Möbius Transformations

Bachelor Thesis

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Abstract

In this thesis the Möbius transformations are studied along with two of its applications. Firstly, Möbius transformations are characterized as transformations of the extended complex plane to itself $M : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$. Some of their geometric properties, as preservation of generalized circles and angles are studied. Their fixed points are then reviewed, and with the help of the concept of cross-ratio, a way to find Möbius transformations with specific behaviour is found. Lastly, the set of all Möbius transformations $\mathcal{M}$ is studied as a group isomorphic to $PSL(2, \mathbb{C})$, and a classification into four categories is made.

In the second part, their role in the Poincaré half-plane model of hyperbolic geometry is studied. Such model is briefly introduced, defining the concepts of hyperbolic length and distance in the half-plane, as well as lines and geodesics. Using the geometric properties of Möbius transformations, the lines of the model are derived.

Lastly, the connection of the Möbius group and the Lorentz transformations is reviewed. The foundamentals of special relativity are briefly introduced. Then, the isomorphism between the Möbius group and a subset of the Lorentz gorup, the restricted Lorentz group, is fully reviewed. This will allow the study of transformations of the celestial sphere using Möbius transformations.

A todos los que me han aguantado estos cuatro años.

Litany Against Fear

I must not fear.
Fear is the mind-killer.
Fear is the little-death that brings total obliteration.
I will face my fear.
I will permit it to pass over me and through me.
And when it has gone past I will turn the inner eye to see its path.
Where the fear has gone there will be nothing. Only I will remain.

Frank Herbert
Dune

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Chapter 0

Introduction

Möbius transformations are a set of transformations of the extended complex plane into itself $M : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ with a series of beautiful properties and deep connections to both non-euclidean geometry and physics. These properties are in stark contrast with their simple appearance. The porpoise of this thesis is to give a general overview of the Möbius transformations, as well as a review of two of their applications.The thesis, then, is naturally divided in three sections: the first one covering their general properties and the other two reviewing their application in the Poincaré half-plane model of hyperbolic geometry and in the special theory of relativity.

The first part is mostly based on the third chapter of *Visual complex analysis* [5], a magnificent manual covering extensively the Möbius transformation. In this section we define Möbius transformations acting on the extended complex plane, which is introduced through the concept of the Riemann sphere, adding another point, the infinity ∞, to the complex plane. After this, Möbius transformations are proven to leave angles and circles invariant, properties that will be key in the next sections. Studying them as a group, we will prove the isomorphism between them and $PSL(2, \mathbb{C})$ and then we will classify them into four distinct categories.

In the second section we find our first example of the applicability of the Möbius transformations. Here we review the basics of the Poincaré half-plane model of hyperbolic geometry and the key role the Möbius transformations in the derivation of its lines. For this part, we have mainly used *Geometría Hiperbólica I. Movimientos rígidos y recetas hiperbólicas* [1], an article appearing in an Argentinian publication which covers precisely the topic. The Poincaré half-plane in this thesis is defined as the set of complex numbers with positive real part. The particular definitions of length and distance of the hyperbolic half-plane model allows us to define lines that are neither parallel nor secant. For their derivations we will use a specific subset of the Möbius transformations that preserve the half-plane

Last, but no least, the third part covers the unexpected link between the Möbius transformations and the special theory of relativity. The link between the Lorentz transformations, which relate different inertial systems of reference in this theory, and the Möbius transformations was studied by Roger Penrose in his book *Spinors and space-time*[8], co-authored by Wolfgang Ridler. This is the main reference we have used for this chapter. Here, after we introduce the basics of special relativity, we derive the isomorphism between both the Möbius and restricted Lorentz group, consisting the first on the set of all Möbius transformations and the second on a certain subset of the Lorentz transformations. This connection will allow us to divide the restricted Lorentz group into the four distinct categories we found in the first part.

Chapter 1

Meeting Möbius

1.1 Definition

Möbius transformations are defined as follows:

Definition 1.1.1. A *Möbius transformation* $M : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ is a map such as

$$M(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0$$

We impose $ac - bd \neq 0$ for them not to be singular. Here, $\mathbb{C}_\infty$ is the extended complex plane, an "enlarged" version of the complex plane whose concrete definition will be given soon.

It is easy to realize that more than one Möbius transformation yield the same mapping, as it can be easily seen if the coefficients a, b, c, d are multiplied by the same constant. To solve this, we introduce a *normalization*.

Definition 1.1.2. A *normalized Möbius transformation* $M : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ is a map such as

$$M(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc = 1.$$

Later we will prove that there is exactly one normalized Möbius transformation taking determined three points in $\mathbb{C}_\infty$ to other determined three points $\mathbb{C}_\infty$. *From now on, every time we refer to a Möbius transformation we are going to be refering to a normalized Möbius transformation unless explicitly stated.*

1.2 The Riemann sphere

The Riemann sphere is intimately related to the stereographic projection, which is a mapping between a 2-sphere and a two-dimensional plane. For this context,

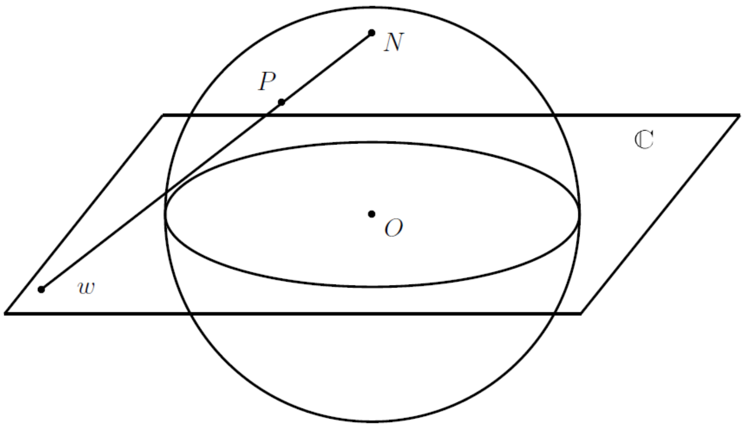

Figure 1.1: Riemann sphere and stereographic proyection [7]

such plane is going to be the complex plane.
Let S^2 be the unit 2-sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

centered in the origin of the complex plane and the unit circle in the complex plane coincides with its *equator*, i.e., $x^2 + y^2 = 1$ and $(1, 0, 0)$ with $z = 1$ and $(0, 1, 0)$ with $z = i$. From now on, we are going to call the points $(0, 0, 1)$ *north pole* or N and $(0, 0, -1)$ *south pole* or S. We are going to derive now a bijection between $\mathbb{C}$ and S^2.

Definition 1.2.1. The *stereographic projection*, denoted as SP, is the map

$$S^2 \setminus N \mapsto \mathbb{C}$$

so that assigns to every point $P \in S^2$ the point in C given by $\text{span}(NP) \cap \mathbb{C}$

Such map is represented in figure (1.1). It can intuitively be seen that such map is bijective, as it is equally possible for every point in $\mathbb{C}$ to draw a line from such point and N, which would intersect with exactly one point in S^2.
A set of formulae can be derived for the stereographic projection and its inverse.

Theorem 1.2.1. *For every $P = (x, y, z) \in S^2$ and $w = u + iv \in \mathbb{C}$, the stereographic projection SP has the form*

$$SP(P) = \frac{x}{1-z} + i\frac{y}{1-z}, \tag{1.1}$$

and the inverse transformation SP^{-1}

$$SP^{-1}(w) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \tag{1.2}$$

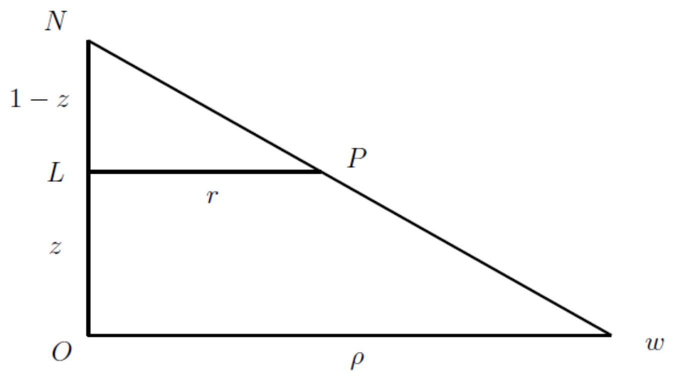

Figure 1.2: Vertical section of figure 1.1. Here $\rho = u^2 + v^2$ and $r = x^2 + y^2$ [7]

Proof. Let $P = (x, y, z) \in S^2$ a point in the sphere. consider a vertical section as seen in image 1.2. As the triangles NLP and $N0z$ are similar and their dimensions are the same as $N0u$, NLx are similar, then it follows:

$$\frac{x^2 + y^2}{|w|} = \frac{1 - z}{1} = \frac{x}{u} = \frac{y}{v}$$

Solving we get:

$$\frac{1 - z}{1} = \frac{x}{u} \Rightarrow u = \frac{x}{1 - z} \frac{1 - z}{1} = \frac{y}{v} \Rightarrow v = \frac{y}{1 - z}$$

So, we have

$$SP(x, y, z) = u + iv = \frac{x}{1 - z} + i\frac{y}{1 - z}.$$

Now, we are left to prove the inverse formula. Solving for $u + iv = SP(x, y, z)$, we have

$$u^2 + v^2 = \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 - z^2}{1 + z} = -1 + \frac{2}{1 - z} \Rightarrow 1 - z = \frac{2}{u^2 + v^2 + 1} \Rightarrow z = \frac{u^2 + v^2 + 1}{u^2 + v^2 - 1}$$

$$u + iv = \frac{x + iy}{1 - z} \Rightarrow x + iy = \frac{2u + i2v}{u^2 + v^2 + 1}$$

Identifying real and complex part:

$$x = \frac{2u}{u^2 + v^2 + 1} \quad y = \frac{2v}{u^2 + v^2 + 1}$$

□

We have been considering the set $S^2 \setminus N$ for these formulae. However, it feels natural to assign another complex number to N. Such number is going to be ∞, as points in S^2 increasingly closer to N correspond to complex increasingly larger in modulus. Even more, the identities $1/0 = \infty$ and $1/\infty = 0$ must be true for such point. This point allows us to expand the deffinition of the stereographic projection such

$$SP : S^2 \mapsto \mathbb{C}_\infty$$

with

$$SP(N) = \infty$$

Definition 1.2.2. The *extended complex plane* $\mathbb{C}_\infty$ is the set $\mathbb{C} \cup \infty$.

Thanks to this concept and the bijectivity of the stereographic projection, the next definition follows.

Definition 1.2.3. The identification between $\mathbb{C}_\infty$ and S^2, with ∞ corresponding to N and every other point in S^2 to its stereographic projection is known as the *Riemann Sphere.*

This representation of $\mathbb{C}_\infty$ allows us to study the behaviour of functions ant ∞, as well as introducing the concept of angle at which two curves intersect at ∞.

It is important to note the following properties of the stereographic projection

Proposition 1.2.1. The stereographic projection is conformal, i.e. it preserves angles.

Remark. It is important to note that here we are defining angles as seen by an observer "inside" the sphere. If we "measure" angles from outside the sphere, then it would give anti-conformal, i.e. angles would change their sign.

Proof. As we can see in figure 1.3, tangent lines of curves in $\mathbb{C}$ are parallel to those in the sphere, thus the angles are preserved. □

Proposition 1.2.2. The stereographic projection preserves generalized circles, i.e. circles and lines.

Proof. Geometrically, we can see that lines in C correspond to the intersection of the plane that contains such line and N and the sphere, which clearly are circles. For circles, the proof is less straightforward. Let $P = (x, y, t)$ such that $SP(P) = w$. It can be proven that the general equation for a circle in the complex plane is

$$Aw\overline{w} + Bw + \overline{B}\overline{w} + C = 0, \quad A, C \in \mathbb{R}; B \in \mathbb{C}. \tag{1.3}$$

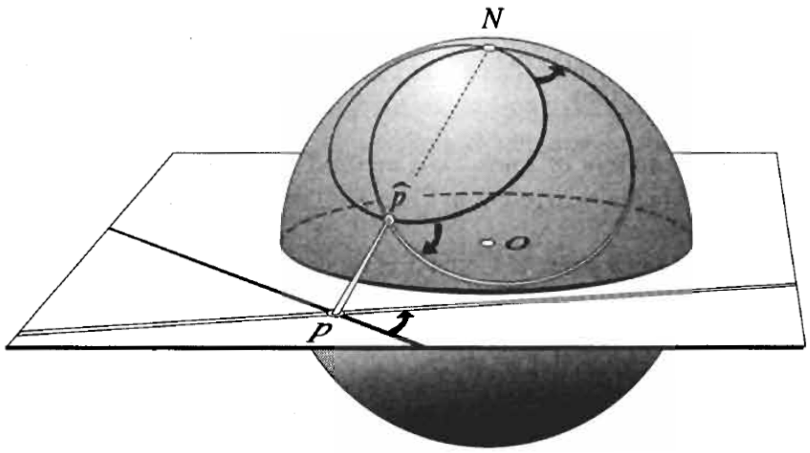

Figure 1.3: Preservation of angles on the Riemann sphere. [5]

A circle in the sphere is the intersection between a plane and the sphere, which means that its points must fulfill simultaneously

$$\begin{cases} \alpha x + \beta y + \gamma z + \delta = 0 \\ x^2 + y^2 + z^2 \qquad = 1 \end{cases}$$

Considering equation (1.1), we have

$$w\overline{w} = \frac{x^2+y^2}{(1-z^2)} = \frac{1-z^2}{(1-z^2)} = \frac{1+z}{1-z} = \frac{-1}{\gamma}\frac{1}{1-z}(1+\alpha x+\beta y+\delta)$$

As

$$\frac{1}{2}(\alpha - i\beta)(x+iy) + \frac{1}{2}(\alpha+i\beta)(x-iy) = \alpha x + \beta y$$

then

$$w\overline{w} = \frac{-1}{\gamma}\frac{1}{1-z}\frac{1}{2}[(\alpha - i\beta)(x+iy) + \frac{1}{2}(\alpha+i\beta)(x-iy) + \delta + 1] \Rightarrow$$

$$Aw\overline{w} + Bw + \overline{B}\overline{w} + C = 0$$

With $A = \gamma$, $B = (\alpha - \beta)/2$ and $C = \delta + 1$ □

1.3 Basic results

Now, a series of basic ressults of Möbius transformations will be discussed.

Theorem 1.3.1. *Any Möbius transformation is the composition of:*

- $T_1(z) = z + \frac{d}{c}$, *a translation,*

- $I(z) = \frac{1}{z}$, *a complex inversion,*
- $R(z) = -\frac{ad - cd}{c^2} z$, *an expansion-rotation,*
- $T_2(z) = z + \frac{a}{c}$, *another translation.*

Proof. Immediate □

Theorem 1.3.2. *Möbius transformations map* $\mathbb{C}_\infty$ *one-to-one onto itself and are continuous. For every Möbus transformation* M *there is an inverse Möbius transformation* M^{-1}*, which is*

$$M^{-1}(w) = \frac{-dw + b}{cw - a} \tag{1.4}$$

Proof. Let $M(z) = \frac{az + b}{cz + d}$ be a Möbius transformation. For $z \neq -\frac{d}{c}, \infty$ it is easy to see that $M(z)$ is continuous, as both $f(z) = az + b$ and $g(z) = cz + d$ are. For $z = d/c$, $|M(z)|$ goes to infinity as we get closer to $z = d/c$, and $M(d/c) = \infty$, so it is also continuous there. Also for z of increasing modulus it is also easy to notice that $M(z)$ goes to a/c, which makes $M(z)$ continious in all its domain. If we equate $M(z) = w$, then

$$w = \frac{az - b}{cz - d} \Rightarrow z(cw - a) = -wd + b \Rightarrow z = \frac{-wd + b}{cw - a}$$

As such inverse exist, and it is well defined, the map is one-to-one onto itself and continuous □

One of the most important features of Möbius transformations folowws.

Theorem 1.3.3. *Möbius transformations map generalized circles into generalized circles.*

Proof. The formula for generalized circles in $\mathbb{C}$ is

$$Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0, \quad A, C \in \mathbb{R}; B \in \mathbb{C}$$

As a Möbius transformations can be expressed as the simpler transformations described in theorem 1.3.1, it is enough to proof that for the rotation-expansion-translation $R(z) = \alpha z + \beta, \alpha,\ \beta \in \mathbb{C}$ and the inversion $I(z) = 1/z$, we can get

the same equation.
For $w = R(z)$, we have

$$Az\overline{z}+Bz+\overline{B}\overline{z}+C = 0 \Rightarrow A\left(\frac{w-\beta}{\alpha}\right)\left(\frac{\overline{w}-\overline{\beta}}{\overline{\alpha}}\right)+B\left(\frac{w-\beta}{\alpha}+\overline{B}\right)\left(\frac{\overline{w}-\overline{\beta}}{\overline{\alpha}}\right)+C = 0$$

$$\Rightarrow \frac{A}{\alpha\overline{\alpha}}w\overline{w} + \frac{B}{\alpha}w + \frac{\overline{B}}{\overline{\alpha}}\overline{w} + C - \left(\frac{B\beta}{\alpha} + \frac{\overline{B\beta}}{\overline{\alpha}}\right) = A'w\overline{w} + B'w + \overline{B'}\overline{w} + C' = 0$$

with $A' = A/(\alpha\overline{\alpha})$, $B' = B/\alpha$ and $C' = C - B\beta/\alpha - \overline{B}\overline{\beta/\overline{\alpha}}$.
For $w = I(z)$, then

$$A\frac{1}{w\overline{w}} + B\frac{1}{w} + \overline{B}\frac{1}{\overline{w}} + C = 0 \Rightarrow A + B\overline{w} + \overline{B}w + Cw\overline{w} = 0$$
$$\Rightarrow A'w\overline{w} + B'w + \overline{B'}\overline{w} + C' = 0$$

with $A' = C$, $B' = \overline{B}$ and $C' = A$. □

This result is key to understand the geometrical significance of Möbius transformations. First, as the stereographical projection also maps circles to circles, then *Möbius transformations preserve circles in the Riemann sphere.* Actually, if Möbius transformations are conformal, then they also preserve angles in the Riemann sphere. In fact, we are going to prove that they are:

Theorem 1.3.4. *Möbius transformations are conformal.*

Proof. Let $\gamma : (a_1, c_1) \mapsto \mathbb{C}_\infty$, $\epsilon : (a_2, c_2) \mapsto \mathbb{C}_\infty$ be curves so that $\gamma(b_1) = \epsilon(b_2)$ with $a_1 < b_1 < c_1$ and $a_2 < b_2 < c_2$ in the extended complex plane. Möbius transformations are conformal if they preserve the quotient $\dfrac{\gamma'(b_1)}{\epsilon'(b_2)}$. Again, we are going to prove this for an inversion and a rotation-expansion-traslation.

For $M[\gamma(t_1)] = \alpha\gamma(t_1) + \beta$, then $M'[\gamma(t_1)] = \alpha\gamma(t_1)$, so

$$\frac{M'[\gamma(b_1)]}{M'[\epsilon(b_2)]} = \frac{\alpha\gamma'(b_1)}{\alpha\epsilon'(b_2))} = \frac{\gamma'(b_1)}{\epsilon'(b_2)}$$

For $I(z) = \dfrac{1}{z}$, then $I'(\gamma(t_1) = -\dfrac{\gamma'(t_1)}{\gamma^2(t_1)}$, so

$$\frac{I[\gamma(b_1)]}{I'[\epsilon(b_2)]} = \frac{-\gamma'(b_1)\epsilon^2(b_2)}{-\epsilon'(b_2)\gamma^2(b_1} = \frac{\gamma'(b_1))}{\epsilon'(b_2))}$$

□

These basic propierties are going to be useful to apply Möbius transformations in different fields, as we will do in following sections. Now we are going to find a way to characterize each transformation by how they transform a set of three points.

1.4 Fixed points and the cross-ratio

Definition 1.4.1. Let $z, q, r, s \in \mathbb{C}_\infty$. The cross ratio of these numbers, expressed as $[z, q, r, s]$, is the quantity

$$[z,q,r,s] := \begin{cases} \dfrac{z-q}{z-s}\dfrac{r-s}{r-q}, & \text{if } z,q,r,s \neq \infty; \\ \dfrac{r-s}{r-q}, & \text{if } z=\infty;\ q,r,s \neq \infty; \\ \dfrac{r-s}{z-s}, & \text{if } q=\infty; z,r,s \neq \infty; \\ \dfrac{z-q}{z-s}, & \text{if } r=\infty; z,q,s \neq \infty; \\ \dfrac{z-q}{r-q}, & \text{if } s=\infty; z,q,r \neq \infty; \end{cases} \tag{1.5}$$

This quantity will come handy to find Möbius transformations sending a set of points to another set of points in $\mathbb{C}_\infty$. First, we will take notice of the next propierty.

Proposition 1.4.1. The Möbius transformation

$$M(z) = [z, q, r, s]$$

maps q to 0, s to ∞ and r to 1

Proof. Immediate □

Remark. It is important to note that this Möbius transformation is, in general, not normalized, so we would need to multiply all coefficients by a normalization constant.

Another interesting result is:

Proposition 1.4.2. Möbius transformations leave cross ratios invariant, i.e

$$[z, q, r, s] = [M(z), M(q), M(r), M(s)]$$

Proof. This can be proven by simple algebra. □

Let's now consider the points that a Möbius transformation leaves invariant, i.e., its fixed points.

Theorem 1.4.1. *A Möbius transformation* $M(z) = \dfrac{az+b}{cz+d}$ *that is not the identity has at most two fixed points, which are*

$$\xi^{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

Proof. Fix points fullfil $M(\xi) = \xi$. Then

$$\begin{aligned}
\xi = \frac{a\xi + b}{c\xi + d} &\Rightarrow c\xi^2 + (d-a)\xi - b = 0 \\
&\Rightarrow \xi^{\pm} = \frac{(a-d) \pm \sqrt{(a-d)^2 - 4cd}}{2c} = \frac{(a-d) \pm \sqrt{a^2 + d^2 - 2ad + 4cd}}{2c} \\
&= \frac{(a-d) \pm \sqrt{a^2 + d^2 + 2ad - 4ad + 4cd}}{2c} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}
\end{aligned}$$

□

From this, we can establish a way to fully characterize a Möbius transformation

Theorem 1.4.2. *There is exactly one Möbius transformation* $M(z) = w$ *transforming three determined points into three other determined points, i.e. for* $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{C}_\infty$ *then* $M(z_1) = w_1, M(z_2) = w_2, M(z_3) = w_3$. *Such transformation is given by solving*

$$[w, w_1, w_2, w_3] = [z, z_1, z_2, z_3]$$

with $w = M(z)$

Proof. Let's consider the Möbius transformations $f(z) = [z, z_1, z_2, z_3]$ and $g(w) = [w, w_1, w_2, w_3]$. As $f(z_1) = g(w_1) = 0$, $f(z_2) = g(w_2) = 1$ and $f(z_3) = g(w_3) = \infty$, then the transformation $M(z_i) = g^{-1} \circ f(z_i) = w_i$ for $i = 1, 2, 3$.
To check its uniqueness, let's suppose there are two Möbius transformations $M_1(z)$ and $M_2(z)$ so that $M_1(z_i) = M_2(z_i) = w_i$. Then the transformation $M(z) = M_1^{-1} \circ M_2(z)$ has z_1, z_2, z_3 as three fixed points. As a non-identity transformation has at most two fixed points, then M is the identity and $M_1 = M_2$.

□

This property will allow us to find specific transformations that will send points to other points at our will.

Lemma 1.4.3. *A Möbius transformation with a fixed point at ∞ must have the form*

$$M(z) = Az + B, \quad A, B \in \mathbb{C},\ A \neq 0$$

Proof. As stated in theorem 1.4.1, ∞ is a fixed point if $c = 0$ for $M(z) = \dfrac{az+b}{cz+d}$. On the other hand, if ∞ is a fixed point, then $M(\infty) = a/c = \infty$, so $c = 0$. □

Actually, more restrictive statements can be made.

Lemma 1.4.4. *If a Möbius transformation has a fixed point at 0 and another at ∞ then it has the form*

$$M(z) = Az, \quad A \in \mathbb{C},\ A \neq 0.$$

If its only fixed point is ∞, then it is a translation

$$M(z) = z + B$$

Proof. According to lemma 1.4, if ∞ is a fixed point, then $M(z) = Az+B,\ A, B \in \mathbb{C},\ A \neq 0$. If 0 is a fixed point, then $M(0) = B = 0$. Then, $M(z) = Az$
If ∞ is the only fixed point and $A \neq 1$, then $z = B/(1 - A)$ would be a fixed point to, so $A = 1$. □

Let's now consider the cross ratio $[z, \xi^+, \xi^-, \infty]$, being $\xi^\pm$ the two fixed points of a Möbius transformation $M(z)$ with two distinct fixed points none of them being ∞. According to proposition 1.4.2, applying $M(z)$, we have:

$$\begin{aligned} [z, \xi^+, \xi^-, \infty] =& [w, \xi^+, \xi^-, a/c] \Rightarrow \frac{z-\xi^+}{z-\xi^-} = \frac{a/c - \xi^-}{a/c - \xi^+}\frac{z-\xi^+}{z-\xi^-} \\ &\Rightarrow \frac{z-\xi^+}{z-\xi^-} = \frac{\xi^- - a/c}{\xi^+ - a/c}\frac{z-\xi^+}{z-\xi^-} \end{aligned} \tag{1.6}$$

with $m = (\xi^- - a/c)/(\xi^+ - a/c)$ being the *multiplier*. This way of writing a Möbius transformation is called the *normal form*. This way of expressing a Möbius transformation is particularly visual, as successive applications of the same Möbius transformations can be seen as multiplying the ratio $(z - \xi^+)/(z - \xi^-)$ by m that many times. In fact, we will see that we can understand the fixed points as "sources" or "sinks", as multiple applications of $M(z)$ bring closer or take further the other points to them.

The cross ratio also allows us to determine whether four points lie in a circle:

Theorem 1.4.5. *Let $z, q, r, s \in \mathbb{C}_\infty$. All four points lie in a circle C if and only if*

$$\Im[z, q, r, s] = 0$$

Proof. As stated in proposition 1.4.1, the Möbius transformation $M(z) = [z, q, r, s]$ sends q to 0, r to 1 and s to ∞. According to theorem 1.3.3, Möbius transformations preserve generalised circles. As three points determine a circle in $\mathbb{C}_\infty$, then M sends the circle C that contains q, r, s to the real axis. Consequently, if $z \in C$, it is mapped to the real axis, so $\Im[z, q, r, s] = 0$.
On the other hand, if $\Im[z, q, r, s] = 0$, that means that $\Im M(z) = 0$, with $M(z) = [z, q, r, s]$. As Möbius transformations preserve circles, and the circle C containing q, r, s is mapped to the real axis, then $\Im[z, q, r, s] = 0$ only if z is in such circle. □

Let $\mathcal{O} = \{q, r, s\}$ with $q, r, s \in \mathbb{C}$ be a set of three ordered points.Then we can assign a positive orientation to $\mathcal{O}$, which consists on going over a circle from q to s passing through r. This divides the extended complex plane into three regions:

- The circle containing such points C.
- The region *to the left* defined as

$$L(C) = \{z \in \mathbb{C}_\infty \mid \Im[z_1, q, r, s] > 0\}$$

- The region left *to the right.*

$$R(C) = \{z \in \mathbb{C}_\infty \mid \Im[z_1, q, r, s] < 0\}$$

Regions to the left and to the right can easily be seen to be simply connected. This property allows us to determine an important geometric result.

Theorem 1.4.6. *Möbius transformations preserve orientation of circles, i.e. map left regions to left regions and right regions to right regions.*

Proof. It can be proven that Möbius transformations map connected regions of $\mathbb{C}_\infty$ into connected regions. Möbius transformations leave cross ratios invariant, so left regions are mapped to left regions and right regions to right regions.

□

1.5 Möbius transformations as matrices

Möbius transformations have deeper connections to complex algebra as it may seem. To start, we are going to prove that the set of all Möbius transformations has group structure and then we will proceed to identify it with a very important group of linear algebra.

Theorem 1.5.1. *The set of all Möbius transformations*

$$\mathcal{M} = \left\{ M(z) = \frac{az+b}{cz+d} \mid ac - bd = 1 \right\}$$

is a group under function composition. Such group is known as the Möbius group.

Proof. All group properties must be fulfilled.

Closure

For $M_1 = (a_1 z + b_1)/(c_1 z + d_1)$ and $M_2 = (a_2 z + b_2)/(c_2 z + d_2)$,

$$M_2 \circ M_1 = \frac{(a_2 a_1 + b_2 c_1)z + a_2 b_1 + b_2 d_1}{(c_2 a_1 + d_2 c_1)z + c_2 b_1 + d_2 d_1} \in \mathcal{M}$$

which can easily be proven to be normalized.

Associativity

As function composition is associative, it is fulfilled.

Identity element

The Möbius transformations $I(z) = z$ acts as the identity.

Inverse

We proved in thorem 1.3.2 that there exist an inverse for every Möbius transformation. □

This vision of the Möbius transformations allow us to formulate the next map.

Definition 1.5.1. The map $\Gamma : SL(2, \mathbb{C}) \mapsto \mathcal{M}$ acts on 2×2 with determinant 1 matrices as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

Remark. It is easy to check that, as the matrices have determinant one, these Möbius transformations are normalized. We could also define the map $\Gamma' : GL(2,\mathbb{C}) \mapsto \mathcal{M}$, where we would get not-normalized Möbius transformations.

Through this map, we are going to prove that we can work equivalently with Möbius transformations and members of $SL(2,\mathbb{C})$

Proposition 1.5.1. The map Γ is a surjective homeomorphism

Proof. To do this, first we prove that the dentity matrix corresponds to the Möbius transformation. In fact:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \frac{1z}{1} = z$$

Then the product of two matrices corresponds to the composition of the corresponding Möbius transformatios

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix} \mapsto \frac{(a_2a_1 + b_2c_1)z + a_2b_1 + b_2d_1}{(c_2a_1 + d_2c_1)z + c_2b_1 + d_2d_1}$$

And, lastly, that the inverse matrix is maped to the inverse Möbius transformation

$$\begin{pmatrix} -d & b \\ c & -a \end{pmatrix} \mapsto \frac{-dz + b}{cz - a}.$$

The fact that the homeomorpism is surjetive comes from the fact that exactly 2 matrix are mapped to the same Möbius transformation

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d}. \tag{1.7}$$

□

Equation (1.7) allows us to determine a isomorphism of the Möbius transformations with a more restricted group.

Theorem 1.5.2. *The Möbius group and the projective special linear group,* $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I$ *are isomorphic, i.e.*

$$\mathcal{M} \cong PSL(2,\mathbb{C})$$

Proof. As every member in $PSL(2,\mathbb{C})$ corresponds to $\pm\mathbf{A}, \mathbf{A} \in SL(2,\mathbb{C})$, then every member of M corresponds to a member of $PSL(2,\mathbb{C})$ and vice-versa. □

As working with this group is a bit less intuitive, for the next discussions we are going to use the homeomorphism with $SL(2, \mathbb{C})$.

Actually, there is a way to characterize the action of Möbius transformations through the action of 2-column vectors.

Definition 1.5.2. The *homogeneous coordinates* of $z \in \mathbb{C}_\infty$ are is defined as a pair $(\delta, \eta) \in \mathbb{C}^2$ such that $z = \delta/\eta$.

Actually, $\lambda(\delta, \eta), \lambda \in \mathbb{C}^2$ determine the same complex number. To solve this, there exist the concept of *normalized homogeneous coordinates*, where λ is chosen so that $\delta^2 + \eta^2 = 1$. However, we are not going to use them, as it does not provide any advantage in this context

Proposition 1.5.2. The action of $PSL(2, \mathbb{C})$ on $\mathbb{C}^2$ is equivalent to the action of $\mathcal{M}$ on $\mathbb{C}$

Proof. If we consider a member of $\mathbb{C}^2$ as complex homogeneus coordinates of $z \in \mathbb{C}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix} = \begin{pmatrix} a\delta + b\eta \\ c\delta + d\eta \end{pmatrix} \mapsto \frac{a\delta + b\eta}{c\delta + d\eta} = \frac{a\delta/\eta + b}{c\delta\eta + d} = \frac{az + b}{cz + d} = M(z)$$

□

From this, we can establish another way of seeing fixed points.

Lemma 1.5.3. *Fix points of a Möbius transformation* $M(z) = \dfrac{az + b}{cz + d}$ *are given by the eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Proof. The eigenvectors of $\mathbf{A}$ are the vectors $\mathbf{v} = (\delta_1, \delta_2)^T, \delta_1, \delta_2 \in \mathbb{C}$ such as $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, for some $\lambda \in \mathbb{C}$. That corresponds to $M(z) = M(\delta_1/\delta_2) = \lambda\delta_1/(\lambda\delta_2) = z$ □

To find this eigenvectors, we have first to solve the characteristic equation $|\mathbf{A} - \lambda\mathbb{1}| = 0$. The eigenvalues will have a very relevant geometric role that we will discuss in next sections. Also, diagonalizing this matrices will help us classify Möbius transformations according to their effect on the Riemann sphere.

1.6 Classification

As we have previously stated, Möbius transformations have at most two fixed points. Let $F(z)$ be a Möbius transformations such

$$F(z) := \frac{z - \xi^+}{z - \xi^-}, \tag{1.8}$$

where ξ^+ and ξ^- are fix points of a Möbius transformation $M(z)$ with two distinct and finite fixed points. This transformation is, in general, not normalized. It is easy to see that $F(z)$ maps ξ^+ to 0 and ξ^- to ∞. Now, let's consider the transformation:

$$\tilde{M}(z) := F\{M[F^{-1}(z)]\} = F \circ M \circ F^{-1}(z). \tag{1.9}$$

It can be seen that in this new transformation, the fixed points are 0 and ∞. So as we stated in lemma 1.4.4, $\tilde{M}(z)$ must have the form:

$$\tilde{M}(z) = mz,$$

where $m = \rho e^{i\phi}$ is the multiplier that we derived in previous sections. We can see this as from (1.9) we can get to the normal form of the Möbius transform

$$F \circ M = \tilde{M} \circ F \Rightarrow \frac{w - \xi^+}{w - \xi^-} = m\frac{z - \xi^+}{z - \xi^-}$$

where $M(z) = w$. From this fact we can now classify Möbius Transformations according to the form of m.

1.6.1 Hyperbolic transformations

Hyperbolic transformations are characterized by

$$m = \rho \neq 1$$

If $\rho > 1$, applying $\tilde{M}$ in the Riemann sphere will cause flow lines to go perpendicular to its latitudes from S to N, whereas for $\rho < 1$ they flow from N to S. For $M(z)$, ξ^- will act as a sink, whereas ξ^+ will be a source, as shown in figure 1.4

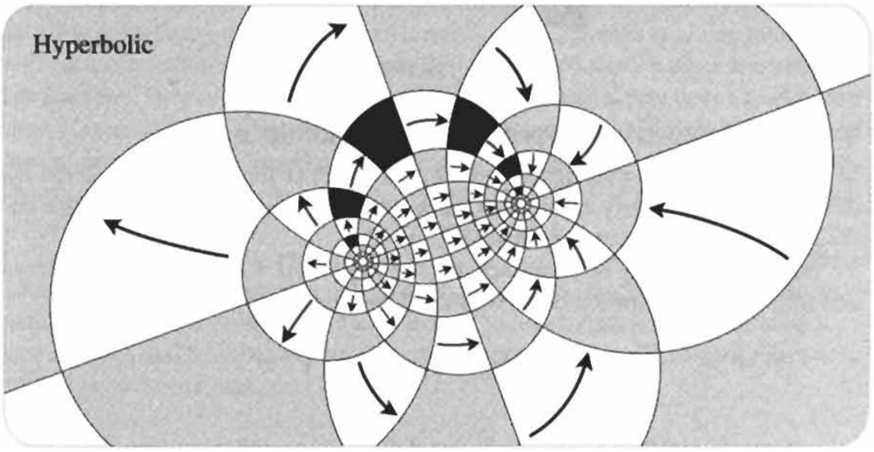

Figure 1.4: Hyperbolic transformation [5]

1.6.2 Elliptic transformations

Elliptic transformations fulfill

$$m = e^{i\phi}.$$

In the Riemann sphere, $\tilde{M}$ equates to a rotation along the z-axis. Applying $M(z)$ will result on points describing elliptic orbits around ξ^- and ξ^+, without being neither of them either a source nor a sink, as shown on figure 1.5

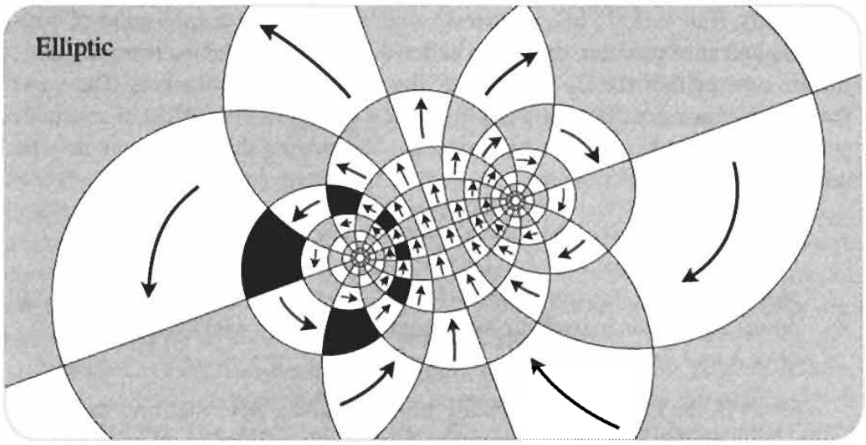

Figure 1.5: Elliptic transformation [5]

1.6.3 Loxodromic transformations

Loxodromic are a combination of the last two, where

$$m = \rho e^{i\phi}.$$

In the Riemann sphere, applying $\tilde{M}$ make points in spiral trajectories form S to N. As is the case with elliptic transformations, applying $M(z)$ will result on ξ^-

acting as a sink and ξ^+ as a source, as shown in figure 1.6

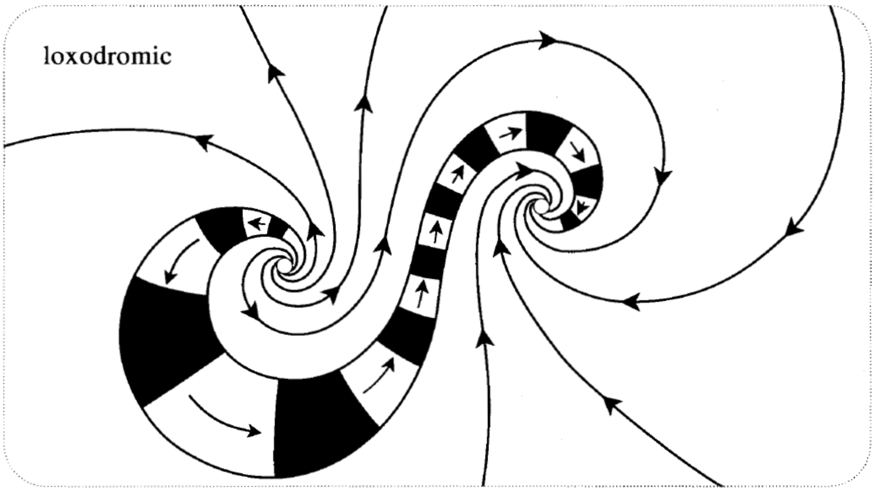

Figure 1.6: Loxodromic transformation.

1.6.4 Parabolic transformations.

Parabolic transformations correspond to the case where there is only one fix point. Thus, to follow the same procedure we have been using, we must now apply:

$$\tilde{M}(z) = G^{-1} \circ M \circ G(z)$$

where

$$G(z) = \frac{1}{z - \xi}$$

being ξ the only fixed point. Thus will have now a fixed point in ∞. As we know, this means that, according to lemma 1.4.4,

$$M(z) = z + T.$$

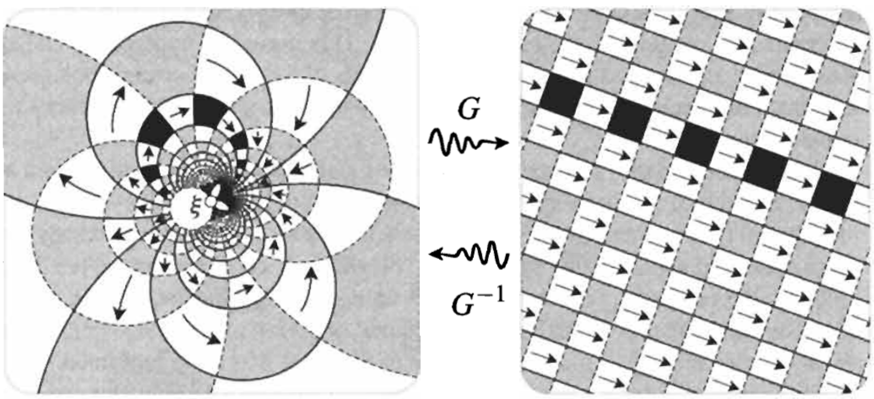

Figure 1.7: Parabolic transformation [5]

In the Riemann sphere, $\tilde{M}$ acts moving points within circles passing through N, as lines in $\mathbb{C}$ are mapped to them. As $M(z)$ is a simple translation, moving all points the same quantity in a given direction, as shown in figure

The effect of all transformations on the Riemann sphere with 0 (except for parabolic transformations) and ∞ as fixed points is shown in 1.8

With the help of the matrix form, we can find where does any Möbius transformation belong just by looking at their coefficients.

We can express 1.9 with matrices as

$$\tilde{\mathbf{M}} = \mathbf{F^{-1}MF}.$$

A result of linear algebra states that this transformations leave traces invariant, i.e. $\text{Tr}(\mathbf{M}) = \text{Tr}(\tilde{\mathbf{M}})$. Then, for every Möbius transformation, we have.

$$\sqrt{m} + \frac{1}{\sqrt{m}} = a + d \tag{1.10}$$

In fact, we can understand $\sqrt{m}$ and $1/\sqrt{m}$ as the eigenvalues of parabolic, elliptic

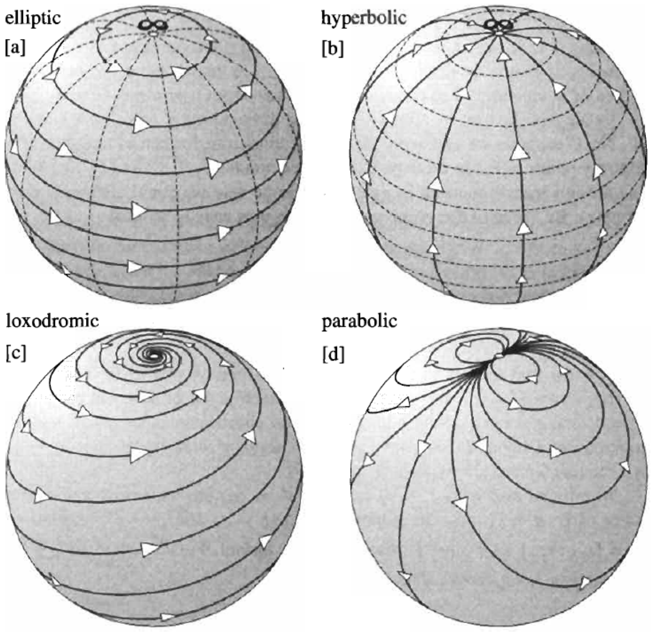

Figure 1.8: Different transformations acting on the Riemann sphere with fixed points at 0 and ∞ [5]

and loxodromic transformations, as they are the values of the diagonal entries in the diagonalized matrices. Looking at (1.10) and knowing the form of every multiplier, we get to the result.

Theorem 1.6.1. *A Möbius transformation is*

- *elliptic if* $(a + d)$ *is real and* $|a + d| < 2$
- *parabolic if* $(a + d) = \pm 2$
- *hyperbolic if* $(a + d)$ *is real and* $|a + d| > 2$
- *loxodromic if* $(a + d)$ *is complex*

Chapter 2

Poincaré half-plane

The Möbius transformations can be applied to derive the geoesics and hyperbolic lines in the Poicaré half-plane model of two-dimensional hyperbolic geometry. In this section such model and its motivation are briefly introduced and then all its geodesic lines are computed.

2.1 Brief introduction to hyperbolic geometry

Hyperbolic geometry was developed mainly in the XIX century as an attempt to disproof Euclid's fifth postulate. Said postulate, in its modern version, states that *in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point* [1].
There exist several models to represent the Hyperbolic plane. We focus here only in the half-plane Poincaré model, where an open euclidean half-plane is used, representing the hyperbolic lines by lines and semicircles perpendicular to the plane's border. For our description we will members of $\mathbb{C}_\infty$ with positive imaginary part as the half-plane, so the real axis will be its border.

2.2 The Poincaré half-plane model

Definition 2.2.1. The Poincaré half-plane is defined as the set

$$\mathcal{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}. \tag{2.1}$$

As stated before, $\mathcal{H}$ is the hyperbolic plane in this model: each of the elements z correspond to a point in the plane. We have yet to define a metric. For this, it is necessary to define curves joining different points.

[1]As expressed by John Playfair

A curve in the complex plane is a continuously differentiable function $\gamma : (a,b) \mapsto \mathbb{C}$ defined on an open or cosed interval. For a curve $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ continuous differentiability implies that $\gamma_1(t) = \Re(\gamma(t))$, $\gamma_2(t) = \Im(\gamma(t))$ must be continuously differentiable. Its *tangent vector* in t_0 is defined as $\gamma_1'(t_0) + i\gamma_2'(t_0), \forall a < t_0 < b$

Definition 2.2.2. Let γ be a curve in $\mathcal{H}$. The *hyperbolic length* of γ is defined as

$$l(\gamma) := \int_a^b \frac{\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}}{\gamma_2(t)} dt \tag{2.2}$$

This quantity, as it can be easily seen, is positive defined. To picture its meaning, it is useful to note that euclidean segments parallel to the real axis with the same euclidean length have greater hyperbolic length the closer they are to the real axis. See figure 2.1

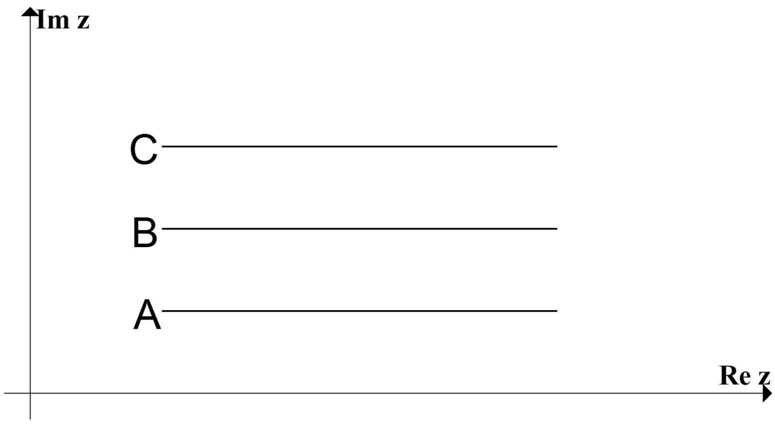

Figure 2.1: $l(\text{A}) > l(\text{B}) > l(\text{C})$

From the concept of hyperbolic length follows a definition of distance:

Definition 2.2.3. Let $z_1, z_2 \in \mathcal{H}$. The *hyperbolic distance* between z_1 and z_2, denoted as $\rho(z_1, z_2)$, is defined as

$$\rho(z_1, z_2) := \inf\{l(\gamma) \mid \gamma : [a,b] \mapsto \mathcal{H}, \gamma(a) = z_1, \gamma(b) = z_2\} \tag{2.3}$$

It can be easilly proven that this distance is well defined i.e. it fulfills the conditions

- $\rho(z_1, z_2) \geq 0, \forall z_1, z_2 \in \mathcal{H}$
- $\rho(z_1, z_2) = \rho(z_2, z_1), \forall z_1, z_2 \in \mathcal{H}$
- $\rho(z_1, z_2) = 0 \iff z_1 = z_2, \forall z_1, z_2 \in \mathcal{H}$

- $\rho(z_1, z_2) + \rho(z_2, z_1) \geq \rho(z_1, z_3), \forall z_1, z_2 \in \mathcal{H}$.

The first three conditions are trivially derived from the previous definition. The fourth one is a bit less straightforward: the distance $4)\rho(z_1, z_2) + \rho(z_2, z_3)$ is the shortest distance between z_1 and z_3 traversing z_2, which is going to be equal or greater than $\rho(z_1, z_3)$.

These are the basic ingredients of the model. We will proceed now to find its geodesics and lines, which will be defined in the next section. For this, the Möbius transformations will prove essential.

2.3 Geodesics and hyperbolic lines

Definition 2.3.1. A *geodesic* is defined as a curve $\gamma : (a, b) \mapsto \mathcal{H}$ that provides the path of integration used to compute the distance between each of its points. An *hyperbolic line arc* is the image of a geodesic $[\gamma]$. A *hyperbolic line* is a hyperbolic line arc which is not contained in any other hyperbolic line arc.

We will now provide the simplest geodesic in the model, which will be crucial to derive the rest.

Proposition 2.3.1. Let $z_1 = ai, z_2 = bi$ with $a, b > 0$. The hyperbolic distance between them is $\rho(z_1, z_2) = |\log(b/a)|$

Proof. Let's asume $a < b$ (the case $b < a$ is analogous).The hyperbolic length of a curve $\gamma : [t_1, t_2] \mapsto \mathcal{H}$ such us $\gamma(t_1) = ai$,$\gamma(t_2) = bi$ is bounded as

$$l(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}}{\gamma_2(t)} dt \geq \int_{t_1}^{t_2} \frac{|\gamma_2'(t)|}{\gamma_2(t)} dt \geq \int_{t_1}^{t_2} \frac{\gamma_2'(t)}{\gamma_2(t)} dt = \log \frac{\gamma_2(t_2)}{\gamma_2(t_1)} \tag{2.4}$$

If we use the curve $\beta(t) = it$ with $a \leq t \leq b$, then we get the length

$$l(\beta) = \int_a^b \frac{dt}{t} = \log \frac{b}{a}. \tag{2.5}$$

which coincides with the smallest value of the length of a curve joining the points ai and bt. This is the definition of the hyperbolic distance, so $\rho(z_1, z_2) = log(b/a)$ □

As the points chosen on the imaginary axis were arbitrary, it follows inmediately that the curve $\beta(t) = it$ with $0 < t < \infty$ is a geodesic. It remains to be proven whether it is also an hyperbolic line. With this in mind, a sufficient condition for a curve to be a hyperbolic line is derived.

Proposition 2.3.2. Let $\gamma : (a, b) \mapsto \mathcal{H}$ be a geodesic such as for a point $c \in (a, b)$ the curves $\gamma|_{(a,c)} : (a, c) \mapsto \mathcal{H}$ and $\gamma|_{(c,b)} : (c, b) \mapsto \mathcal{H}$ have infinite length. Then $\gamma : (a, b) \mapsto \mathcal{H}$ is an hyperbolic line.

Proof. Reductio ad absurdum. Let $\sigma : (a', b') \mapsto \mathcal{H}$ be a geodesic such as $[\gamma] \subset [\sigma]$. Let's assume $\exists \sigma(t_0) = z_0 \notin [\gamma]$. Let z_1 be a point contained in both $[\sigma]$ and $[\gamma]$ such as $\gamma(c) = \sigma(t_1) = z_1$. Let assume $t_0 < t_1$ (for $t_0 > t_1$ the procedure would be analogous). As the restricted curve $\sigma|_{(t_0,t_1)}$ is a geodesic, its length $l(\sigma\|_{(t_0,t_1)})$ must be finite, which would imply $l(\gamma|_{(a,c)}) < \infty$, as it is contained in the other curve. This contradicts the initial hypothesis. □

With this result we can establish the next fundamental result

Theorem 2.3.1. *The image* $[\beta]$ *of the curve*

$$\beta(t) = it, \quad 0 < t < \infty \tag{2.6}$$

is an hyperbolic line.

Proof. We already proved that the length curve $\sigma(t) = it$ for $a < t < b$ is $l(\sigma) = log(b/a)$. For an arbitrary point $z_1 = ct$ with $c \in (0, \infty)$ the lengths of the curves $\sigma(t)|_{(0,c)}$ and $\sigma(t)|_{(c,\infty)}$ can easily be seen to be infinite. □

In the next sections we will use this hyperbolic line to find all the others in the model.

2.4 Isometries of the Half-plane

For now, we know the distance between points lying in a particular line of our model. We are interested now to find a way to move any pair of points outside this particular line into it in such a way that the distance between them does not vary. We establish now which are the transformations that will allow us to leave the distance between points invariant.

Definition 2.4.1. An *isometry* in $\mathcal{H}$ is a bijective transformation $T : \mathcal{H} \mapsto \mathcal{H}$ that preserves distances i.e.

$$\rho(T(z_1), T(z_2)) = \rho(z_1, z_2)$$

It will be proven that some subset of the Möbius transformations are isometries of $\mathcal{H}$.

Definition 2.4.2. We define $\mathcal{T}$ as the set consisting on the normalized Möbius transformations with real coefficients

$$\mathcal{T} = \{T(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}, ab - cd = 1\}$$

It can be easily proven that $\mathcal{T}$ is a subgroup of $\mathcal{M}$ under composition.

Lemma 2.4.1. *A Möbius transformation belonging to $\mathcal{T}$ preserves $\mathcal{H}$.*

Proof. Let $z = x + iy \in \mathcal{H}$. Appliying a transformation $M(z)$ we get

$$T(z) = \frac{az+b}{cz+d} = \frac{ac|z|^2 + bc\overline{z} + adz + bdz}{|cz+d|^2}.$$

with real and imaginary part

$$\Re[T(z)] = \frac{|z|^2 + bc(|z|^2+1)}{|cz+d|^2},$$

$$\Im[T(z)] = \frac{(ac-dc)\Im z}{|cz+d|^2} = \frac{\Im z}{|cz+d|^2} > 0$$

So $G(z) \in \mathcal{H}$. □

It is also worth noting that with this same proof it can be seen that transformations of real coefficients with $ab - cd > 0$ also preserve the $\mathcal{H}$. These also form a subgroup under composition. It will be proved now that these are, in fact, isometries of the model, although not the only ones for $\mathcal{H}$.

Proposition 2.4.1. The transformations $T \in \mathcal{T}$ preserve the hyperbolic length

Proof. As it was shown in theorem 1.3.1, Möbius transformations can be expressed as compositions of two translations, an a rotation-expansion and a complex inversion. If we prove lengths to be constant under this transformations, then they are also for a general transformation T. As the coefficients of T are real, the rotation-expansion becomes a simple expansion.For expansions and rotations deducing this from the formula (2.2) is trivial. For an inversion this is a bit more tricky. Computing from $M(\gamma(t)) = 1/\gamma(t)$ with $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ we obtain

$$\Re[M(\gamma(t))] = \frac{\gamma_1(t)}{|\gamma(t)|^2} \quad \Im[M(\gamma(t))] = -\frac{\gamma_2(t)}{|\gamma(t)|^2}$$

$$[\Re[(M(\gamma(t))]' = \frac{\gamma_1'(t)|\gamma(t)|^2 - 2\gamma_1(t)[\gamma_1(t)\gamma_1(t)' + \gamma_2(t)\gamma_2'(t)]}{|\gamma(t)|^2}$$

$$[\Im[(M(\gamma(t))]' = \frac{\gamma_2'(t)|\gamma(t)|^2 - 2\gamma_2(t)[\gamma_1(t)\gamma_1(t)' + \gamma_2(t)\gamma_2'(t)]}{|\gamma(t)|^2}$$

$$[\Re[(M(\gamma(t))']^2 + [\Im[(M(\gamma(t))']^2 = \frac{[\gamma_1'(t)]^2 + [\gamma_2'(t)]^2}{|\gamma(t)|^4}.$$

Applying this in (2.2), we can see that the length is preserved. □

One can intuitively see (as it can also be proven) that for every pair of points in $\mathcal{H}$ there is exactly one semicircle or line both orthogonal to the real axis that contain them. This, and the fact that Möbius transformations transform generalized circles into generalized circles are used now to derive the next result.

Proposition 2.4.2. Let $z_1, z_2 \in \mathcal{H}$. There exist a Möbius transformation belonging to $\mathcal{T}$ such as $M(z_1) = ai, M(z_2) = bi$ with $a, b > 0$.

Proof. If z_1 and z_2 lie in a line orthogonal to the real axis, then such transformation is $G(z) = z - \alpha$, being α the euclidean distance between the line and the imaginary axis.
If the points lie on the a circle orthogonal to the real axis, then the transformation is more complex. First, the transformation $G_1(z) = z - \alpha$, being α the distance between the complex axis and the left intersection between the circle and the real axis, as applied. This transformation moves the semicircle so its left extreme touches the origin. Then, the transformation $G_2 = -1/z$ is applied, turning the semicircle into a line orthogonal to the real axis at $x = 1/d = \beta$, being d its diameter. The sign minus is added so the transformation preserves $\mathcal{H}$. Finally, the transformation $G_3 = z - \beta$ finally takes the points to the real axis. Then, the transformation we are looking for is $G(z) = G_1 \circ G_2 \circ G_3 = \frac{1}{z-\alpha} - \beta$
For both cases, we have only used Möbius transformations with real numbers, all of them preserving the real axis, so the condition $ab - cd > 0$ must be fulfilled. To get the corresponding transformation belonging to $\mathcal{T}$, we just have to normalize the coefficients. □

This is the last ingredient we needed to get to establish $\mathcal{T}$ as a set of isometries of the model.

Theorem 2.4.2. *The elements of $\mathcal{T}$ are isometries of the Poincaré half-plane model.*

Proof. The elements of $\mathcal{T}$ preserve lengths and for every pair of points there exist a transformation that places them in the imaginary axis, where we know the distance between two points, so distances are preserved □

From this fact it is possible to derive a general formula for the distance between each pair of points.

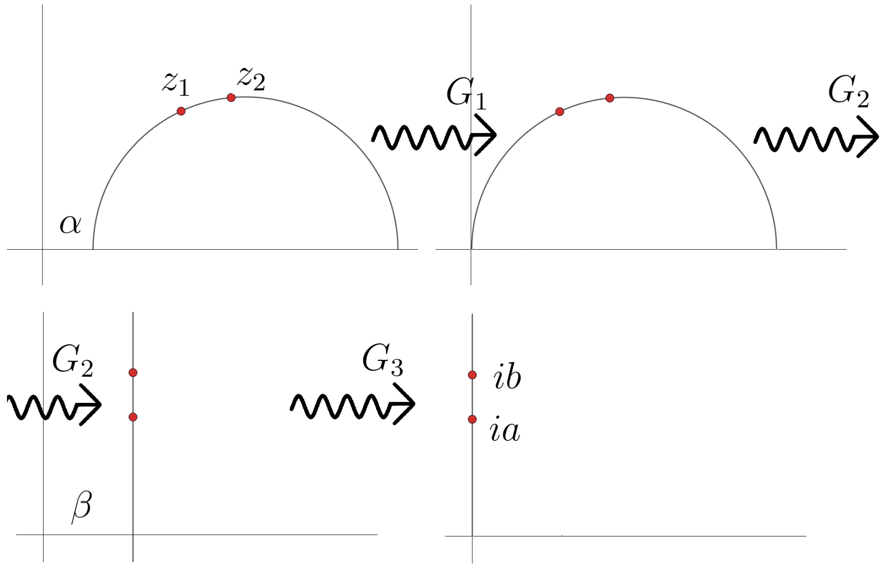

Figure 2.2: Caption

2.5 Derivation of lines

With this, and using the circle preserving property of the Möbius transformations we can establish all the lines of the model.

As $\mathcal{T}$ are isometries of the plane, a geodesic α is such if and only if it can be written as $P \circ \alpha$ with $P \in G$. The same can be said of the hyperbolic lines. Then, the next theorem is derived.

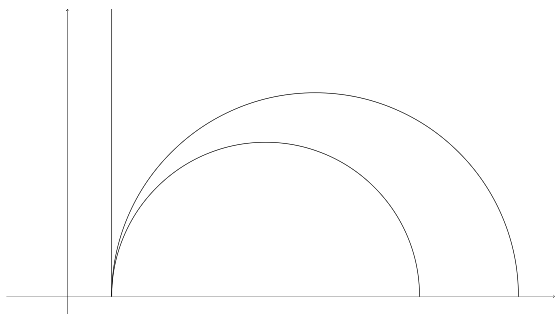

Figure 2.3: Paralel lines

Theorem 2.5.1. *Hyperbolic lines are euclidean lines and semicircles perpendicular to the real axis. Arcs of hyperbolic lines are arcs of circles and segments of these hyperbolic lines and geodesics the curves they are the image of.*

Proof. Möbius transformation are conformal and every hyperbolic line $[\beta]$ can be expressed as $[\beta] = T \circ [\alpha]$ for every other α. We already know that the complex axis is an hyperbolic line, which is perpendicular with the real axis at $x = 0$ and for ∞. This can be easily seen if we represent both axis over the Riemann sphere.

As Möbius transformations transform lines into lines or circles, any other line will be a semicircle or a line perpendicular to the real axis. □

This notion of line will allow us to define parallel, secant and perpendicular lines in such a way that Euclid's fifth postulate is not fulfilled.

Definition 2.5.1. Two lines are parallel if they intersect on the real axis.

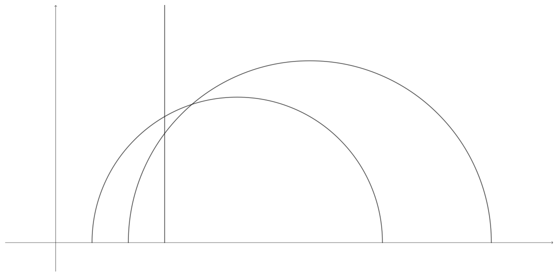

Figure 2.4: Secant lines

Definition 2.5.2. Two lines are secant if they intersect in any point other than the real axis.

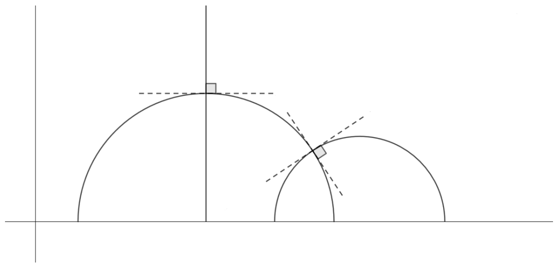

Figure 2.5: Caption

Definition 2.5.3. Two lines are perpendicular if the intersect with a straight angle.

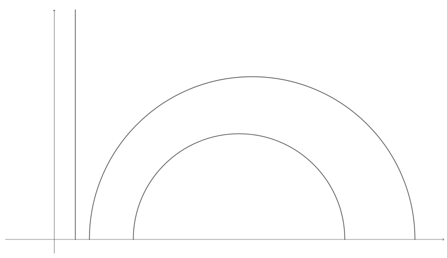

Figure 2.6: Not secants nor parallel lines

One can easily realize, as shown in figure (2.6) that this definition allows the existence of lines that are neither parallel nor secant, which contradicts Euclid's fifth postulate. It can be proven that the model fulfills the other four postulates opening, making it valid for hyperbolic geometry.

Chapter 3

Möbius as Lorentz

Within the framework of special relativity, events are represented as points in the four-dimensional Minkowsky space-time. Lorentz transformations substitute Galilean transformations as the coordinate transformations between inertial frames of reference. The equivalence between the Möbius group acting on $\mathbb{C}_\infty$ and the proper orthochronous Lorentz group on world vectors is studied. This relation will be used to characterize how celestial spheres change with Lorentz transformations. As the focus of this work is not the exhaustive study of the Lorentz group but its relation with the Möbius transformations, only a quick review of its properties and subgroups is presented here. More information about this can be found on the extensive existing literature about special relativity.

3.1 Framework

Special relativity was developed in the early XX century mainly by Albert Einstein. Its initial goal was to find a formalism where the equations of electromagnetism are invariant, as they are not under Galilean transformations. It is built from two postulates:

- The laws of Physics are invariant in all inertial systems.
- The speed of light in vacuum c is a invariant for all inertial observers.

These postulates imply that time and space are not absolute anymore. A series of new equivalencies and conservation laws can be deduced from them. The most important, which is going to be key for the next results, is that the quantity known as space time interval,i.e.

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{3.1}$$

is the same for all inertial observers, being Δt the difference of time and $\Delta x, \Delta y, \Delta x$ differences of distances in a Cartesian coordinate system between two events measured by an observer O.
From now on, we are going to use natural units, so $c = 1$, which will simplify all expressions.

3.2 Minkowsky space-time and Lorentz transformations

In special relativity, events are represented as elements of a four-dimensional vector space, with one temporal coordinate and three spatial ones.

Definition 3.2.1. Minkowski space is defined as the vector space

$$\mathbb{M} := \{\mathbf{X} = (t, x, y, z) \in \mathbb{R}^4\}$$

alongside the inner product

$$(\mathbf{X}_1, \mathbf{X}_2) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2, \quad \mathbf{X}_1, \mathbf{X}_2 \in \mathbb{M}.$$

Vectors in M are known as *world vectors.*

Each world vector represents an event in space time. This vectors are equivalent to the position vector in Newtonian mechanics. The introduced inner product allows the definition of the quantity $\mathbf{X}^2 = (\mathbf{X}, \mathbf{X})$, which coincides with the definition of space-time interval (3.1). This quantity is known as the *Lorentz norm*[1]
The previously introduced inner product can be also expressed as:

$$(\mathbf{X}_1, \mathbf{X}_2) = g_{\mu v} x_1^\mu x_2^v \tag{3.2}$$

where $g_{\mu v}$ is the *metric tensor*, with $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$ and $g_{\mu v} = 0$ for $\mu \neq v$ .Here, $x_0 = t$, and we are using Einstein's summation convention, were repeated indices imply summation from 0 to 3, for example, $a_i b_i = \sum_{i=0}^{3} a_i b_i$. The metric tensor can be expressed as a 4×4 matrix

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.3}$$

[1] Although we call it *norm*, it is not one, as it can have negative values.

The special relativity postulates allow us to define a set of transformations that preserve the space-time interval.

Definition 3.2.2. *Lorentz transformations* are linear and real transformations $\mathbf{\Lambda} : \mathbb{R}^4 \mapsto \mathbb{R}^4$ acating on world vectors as:

$$\tilde{x}^\mu = \Lambda^\mu{}_\upsilon x^\upsilon$$

that preserve the Lorentz norm, i.e. $(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}) = (\mathbf{X}, \mathbf{X})$

Lorentz transformations can also be seen as 4×4 matrices acting on world vectors

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \mathbf{\Lambda} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{3.4}$$

All objects that transform with Lorentz transformation as world vectors are known as *four-vectors*. It can be shown that every Lorentz transformation fulfill a certain condition

Proposition 3.2.1. For every Lorentz transformation $\mathbf{\Lambda}$,

$$\mathbf{g} = \mathbf{\Lambda}^\intercal \mathbf{g} \mathbf{\Lambda} \tag{3.5}$$

Proof. A lorentz transformation fulfils $\tilde{\mathbf{X}}^2 = \mathbf{X}^2$. Using Einstein's notation:

$$\tilde{\mathbf{X}}^2 = g_{\mu\upsilon} \tilde{x}^\upsilon \tilde{x}^\mu = g_{\mu\upsilon} \Lambda^\upsilon{}_\alpha x^\alpha \Lambda^\mu{}_\beta x^\beta$$

$$\mathbf{X}^2 = g_{\alpha\beta} x^\beta x^\alpha$$

$$\tilde{\mathbf{X}}^2 = \mathbf{X}^2 \Rightarrow g_{\mu\upsilon} \Lambda^\upsilon{}_\alpha x^\alpha \Lambda^\mu{}_\beta x^\beta = g_{\alpha\beta} x^\beta x^\alpha \Rightarrow g_{\mu\upsilon} \Lambda^\upsilon{}_\alpha \Lambda^\mu{}_\beta = g_{\alpha\beta} \Rightarrow \mathbf{g} = \mathbf{\Lambda}^\intercal \mathbf{g} \mathbf{\Lambda}$$

□

From this, we can go further and generalize the idea:

Definition 3.2.3. The Lorentz group $\mathcal{L}$ is defined as

$$\mathcal{L} = \{\mathbf{\Lambda} \mid \mathbf{g} = \mathbf{\Lambda}^\intercal \mathbf{g} \mathbf{\Lambda}\}.$$

It can be proven that matrices fulfilling this condition conform a group under matrix multiplication. This group also contains time reversal $(t, x, y, z) \mapsto (-t, x, y, z)$ and space parity $(t, x, y, z) \mapsto (t, -x, -y, -z)$ between others.

3.3 Light cones

Vectors in Minkowski space-time can be classified into three categories according to their Lorentz norm and their time coordinate:

Definition 3.3.1. A world vector $\mathbf{X}$ is called *time-like* if $\mathbf{X}^2 > 0$, *null* or *light-like* if $\mathbf{X}^2 = 0$ and *space-like* if $\mathbf{X}^2 < 0$.

Definition 3.3.2. Vectors in Minkowski space-time are *future-pointing* if $t > 0$ and *past-pointing* if $t < 0$.

In Minkowski space, null vectors conform two three dimensional cones with their vertex in the origins as shown in figure (3.1): Future-pointing light-like vectors lie inside the future cone, whereas past-pointing light-like vectors lie inside the past cone.

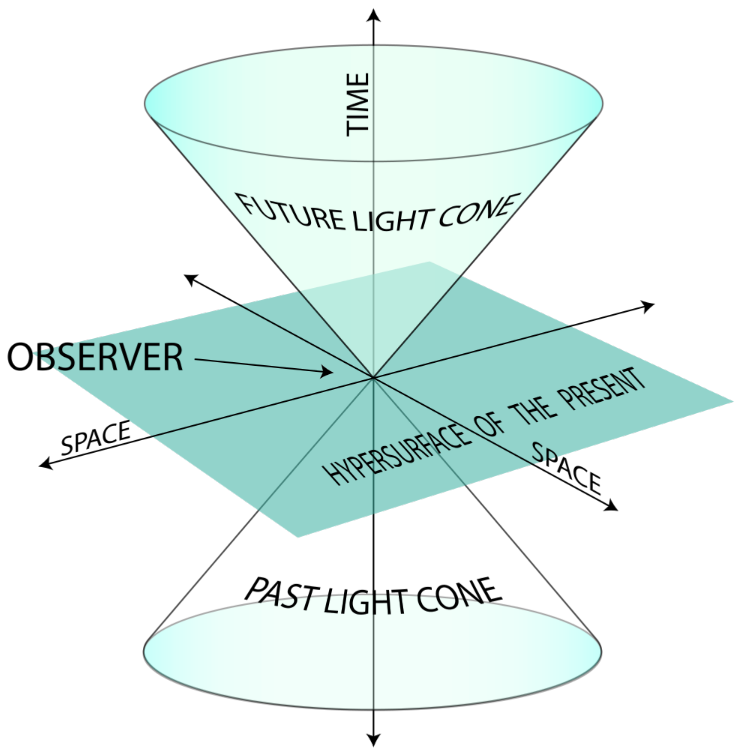

Figure 3.1: Light cones. Here, only two spatial and one temporal dimensions are represented. Minkowski space-time is four-dimensional.[3]

3.4 Classification of Lorentz transformations

The Lorentz group can be classified into four disjoint subgroups. To introduce this classification, let's review some characteristics of their members.

Proposition 3.4.1. Let $\mathbf{\Lambda} \in \mathcal{L}$. Then its determinant is either 1 or -1

Proof. From (3.3), we can see that $\det(\mathbf{g}) = -1$. From equation (3.5), we get

$$\det(\mathbf{g}) = \det(\mathbf{\Lambda}^\intercal \mathbf{g} \mathbf{\Lambda}) = \det(\mathbf{\Lambda}^\intercal)\det(\mathbf{g})\det(\mathbf{\Lambda}) \Rightarrow [\det(\mathbf{\Lambda})]^2 = 1 \Rightarrow \det(\mathbf{\Lambda}) = \pm 1$$

□

Proposition 3.4.2. Let $\Lambda \in \mathcal{L}$. Its $\Lambda^0{}_0$ component is either bigger or equal than 1 or smaller or equal than -1

Proof. From 3.5 we have

$$\mathbf{g} = \mathbf{\Lambda}^\intercal \mathbf{g} \mathbf{\Lambda} \Rightarrow g_{00} = \Lambda^\mu{}_0 g_{\mu v} \Lambda^v{}_0 \Rightarrow 1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \Rightarrow$$

$$\Rightarrow \Lambda^0{}_0 = \pm \sqrt{1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2}$$

□

With this properties derived, we can make a classification based on their determinant and their $\Lambda^0{}_0$ component. Each of the four resulting subgroups is listed in table 3.1.

Subgroup	**Symbol**	**Properties**	**Continuous with**
Proper orthochronous (Restricted)	$\mathcal{L}_+^\uparrow$	$\Lambda^0{}_0 \geq 1$ $\det\Lambda = 1$	$\mathbb{1}$
Proper not-orthochronous	$\mathcal{L}_+^\downarrow$	$\Lambda^0{}_0 \leq -1$ $\det\Lambda = 1$	$-\mathbb{1}$
Improper orthochronous	$\mathcal{L}_-^\uparrow$	$\Lambda^0{}_0 \geq 1$ $\det\Lambda = -1$	$\begin{pmatrix} 1 & \\ & -\mathbb{1} \end{pmatrix}$
Improper not-orthochronous	$\mathcal{L}_-^\downarrow$	$\Lambda^0{}_0 \geq 1$ $\det\Lambda = -1$	$\begin{pmatrix} -1 & \\ & \mathbb{1} \end{pmatrix}$

Table 3.1: Classification of the Lorentz group

The proper orthochronous $\mathcal{L}_+^\uparrow$ is the only subgroup that preserves both time and space orientation. That means that its transformations preserve the future light cone and that every three vectors arranged in with right-hand orientation

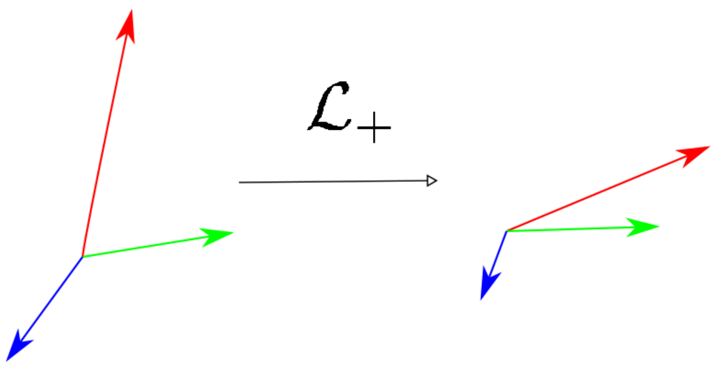

Figure 3.2: Transformation that preserves space orientation

will preserve such orientation, as seen in figure 3.2. These transformations are also known as the *restricted Lorentz group*.
As the porpoise of this work is not to give an exhaustive overview of Lorentz transformations, a series of results are given now. Detailed derivations are presented in the references.[10]
First, Lorentz transformation can be continuously derived from the different matrices listed in table 3.1. The restricted Lorentz group can be obtained continuously from the identity, which will be key to make the connection with Möbius transformations. A formal deffinition of continuity between transformations follows:

Definition 3.4.1. Two Lorentz transformations $\Lambda_1, \Lambda_2 \in \mathcal{L}$ are *continuously connected* if there exist an continuous application $\Xi[a, b] \mapsto \mathcal{L}$ with $a, b \in \mathbb{R}$ such $\Xi(a) = \Lambda_1$ and $\Xi(b) = \Lambda_2$.

The fact that such concept of continuity can be applied here has to do with the fact that the Lorentz group is a Lie Group, i.e. can be identified with a differentiable manifold. However, in this work we will not focus on this characterization, as it unnecessarily complex.
Second, the components of restricted Lorentz transformations consist on two sets of transformations

Proposition 3.4.3. Restricted Lorentz transformations are either *boosts* or *rotations*

Boosts acting on world vectors determine how they change for an observer with a constant velocity v with respect to the original observer. The general form for a boost in the z-direction is

$$\mathbf{\Lambda}^{z}_{Boost} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma v & 0 & 0 & -\gamma \end{pmatrix} \tag{3.6}$$

where $\gamma = (1 - v^2)^{-1/2}$. A rotation corresponds to an ordinary rotation in the 3D space, and has the form

$$\mathbf{\Lambda}_{Rot} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbf{R}_{3\times 3} & \\ 0 & & & \end{pmatrix} \tag{3.7}$$

where $\mathbf{R}_{3\times3}$ is one of the well-known 3D ratations matrices. For a boost over an arbitrary axis Γ, the transformation would be

$$\mathbf{\Lambda}^{\Gamma}_{Boost} = \mathbf{\Lambda}^{-1}_{Rot,\Gamma}\mathbf{\Lambda}^{z}_{Boost}\mathbf{\Lambda}_{Rot,\Gamma} \tag{3.8}$$

being $\Lambda_{Rot,\Gamma}$ the rotation that sends the z axis to such axis

3.5 Null directions and the celestial sphere

Definition 3.5.1. A *null direction* is a set of world vectors such

$$N_{\mathbf{X}} = \{\lambda \mathbf{X} : \lambda \in \mathbb{R}^+\}$$

such as $\mathbf{X}$ is null.

Null lines can be seen as trajectories of light rays in Minkowski space. The intersection of the plane $T = 1$ with the future light cones is a three dimensional sphere where each point corresponds to a null direction. The same can be said about the intersection between the past cone and the plane $T = -1$. See figure 3.3.

Definition 3.5.2. The intersection between the future light cone and the plane $T = 1$ is called S^+. The intersection between the past light cone and the plane $T = -1$ is called S^- or *celestial sphere*. The map $\mathbf{\Sigma} : (T, X, Y, Z) \mapsto (-T, -X, -Y, -Z)$ is called the *anti-sky mapping*.

The sphere S^- can be interpreted as the sky seen by an observer looking at distant objects, like stars. In fact, every point of the sphere corresponds to the

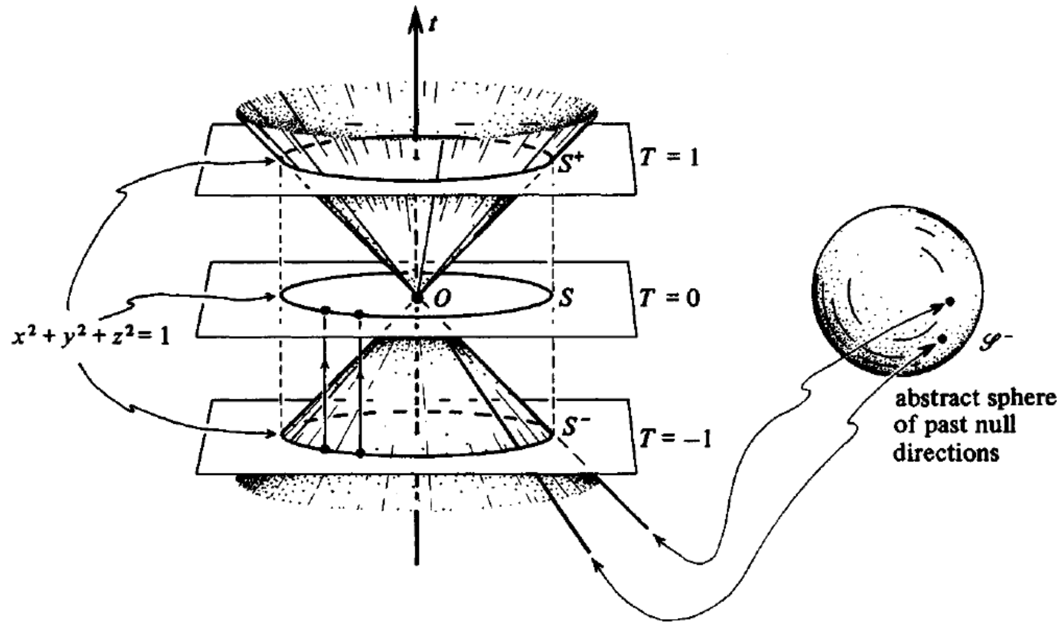

Figure 3.3: The celestial sphere is the intersection of the past light-cone with the plane $T = -1$. [8]

set of all light rays coming from a certain spatial directions. That is why S^- is alternatively called *celestial sphere*. Every point of the celestial sphere can be mapped to a point in S^+ by the antipodal mapping Σ, representing both the same null direction. As such map is simple and well known, we are going to work from now on in S^+, as it will allow us to get rid the minus sign.

As S^+ is a 3D sphere of radius 1, we can establish a correspondence with its points and those in the Riemann sphere. Thus, every point corresponds to a complex number in the extended plane, with $\mathbf{X} = (1, 0, 0, 1)$ corresponding to the north pole. Then, the expressions in section 1.1 will be valid to refer to every null direction.

3.6 Equivalence of Möbius and Lorentz transformations

Each point in the sphere can be expressed through the stereographic projection. Expressing every complex number with homogeneous coordinates $\zeta = \delta/\eta$, equation 1.1 can be written as

$$x = \frac{\delta\overline{\eta} + \eta\overline{\delta}}{\eta\overline{\eta} + \delta\overline{\delta}}, \quad y = \frac{\delta\overline{\eta} - \eta\overline{\delta}}{i(\eta\overline{\eta} + \delta\overline{\delta})}, \quad z = \frac{\eta\overline{\eta} + \delta\overline{\delta}}{i(\eta\overline{\eta} + \delta\overline{\delta})} \tag{3.9}$$

As the null world vectors $\mathbf{X}$ and $\lambda\mathbf{X}$ with $\lambda \in \mathbb{R}$ are contained in the same null direction, then the vector with coordinates $[\sqrt{2}(\eta\overline{\eta}+\delta\overline{\delta})]^{-1}(1,x,y,z)$, i.e.

$$T=\frac{1}{\sqrt{2}(\eta\overline{\eta}+\delta\overline{\delta})},\ X=\frac{1}{\sqrt{2}}(\delta\overline{\eta}+\eta\overline{\delta}),\ Y=\frac{1}{i\sqrt{2}}(\delta\overline{\eta}-\eta\overline{\delta}),\ Z=\frac{1}{\sqrt{2}}(\eta\overline{\eta}+\delta\overline{\delta}) \tag{3.10}$$

corresponds to the same point in S^+ as $(1,x,y,z)$.

We consider now the next map:

Definition 3.6.1. The map $\Upsilon:\mathbb{R}^4\mapsto H(2)$, being $H(2)$ the set of 2×2 hermitian matrices, acts on world-vectors as

$$\Upsilon(T,X,Y,Z)\mapsto\frac{1}{\sqrt{2}}\begin{pmatrix}T+Z & X+iY\\ X-iY & T-Z\end{pmatrix} \tag{3.11}$$

Remark. It can easily be noted that Υ is linear and bijective, as for every member of $\mathbb{R}^4$ there is one and only one member of $H(2)$ and vice-versa.

This will alow us to proove the next statment

Proposition 3.6.1. Every Möbius transformation corresponds to a an unique Lorentz transformation.

Proof. For some $\mathbf{A}\in PSL(2,\mathbb{C})$, we want to obtain a Lorentz transformation $\Lambda_{\mathbf{A}}\in\mathcal{L}$. Let $\mathbf{Q}\in PSL(2,\mathbb{C})$ be the image of a world-vector $\mathbf{Q}=\Upsilon(T,X,Y,Z)$. Then

$$\mathbf{Q}=\frac{1}{\sqrt{2}}\begin{pmatrix}T+Z & X+iY\\ X-iY & T-Z\end{pmatrix}=\begin{pmatrix}\delta\overline{\delta} & \delta\overline{\eta}\\ \eta\overline{\delta} & \eta\overline{\eta}\end{pmatrix}=\begin{pmatrix}\delta\\ \eta\end{pmatrix}\begin{pmatrix}\overline{\delta} & \overline{\eta}\end{pmatrix}. \tag{3.12}$$

It can be easily seen that $|Q|=\mathbf{X}^2/2$.
Now, let's consider the matrix product

$$\mathbf{AQA}^*.$$

As we already have established, the action of a Möbius transformation on a complex number $\zeta\in\mathbb{C}_\infty$ can be expressed as the action of a member of $SL(2,\mathbb{C})$ on that number's homogeneous coordinates (δ,η). Then,

$$\begin{aligned}\mathbf{AQA}^*=\mathbf{A}\begin{pmatrix}\delta\\ \eta\end{pmatrix}\begin{pmatrix}\overline{\delta} & \overline{\eta}\end{pmatrix}\mathbf{A}^*=\begin{pmatrix}\tilde{\delta}\tilde{\overline{\delta}} & \tilde{\overline{\delta}}\tilde{\eta}\\ \tilde{\eta}\tilde{\overline{\delta}} & \tilde{\eta}\tilde{\overline{\eta}}\end{pmatrix}=\\ =\frac{1}{\sqrt{2}}\begin{pmatrix}\tilde{T}+\tilde{Z} & \tilde{X}+i\tilde{Y}\\ \tilde{X}-i\tilde{Y} & \tilde{T}-\tilde{Z}\end{pmatrix}=\tilde{\mathbf{Q}}\end{aligned} \tag{3.13}$$

As the action of $\mathbf{A}$ over $\mathbf{Q}$ can easily be seen to be linear, then such action corresponds to a linear transformation of a world vector (T, X, Y, Z). Lorentz transformations are those leaving the Lorentz norm invariant. As $|\mathbf{A}| = 1$, then $|\mathbf{Q}| = |\tilde{\mathbf{Q}}|$, which means that the Lorentz norm is preserved. As Lorentz transformations are bijective and linear, then $\mathbf{A}$ corresponds to an unique Lorentz transformation. Every $\mathbf{A} \in SL(2, \mathbb{C})$ corresponds to a Möbius transformation, as it was shown in proposition 1.5.1.

□

It can be also shown that there is a more restrictive equivalence.

Proposition 3.6.2. Every Möbius transformation corresponds to a an unique Lorentz restricted transformation.

Proof. A Lorentz transformation is restricted if and only if they are continuous with the identity. It can be easily seen that the identity Lorentz transformation corresponds to the identity in $SL(2, \mathbb{C})$. Let's consider the matrix $\mathbf{B} = \lambda \mathbf{I} + (1 - \lambda)\mathbf{A})$ with $\lambda \in \mathbb{C}$. There are exactly two values of λ for $\mathbf{B}$ not being singular, which are not 1 nor 0. Then, there is a complex curve $\lambda(t) : (a, b) \mapsto \mathbb{C}_\infty$ so that $\lambda(a) = 0, \lambda(b) = 1$. It is clear that the matrix $\mathbf{C} = |\mathbf{B}|^{-1/2}\mathbf{B}$ is a member of $SL(2, \mathbb{C})$ so that $\mathbf{C}(\lambda = 0) = \mathbf{A}$ and $\mathbf{C}(\lambda = 1) = \mathbf{I}$. As $\lambda(t)$ goes smoothly from 0 to 1, $\mathbf{A}$ is continuous to identity. □

We have proven that every Möbius transformation corresponds to a restricted Lorentz transformation. Now we are going to proof the inverse relation.

Restricted Lorentz transformations consist on boosts and rotations. A boost on an arbitrary direction can be expressed as the consecutive application of a rotation so that such direction is "placed" in the z axis, a boost in the z axis and a rotation such as the z axis is placed on the original direction. Then, we are going to find now equivalences of spatial rotations and boosts in the z direction with matrices in $SL(2, \mathbb{C})$.

The next result will help to find the expression of rotations

Proposition 3.6.3. Every unitary matrix in $SL(2, \mathbb{C})$ corresponds to a spatial rotation.

Proof. Unitary matrices (that is $\mathbf{A}^{-1} = \mathbf{A}^*$) leave traces invariant, then applying one to the matrix 3.12 leaves its trace $\operatorname{Tr}\mathbf{Q} = \sqrt{2}T$ invariant. As we have already established, $\mathbf{A}$ corresponds to a Lorentz transformation, then it leaves the quantity $X^2 + Y^2 + Z^2$ invariant, which is the property that characterizes a spatial rotation. □

If we find an unitary matrix for rotations along every spatial axis, then we would be able to find a matrix corresponding to any rotation, as an arbitrary rotation can be expressed composition of rotation along the three axis.

A rotation along the z axis of S^+, and thus the Riemann sphere, has already been derived. It corresponds to the elliptic hyperbolic transformation

$$\tilde{\zeta} = e^{i\phi}\zeta.$$

This can be expressed with the two matrices

$$\mathbf{R}_z = \pm \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

as every Möbius transformation corresponds to two matrices, one the opposite of the other.
Now, we need to find a transformation for rotations along the Y and X axis. The matrices

$$\mathbf{R}_y^{\pm} = \pm \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

are unitary and leave the quantity $\xi\overline{\eta} - \eta\overline{\xi}$ invariant, thus, according to equation 3.10, Y is preserved, which corresponds to a rotation on the y-axis. Computing the expression 3.13, it can be shown that it is a rotation of an angle θ. The same procedure can be applied to verify that the matrices

$$\mathbf{R}_x^{\pm} = \pm \begin{pmatrix} \cos\chi/2 & i\sin\chi/2 \\ i\sin\chi/2 & \cos\chi/2 \end{pmatrix}$$

corresponds to a rotation of an angle χ along the x axis.All of this allow us to state the next result.

Proposition 3.6.4. Every spatial rotation corresponds to an unitary matrix in $SL(2,\mathbb{C})$.

Proof. As we have found unitary matrices corresponding to rotations along the three axis, every other rotation must be the product of these matrices. As the product of unitaries matrices are unitary matrices, every rotation corresponds to an unitary matrix. □

As we said, we have to still proof that a boost corresponds to a matrix in $SL(2,\mathbb{C})$. A boost in the z axis corresponds to how coordinates change for an observer traveling at a constant velocity v. Its explicit form is

$$\tilde{T} = (1-v^2)^{-1/2}(T+vZ), \quad \tilde{X} = X, \tilde{Y} = Y, \quad \tilde{Z} = (1-v^2)^{-1/2}(Z+vT) \tag{3.14}$$

This transformation can also be written as

$$\tilde{T} + \tilde{Z} = w(T+Z), \quad \tilde{T} - \tilde{Z} = w^{-1}(T-Z), \tilde{X} = X, \quad \tilde{Y} = Y \tag{3.15}$$

with

$$w = \left(\frac{1+v}{1-v}\right)^{1/2}. \tag{3.16}$$

Looking at the matrix 3.10, it can be seen that this corresponds to the application of the matrix

$$\mathbf{B}^{\pm} = \pm \begin{pmatrix} w^{1/2} & 0 \\ 0 & w^{-1/2} \end{pmatrix}, \tag{3.17}$$

which, in turn, corresponds to the Möbius transformation

$$\tilde{\zeta} = w\zeta \tag{3.18}$$

which lead us to the conclusion

Proposition 3.6.5. Every Lorentz boost corresponds to two hermitian matrices in $PSL(2,\mathbb{C})$ and every hermitian matrix in $PSL(2,\mathbb{C})$ to a Lorentz Boost .

Proof. As we stablished before, we can get a Lorentz boost in any direction by a combination of rotations and boosts in the z axis as we stated before. A boost in the z direction is unequivocally determined by the two matrices (3.17), which is hermitian. As the product $\mathbf{U}^{-1}\mathbf{B}\mathbf{U}$ is hermitian if $\mathbf{B}$ is hermitian and $\mathbf{U}$ unitary, then every boost corresponds to two hermitian matrices. Also, every hermitian matrix $\mathbf{H}$ can be diagonalized by applying an unitary transform $\mathbf{A}$ as $\mathbf{A}^{-1}\mathbf{H}\mathbf{A}$, which can be equated to (3.17), as determinant and hermiticity must be preserved by unitary transformations. □

We have proven all the relations we wanted to proof. As there is a one to one correspondence between the restricted Lorentz group and the Möbius group, then it follows:

Theorem 3.6.1. *The restricted Lorentz group and the Möbius group are isomorphic,*

$$\mathcal{M} \cong \mathcal{L}_+^{\uparrow}.$$

This total correspondence allows us to identify every Lorentz transformation with each of the different types of Möbius transformations.

- Hyperbolic transformations to boosts in any direction
- Elliptic transformations correspond to rotations along any axis
- Loxodromic transformations correspond to compositions of boosts and rotations in the same axis.
- Parabolic transformations correspond to a composition of boosts and rotations such as only one point is left invariant in the celestial-sphere.

3.7 Transforming the celestial sphere

We got a one to one correspondence with the Möbius Group and the restricted Lorentz. This will allow us to study how cellestial spheres changes for observers going through rotations and/or Lorentz Boosts.
Let's supose we are on a space-ship and from its windows we can see distant stars. We can see each of the stars as points in the cellestial sphere. Then, the algorithm we must follow is the following

- Apply the anti-sky mapping Σ to the cellestial sphere to go to S^+
- Using the stereographic formulae to assign a complex point to each point in S^+, i.e. mapping S^+ to the Riemann sphere.
- Applying the corresponding Möbius transformations to the Riemann sphere.
- Apply the inverse stereographic map and the inverse anti-sky mapping to get to the transformed S^-

The process followed is shown in figure 3.4, where we apply this process for a boost in the z-direction. This simple treatment only covers how the position of light sources would change in the sky. Other effects, as red shift, are of course not included in this treatment.

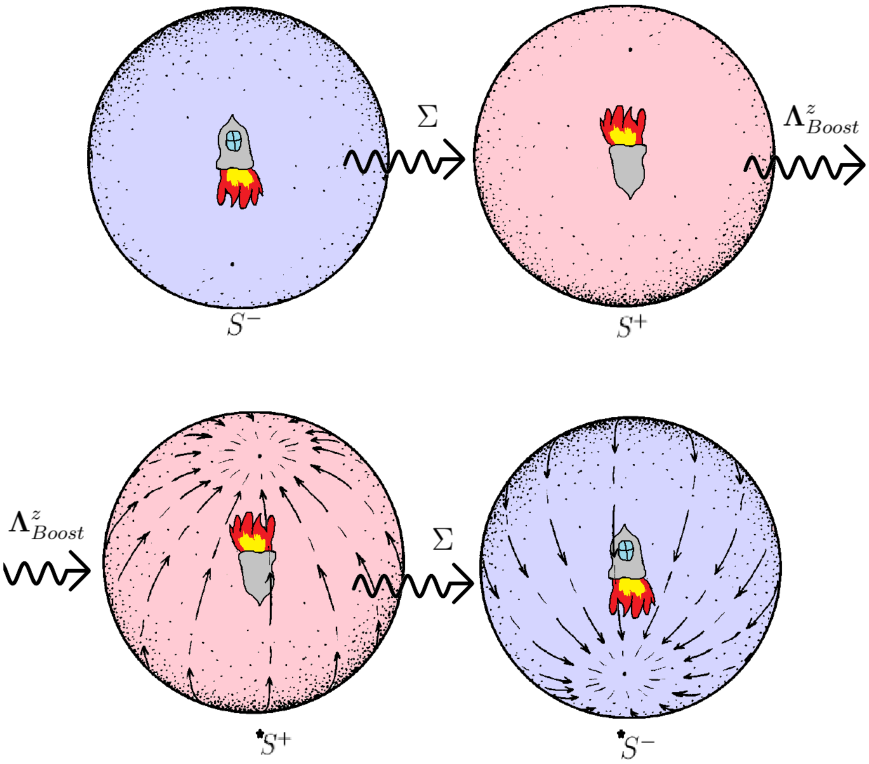

Figure 3.4: Boost as seen by an spaceship. Figure heavily modified taken from [8]

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