaces. In: Kirk, A., Sims, B. (eds.) Handbook of Metric Fixed Point Theory, pp. 391–435. Kluwer Academic Publishers, Dordrecht (2001)

[13] Fabian, M., Habala, P., Hájek, P., Montesinos, V., Pelant, J., Zizler, V.: Functional analysis and infinite-dimensional geometry. CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC, vol. 8. Springer-Verlag, New York (2001)

[14] Gallagher, T., Lennard, C., Popescu, R.: Weak compactness is not equivalent to the fixed point property in c. J. Math. Anal. Appl. **431**(1), 471–481 (2015)

[15] Goebel, K., Kirk, W.A.: *Topics in Metric Fixed Point Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press (1990)

[16] Japón, M.: Fixed-point characterizations of weak compactness for closed convex subsets of c. J. Nonlinear Convex Anal. **18**, 241–249 (2017)

[17] Khamsi, M., Kirk, W.: *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley Sons (2001)

[18] Khamsi, M.A.: On metric spaces with uniform normal structure. Proc. Am. Math. Soc. **106**, 641–644 (1989)

[19] Khamsi, M.A., Reich, S.: Nonexpansive mappings and semigroups in hyperconvex spaces. Math. Jpn. **35**(3), 467–471 (1990)

[20] Kirk, W., Sims, B. (eds.): Handbook of Metric Fixed Point Theory. Springer, Dordrecht (2001)

[21] Kirk, W.A.: Nonexpansive mappings in metric and Banach spaces. Rend. Sem. Mat. Fis. Milano **51**(133–144), 1981 (1983)

[22] Nakano, H.: Modulared sequence spaces. Proc. Jpn. Acad. **27**, 508–512 (1951)
[23] Reich, Simeon: Research Problems: The fixed point property for non-expansive mappings. Am. Math. Mon. **83**(4), 266–268 (1976)
[24] Reich, Simeon: Research Problems: The fixed point property for non-expansive mappings. II. Am. Math. Mon. **87**(4), 292–294 (1980)
[25] Soardi, P.: Existence of fixed points of nonexpansive mappings in certain Banach lattices. Proc. Am. Math. Soc. **73**, 25–29 (1979)
[26] Wiśnicki, A., Wośko, J.: Uniformly Lipschitzian group actions on hyperconvex spaces. Proc. Am. Math. Soc. **144**(9), 3813–3824 (2016)

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