

On the vulnerability of some families of graphs

Rocío M. Casablanca, Ana R. Diáñez and Pedro García-Vázquez
Universidad de Sevilla
Sevilla

Abstract

The toughness of a noncomplete graph G is defined as $\tau(G) = \min\{|S|/\omega(G - S)\}$, where the minimum is taken over all cutsets S of vertices of G and $\omega(G - S)$ denotes the number of components of the resultant graph $G - S$ by deletion of S . In this paper, we investigate the toughness of the corona of two connected graphs and obtain the exact value for the corona of two graphs belonging to some families as paths, cycles, wheels or complete graphs. We also get an upper and a lower bounds for the toughness of the cartesian product of the complete graph K_2 with a predetermined graph G .

1 Introduction

Throughout this paper, all the graphs are simple, that is, without loops and multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [3].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The graph G is called *connected* if every pair of vertices is joined by a path. A *cutset* in a graph G is a subset $S \subset V(G)$ of vertices of G such that $G - S$ is not connected.

The existence of a cutset is always guaranteed in every graph different from a complete graph K_n . The index of *connectivity* of G , denoted by $\kappa(G)$, is defined as the minimum cardinality over all cutsets of G , if G is a noncomplete graph, or $|V(G)| - 1$, otherwise.

There are several measures of vulnerability of a network. The vulnerability parameters one generally encounters are the indices of connectivity

and edge-connectivity. These two parameters give the minimum cost to disrupt the network, but they take no account of what remains after the destruction. To measure the vulnerability of networks more properly, some vulnerability parameters have been introduced and studied. Among them are toughness, integrity, scattering number, tenacity and several variants of connectivity and edge-connectivity called conditional connectivity, each of which measures not only the difficulty of breaking down the network but also the damage caused. In general, for most of the aforementioned parameters, the corresponding computational problem is *NP*-hard. So it is of interest to give the formulae or algorithms for computing these parameters for special classes of graphs. For our purpose, we deal with the notion of *toughness*, introduced by Chvátal [4], which pays special attention to the relationship between the cardinality of the rupture set in the network and the number of components after the rupture. The parameter is defined as

$$\tau(G) = \min \{|S|/\omega(G - S) : S \subseteq J(G)\},$$

where

$$J(G) = \{S \subset V(G) : S \text{ is a cutset of } G \text{ or } G - S \text{ is an isolated vertex}\},$$

and $\omega(G - S)$ denotes the number of components in the resultant graph $G - S$ by removing S .

Since this parameter was introduced, lots of research has been done, mainly relating toughness conditions to the existence of cycle structures. Historically, most of the research was based on a number of conjectures in [4]. Some of most interesting results are [1, 2, 5]. However, exact values of $\tau(G)$ are known only for a few families of graphs as paths and cycles [4], the cartesian product of two complete graphs [4] and of paths and/or cycles [7], and the composition of two graphs, one of them being a path, a cycle or a complete bipartite graph [7]. In this paper we focus on the toughness of two families of graphs: the *corona* $G \circ H$ of two graphs [6] and the *cartesian product* $K_2 \times G$.

If for each vertex x in a graph G , we introduce a new vertex x' and join x and x' by an edge, the resulting graph is called the *corona* of G . The operation of adding one vertex for each vertex of G and connecting them by an edge can be generalized as follows. The corona of any two graphs G and H , denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H , and then joining the i th vertex of G to every vertex

in the i th copy of H . Observe that the particular case in which $H = K_1$, the graph $G \circ K_1$ is called the corona of G . The *cartesian product* $K_2 \times G$ of the complete graph K_2 and any graph G is the graph with vertex set $V(K_2) \times V(G)$ in which vertex (i, u) , for $i = 1, 2$, is adjacent to vertex (j, v) whenever $i = j$ and $uv \in E(G)$, or $i \neq j$ and $u = v$ [6].

There exists several kinds of interconnection networks whose structure can be modeled in terms of the cartesian product or the corona of two predetermined networks. The cartesian product of graphs seeks to establish parallel connections between identical structures, minimizing the cost of such connections. The corona of two predetermined graphs is often present in electric networks distributed in a big city where each transformer must guarantee the energy supply of its catchment area. In order to optimize resources, the distribution of transformers is made by dividing the city in catchment areas of the same entity. Thus, in terms of Graph Theory, the structure to be analyzed consists of a network transformers, modeled by a graph, G where each transformer is connected with its catchment area, modeled by the graph H . The resultant graph is the corona $G \circ H$ of G and H . In the maintenance of electric networks is relevant to avoid the disruption of the energy supply, but when the failure in some nodes produces the rupture of the network, the greater the number of fragments in which the network has been divided, the greater the cost of reconstruction.

The relationship between the cardinality of a cutset of a graph G and the remaining component after disruption is analyzed by the notion of toughness, defined above. So our aim in this work is to determine the toughness of the corona $G \circ H$ of two connected graphs G and H in terms of known parameters of them. As a consequence, we will deduce the exact value of the corona of some families of graphs involving stars, paths, cycles, wheels or complete graphs. We will also find an upper and a lower bounds for the toughness of $K_2 \times G$, for any arbitrary graph G .

2 The toughness of the corona of two graphs

2.1 Notations and remarks

Let G, H be two connected graphs on m and n vertices, respectively. Let us set $V(G) = \{v_1, \dots, v_m\}$ and denote by H_i the copy of H that is joined to vertex v_i of G in $G \circ H$. Thus, every cutset S of $G \circ H$ will henceforth expressed as $S = S_0 \cup \bigcup_{i=1}^m S_i$, where $S_0 \subseteq V(G)$ and $S_i \subseteq V(H_i)$, for

$i = 1, \dots, m$. We denote by $\omega_0 = \omega(G - S_0)$, $\omega_i = \omega(H_i - S_i)$, $i = 1, \dots, m$, that is, the number of component of $G - S_0$ and $H_i - S_i$, $i = 1, \dots, m$, respectively.

A cutset of $G \circ H$ such that $|S|/\omega(G \circ H - S) = \tau(G \circ H)$ will be called a τ -cut of $G \circ H$. Let us see some remarks on the τ -cut of the corona of two graphs.

Remark 1 If $S = S_0 \cup \bigcup_{i=1}^m S_i$ is a cutset of the corona $G \circ H$ of two connected graphs G, H , then $S_0 \neq \emptyset$.

Proof: If $S_0 = \emptyset$ then every vertex of $G \circ H - S$ either is in $V(G)$ or is adjacent to one vertex of G , hence, $G \circ H - S$ is connected, against the fact that S is a cutset of $G \circ H$. \square

Remark 2 Let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of the corona $G \circ H$ of two connected graphs G, H . If $v_j \in S_0$ then either $S_j = \emptyset$ or S_j is a cutset of H_j .

Proof: Let $v_j \in S_0$ and suppose by way of contradiction that $S_j \neq \emptyset$ is not a cutset of H_j . Let us consider the set $S^* = S \setminus S_j$. Observe that either $H_j - S_j$ is a component of $G \circ H - S$ or $S_j = V(H_j)$ and H_j is a component of $G \circ H - S^*$. Thus, $\omega(G \circ H - S^*) \geq \omega(G \circ H - S)$ and therefore,

$$\frac{|S^*|}{\omega(G \circ H - S^*)} \leq \frac{|S| - n}{\omega(G \circ H - S)} < \frac{|S|}{\omega(G \circ H - S)} = \tau(G \circ H - S),$$

which contradicts the hypothesis that S is a τ -cut of $G \circ H$. Then either $S_j = \emptyset$ or S_j is a cutset of H_j . \square

Remark 3 Let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of the corona $G \circ H$ of two connected graphs G, H . If $v_j \notin S_0$ then $S_j = \emptyset$.

Proof: Let $v_j \notin S_0$ and suppose by way of contradiction that $S_j \neq \emptyset$. Let us consider the set $S^* = S \setminus S_j$. Observe that either $H_j - S_j$ belongs to the component of $G \circ H - S$ that contains vertex v_j or $S_j = V(H_j)$ and H_j belongs to the component of $G \circ H - S^*$ that contains vertex v_j . Thus, $\omega(G \circ H - S^*) = \omega(G \circ H - S)$ and therefore,

$$\frac{|S^*|}{\omega(G \circ H - S^*)} = \frac{|S| - n}{\omega(G \circ H - S)} < \frac{|S|}{\omega(G \circ H - S)} = \tau(G \circ H - S),$$

which is again a contradiction with the fact that S is a τ -cut of $G \circ H$. Then $S_j = \emptyset$. \square

Let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of $G \circ H$. From now on, we may assume without loss of generality that the vertices of the set $V(G) = \{v_1, \dots, v_m\}$ are ordered so that $|S_1| \geq \dots \geq |S_m|$. Let $k \in \{1, \dots, m\}$ be the maximum integer such that $S_i \neq \emptyset$ for all $i = 1, \dots, k$. Then, as an immediate consequence of Remark 1, Remark 2 and Remark 3, it follows that $|S| = |S_0| + \sum_{i=1}^k |S_i|$ and $\omega(G \circ H - S) = \omega_0 + \sum_{i=1}^k \omega_i + |S_0| - k$.

2.2 Main results

Let G, H be two connected graphs on m and n vertices, respectively. Our purpose is to determine the toughness of the corona $G \circ H$ of G and H . To begin with, given a τ -cut of $G \circ H$, the first question that we must answer is whether every copy of graph H can be disconnected in the same way. The following lemma provides an answer to this question.

Lemma 4 *Let G, H be two connected graphs of order m and n , respectively, and let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of $G \circ H$ of minimum cardinality. If $S_i \neq \emptyset, S_j \neq \emptyset$, for $i, j = 1, \dots, m$ with $i \neq j$, then $|S_i| = |S_j|$ and $\omega_i = \omega_j$.*

Proof: Let us consider the vertex set $V(G) = \{v_1, \dots, v_m\}$ ordered so that $|S_1| \geq \dots \geq |S_m|$, and let $k \in \{1, \dots, m\}$ be the maximum integer such that $S_i \neq \emptyset$ for all $i = 1, \dots, k$. Thus, $|S| = |S_0| + \sum_{i=1}^k |S_i|$. Since S is a τ -cut of $G \circ H$, we have

$$\tau(G \circ H) = \frac{|S_0| + \sum_{i=1}^k |S_i|}{\omega_0 + \sum_{i=1}^k \omega_i + |S_0| - k} \leq \frac{|S_0| + k|S_\ell|}{\omega_0 + k\omega_\ell + |S_0| - k}, \quad (1)$$

for every $\ell = 1, \dots, k$,

yielding to

$$\begin{aligned}
 & \left(|S_0| + \sum_{i=1}^k |S_i| \right) k\omega_\ell + (\omega_0 + |S_0| - k) \sum_{i=1}^k |S_i| \\
 & \leq |S_0| \sum_{i=1}^k \omega_i + \left(\omega_0 + \sum_{i=1}^k \omega_i + |S_0| - k \right) k|S_\ell|, \text{ for } \ell = 1, \dots, k.
 \end{aligned} \tag{2}$$

By taking summation in (2) we deduce that

$$\begin{aligned}
 & \left(|S_0| + \sum_{i=1}^k |S_i| \right) k \sum_{\ell=1}^k \omega_\ell + k(\omega_0 + |S_0| - k) \sum_{i=1}^k |S_i| \\
 & \leq k|S_0| \sum_{i=1}^k \omega_i + \left(\omega_0 + \sum_{i=1}^k \omega_i + |S_0| - k \right) k \sum_{\ell=1}^k |S_\ell| \\
 & = \left(|S_0| + \sum_{\ell=1}^k |S_\ell| \right) k \sum_{i=1}^k \omega_i + k(\omega_0 + |S_0| - k) \sum_{\ell=1}^k |S_\ell|,
 \end{aligned}$$

which implies that all the inequalities of (2) become equalities, and therefore, all the inequalities of (1) become equalities. Thus,

$$\begin{aligned}
 \tau(G \circ H) &= \frac{|S_0| + k|S_i|}{\omega_0 + k\omega_i + |S_0| - k} = \frac{|S_0| + k|S_j|}{\omega_0 + k\omega_j + |S_0| - k}, \\
 & \text{for all } i, j = 1, \dots, k,
 \end{aligned} \tag{3}$$

which means that the set $S^* = S_0 \cup \bigcup_{i=1}^k S_i^*$, where $S_i^* = S_k$, for all $i = 1, \dots, k$, is also a τ -cut. Hence,

$$|S| = |S_0| + \sum_{i=1}^k |S_i| \geq |S_0| + k|S_k| = |S^*|,$$

yielding to $|S_1| = \dots = |S_k|$ because S has minimum cardinality. Moreover, given any two subsets S_i, S_j , with $i, j \in \{1, \dots, k\}$ and $i \neq j$, from (3) it is clear that $\omega_i = \omega_j$. Then the result holds. \square

Given a τ -cut $S = S_0 \cup \bigcup_{i=1}^m S_i$ of $G \circ H$ with minimum cardinality, by Lemma 4 we may assume without loss of generality that for each $i = 1, \dots, m$, either $S_i = \emptyset$ or $S_i = S_H$, for some $S_H \subset V(H)$. Furthermore, it

follows that either $\omega(H_i - S_i) = 1$ (if $S_i = \emptyset$) or $\omega(H_i - S_i) = \omega(H - S_H)$ (if $S_i = S_H$).

To upper bound the index of toughness of $G \circ H$, it is enough to find a cutset S of $G \circ H$ and compute $|S|/\omega(G \circ H - S)$. There are some alternatives in the choice of such a cutset, as the following proposition shows.

Proposition 5 *Let G, H be two connected graphs of order m and n , respectively. Let $S_H \subset V(H)$ be any cutset of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Then*

$$\tau(G \circ H) \leq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)}, \frac{1 + p}{1 + q}, \frac{1 + p}{\frac{1}{\tau(G)} + q} \right\}.$$

Proof: First, let v_j be any vertex of $V(G)$ and let us consider the set $S = \{v_j\}$ in $G \circ H$. Then S is a cutset and $G \circ H - S$ since v_j separates the copy H_j of H from $G \circ H - (\{v_j\} \cup V(H_j))$. Furthermore, $G \circ H - S$ has at least two components, i.e., $\omega(G \circ H - S) = 1 + \omega(G \circ H - (\{v_j\} \cup V(H_j))) \geq 2$, yielding to $\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{1}{2}$.

Second, let $S \subset V(G)$ be a τ -cut of G . Then S is a cutset of $G \circ H$ and $\omega(G \circ H - S) = \omega(G - S) + |S|$ and therefore,

$$\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{|S|}{\omega(G - S) + |S|} = \frac{\frac{|S|}{\omega(G - S)}}{1 + \frac{|S|}{\omega(G - S)}} = \frac{\tau(G)}{1 + \tau(G)}.$$

Third, let $S_H \subset V(H)$ be any cutset of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Take any vertex $v_j \in V(G)$ and set $S_j = S_H \subset V(H_j)$. Let us consider the vertex set $S = \{v_j\} \cup S_j$ and observe that S is a cutset of $G \circ H$. Indeed, $\omega(G \circ H - S) = \omega(G - v_j) + \omega(H_j - S_j) \geq 1 + \omega(H_j - S_j)$. Thus, if we denote by $p = |S_j|$ and denote by $q = \omega(H - S_H)$, it follows that

$$\tau(G \circ H) \leq \frac{|S|}{\omega(G \circ H - S)} \leq \frac{1 + |S_j|}{1 + \omega(H_j - S_j)} = \frac{1 + p}{1 + q}.$$

Finally, take any cutset $S_H \subset V(H)$ of H of cardinality $|S_H| = p$ and denote by $q = \omega(H - S_H)$. Let $S_0 = \{w_1, \dots, w_{|S_0|}\} \subset V(G)$ be a τ -cut of G and denote by H_i the copy of H joined to vertex w_i in $G \circ H$, for

$i = 1, \dots, |S_0|$. Let us consider the vertex set $S = S_0 \cup \bigcup_{i=1}^{|S_0|} S_i$, where $S_i = S_H$, for every $i = 1, \dots, |S_0|$. Clearly S is a cutset of $G \circ H$ and $\omega(G \circ H - S) = \omega(G - S_0) + |S_0|\omega(H - S_H)$. Hence,

$$\begin{aligned}
 \tau(G \circ H) &\leq \frac{|S|}{\omega(G \circ H - S)} = \frac{|S_0| + |S_0||S_H|}{\omega(G - S_0) + |S_0|\omega(H - S_H)} \\
 &= \frac{|S_0|(1+p)}{\omega(G - S_0) + |S_0|q} \\
 &= \frac{\tau(G)(1+p)}{1 + \tau(G)q} \\
 &= \frac{1+p}{1/\tau(G) + q}.
 \end{aligned}$$

Thus, $\tau(G \circ H) \leq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1+\tau(G)}, \frac{1+p}{1+q}, \frac{1+p}{\tau(G)+q} \right\}$ and the result holds. \square

The next result gives a necessary condition for a τ -cut of $G \circ H$ to contain vertices of some copy H_i .

Lemma 6 *Let G, H be two connected graphs of order m and n , respectively, and let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of $G \circ H$ of minimum cardinality. If $S_j \neq \emptyset$ for some $j = 1, \dots, m$, then $|S_j|/\omega(H_j - S_j) < 1/2$.*

Proof: From Lemma 4 there exists a vertex set $S_H \subset V(H)$ such that either $S_i = \emptyset$ or $S_i = S_H$, for every $i = 1, \dots, m$. So without loss of generality we may assume that there is an integer $k \in \{1, \dots, m\}$ such that $S = S_0 \cup \bigcup_{i=1}^k S_H$; that is, $S_i = S_H$ if $i \in \{1, \dots, k\}$ and $S_i = \emptyset$ otherwise. Therefore, it is enough to us to prove that $|S_H|/\omega(H - S_H) < 1/2$. To clarify expressions, denote by $\omega_0 = \omega(G - S_0)$ and $\omega_H = \omega(H - S_H)$. By applying Remark 1, we know that $S_0 \neq \emptyset$, and from Remark 2 and Remark 3 it follows that $k \leq |S_0|$. Thus, $|S| = |S_0| + k|S_H|$ and $\omega(G \circ H - S) = \omega_0 + k\omega_H + |S_0| - k$. By applying Proposition 5 we know that $\tau(G \circ H) \leq 1/2$, which implies that

$$\frac{|S|}{\omega(G \circ H - S)} = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k} \leq \frac{1}{2},$$

yielding to

$$\frac{|S_H|}{\omega_H} \leq \frac{1}{2} + \frac{\omega_0 - (|S_0| + k)}{2k\omega_H}. \quad (4)$$

Since $S_0 \neq \emptyset$ because of Remark 1, and $k \geq 1$, if S_0 is not a cutset of G then $\omega_0 \leq 1$ (i.e., $\omega_0 = 0$ if $S_0 = V(G)$, and $\omega_0 = 1$ otherwise). Hence, applying inequality $\omega_0 - (|S_0| + k) < 0$ in (4), we have $\frac{|S_H|}{\omega_H} < \frac{1}{2}$. Thus, suppose that $S_0 \subset V(G)$ is a cutset of G .

First assume that $|S_0|/\omega_0 \geq 1$. This means that $\omega_0 - (|S_0| + k) < \omega_0 - |S_0| \leq 0$, yielding in (4) to $\frac{|S_H|}{\omega_H} < \frac{1}{2}$.

Second assume that $|S_0|/\omega_0 < 1$. Since S_0 is a cutset of G then it is also a cutset of $G \circ H$ and $\omega(G \circ H - S_0) = \omega_0 + |S_0|$. Therefore, by using that S is a τ -cut of $G \circ H$, it follows that

$$\frac{|S_0|}{\omega_0 + |S_0|} \geq \tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k} > \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0|}. \quad (5)$$

Combining the first and the last members of (5) we deduce that

$$\frac{|S_H|}{\omega_H} < \frac{|S_0|}{\omega_0 + |S_0|} = \frac{\frac{|S_0|}{\omega_0}}{1 + \frac{|S_0|}{\omega_0}} < \frac{1}{2},$$

because $|S_0|/\omega_0 < 1$. This concludes the proof. \square

From these previous results it follows the next theorem where the toughness of the corona $G \circ H$ of two connected graphs is determined in terms of some parameter of G and H .

Theorem 7 *Let G, H be two connected graphs of order m and n , respectively. Then the following assertions holds:*

(i) *If $\tau(G) \geq 1$ and $\tau(H) \geq 1/2$, then $\tau(G \circ H) = \frac{1}{2}$.*

(ii) *If $\tau(G) < 1$ and $\tau(H) \geq 1/2$, then $\tau(G \circ H) = \frac{\tau(G)}{1 + \tau(G)}$.*

(iii) *If $\tau(G) \geq 1$ and $\tau(H) < 1/2$, then*

$$\tau(G \circ H) = \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega(H - S_H)} \right\}.$$

(iv) *If $\tau(G) < 1$ and $\tau(H) < 1/2$, then*

$$\tau(G \circ H) = \min \left\{ \frac{\tau(G)}{1 + \tau(G)}, \min_{S_H \in J(H)} \frac{1 + |S_H|}{\frac{1}{\tau(G)} + \omega(H - S_H)} \right\}.$$

Proof: Let $S = S_0 \cup \bigcup_{i=1}^m S_i$ be a τ -cut of $G \circ H$. Without loss of generality we may assume that $V(G) = \{v_1, \dots, v_m\}$, where the vertices are numbered so that $|S_i| \geq |S_{i+1}|$, for all $i = 1, \dots, m-1$. We also may suppose that S has minimum cardinality over all the τ -cuts of $G \circ H$.

First, assume that $\tau(H) \geq 1/2$. Then by applying Lemma 6 we deduce that $S_i = \emptyset$, for all $i = 1, \dots, m$, hence, $S = S_0$. This implies that $\tau(G \circ H) = \frac{|S|}{\omega(G \circ H - S)} = \frac{|S_0|}{\omega_0 + |S_0|}$. Note that $S_0 \neq V(G)$, because otherwise, we have $\omega(G \circ H - S_0) = 0$ and therefore, $\tau(G \circ H) = 1$, which is a contradiction with Proposition 5, thus, $S_0 \subset V(G)$, which means that $\omega(G \circ H - S_0) \geq 1$. If S_0 is not a cutset of G then $\omega(G \circ H - S_0) = 1$ and therefore, $\tau(G \circ H) = \frac{|S_0|}{\omega_0 + |S_0|} = \frac{|S_0|}{1 + |S_0|} \geq \frac{1}{2}$. If S_0 is a cutset of G then $\omega(G \circ H - S_0) \geq 2$ and therefore,

$$\tau(G \circ H) = \frac{|S_0|}{\omega_0 + |S_0|} = \frac{\frac{|S_0|}{\omega_0}}{1 + \frac{|S_0|}{\omega_0}} \geq \frac{\tau(G)}{1 + \tau(G)}.$$

Hence, $\tau(G \circ H) \geq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)} \right\}$. Moreover, by Proposition 5 we have $\tau(G \circ H) \leq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)} \right\}$, yielding to

$$\tau(G \circ H) = \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)} \right\} = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1 \\ \frac{\tau(G)}{1 + \tau(G)}, & \text{if } \tau(G) < 1 \end{cases},$$

which proves items (i) and (ii).

Second, assume that $\tau(H) < 1/2$. If $S_1 = \emptyset$ then $S_i = \emptyset$ for every $i = 1, \dots, m$, and reasoning as above, we prove that

$$\tau(G \circ H) \geq \min \left\{ \frac{1}{2}, \frac{\tau(G)}{1 + \tau(G)} \right\} = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1 \\ \frac{\tau(G)}{1 + \tau(G)}, & \text{if } \tau(G) < 1 \end{cases} \quad (6)$$

Thus, suppose that $S_1 \neq \emptyset$, then by Lemma 4 we may assume that there exist an integer $k \in \{1, \dots, m\}$ and a nonempty vertex set $S_H \subset V(H)$ such that $S_i = S_H$ if $i \leq k$, and $S_i = \emptyset$ otherwise. Further, from Lemma 6, it follows that $|S_H|/\omega(H - S_H) < 1/2$. Again to clarify expressions, denote by $\omega_H = \omega(H - S_H)$. Notice that $k \leq |S_0|$ because of Remark 2 and Remark 3 and therefore, $\tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k}$. Since S_0 is also a cutset of $G \circ H$, $|S_0| > |S|$ and S is a τ -cut of $G \circ H$ of minimum cardinality, then

$$\frac{|S_0|}{\omega_0 + |S_0|} > \tau(G \circ H) = \frac{|S_0| + k|S_H|}{\omega_0 + k\omega_H + |S_0| - k},$$

yielding to

$$|S_H|(\omega_0 + |S_0|) - |S_0|(\omega_H - 1) < 0. \quad (7)$$

The function $f(k) = \frac{|S_0|+k|S_H|}{\omega_0+k\omega_H+|S_0|-k}$ has derivate

$$\frac{df}{dk} = \frac{|S_H|(\omega_0 + |S_0|) - |S_0|(\omega_H - 1)}{(\omega_0 + k\omega_H + |S_0| - k)^2},$$

and by (7), we deduce that $f(k)$ is decreasing in k . Hence,

$$\tau(G \circ H) = f(k) \geq f(|S_0|) = \frac{|S_0|(1 + |S_H|)}{\omega_0 + |S_0|\omega_H}. \quad (8)$$

If S_0 is not a cutset of G then $\omega_0 \leq 1$ ($\omega_0 = 0$ if $S_0 = V(G)$, and $\omega_0 = 1$ otherwise), and from (8) we have

$$\begin{aligned} \tau(G \circ H) &\geq \frac{|S_0|(1 + |S_H|)}{1 + |S_0|\omega_H} = \frac{1 + |S_H|}{1/|S_0| + \omega_H} \\ &\geq \frac{1 + |S_H|}{1 + \omega_H} \\ &\geq \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega_H} \right\}. \end{aligned} \quad (9)$$

If S_0 is a cutset of G then $|S_0|/\omega_0 \geq \tau(G)$ and therefore, from (8) it follows that

$$\begin{aligned} \tau(G \circ H) &\geq \frac{|S_0|(1 + |S_H|)}{\omega_0 + |S_0|\omega_H} = \frac{1 + |S_H|}{\omega_0/|S_0| + \omega_H} \\ &\geq \frac{1 + |S_H|}{1/\tau(G) + \omega_H} \\ &\geq \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1/\tau(G) + \omega_H} \right\}. \end{aligned} \quad (10)$$

(iii) Suppose that $\tau(G) \geq 1$, then combining (6), (9) and (10), we deduce that

$$\begin{aligned} \tau(G \circ H) &\geq \min \left\{ \frac{1}{2}, \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega_H} \right\}, \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1/\tau(G) + \omega_H} \right\} \right\} \\ &= \min \left\{ \frac{1}{2}, \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega_H} \right\} \right\}. \end{aligned}$$

Since $\tau(H) > 1/2$ then there exists a cutset $S_H \subset V(H)$ such that $|S_H|/\omega_H < 1/2$, which implies that $2|S_H| + 1 \leq \omega_H$. Then

$$\frac{1 + |S_H|}{1 + \omega_H} \leq \frac{\omega_H - |S_H|}{1 + \omega_H} = \frac{1 + \omega_H - (1 + |S_H|)}{1 + \omega_H} = 1 - \frac{1 + |S_H|}{1 + \omega_H},$$

which means that $\frac{1 + |S_H|}{1 + \omega_H} \leq 1/2$ and therefore,

$$\tau(G \circ H) = \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega_H} \right\}.$$

(iv) Now suppose that $\tau(G) < 1$, then from (6), (9) and (10) it follows that

$$\tau(G \circ H) \geq \min \left\{ \frac{\tau(G)}{1 + \tau(G)}, \min_{S_H \in J(H)} \left\{ \frac{1 + |S_H|}{1/\tau(G) + \omega_H} \right\} \right\}. \quad \square$$

\circ	S_n	P_n	C_n	$W_{1,n}$	K_n
S_m	$\begin{cases} \frac{1}{m}, & \text{si } m > n - 2, \\ \frac{2}{m+n-2}, & \text{si } m \leq n - 2. \end{cases}$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$
P_m	$\begin{cases} \frac{1}{3}, & \text{si } n < 5, \\ \frac{2}{n+1}, & \text{si } n \geq 5. \end{cases}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
C_m	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$W_{1,m}$	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
K_m	$\frac{2}{n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 1: The toughness of the corona of some families of graphs.

As a consequence of Theorem 2, the toughness of the corona of some families of graphs can be derived. Let $n \geq 3$ be an integer. Let us denote by P_n and C_n the path and the cycle with n vertices, respectively; by S_n the complete bipartite graph $K_{1,n-1}$; by $W_{1,n}$ the wheel with $n+1$ vertices; and by K_n the complete graph of order n . As a consequence of Theorem 2, the

toughness of the corona of two graphs, one of them being a complete graph is deduced. Further, in Table 1 we can find the toughness of the corona of two graphs belonging to some of these families: stars, paths, cycles, wheels and complete graphs.

Corollary 8 *Let $m \geq 3$, $n \geq 3$ be two integers and let G , H be two connected graphs. Then the following assertions hold:*

$$(i) \quad \tau(G \circ K_n) = \begin{cases} \frac{1}{2}, & \text{if } \tau(G) \geq 1, \\ \frac{\tau(G)}{1+\tau(G)}, & \text{if } \tau(G) < 1. \end{cases}$$

$$(ii) \quad \tau(K_m \circ H) = \begin{cases} \frac{1}{2}, & \text{if } \tau(H) \geq 1/2, \\ \min_{S_H \subset J(H)} \left\{ \frac{1 + |S_H|}{1 + \omega(H - S_H)} \right\}, & \text{if } \tau(H) < 1/2. \end{cases}$$

3 The toughness of the cartesian product $K_2 \times G$

The main result in this section is the following theorem in which the toughness of $K_2 \times G$ is determined in terms of the some invariants of graph G .

Theorem 9 *Let G be a connected graph of minimum degree δ and independence number β . Then*

$$\min \left\{ \tau(G), \frac{|V(G)|}{1 + \beta} \right\} \leq \tau(K_2 \times G) \leq \min \left\{ 2\tau(G), \frac{\delta + 1}{2} \right\}.$$

Acknowledgement

This research is supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02 and under the Andalusian Government project P06-FQM-01649.

References

- [1] D. Bauer, S.L. Hakimi and E. Schmeichel. Recognizing tough graphs is NP-hard. *Discrete Appl. Math.*, 28:191–195, 1990.

- [2] H.J. Broersma, E. Engels and H. Trossel. Various results on the toughness of graphs. *Networks*, 33:233–238, 1999.
- [3] Gary Chartrand and Linda Lesniak. *Graphs and Digraphs*. Chapman and Hall/CRC, United States of America, 2005.
- [4] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:215–228, 1973.
- [5] R. Faudree, R. Gould, M. Jacobson, L. Lesniak and A. Saito. Toughness, degrees and 2-factors. *Discrete Math.*, 286:245–249, 2004.
- [6] R. Frucht and F. Harary. On the corona of two graphs. *Aequationes Math.*, 4:322–325, 1970.
- [7] W.D. Goddard and H.C. Swart. On the toughness of a graph. *Quaestiones Math.*, 13:217–232, 1990.
- [8] R.E. Pippert. On the toughness of a graph. *Lecture Notes in Math.*, 303:225–233, 1972.