Persistence Partial Matchings Induced by Morphisms between Persistence Modules

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Abstract

The notion of persistence partial matching, as a generalization of partial matchings between persistence modules, is introduced. We study how to obtain a persistence partial matching $\mathcal{G}_f$, and a partial matching $\mathcal{M}_f$, induced by a morphism f between persistence modules, both being linear with respect to direct sums of morphisms. Some of their properties are also provided, including their stability after a perturbation of the morphism f, and their relationship with other induced partial matchings already defined in TDA.

1 Introduction

Persistence modules [1, 2] are of vital importance in topological data analysis (TDA). Specifically, *persistence modules indexed over a poset* P are functors from P to the category of vector spaces, **vect**. When P is a subset of $\mathbb{R}$, together with some minor assumptions, a persistence module can be completely described by a multiset of intervals of $\mathbb{R}$ called its *(persistence)* barcode.

In practice, TDA softwares take discrete data as an input, and give a barcode as the output. In some situations, the user may want to repeat the procedure, after applying some minor modifications to the original data. In such case, two questions arise. Could we reuse the calculations previously made to obtain the new barcode, taking advantage of the similarities in the input? Is there any relation between both barcodes?

Answering the first questing would speed up the calculations significantly. Answering the second question would allow, for example, to describe how intervals in the barcode change (or are kept unchanged) when the data is modified. More concretely, if a change in data induces a morphism $f : V \to U$ between persistent modules indexed over $\mathbb{R}$, answering the second question means to know how f induces a partial matching $\mathcal{B}(V) \to \mathcal{B}(U)$ between the corresponding barcodes. It is known that such partial matching cannot be functorial [3].

To try to answer both questions, we could think of two possible research directions: (1) To consider the morphism f as a persistence module by its own right (seen as a functor from a poset contained in $\mathbb{R}^2$ to **vect**) and to try to describe it in terms of “simple peaces”, that may have an interpretation at the barcode level. (2) To try to define rules that produce a partial matching induced by f, guaranteeing that it satisfies some desirable properties. Before going into details, let us comment on the state of the art in both directions.

1. Decomposition of persistence modules We say that a persistence module V is decomposable when $V \simeq U \oplus W$ with $U, W \neq 0$. Otherwise, V is said to be indecomposable. Indecomposable modules indexed over $\mathbb{R}$ are well-known and called interval modules [1, 4]. Indecomposable modules of the form

$$\begin{array}{c} U \\ {\scriptstyle f}\big\uparrow \\ V \end{array} \simeq \begin{array}{ccccccc} U(1) & \longrightarrow & U(2) & \longrightarrow & \dots & \longrightarrow & U(n) \\ \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow \\ V(1) & \longrightarrow & V(2) & \longrightarrow & \dots & \longrightarrow & V(n) \end{array} \tag{1}$$

being f a morphism between persistence modules, are also well understood when $n \leq 4$. When $n > 4$, the theory becomes increasingly complex, and for $n \geq 6$ there is no way to parametrize the set of indecomposable modules since the underlying graph (the quiver) is of "wild" type [5, 6]. Recall that the category of modules of the form (1), also known as ladder modules, is isomorphic to the category of morphisms between persistence modules indexed over the poset $\mathbf{n} = \{1, \dots, n\}$ (see [7]).

In general, finding a description for all indecomposable modules indexed over a poset P, when P is a subset of $\mathbb{R}^2$, is quite complex. Nevertheless, there are families of persistence modules indexed over P which can be built by adding a well-knwon class of indecomposable modules like rectangle-decomposable modules [6] and block-decomposable modules [8].

2. The induced partial matching χ_f In [3] and [9], given a ladder module $L = V \xrightarrow{f} U$, the authors provided a procedure to construct a partial matching between the two barcodes $\mathcal{B}(V)$ and $\mathcal{B}(U)$. Such partial matching, denoted by χ_f, is induced by the ladder module L. The aim of providing such construction was to give an explicit proof of the *Stability Theorem* for persistence barcodes [3].

In this paper, we follow the second research direction, although knowing that one is not independent of the other. Specifically, our aim is to define a partial matching induced by a morphism between persistence modules, compatible with the decomposition of the morphism, seen as a ladder module, in simpler ladder modules such as the indecomposable ones. See Example 1.1 where we describe the partial matching induced by a specific morphism f that we should obtain to be compatible with the given decomposition of f.

Example 1.1. *Consider a morphism between persistence modules*

$$\begin{array}{c} U \\ {\scriptstyle f}\big\uparrow \\ V \end{array} \simeq \begin{array}{ccccc} U(1) & \longrightarrow & U(2) & \longrightarrow & U(3) \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ V(1) & \longrightarrow & V(2) & \longrightarrow & V(3) \end{array}$$

that can be expressed by the following direct sum:

$$\begin{array}{c} U \\ {\scriptstyle f}\big\uparrow \\ V \end{array} \simeq \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ 0 & \longrightarrow & k & \xrightarrow{\mathrm{Id}} & k \end{array} \oplus \begin{array}{ccccc} k & \xrightarrow{\mathrm{Id}} & k & \longrightarrow & 0 \\ \big\uparrow & & \big\uparrow{\scriptstyle \mathrm{Id}} & & \big\uparrow \\ 0 & \longrightarrow & k & \longrightarrow & 0. \end{array} \tag{2}$$

being Id *the identity map. We will see, in Example 2.8, that the barcode of the persistence module V is composed by the intervals* $[2, 3]$ *and* $[2, 2]$ *while the barcode of the persistence module U is just the interval* $[1, 2]$*. Looking at the direct sum in (2), one would expect that the partial matching induced by f would match* $[2, 2]$ *with* $[1, 2]$ *and leave* $[2, 3]$ *unmatched. Nevertheless, as we will see in Example 2.10, the partial matching χ_f behaves differently.*

In this paper, we introduce a new partial matching, denoted by $\mathcal{M}_f$, induced by a morphism f between persistence modules, that will be linear with respect to the decomposition of ladder modules and, in particular, it will produce the expected matching in Example 2. The partial matching $\mathcal{M}_f$ will be defined by counting the intervals of a particular barcode that we call the *persistence partial matching* induced by f. Through this barcode, denoted by $\mathcal{G}_f$, we will be able to study the stability of $\mathcal{M}_f$ in the sense that we could detect the matchings of $\mathcal{M}_f$ which will remain (or not) after a perturbation of f.

The paper is organized as follows. In Section 2, the background needed to follow this paper is introduced, including some lemmas proved in (or which are direct consequences of) other papers. In Section 3, the new notion of persistence partial matching is defined. We will also introduce the new notions of persistence partial matching $\mathcal{G}_f$, and partial matching $\mathcal{M}_f$, induced by a morphism f between persistence modules. At the end of the section, we will prove that $\mathcal{G}_f$ and $\mathcal{M}_f$ are well-defined. Then, in Section 4, some properties of $\mathcal{G}_f$ and $\mathcal{M}_f$ are proven. In particular, we will prove that $\mathcal{G}_f$ and $\mathcal{M}_f$ are linear with respect to the sum of ladder modules. Some similarities and differences between χ_f and $\mathcal{M}_f$ are discussed in Section 5. Lastly, in Section 6, we will discuss the stability of $\mathcal{G}_f$ and $\mathcal{M}_f$. The main conclusions and open questions arising from this paper are discussed in Section 7.

2 Preliminaries

Let us introduce the background and minor lemmas needed in further sections. We first introduce the category of persistence modules. Then, we recall the concept of totally ordered set of decorated endpoints needed to explain the decomposition theorem of persistence modules. A subsection explaining the concepts of barcodes and partial matchings between them has been added as well. Besides, direct limits of persistence modules will be recalled. Finally, ladder modules and their decomposition will be discussed.

2.1 Persistence modules

All vector spaces considered in this paper will be defined over a fixed field k with unit denoted by 1_k. Vectors will be expressed in column form.

As said in the introduction, a persistence module V indexed over a poset P is a functor from P to **vect**. Then, V consists of a set of vector spaces $V(p)$ for $p \in T$ and a set of linear maps $\rho_{pq} : V(p) \to V(q)$ for $p \leq q$ satisfying that $\rho_{ql}\rho_{pq} = \rho_{pl}$ if $p \leq q \leq l$; and ρ_{pp} being the identity map. The set of linear maps $\{\rho_{pq}\}_{p \leq q}$ will be denoted simply by ρ and called the *structure maps* of V. The direct sum of persistence modules together with the intersection and quotient of persistence modules are also persistence modules.

We will also consider the category of persistence modules with morphisms given by natural transformations. In other words, given two persistence modules (V and U) with structure maps (ρ and ϕ), a morphism $f \in \mathit{Hom}(V, U)$ (denoted also by $f : V \to U$) is given by a set of linear maps $\{f_a\}_{a \in T}$, such that $f_b \rho_{ab} = \phi_{ab} f_a$ if $a \leq b$. A morphism f is *injective* (*surjective*) if all its linear maps f_a, $a \in T$, are injective (surjective). Notice that $\operatorname{Im} f$ and $\operatorname{Ker} f$ are particular cases of persistence modules. We will impose that all persistence modules that appear in this paper satisfy the descending chain condition for images and kernels. In other words, for $t \geq s_1 \geq s_2 \geq \ldots$ and $t \leq r_1 \leq r_2 \leq \ldots$, the chains

$$V(t) \supset \operatorname{Ker} \rho_{s_1 t} \supset \operatorname{Ker} \rho_{s_2 t} \supset \ldots$$

and

$$V(t) \subset \operatorname{Im} \rho_{t r_1} \supset \operatorname{Im} \rho_{t r_2} \supset \ldots$$

stabilizes. Nevertheless, for some results, we need persistence modules to be pointwise finite dimensional, which means that all vector spaces $V(t)$ are finite dimensional.

In this paper, we will restrict ourselves to persistence modules indexed over a totally ordered set. Most of the time, we will deal with persistence modules indexed over $\mathbb{R}$ or a subset of $\mathbb{R}$. We will point out the index poset when it is not clear from the context.

2.1.1 Interleaving of persistence modules

Consider V_ε to be the persistent module V indexed over $\mathbb{R}$ shifted ε to the left. That is, the vector spaces of V_ε are $V_\varepsilon(t) := V(t+\varepsilon)$, $t \in \mathbb{R}$, and the structure maps of V_ε are $\rho'_{pq} := \rho_{p+\varepsilon,q+\varepsilon}$, $p \leq q$. Let $\psi : V \to V_\varepsilon$ denote the set of linear isomorphisms $\{\psi_t : V(t) \to V_\varepsilon(t-\varepsilon)\}_{t\in\mathbb{R}}$ induced by the identity map. Then, the morphism of persistence modules $1^V_\varepsilon : V \to V_\varepsilon$ is defined as follows:

$$\{(1^V_\varepsilon)_t := \psi_{t+\varepsilon} \circ \rho_{t,t+\varepsilon}\}_{t\in\mathbb{R}}.$$

Besides, the persistence module V is said to be ε-trivial if $\operatorname{Im} 1^V_\varepsilon = 0$.

Two persistence modules V and U indexed over $\mathbb{R}$ are said to be ε-interleaved if there exist two morphisms $f : V \to U_\varepsilon$ and $g : U \to V_\varepsilon$ such that

$$f^\varepsilon \circ g = 1^U_{2\varepsilon} \text{ and } g^\varepsilon \circ f = 1^V_{2\varepsilon}.$$

Given a morphism $f : V \to U$ between persistence modules indexed over $\mathbb{R}$, the morphism $f^\varepsilon : V_\varepsilon \to U_\varepsilon$, is defined by the set of linear maps $\{f^\varepsilon_t := f_{t+\varepsilon}\}_{t\in\mathbb{R}}$. Lastly, the functor $\mathbf{1}_\varepsilon$ is defined by

$$\mathbf{1}_\varepsilon(V) := \operatorname{Im} 1^V_\varepsilon \text{ and } \mathbf{1}_\varepsilon(f) := f^\varepsilon|_{\operatorname{Im} 1^V_\varepsilon}.$$

Example 2.1. *Let $\varepsilon < 1$ and let $f : V \to U$ be the morphism between persistence modules given as follows:*

$$\begin{tikzcd}
U(1) = \oplus_2 k \arrow[r, "\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]"] & U(2) = \oplus_3 k \arrow[r, "\mathrm{Id}"] & U(3) = \oplus_3 k \arrow[r, "{[0\ 1\ 0]}"] & U(4) = k \\
V(1) = k \arrow[u, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & V(2) = \oplus_2 k \arrow[u, "\left[\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right]"] \arrow[r, "\mathrm{Id}"] & V(3) = \oplus_2 k \arrow[u, "\left[\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right]"] \arrow[r, "{[0\ 1]}"] & V(4) = k \arrow[u, "\mathrm{Id}"]
\end{tikzcd}$$

Then, $\mathbf{1}_\varepsilon(f)$ is given by

$$\begin{tikzcd}
U(1) = \oplus_2 k \arrow[r, "\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]"] & U(2) = \oplus_3 k \arrow[r, "\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]"] & U(3-\varepsilon) = \oplus_3 k \arrow[r, "{[0\ 1\ 0]}"] & U(4-\varepsilon) = k \\
V(1) = k \arrow[u, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & V(2) = \oplus_2 k \arrow[u, "\left[\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right]"] \arrow[r, "\mathrm{Id}"] & V(3-\varepsilon) = \oplus_2 k \arrow[u, "\left[\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right]"] \arrow[r, "{[0\ 1]}"] & V(4-\varepsilon) = k \arrow[u, "\mathrm{Id}"]
\end{tikzcd}$$

2.2 Decorated points

In this subsection, we will use the notational convention that appears in [3] based on the one introduced in [1].

Let $\mathbb{E}$ denote the set of *decorated endpoints* defined as $\mathbb{E} := \mathbb{R} \times D \cup \{-\infty, \infty\}$, where $D = \{-, +\}$. In the sequel, decorated points $(r, -)$ and $(r, +)$ will be denoted by r^- and r^+, respectively. Note that $\mathbb{E}$ can be seen as a totally ordered set stating that $r^- < r^+$

together with the order inherited by the reals. The sum $(\cdot)+(\cdot) : \mathbb{E}\times\mathbb{R}\to\mathbb{E}$ is defined as $r^{\pm}+s := (r+s)^{\pm}$. It can be proved that there is a bijection between $\mathbb{E}$ and the *cuts* of $\mathbb{R}$ defined in [4]. Besides, there is also a bijection between the pairs $\{(a,b)\in\mathbb{E}\times\mathbb{E} : a<b\}$ and the intervals of $\mathbb{R}$. The following table show all possible cases:

	s^-	s^+	∞
$-\infty$	$(-\infty,s)$	$(-\infty,s]$	$(-\infty,\infty)$
r^-	$[r,s)$	$[r,s]$	$[r,\infty)$
r^+	(r,s)	$(r,s]$	(r,∞)

From now on, an interval of $\mathbb{R}$ represented by $(a,b)\in\mathbb{E}\times\mathbb{E}$, with $a<b$, will be denoted by $\langle a,b\rangle$. Intervals of any other totally ordered set may turn out. In such case we will point it out.

Finally, let us point out that we will use the letters a,b,c and d to denote elements of $\mathbb{E}$; the letters r,s,t and ε to denote elements of $\mathbb{R}$; and the letters p,q and l to denote elements of a general totally ordered set T.

2.3 Sections and decomposition of persistence modules

The results and constructions that appear in this section, except for Lemma 2.4, are directly taken from [4].

Let $c\in\mathbb{E}$ and $t\in\langle c,\infty\rangle$. Let us consider the following operators from a persistence module V indexed over $\mathbb{R}$ with structure maps ρ, to a vector space:

$$\mathrm{Im}^+_{ct}(V) := \bigcap_{s\in\langle c,t^+\rangle}\mathrm{Im}\,\rho_{st},\quad \mathrm{Im}^-_{ct}(V) := \bigcup_{s\in\langle -\infty,c\rangle}\mathrm{Im}\,\rho_{st},\quad \text{for } t\in\langle c,\infty\rangle;$$

$$\mathrm{Ker}^+_{ct}(V) := \bigcap_{r\in\langle c,\infty\rangle}\mathrm{Ker}\,\rho_{tr},\quad \mathrm{Ker}^-_{ct}(V) := \bigcup_{r\in\langle t^-,c\rangle}\mathrm{Ker}\,\rho_{tr},\quad \text{for } t\in\langle -\infty,c\rangle.$$

By convention, $\mathrm{Im}^-_{ct}(V) := 0$ if $c=-\infty$, and $\mathrm{Ker}^+_{ct}(V) := V(t)$ if $c=\infty$.
Now, let $I=\langle a,b\rangle$ be an interval of $\mathbb{R}$. For $t\in I$, let:

$$\begin{aligned} V^+_{It} &:= \mathrm{Im}^+_{at}(V)\cap\mathrm{Ker}^+_{bt}(V)\\ V^-_{It} &:= \mathrm{Im}^-_{at}(V)\cap\mathrm{Ker}^+_{bt}(V)+\mathrm{Im}^+_{at}(V)\cap\mathrm{Ker}^-_{bt}(V)\\ V_{It} &:= V^+_{It}/V^-_{It}\end{aligned}$$

and, for $t\notin I$, let them be 0. Note that $V^-_{It} = \big(\mathrm{Im}^-_{at}(V)+\mathrm{Ker}^-_{bt}(V)\big)\cap V^+_{It}$.

Lemma 2.2 (Lemmas 2.1 and 3.1 of [4])**.** *Let $c\in\mathbb{E}$ and let V be a persistence module indexed over $\mathbb{R}$ with structure maps ρ.*

(a) If $t\in\langle c,\infty\rangle$ and $c\neq-\infty$ then $\mathrm{Im}^+_{ct}(V)=\mathrm{Im}\,\rho_{st}$ for some $s\in\langle c,t^+\rangle$.

(b) If $t\in\langle -\infty,c\rangle$ and $c\neq\infty$ then $\mathrm{Ker}^+_{ct}(V)=\mathrm{Ker}\,\rho_{tr}$ for some $r\in\langle c,\infty\rangle$.

(c) If $s\leq t$ in I then $\rho_{st}V^{\pm}_{Is}=V^{\pm}_{It}$.

(d) If $s\leq t$ in I then the map induced by ρ_{st} is an isomorphism between V_{Is} and V_{It}.

Let us introduce now the concept of section that will be key in some of our arguments. Let us also introduce how sections can be used to prove the decomposition theorem for persistence modules.

A *section* of a vector space A is a pair of vector spaces $\{F^+, F^-\}$ such that $F^- \subseteq F^+ \subseteq A$. We say that a set $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$ of sections (with Λ its index set) of A *is disjoint* if, for all $\lambda \neq \mu$, either $F_\lambda^+ \subseteq F_\mu^-$ or $F_\mu^+ \subseteq F_\lambda^-$; and that it *covers* A provided that for any subspace $B \subset A$ with $B \neq A$ there is some λ satisfying that

$$B + F_\lambda^- \neq B + F_\lambda^+.$$

Lemma 2.3 (Lemma 6.1 of [4]). *Suppose that $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$ is a set of sections of A which is disjoint and covers A. For each $\lambda \in \Lambda$, let W_λ be a subspace of A with $F_\lambda^+ = W_\lambda \oplus F_\lambda^-$. Then $A = \bigoplus_{\lambda \in \Lambda} W_\lambda$.*

We will also deal with the following less restrictive situation.

Lemma 2.4. *Suppose that $\{(F_\lambda^-, F_\lambda^+) : \lambda \in \Lambda\}$ is a set of sections of A which is disjoint. For each $\lambda \in \Lambda$, let W_λ be a subspace of A with $F_\lambda^+ = W_\lambda \oplus F_\lambda^-$. Then $\bigoplus_{\lambda \in \Lambda} W_\lambda \subset V$.*

Proof. The proof of this lemma is given at the beginning of the proof of Theorem 6.1 of [4]. □

Given an interval $I = \langle a, b\rangle$ of $\mathbb{R}$, the *interval module over* I, denoted as k_I, is the persistence module indexed over $\mathbb{R}$ defined as follows: $k_I(t) = k$ if $t \in I$, $k_I(t) = 0$ if $t \notin I$ and $\rho_{st} = \mathrm{Id}$ (the identity map) when $s, t \in I$.

Proposition 2.5 (Lemmas 5.2 and 5.3 of [4]). *For any interval I of $\mathbb{R}$ and any persistence module V indexed over $\mathbb{R}$, we have that $V_{It}^+ = W_{It} \oplus V_{It}^-$ where:*

- *$W_I \simeq \oplus_{mI} k_I$ with mI being the multiplicity of the interval module k_I.*
- *There exists a set of sections $\{(F_{It}^-, F_{It}^+) : t \in \mathbb{R}\}$ which is disjoint and covers V, such that $F_{It}^+ = W_{It} \oplus F_{It}^-$ for all $t \in \mathbb{R}$.*

Combining Lemma 2.3 and Proposition 2.5, the following decomposition theorem can be obtained.

Theorem 2.6 (Theorem 1.2 of [4]). *For any persistence module V indexed over $\mathbb{R}$ satisfying the descending chain condition for images and kernels, we have:*

$$V \simeq \bigoplus_{I \in S} W_I$$

being S a set of intervals of $\mathbb{R}$.

2.4 Basis of persistence modules

A *persistence basis* [10, 1] for a persistence module V is an isomorphism

$$\beta : \oplus_{i \in \mathcal{I}} k_{\langle a_i, b_i\rangle} \to V,$$

being $\mathcal{I}$ an index set. The *persistence generator* $\beta_i : k_{\langle a_i, b_i\rangle} \to V$ is defined as the restriction morphism $\beta|_{k_{\langle a_i, b_i\rangle}}$. When we write $\beta_i \sim \langle a, b\rangle$, we mean that $k_{\langle a, b\rangle}$ is the domain of β_i. In particular, $V(t) = \mathrm{span}\{(\beta_i)_t(1_k) : i \in \mathcal{I}\}$. Extending the use of the operator span to the whole persistence module, then V can be expressed as $\mathrm{span}\{\beta_i : i \in \mathcal{I}\}$.

Lemma 2.7. *Let V be a persistence module indexed over $\mathbb{R}$ and let $c \in \mathbb{E}$. Let $\beta : \oplus_{i\in\mathcal{J}} k_{\langle a_i, b_i\rangle} \to V$ be a persistence basis for V. If $t \in \langle c, \infty\rangle$ then:*

$$\dim \operatorname{Im}^{+}_{ct}(V) = \#\left\{\beta_i : \beta_i \sim \langle a, b\rangle \textit{ with } a \le c,\, t \in \langle a, b\rangle\right\},$$

$$\dim \operatorname{Im}^{-}_{ct}(V) = \#\left\{\beta_i : \beta_i \sim \langle a, b\rangle \textit{ with } a < c,\, t \in \langle a, b\rangle\right\};$$

and if $t \in \langle -\infty, c\rangle$ then:

$$\dim \operatorname{Ker}^{+}_{ct}(V) = \#\left\{\beta_i : \beta_i \sim \langle a, b\rangle \textit{ with } b \le c,\, t \in \langle a, b\rangle\right\},$$

$$\dim \operatorname{Ker}^{-}_{ct}(V) = \#\left\{\beta_i : \beta_i \sim \langle a, b\rangle \textit{ with } b < c,\, t \in \langle a, b\rangle\right\}.$$

Proof. For the first statement, note that $V(s) = \beta(\oplus_{i\in\mathcal{J}} k_{\langle a_i, b_i\rangle}(s))$ since β is an isomorphism. Then,

$$\operatorname{Im}^{+}_{ct}(V) = \bigcap_{s\in\langle c, t^{+}\rangle} \operatorname{Im} \rho_{st} = \bigcap_{s\in\langle c, t^{+}\rangle} \rho_{st}\beta(\oplus_{i\in\mathcal{J}} k_{\langle a_i, b_i\rangle}(s)). \tag{3}$$

In particular, $x \in \operatorname{Im}^{+}_{ct}(V)$ if $\forall s \in \langle c, t^{+}\rangle$ there is a finite set $\mathcal{I}_x \subset \mathcal{J}$ such that

$$x \in \rho_{st}\beta(\oplus_{i\in\mathcal{I}} k_{\langle a_i, b_i\rangle}(s)) = \oplus_{i\in\mathcal{I}_x} \rho_{st}\beta_i(k_{\langle a_i, b_i\rangle}(s)) = \oplus_{i\in\mathcal{I}_x} \beta_i(k_{\langle a_i, b_i\rangle}(t)).$$

Note that $\mathcal{I}_x$ depends on x and not on s then we can commute the direct sum with the intersection in (3), and, for $\mathcal{I} = \bigcup_{x\in\operatorname{Im}^{+}_{ct}(V)} \mathcal{I}_x$, we have:

$$\operatorname{Im}^{+}_{ct}(V) = \bigoplus_{i\in\mathcal{I}} \bigcap_{s\in\langle c, t^{+}\rangle} \rho_{st}\beta_i(k_{\langle a_i, b_i\rangle}(s)).$$

In particular, fixed $i \in \mathcal{I}$, the addend $\bigcap_{s\in\langle c, t^{+}\rangle} \rho_{st}\beta_i(k_{\langle a_i, b_i\rangle}(s))$ is not zero if and only if $s \in \langle a_i, b_i\rangle$ for any $s \in \langle c, t^{+}\rangle$. In other words, $a_i \le c$, $t \in \langle a_i, b_i\rangle$ and the intersection is not null. The other cases can be proven in a similar way. $\square$

2.5 Barcodes and partial matchings

A *multiset* is a pair (S, m) where S is a set and $m : S \to \mathbb{N} \cup \{\infty\}$ represents the multiplicity of the elements of S. An element of the multiset (S, m) will be denoted by the pair $(s, m(s))$ where $s \in S$. A *representation of a multiset* (S, m) is the set

$$\mathit{Rep}(S, m) = \{(s, l) \in S \times \mathbb{N} : l \le m(s)\}$$

From now on, we will use the notation s_l instead of (s, l).

The *(persistence)* barcode of a persistence module V is the multiset $\mathcal{B}(V) = (S, m)$ where S is the set of the intervals that appear in the decomposition of V, being $mI = m(I)$ the multiplicity of $I \in S$ in the decomposition of V. We will denote the collection of all possible barcodes by **B**.

Note that given the barcode $\mathcal{B}(V)$ of a persistence module V and a persistence basis β for V, we can define a (non-unique) bijection $\sigma : \{\beta_i\} \to \mathit{Rep}(\mathcal{B}(V))$ such that $\sigma(\beta_i) = I_l$ if and only if $\beta_i \sim I$.

Example 2.8. *Consider the ladder module $L : V \xrightarrow{f} U$ showed in Example 1.1. The barcodes $\mathcal{B}(V)$, $\mathcal{B}(U)$ and $\mathcal{B}(\operatorname{Im} f)$ are, respectively, $\mathcal{B}(V) = \{([2,3],1),([2,2],1)\}$, $\mathcal{B}(U) = \{[(1,2],1)\}$ and $\mathcal{B}(\operatorname{Im} f) = \{([2,2],1)\}$.*

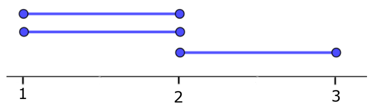
Example 2.9. *Consider a persistent module U isomorphic to*

$$k_{[1,2]} \oplus k_{[1,2]} \oplus k_{[2,3]}$$

through the basis β given by $\beta_1 \sim [1,2]$, $\beta_2 \sim [1,2]$ and $\beta_3 \sim [2,3]$. The persistence module U can be expressed as:

$$U \simeq \oplus_2 k \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \oplus_3 k \xrightarrow{[0\ 0\ 1]} k.$$

Its barcode is $\mathcal{B}(U) = \{([1,2],2),([2,3],1)\}$, and the representation of its barcode is $\mathrm{Rep}(\mathcal{B}(U)) = \{[1,2]_1, [1,2]_2, [2,3]_1\}$, *as shown in the following picture:*



A *partial matching between two barcodes* $\mathcal{B}_1 = (S_1, m_1)$ and $\mathcal{B}_2 = (S_2, m_2)$ is a function $\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{Z}_{\geq 0}$ such that:

$$\text{if } I \in S_1 \text{ then } \sum_{J \in S_2} \mathcal{M}(I,J) \leq m_1(I)$$

and

$$\text{if } J \in S_2 \text{ then } \sum_{I \in S_1} \mathcal{M}(I,J) \leq m_2(J).$$

A *representation of a partial matching* $\mathcal{M}$ is a an injective function $\sigma : \mathit{Rep}(\mathcal{B}_1) \longrightarrow \mathit{Rep}(\mathcal{B}_2)$ such that

$$\mathcal{M}(I,J) = \#\{(I,\cdot) : \sigma(I,\cdot) = (J,\cdot)\} \text{ for all } I \in S_1 \text{ and } J \in S_2.$$

The reader must be aware that what is commonly denoted as a partial matching in the TDA literature [1, 3] is what we call here a representation of a partial matching.

2.5.1 The Bauer-Lesnick induced matching

In this section, we assume that persistence modules are pointwise finite dimensional. This guarantees that all multiplicities appearing in the barcodes are finite. A method for computing a partial matching induced by a morphism between persistence modules is given in [3]. Such method is introduced in this subsection to show similarities and differences with our approach. First, we need to introduce some notation.

Given the representation of a barcode, $\mathit{Rep}(\mathcal{B})$, we denote the set of intervals ending at $b \in \mathbb{E}$ as $\langle \cdot, b\rangle_*$, and the set of intervals starting at $a \in \mathbb{E}$ as $\langle a, \cdot\rangle_*$. That is,

$$\langle a, \cdot\rangle_* = \{\langle a, c\rangle_n \in \mathit{Rep}(\mathcal{B}) : c \in \mathbb{E} \text{ with } a < c\}.$$

and

$$\langle \cdot, b\rangle_* = \{\langle c, b\rangle_n \in \mathit{Rep}(\mathcal{B}) : c \in \mathbb{E} \text{ with } c < b\}.$$

the following order is fixed in these two sets: The longest the interval is, the earlier it appears in the order. Explicitly, we say $\langle c, b\rangle_n < \langle c', b\rangle_l$ if $c < c'$ or $c = c'$ and $n < l$. Besides, $\langle a, c\rangle_n < \langle a, c'\rangle_l$ if $c > c'$ or $c = c'$ and $n < l$.

Now, let $f : V \to U$ be an injective morphism between persistence modules U and V indexed over $\mathbb{R}$. To compute a partial matching induced by f, consider the ordered sets $Q := \langle \cdot, b\rangle_* \subset Rep(\mathcal{B}(V))$ and $R := \langle \cdot, b\rangle_* \subset Rep(\mathcal{B}(U))$ for $b \in \mathbb{E}$. It was proven in [3] that the number of elements of Q is less or equal than the number of elements of R, that is, $\#Q \leq \#R$, so a partial matching between these two sets can be defined by matching the i-th element of Q with the i-th element of R, for $1 \leq i \leq \#Q$. Putting together the partial matchings obtained for all $b \in \mathbb{E}$, we obtain a new partial matching between $Rep(\mathcal{B}(V))$ and $Rep(\mathcal{B}(U))$ denote by ι_f.

A similar procedure is followed in [3] when f is surjective. In such case, consider the sets $Q := \langle a, \cdot\rangle_* \subseteq Rep(\mathcal{B}(V))$ and $R := \langle a, \cdot\rangle_* \subseteq Rep(\mathcal{B}(U))$ for $a \in \mathbb{E}$. Then, $\#R \leq \#Q$. Again, a partial matching between these two sets can be defined by matching the i-th element of R with the i-th element of Q, for $1 \leq i \leq \#R$. Putting together the partial matchings obtained for all $a \in \mathbb{E}$, we obtain a new partial matching between $Rep(\mathcal{B}(V))$ and $Rep(\mathcal{B}(U))$ denoted by λ_f.

Finally, given a morphism $f : V \to U$ between persistence modules V and U indexed over $\mathbb{R}$, and its descomposition $f = g \circ h$ where h is surjective and g is injective:

$$V \xrightarrow{h} \operatorname{Im} f \xrightarrow{g} U,$$

the *Bauer-Lesnick partial matching (or BL-matching) induced by* f is the partial matching representation $\chi_f = \iota_g \circ \lambda_h$ obtained by the composition of partial matching representations ι_g and λ_h:

$$Rep(\mathcal{B}(V)) \xrightarrow{\lambda_h} Rep(\mathcal{B}(\operatorname{Im} f)) \xrightarrow{\iota_g} Rep(\mathcal{B}(U)).$$

Example 2.10. *Consider the ladder module $L : V \xrightarrow{f} U$ showed in Example 1.1; and the barcodes $\mathcal{B}(V)$, $\mathcal{B}(U)$ and $\mathcal{B}(\operatorname{Im} f)$ given in Example 2.8. Their representations are, respectively:*

$$\begin{aligned}
&\mathrm{Rep}(\mathcal{B}(V)) = \{[2, 3]_1, [2, 2]_1\},\\
&\mathrm{Rep}(\mathcal{B}(U)) = \{[1, 2]_1\}, \text{ and}\\
&\mathrm{Rep}(\mathcal{B}(\operatorname{Im} f)) = \{[2, 2]_1\}.
\end{aligned}$$

Then,

$$\lambda_h([2, 3]_1) = [2, 2]_1, \qquad \iota_g([2, 2]_1) = [1, 2]_1$$

and

$$\chi_f([2, 3]_1) = [1, 2]_1, \qquad \chi_f([2, 2]_1) = \emptyset.$$

Note that the computed BL-matching χ_f is different from the matching that we would expect, discussed in Example 1.1.

Now, recall that if χ_f matches two intervals, they must satisfy the following condition.

Proposition 2.11 (Theorem 6.1 of [3])**.** *Let $a, b, c, d \in \mathbb{E}$ and let χ_f be a BL-matching. If $\chi_f(\langle a, b\rangle) = \langle c, d\rangle$, then $c \leq a < d \leq b$.*

Proposition 2.12 (Proposition 5.4 in [3])**.** *Let $f : V \to U$ be a morphism between two pointwise finite dimensional persistence modules. Then, the BL-matching χ_f is completely determined by $\mathcal{B}(V)$, $\mathcal{B}(U)$ and $\mathcal{B}(\operatorname{Im} f)$. Conversely, χ_f completely determines these three barcodes. In particular, $\mathcal{B}(\operatorname{Im} f) = \{I \cap J : \chi_f(I) = J\}$.*

Lemma 2.13 appears (implicitly) in [3]. Before stating it, we need the concept of ε-matching. Given a barcode $\mathcal{B}$, let

$$\mathcal{B}_\varepsilon := \{(\langle a, b\rangle, m\langle a, b\rangle) \in \mathcal{B} : a + \varepsilon < b\}.$$

A representation of a partial matching $\sigma : Rep(\mathcal{B}^1) \to Rep(\mathcal{B}^2)$ is said to be *a ε-matching* if:

- $Rep(\mathcal{B}^1_{2\varepsilon}) \subset \operatorname{coim} \sigma$,
- $Rep(\mathcal{B}^2_{2\varepsilon}) \subset \operatorname{Im} \sigma$, and
- if $\sigma(\langle a, b\rangle_n) = \langle c, d\rangle_l$ then:

$$\langle a, b\rangle_n \subset \langle c - \varepsilon, d + \varepsilon\rangle_l,$$

$$\langle c, d\rangle_l \subset \langle a - \varepsilon, b + \varepsilon\rangle_n.$$

Lemma 2.13. *Two pointwise finite dimensional persistence modules V and U indexed over $\mathbb{R}$ are ε-interleaved if and only if there is a morphism $f : V \to U_\varepsilon$ such that $\chi_f : \mathcal{B}(V) \to \mathcal{B}(U_\varepsilon)$ is a 2ε-matching.*

Proof. The "only if" part is a direct consequence of Theorem 6.4 in [3], where the expression $r_\varepsilon \circ \chi_f$ being a ε-matching translates directly to χ_f being a 2ε-matching. The "if" part is a consequence of Theorem 8.2 in [3]: If χ_f is a 2ε-matching then $r_\varepsilon \circ \chi_f$ is a ε-matching and, by Theorem 8.2 of [3], we obtain that V and U are ε-interleaved. □

2.6 Direct Limits

Consider now persistence modules indexed over a totally ordered set T. An interval I in T is a subset of T such that, for all $x, y \in I$, if $x \leq z \leq y$ then z is also in I.

Definition 2.14 ([1])**.** *Given a persistence module $V \in \textbf{\textit{vect}}^T$ with structure maps ρ, and an interval I of T, the direct limit of V in I is defined as:*

$$\varinjlim_{t\in I} V(t) := \bigoplus_{t\in I} V(t)/Z$$

where Z is the subspace generated by $v \oplus -\rho_{st}v$ with $s, t \in I$, $s \leq t$, and $v \in V(s)$.

Intuitively, the direct limit of a persistence module in $I = \langle a, b\rangle$ is isomorphic to the vector space generated by the intervals $\langle c, d\rangle$ with $c < b \leq d$.

Example 2.15. *Consider a persistence module V isomorphic to*

$$k_{[1,4)} \oplus k_{(1,4]} \oplus k_{[0,5]}$$

and a persistence basis β for V. Then, if we denote $(\beta_i)_t(1_k)$ as ν_i^t, we have that, for any $v \in V(2)$, $v = \mu_1\nu_1^2 + \mu_2\nu_2^2 + \mu_3\nu_3^2$ for some $\mu_1, \mu_2, \mu_3 \in k$. In particular, the class of v in $\varinjlim_{t\in I} V(t)$, denoted by $\hat{v}$, is the same that the one of $\rho_{2t}v = \mu_1\nu_1^t + \mu_2\nu_2^t + \mu_3\nu_3^t$ for any $t > 2$. In this case, if $I = [1, 4)$ then $\varinjlim_{t\in I} V(t) \simeq V(2)$ since the vector space $V(t)$ is isomorphic to $V(2)$ for any $t \in [2, 4)$. Finally, if $I = [1, 4]$ then $\varinjlim_{t\in I} V(t) \simeq V(4)$ since $\nu_1^4 = 0$.

In general, when $I \subset \mathbb{R}$ is a right-closed interval ending at $s \in \mathbb{R}$ then $\varinjlim_{t\in I} V(t) = V(s)$.

Lemma 2.16. *Let $A, B, C \in \textbf{vect}^T$. If the short exact sequence*

$$0 \to A \to B \to C \to 0,$$

is exact, so is

$$0 \to \varinjlim_T A \to \varinjlim_T B \to \varinjlim_T C \to 0,$$

which is equivalent to saying that the direct limits of persistence modules are exact. Besides, if

$$A \hookrightarrow B,$$

then

$$\varinjlim_T A \hookrightarrow \varinjlim_T B.$$

Proof. Recall that colimits are the categorical definition of direct limits, **vect** is an Abelian category and any totally ordered set is a filtered category. Then, the first result is a direct consequence of the characterization of Abelian category (see [11, Appendix A.4]). The original result comes from [12]. The second result follows directly since injections can be defined in terms of exact sequences. □

Lemma 2.17. *Consider two persistence modules $V, U \in \textbf{vect}^T$ where T is a totally ordered set. Consider three intervals I, J, K with the same right endpoint and such that $K \subset I \cap J$. If V and U are isomorphic, considering that V and U are persistence modules restricted to K, then:*

$$\varinjlim_{t\in I} V(t) \simeq \varinjlim_{t\in J} U(t)$$

Proof. Let us prove that

$$\varinjlim_{t\in I} V(t) \simeq \varinjlim_{t\in K} V(t).$$

By definition of direct limit and since $K \subset I$, we have

$$\varinjlim_{t\in K} V(t) \subset \varinjlim_{t\in I} V(t).$$

Now, for any $\hat{v} \in \varinjlim_{t\in I} V(t)$, there exists $p \in I$ and $w \in V(p)$ such that w is a representative of the class $\hat{v}$. In particular, if we pick $q \in K$ with $p \leq q$, then $\rho_{pq}w$ is also a representative of $\hat{v}$ and lives in the restriction of V to K. Therefore,

$$\varinjlim_{t\in I} V(t) \subset \varinjlim_{t\in K} V(t).$$

Similarly, this property is also satisfied by U, concluding the proof. □

The previous lemmas will be used to prove the main result of this subsection whic is the following.

Lemma 2.18. *Consider a persistence module $V \in \textbf{vect}^{T_{<q}}$ where $T_{<q} = \{p \in T,\, p < q\}$. Then, if all structure maps in V are injective, we have:*

$$V(l) \hookrightarrow \varinjlim_{p\in T_{<q}} V(p), \qquad \forall l \in T_{<q}.$$

Proof. Let $l \in T_{<q}$. Let us consider the persistence module C as the constant functor from $\langle l, q\rangle$ to $V(l)$. Since all structure maps in V are injective, we have

$$C \hookrightarrow V'$$

where V' is the restriction of V to $\langle l, q\rangle$. Then, by Lemma 2.16,

$$\varinjlim_{p\in\langle l,q\rangle} C(p) \hookrightarrow \varinjlim_{p\in\langle l,q\rangle} V'(p).$$

Besides,

$$\varinjlim_{p\in\langle l,q\rangle} C(p) = V(l)$$

by definition, and, finally,

$$\varinjlim_{p\in\langle l,q\rangle} V'(p) = \varinjlim_{p\in T_{<q}} V(p)$$

by Lemma 2.17. □

2.7 Ladder modules

Given the poset with two elements, $\mathbf{2} = \{1, 2\}$, one can define the poset $\mathbb{R} \times \mathbf{2}$ using the induced order, that is, $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. A *ladder modules* is a functor from $\mathbb{R} \times \mathbf{2}$ to $\mathbf{vect}$. It can be proved that the category of ladder modules indexed over $\mathbb{R} \times \mathbf{2}$ is isomorphic to the category of morphisms of persistence modules indexed over $\mathbb{R}$ (see [7] for details). When we want to point out the ladder module structure of a morphism $f : V \to U$, we will denote it by $L = V \xrightarrow{f} U$.

In this paper, some examples will include ladder modules indexed over the set $\{1, 2, 3\}\times \mathbf{2}$. In other words, such ladder modules will represent morphisms between persistence modules of the form $V \simeq V_1 \to V_2 \to V_3$. All possibles indecomposable ladder modules indexed over $\{1, 2, 3\} \times \mathbf{2}$ are described in [5]. There are 29 (non-isomorphic) indecomposable ladder modules. They all can be represented by 2×3 integer matrices. Besides, 27 of them can be represented by 2×3 integer matrices with all entries being 0 or 1. The linear maps are the identity map when possible and the zero map otherwise. For example, $\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix}$ represents the following indecomposable ladder module:

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "\mathrm{Id}"] & k \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k. \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

The other two indecomposable ladder modules can be represented by the matrices $\begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix}$ and $\begin{smallmatrix}1&1&0\\1&2&1\end{smallmatrix}$, which are not made up of only 0 and 1. They represent, respectively, the following indecomposable ladder modules:

$$\begin{tikzcd}
k \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & \oplus_2 k \arrow[r, "{[0\,1]}"] & k \\
0 \arrow[r] \arrow[u] & k \arrow[r, "\mathrm{Id}"] \arrow[u, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & k \arrow[u, "\mathrm{Id}"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
k \arrow[r, "\mathrm{Id}"] & k \arrow[r] & 0 \\
k \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[u, "\mathrm{Id}"] & \oplus_2 k \arrow[r, "{[0\,1]}"] \arrow[u, "{[1\,1]}"'] & k. \arrow[u]
\end{tikzcd}$$

Example 2.19. *The ladder module showed in Example 1.1 and Example 2.10 can be represented as:*

$$\begin{smallmatrix}0&0&0\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&0\\0&1&0\end{smallmatrix}$$

3 Persistence partial matching

Our aim in this section is to obtain (persistence) partial matchings induced by morphisms between persistence modules in such a way that these matchings are linear with respect to direct sum of ladder modules. Along the section, $f : V \to U$ will denote a morphism between persistence modules V and U indexed over $\mathbb{R}$, with structure maps ρ and ϕ, respectively. First, let us see what we mean by persistence partial matchings.

Definition 3.1. *Given two barcodes, $\mathcal{B}_1 = (S_1, m_1)$ and $\mathcal{B}_2 = (S_2, m_2)$, a* persistence partial matching *is a function*

$$\mathcal{G} : S_1 \times S_2 \longrightarrow \boldsymbol{B}$$

such that:

$$\text{if } I \in S_1 \text{ then } \sum_{J \in S_2} \#\mathcal{G}(I, J) \leq m_1 I \tag{4}$$

and

$$\text{if } J \in S_2 \text{ then } \sum_{I \in S_1} \#\mathcal{G}(I, J) \leq m_2 J. \tag{5}$$

To achieve our goal, we define the following vector spaces:

$$Y^+_{IJ}[f](t) := fV^+_{It} \cap U^+_{Jt},$$

$$Y^-_{IJ}[f](t) := fV^-_{It} + U^-_{Jt},$$

$$X_{IJ}[f](t) := \frac{Y^+_{IJ}[f](t)}{Y^-_{IJ}[f](t) \cap Y^+_{IJ}[f](t)}.$$

Notice that when we write $fV^{\pm}_{It}$ we mean $f_t(V^{\pm}_{It})$. Besides, $fV^{\pm}_I(t) := fV^{\pm}_{It}$ and $fV_I := fV^+_I / fV^-_I$. Observe that $Y^{\pm}_{IJ}[f](t)$ and $X_{IJ}[f](t)$ are 0 for $t \notin I \cap J$. We will omit $[f]$ when it is not relevant. Finally, observe that, since Y^+_{IJ}, Y^-_{IJ} and X_{IJ} are made up of sums, intersections and quotient of persistence modules, then they are also persistence modules.

Definition 3.2. *The persistence partial matching, $\mathcal{G}_f$, induced by a morphism between persistence modules, f, is defined as:*

$$\mathcal{G}_f(I, J) := \mathcal{B}(X_{IJ}),$$

and the partial matching, $\mathcal{M}_f$, induced by f, as

$$\mathcal{M}_f(I, J) := \#\mathcal{B}(X_{IJ}).$$

The following lemmas and propositions will be used to prove that (persistence) partial matchings are well-defined (Theorem 3.9). The first step is to relate the barcode $\mathcal{B}(X_{IJ})$ with the direct limit of X_{IJ}.

Lemma 3.3. *If $s \leq t \in I \cap J$ then:*

$$\phi_{st} Y^-_{IJ}(s) = Y^-_{IJ}(t).$$

Proof. Due to Lemma 2.2, we have:

$$\phi_{st}(U^{\pm}_{Js}) = U^{\pm}_{Jt} \text{ and } \phi_{st}(fV^{\pm}_{Is}) = f_s \rho_{st} V^{\pm}_{It}.$$

Then,

$$\phi_{st} Y^-_{IJ}(s) = \phi_{st}(fV^-_{Is} + U^-_{Js}) = f\rho_{st} V^-_{Is} + \phi_{st} U^-_{Js} = fV^-_{It} + U^-_{Jt} = Y^-_{IJ}(t).$$

□

Lemma 3.4. *Let V be a persistence module and let β be a persistence basis for V. Let $b \in \mathbb{E}$. If $V(t) = 0$ for any $t \in \langle b, \infty\rangle$ then, for any $\lambda < b$:*

$$\dim \varinjlim_{t\in\langle\lambda,b\rangle} V(t) = \#\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\}.$$

Proof. Let $W = \operatorname{span}\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\}$. Let $e_i \in \varinjlim_{t\in\langle\lambda,b\rangle} W(t)$ defined as:

$$e_i := \varinjlim_{t\in\langle\lambda,b\rangle} (\beta_i)_t(1_k).$$

Note that $\operatorname{span}\{e_i\} \subset \varinjlim_{t\in\langle\lambda,b\rangle} V(t)$ since $W(t) \subset V(t)$ for any $t \in \langle\lambda, b\rangle$. Now, for any $\hat{v} \in \varinjlim_{t\in\langle\lambda,b\rangle} V(t)$, fix $r \in \langle\lambda, b\rangle$ and choose a representative $v \in V(r)$ of the class $\hat{v}$. Then, we can write v as a finite sum $\sum_i \mu_i(\beta_i)_r(1_k)$. Let $s = \max\{d : \beta_i \sim \langle\cdot, d\rangle,\, d \neq b,\, \mu_i \neq 0\}$. Since $V(t) = 0$ for any $t \in \langle b, \infty\rangle$ then $s < b$. Let us fix $r \in \langle s, b\rangle$ so that $\rho_{tr} v = \sum_i \mu_i(\beta_i)_r(1_k)$ with $\mu_i(\beta_i)_r(1_k) \neq 0$ if and only if $\beta_i \sim \langle\cdot, b\rangle$. Then, $\hat{v} \in \operatorname{span}\{e_i\}$ and, therefore, $\varinjlim_{t\in\langle\lambda,b\rangle} V(t) \subset \operatorname{span}\{e_i\}$, concluding the proof. □

The relationship between $\#\mathcal{B}(X_{IJ})$ and the direct limit of X_{IJ} is given by the following proposition.

Proposition 3.5. *The structure maps of X_{IJ} restricted to $I \cap J$ are all injective. In particular,*

$$\dim \varinjlim_{t\in I\cap J} X_{IJ}(t) = \#\mathcal{B}(X_{IJ})$$

Proof. Given $s \leq t$ with $s, t \in I \cap J$, we need to prove that

$$\phi_{st}^{-1}(Y_{IJ}^-(t) \cap Y_{IJ}^+(t)) \cap Y_{IJ}^+(s) = Y_{IJ}^-(s) \cap Y_{IJ}^+(s).$$

First, note that:

$$\begin{aligned}\phi_{st}^{-1}(Y_{IJ}^-(t) \cap Y_{IJ}^+(t)) \cap Y_{IJ}^+(s) &= \big(\phi_{st}^{-1}Y_{IJ}^-(t) \cap \phi_{st}^{-1}Y_{IJ}^+(t)\big) \cap Y_{IJ}^+(s)\\ &= \phi_{st}^{-1}Y_{IJ}^-(t) \cap Y_{IJ}^+(s)\end{aligned}$$

since $\phi_{st}^{-1}Y_{IJ}^+(t) = Y_{IJ}^+(s)$ by Lemma 2.2. Besides, using Lemma 3.3, we have:

$$\phi_{st}^{-1}Y_{IJ}^-(t) \cap Y_{IJ}^+(s) = \big(Y_{IJ}^-(s) + \operatorname{Ker}\phi_{st}\big) \cap Y_{IJ}^+(s)$$

Since $\operatorname{Ker}\phi_{st} \subset \operatorname{Ker}_{sb'}^-(U)$ where b' is the endpoint of J, we have that $\operatorname{Ker}\phi_{st} \cap U_{Js}^+ \subset U_{Js}^-$ and

$$\begin{aligned}&\big(Y_{IJ}^-(s) + \operatorname{Ker}\phi_{st}\big) \cap Y_{IJ}^+(s) \subset Y_{IJ}^-(s) \cap Y_{IJ}^+(s) + \operatorname{Ker}\phi_{st} \cap U_{Js}^+ \cap fV_{Is}^+\\ &\subset Y_{IJ}^-(s) \cap Y_{IJ}^+(s) + U_{Js}^- \cap fV_{Is}^+ = Y_{IJ}^-(s) \cap Y_{IJ}^+(s).\end{aligned}$$

Since the persistence module X_{IJ} is 0 outside $I\cap J$, Lemma 3.4 tells us that $\dim \varinjlim_{t\in I\cap J} X_{IJ}(t)$ is equal to the number of intervals with the same endpoint that $I \cap J$. In particular, since the structure maps of X_{IJ} restricted to $I \cap J$ are all injective then all the intervals in $\mathcal{B}(X_{IJ})$ have the same endpoint and the result follows. □

In particular, in Proposition 3.5, we have expressed the cardinality of $\mathcal{B}(X_{IJ})$ in terms of the dimension of a particular vector space. We now proceed to prove that $\mathcal{G}_f$ satisfies the inequalities (4) and (5) of Definition 3.1. Again, some intermediate results are needed.

Lemma 3.6. *Let I, J be intervals of $\mathbb{R}$. If $\lambda, \lambda', b \in \mathbb{E}$ with $\lambda < \lambda'$ and $t \in \langle \lambda', b\rangle$ then:*

$$\frac{Y^+_{I\langle\lambda,b\rangle}}{fV^-_I \cap U^+_{\langle\lambda,b\rangle}}(t) \subset \frac{Y^+_{I\langle\lambda',b\rangle} \cap Y^-_{I\langle\lambda',b\rangle}}{fV^-_I \cap U^+_{\langle\lambda',b\rangle}}(t) \subset \frac{Y^+_{I\langle\lambda',b\rangle}}{fV^-_I \cap U^+_{\langle\lambda',b\rangle}}(t).$$

and

$$\frac{Y^+_{\langle\lambda,b\rangle J}}{fV^+_{\langle\lambda,b\rangle} \cap U^-_J}(t) \subset \frac{Y^+_{\langle\lambda',b\rangle J} \cap Y^-_{\langle\lambda',b\rangle J}}{fV^+_{\langle\lambda,b\rangle} \cap U^-_J}(t) \subset \frac{Y^+_{\langle\lambda',b\rangle J}}{fV^+_{\langle\lambda,b\rangle} \cap U^-_J}(t).$$

Proof. Let us prove the first expression, the second is analogous. By definition, $\mathrm{Im}^+_{\lambda t}(U) \subset \mathrm{Im}^-_{\lambda' t}(U)$, then $U^+_{\langle\lambda,b\rangle t} \subset U^-_{\langle\lambda',b\rangle t}$ and then:

$$Y^+_{I\langle\lambda,b\rangle}(t) \subset Y^+_{I\langle\lambda',b\rangle}(t) \cap Y^-_{I\langle\lambda',b\rangle}(t).$$

The result follows since $fV^-_{It} \cap U^+_{\langle\lambda',b\rangle t} \cap Y^+_{I\langle\lambda,b\rangle}(t) = fV^-_{It} \cap U^+_{\langle\lambda,b\rangle t}$. $\square$

Proposition 3.7. *For fixed intervals I and J of $\mathbb{R}$, and $b \in \mathbb{E}$, consider the vector spaces*

$$\tilde{B}(\lambda) := \varinjlim_{t\in\langle\lambda,b\rangle} \frac{Y^+_{I\langle\lambda,b\rangle}}{fV^-_I \cap U^+_{\langle\lambda,b\rangle}}(t) \quad \text{and} \quad \tilde{B}'(\lambda) := \varinjlim_{t\in\langle\lambda,b\rangle} \frac{Y^+_{\langle\lambda,b\rangle J}}{fV^+_{\langle\lambda,b\rangle} \cap U^-_J}(t)$$

where $\lambda \in \mathbb{E}$ with $\lambda < b$. Then, $\tilde{B}$ and $\tilde{B}'$ are persistent modules in $\boldsymbol{vect}^{\mathbb{E}_{<b}}$. Besides,

$$\varinjlim_{t\in\langle-\infty,b\rangle} \oplus_{a<b} W_a(t) \subset \varinjlim_{\lambda<b} \tilde{B}(\lambda) \tag{6}$$

and

$$\varinjlim_{t\in\langle-\infty,b\rangle} \oplus_{a<b} W'_a(t) \subset \varinjlim_{\lambda<b} \tilde{B}'(\lambda) \tag{7}$$

where each W_a is a persistence module isomorphic to $X_{I\langle a,b\rangle}$ and each W'_a is a persistence module isomorphic to $X_{\langle a,b\rangle J}$.

Proof. For each $\lambda \in \mathbb{E}$ with $\lambda < b$, define the following family of persistence modules in $\mathbf{vect}^{\mathbb{R}}$:

$$A_\lambda := \frac{Y^-_{I\langle\lambda,b\rangle} \cap Y^+_{I\langle\lambda,b\rangle}}{fV^-_I \cap U^+_{\langle\lambda,b\rangle}}, \qquad B_\lambda := \frac{Y^+_{I\langle\lambda,b\rangle}}{fV^-_I \cap U^+_{\langle\lambda,b\rangle}}, \qquad C_\lambda := X_{I\langle\lambda,b\rangle}.$$

which are 0 outside $\langle\lambda, b\rangle$. Note that $A_\lambda \subset B_\lambda$ and $C_\lambda \simeq B_\lambda/A_\lambda$. Then, we have the following short exact sequence:

$$0 \to A_\lambda \to B_\lambda \to C_\lambda \to 0.$$

Using Lemma 2.16, the short sequence of vector spaces

$$0 \to \varinjlim_{t\in\langle-\infty,b\rangle} A_\lambda(t) \to \varinjlim_{t\in\langle-\infty,b\rangle} B_\lambda(t) \to \varinjlim_{t\in\langle-\infty,b\rangle} C_\lambda(t) \to 0 \tag{8}$$

is also exact. Now, notice that, for $\lambda < \lambda'$ in $\mathbb{E}$, there exists an injective morphism between D_λ restricted to $\langle\lambda', b\rangle$, $D_\lambda|_{\langle\lambda',b\rangle}$, and $D_{\lambda'}$, for $D = A, B, C$. Then, by Lemma 2.16 and Lemma 3.6, for $D = A, B, C$, we have the following injective morphism:

$$\varinjlim_{t\in\langle-\infty,b\rangle} D_\lambda|_{\langle\lambda',b\rangle}(t) \hookrightarrow \varinjlim_{t\in\langle-\infty,b\rangle} D_{\lambda'}(t).$$

Besides,

$$\varinjlim_{t\in\langle-\infty,b\rangle} B_\lambda|_{\langle\lambda',b\rangle}(t) \hookrightarrow \varinjlim_{t\in\langle-\infty,b\rangle} A_{\lambda'}(t).$$

Moreover, for $D = A, B, C$, since $D_\lambda|_{\langle\lambda',b\rangle}(t) = D_\lambda(t)$ for all $t \in \langle\lambda', b\rangle$ then, by Lemma 2.17, we have:

$$\varinjlim_{t\in\langle-\infty,b\rangle} D_\lambda|_{\langle\lambda',b\rangle}(t) = \varinjlim_{t\in\langle-\infty,b\rangle} D_\lambda(t) \hookrightarrow \varinjlim_{t\in\langle-\infty,b\rangle} D_{\lambda'}(t).$$

Besides,

$$\varinjlim_{t\in\langle-\infty,b\rangle} B_\lambda(t) \hookrightarrow \varinjlim_{t\in\langle-\infty,b\rangle} A_{\lambda'}(t).$$

Then, by the short exact sequence (8), we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{t\in\langle-\infty,b\rangle} A_\lambda(t) \arrow[r] \arrow[d, hook] & \varinjlim_{t\in\langle-\infty,b\rangle} B_\lambda(t) \arrow[r] \arrow[d, hook] \arrow[dl, hook] & \varinjlim_{t\in\langle-\infty,b\rangle} C_\lambda(t) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \varinjlim_{t\in\langle-\infty,b\rangle} A_{\lambda'}(t) \arrow[r] & \varinjlim_{t\in\langle-\infty,b\rangle} B_{\lambda'}(t) \arrow[r] & \varinjlim_{t\in\langle-\infty,b\rangle} C_{\lambda'}(t) \arrow[r] & 0
\end{tikzcd}$$

Where each horizontal short sequence is exact. Ignoring the diagonal arrow, such diagram can be summarized in the following short exact sequence in $\mathbf{vect}^{\mathbb{E}}$:

$$0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are persistence modules indexed over $\mathbb{E}$. Using again Lemma 2.16, we obtain another short exact sequence of vector spaces:

$$0 \to \varinjlim_{\lambda<b} \tilde{A}(\lambda) \to \varinjlim_{\lambda<b} \tilde{B}(\lambda) \to \varinjlim_{\lambda<b} \tilde{C}(\lambda) \to 0.$$

In particular, by Lemma 2.18, $\tilde{A}(\lambda)$ and $\tilde{B}(\lambda)$ are subspaces of $\varinjlim_{\lambda<b} \tilde{B}(\lambda)$ for all λ. Besides, since $\tilde{A}(\lambda) \subset \tilde{B}(\lambda)$ and $\tilde{B}(\lambda) \subset \tilde{A}(\lambda')$ whenever $\lambda < \lambda'$, then $\{\tilde{A}(\lambda), \tilde{B}(\lambda)\}$ is a set of disjoint sections of $\varinjlim_{\lambda<b} \tilde{B}(\lambda)$. Finally, the result

$$\varinjlim_{t\in\langle-\infty,b\rangle} \oplus_{a<b} W_a(t) \subset \varinjlim_{\lambda<b} \tilde{B}(\lambda),$$

with $W_a(t) \simeq \tilde{B}(a)/\tilde{A}(a)$, follows by Lemma 2.4. For the other case, we just have to define

$$A_\lambda = \frac{Y^-_{I\langle\lambda,b\rangle} \cap Y^+_{I\langle\lambda,b\rangle}}{fV^+_I \cap U^-_{\langle\lambda,b\rangle}} \qquad B_\lambda = \frac{Y^+_{I\langle\lambda,b\rangle}}{fV^+_I \cap U^-_{\langle\lambda,b\rangle}} \qquad C_\lambda = X_{I\langle\lambda,b\rangle}.$$

and proceed analogously. $\square$

Lemma 3.8. *Let I and J be intervals of $\mathbb{R}$, let $b \in \mathbb{E}$ and let β be a persistence basis for fV_I. Then,*

$$\dim \varinjlim_{\lambda<b} \varinjlim_{t\in\langle\lambda,b\rangle} \left(\frac{Y^+_{I\langle\lambda,b\rangle}}{(fV^-_I \cap U^+_{\langle\lambda,b\rangle})}(t) \right) \leq \#\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\}$$

and

$$\dim \varinjlim_{\lambda<b} \varinjlim_{t\in\langle\lambda,b\rangle} \left(\frac{Y^+_{\langle\lambda,b\rangle J}}{(fV^+_{\langle\lambda,b\rangle} \cap U^-_J)}(t) \right) \leq \#\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\}$$

Proof. For any $\lambda \in \mathbb{E}_{<b}$, let B_λ be the persistence module defined as

$$B_\lambda := \frac{Y^+_{I\langle\lambda,b\rangle}}{fV^-_I \cap U^+_{\langle\lambda,b\rangle}}.$$

Notice that $fV^-_I \cap Y^+_{I\langle\lambda,b\rangle} = fV^-_I \cap U^+_{\langle\lambda,b\rangle}$ and $Y^+_{I\langle\lambda,b\rangle} \subset fV^+_I$. Therefore, $B_\lambda \subset fV_I$. Besides, since $B_\lambda(t)$ is 0 for $t \in \langle b, \infty\rangle$ then B_λ can be expressed as a submodule of $W = \operatorname{span}\{\beta_i : \beta_i \sim \langle\cdot, d\rangle,\, d \leq b\}$. By Lemma 2.16 and Lemma 2.17, we have:

$$\varinjlim_{t\in\langle\lambda,b\rangle} B_\lambda(t) \subset \varinjlim_{t\in\langle\lambda,b\rangle} W(t) = \varinjlim_{t\in\langle-\infty,b\rangle} W(t).$$

Since $\tilde{B}$, defined as $\tilde{B}(\lambda) := \varinjlim_{t\in\langle\lambda,b\rangle} B_\lambda(t)$, is a persistence module indexed over $\mathbb{E}_{<b}$ then, by Lemma 2.16, we have:

$$\varinjlim_{\lambda<b} \tilde{B}(\lambda) \subset \varinjlim_{\lambda<b} \varinjlim_{t\in\langle-\infty,b\rangle} W(t) = \varinjlim_{t\in\langle-\infty,b\rangle} W(t).$$

Finally, the expression (6) follows from Lemma 3.4. The other case is analogous. □

Finally, let us prove the main result of the section.

Theorem 3.9. *Given a morphism $f : V \to U$ betweeen two persistence modules V and U indexed over $\mathbb{R}$, the function*

$$\mathcal{G}_f(I, J) := \mathcal{B}(X_{IJ})$$

is a persistence partial matching and the function

$$\mathcal{M}_f(I, J) := \#\mathcal{B}(X_{IJ})$$

is a partial matching.

Proof. Let (S, m) be the barcode for V, let $I \in S$ and let β be a persistence basis for fV_I. Let us prove that $\sum_J \#\mathcal{G}_f(I, J) \leq mI$. By Proposition 3.5,

$$\sum_J \#\mathcal{G}_f(I, J) = \sum_{b\in\mathbb{E}} \sum_{a<b} \#\mathcal{B}(X_{I\langle a,b\rangle}) = \sum_{b\in\mathbb{E}} \sum_{a<b} \dim \varinjlim_{t\in\langle\lambda,b\rangle} X_{I\langle a,b\rangle}.$$

Recall that direct limits and direct sums are examples of colimits. Since colimits commute with colimits [13, Warning 6.2.10], we have:

$$\begin{aligned}
\sum_{a<b} \dim \varinjlim_{t\in\langle\lambda,b\rangle} X_{I\langle a,b\rangle} &= \dim \varinjlim_{t\in\langle\lambda,b\rangle} \oplus_{a<b} X_{I\langle a,b\rangle} \\
&\leq \dim \varinjlim_{a<b} \varinjlim_{t\in\langle\lambda,b\rangle} \frac{Y^+_{I\langle a,b\rangle}}{fV^-_I \cap U^+_{\langle a,b\rangle}} \quad \text{(by Proposition 3.7)} \\
&\leq \#\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\} \quad \text{(by Lemma 3.8).}
\end{aligned}$$

Then,

$$\sum_J \#\mathcal{G}_f(I, J) \leq \sum_{b\in\mathbb{E}} \#\{\beta_i : \beta_i \sim \langle\cdot, b\rangle\} \leq \#\mathcal{B}(fV_I)$$

being $\#\mathcal{B}(fV_I) \leq mI$, by definition of fV_I. The proof of $\sum_I \#\mathcal{G}_f(I, J) \leq mJ$ is analogous. □

In the following, the space generated by a set of vectors will be represented using a matrix and angle brackets. For example, given vectors $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ of $\oplus_2 k$, we will write span $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ as $\langle\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle$.

Example 3.10. *Consider the following ladder module, represented by the matrix* $\begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix}$*:*

$$\begin{tikzcd}
U & {} & k \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & \oplus_2 k \arrow[r, "{[0\,1]}"] & k \\
V \arrow[u, "f"] \arrow[r, phantom, "\simeq"] & {} & 0 \arrow[u] \arrow[r] & k \arrow[u, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] \arrow[r, "\mathrm{Id}"] & k. \arrow[u, "\mathrm{Id}"]
\end{tikzcd}$$

Then,

$$\mathcal{B}(V) = \{([2,3],1)\} \qquad \mathcal{B}(U) = \{([1,2],1),([2,3],1)\}.$$

If $I = [2,3]$ *and* $J = [1,2]$ *then*

$$Y^+_{IJ}(2) = fV^+_{I2} \cap U^+_{J2} = \langle\begin{smallmatrix}1\\1\end{smallmatrix}\rangle \cap \langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle = 0 \;\; \text{and} \;\; X_{IJ}(2) = 0.$$

Besides,

$$Y^+_{II}(2) = fV^+_{I2} \cap U^+_{I2} = \langle\begin{smallmatrix}1\\1\end{smallmatrix}\rangle \cap \langle\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\rangle = \langle\begin{smallmatrix}1\\1\end{smallmatrix}\rangle,$$

$$Y^-_{II}(2) \cap Y^+_{II}(2) = (fV^-_{I3} + U^-_{I3}) \cap Y^+_{II}(2) = (\langle\begin{smallmatrix}0\\0\end{smallmatrix}\rangle + \langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle) \cap \langle\begin{smallmatrix}1\\1\end{smallmatrix}\rangle = 0;$$

and

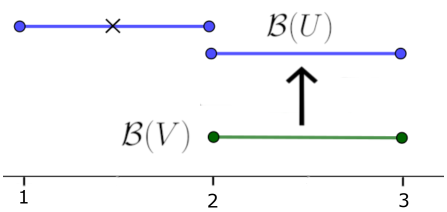
$$Y^+_{II}(3) = fV^+_{I3} \cap U^+_{I3} = \langle 1\rangle \cap \langle 1\rangle = \langle 1\rangle,$$

$$Y^-_{II}(3) \cap Y^+_{II}(3) = (fV^-_{I3} + U^-_{I3}) \cap Y^+_{II}(3) = (\langle 0\rangle + \langle 0\rangle) \cap \langle 1\rangle = \langle 0\rangle.$$

Then, X_{II} *is the submodule* $0 \longrightarrow \langle\begin{smallmatrix}1\\1\end{smallmatrix}\rangle \xrightarrow{[01]} 1$ *and the only case where* $\mathcal{G}_f$ *is not empty is:*

$$\mathcal{G}_f(I,I) = \mathcal{B}(X_{II}) = \{([2,3],1)\}.$$

A representation of $\mathcal{M}_f$ *is as follows:*



where an arrow between two intervals denotes a matching and a cross on an interval denotes it is unmatched.

4 Properties of induced persistence partial matchings

In this section, we will introduce some properties of persistence partial matchings induced by morphisms between persistence modules, defined in this paper. These properties will help us to develop an intuition about how induced persistence partial matchings work. Consider a morphism $f : V \to U$ between two persistence modules V and U indexed over $\mathbb{R}$. Let $I = \langle a,b\rangle$ and $J = \langle c,d\rangle$ be intervals of $\mathbb{R}$. We say that $J \leq I$ if $c \leq a < d \leq b$.

Proposition 4.1. *The persistence module* X_{IJ} *is zero unless* $J \leq I$.

Proof. Note that $f\,\mathrm{Im}^{\pm}_{at}(V) \subset \mathrm{Im}^{\pm}_{at}(U)$ for $t \in \mathbb{R}$. If $a < c$ and $t \in \langle a, c\rangle$ then $\mathrm{Im}^{+}_{at}(U) \subset \mathrm{Im}^{-}_{ct}(U)$ and, in particular,

$$fV^{+}_{It} \subset f\,\mathrm{Im}^{+}_{at}(V) \subset \mathrm{Im}^{+}_{at}(U) \subset \mathrm{Im}^{-}_{ct}(U).$$

Then,

$$\begin{aligned} Y^{+}_{IJ}(t) = fV^{+}_{It} \cap U^{+}_{Jt} &\subset fV^{+}_{It} \cap \mathrm{Im}^{-}_{ct}(U) \cap \mathrm{Ker}^{+}_{dt}(U) \\ &\subset fV^{+}_{It} \cap U^{-}_{Jt} \subset Y^{+}_{IJ}(t) \cap Y^{-}_{IJ}(t). \end{aligned}$$

Therefore, if $a < c$ then $X_{IJ} = 0$. There is a similar proof for the case $b < d$ using Ker instead of Im. ☐

This last result can be seen as a counterpart of the following.

Remark 4.2. *Any morphism between k_I and k_J is zero unless $J \leq I$.*

Proof. Let ρ, ϕ be the structure maps of k_I and k_J, respectively. Let $f \in \mathit{Hom}(k_I, k_J)$. If $a < c$, let us prove that $f = 0$. Let $t \in I \cap J$, $v^t \in k_I(t)$ and $s \in \langle a, c\rangle$. Then, there exists $v^s \in k_I(s)$ such that $\rho_{st}(v^s) = v^t$. If $f \in \mathit{Hom}(k_I, k_J)$, since $k_J(s) = 0$, then:

$$f_t v^t = f_t \rho_{st} v^s = \phi_{st} f_s v^s = 0.$$

A similar reasoning exists for $b < d$. ☐

Actually, if $J \leq I$ then any morphism $f \in \mathit{Hom}(k_I, k_J)$ is determined by the image of $f_t(1_k)$ in any $t \in I \cap J$. This can also be seen in terms of X_{IJ}.

Proposition 4.3. *If $f : k_I \to k_J$ is not zero then $X_{IJ} = k_{I\cap J}$.*

Proof. Observe that, for $t \in I \cap J$, if f is not zero then $fV^{+}_{It} \cap U^{+}_{Jt} = k$ and $fV^{-}_{It} \cap U^{+}_{Jt} = fV^{+}_{It} \cap U^{-}_{Jt} = 0$. Then $X_{IJ}[f](t) = k$ for any $t \in I \cap J$. The proof concludes since $V_{It} \cap U_{Jt}$ is 0 when $t \notin I \cap J$. ☐

Then, induced persistence partial matchings take a simpler form for this kind of morphisms. Fortunately, induced persistence partial matchings are also linear with respect to direct sum of ladder modules.

Proposition 4.4. *Let I and J be intervals of $\mathbb{R}$. If $L = V \xrightarrow{f} U$ can be decomposed as $L_1 \oplus L_2 = V_1 \oplus V_2 \xrightarrow{f_1 \oplus f_2} U_1 \oplus U_2$, where V_i and U_i, $i = 1, 2$, then:*

$$\mathrm{Im}^{\pm}_{I}(V_1 \oplus V_2) = \mathrm{Im}^{\pm}_{I}(V_1) \oplus \mathrm{Im}^{\pm}_{I}(V_2), \ \ \mathrm{Ker}^{\pm}_{I}(V_1 \oplus V_2) = \mathrm{Ker}^{\pm}_{I}(V_1) \oplus \mathrm{Ker}^{\pm}_{I}(V_2),$$

$$Y^{\pm}_{IJ}[f_1 \oplus f_2] = Y^{\pm}_{IJ}[f_1] \oplus Y^{\pm}_{IJ}[f_2], \ \ X_{IJ}[f_1 \oplus f_2] = X_{IJ}[f_1] \oplus X_{IJ}[f_2].$$

Proof. Let us prove that $\mathrm{Im}_I(V_1 \oplus V_2) = \mathrm{Im}_I(V_1) \oplus \mathrm{Im}_I(V_2)$. The other similar expressions follow analogously. Let $t \in I$. Since the persistence modules V_1 and V_2 satisfy the descending chain condition for images and kernels, we have that $\mathrm{Im}^{+}_{It}(V_1 \oplus V_2) = \rho_{st}(V_1 \oplus V_2)(t)$, $\mathrm{Im}^{+}_{It}(V_1) = \rho_{s_1 t} V_1(t)$ and $\mathrm{Im}^{+}_{It}(V_2) = \rho_{s_2 t} V_2(t)$ for some $s, s_1, s_2 \in I$. Let $r = \min\{s, s_1, s_2\}$. Then,

$$\mathrm{Im}^{+}_{It}(V_1 \oplus V_2) = \rho_{rt}(V_1 \oplus V_2)(t) = \rho_{rt} V_1(t) \oplus \rho_{rt} V(t_2) = \mathrm{Im}^{+}_{It}(V_1) \oplus \mathrm{Im}^{+}_{It}(V_2).$$

Now, let us prove that $Y^+_{IJ}[f_1 \oplus f_2](t) = Y^+_{IJ}[f_1](t) \oplus Y^+_{IJ}[f_2](t)$. The other similar expressions follow analogously. Note that:

$$\begin{aligned}
(f_1 \oplus f_2)(V^+_{It}) =& (f_1 \oplus f_2)(\mathrm{Im}^+_{It}(V) \cap \mathrm{Ker}^+_{It}(V)) \\
=& (f_1 \oplus f_2)\left((\mathrm{Im}^+_{It}(V_1) \oplus \mathrm{Im}^+_{It}(V_2)) \cap (\mathrm{Ker}^+_{It}(V_1) \oplus \mathrm{Ker}^+_{It}(V_2))\right) \\
=& (f_1 \oplus f_2)\left((\mathrm{Im}^+_{It}(V_1) \cap \mathrm{Ker}^+_{It}(V_1)) \oplus (\mathrm{Im}^+_{It}(V_2) \cap \mathrm{Ker}^+_{It}(V_2))\right) \\
=& f_1(\mathrm{Im}^+_{It}(V_1) \cap \mathrm{Ker}^+_{It}(V_1)) \oplus f_2(\mathrm{Im}^+_{It}(V_2) \cap \mathrm{Ker}^+_{It}(V_2)) \\
=& f_1(V_1)^+_{It} \oplus f_2(V_2)^+_{It}
\end{aligned}$$

and

$$\begin{aligned}
Y^+_{IJ}[f_1 \oplus f_2](t) &= \left(f_1(V_1)^+_{It} \oplus f_2(V_2)^+_{It}\right) \cap ((U_1)^+_{Jt} \oplus (U_2)^+_{Jt}) \\
&= (f_1(V_1)^+_{It} \cap (U_1)^+_{Jt}) \oplus (f_2(V_2)^+_{It} \cap (U_2)^+_{Jt}) \\
&= Y^+_{IJ}[f_1](t) \oplus Y^+_{IJ}[f_2](t).
\end{aligned}$$

□

Now, let us introduce the main result of this section.

Theorem 4.5. *Let I and J be intervals of $\mathbb{R}$. If $L = V \xrightarrow{f} U$ can be decomposed as $L_1 \oplus L_2 = V_1 \oplus V_2 \xrightarrow{f_1 \oplus f_2} U_1 \oplus U_2$, then,*

$$\mathcal{G}_f(I,J) = \mathcal{G}_{f_1}(I,J) \cup \mathcal{G}_{f_2}(I,J)$$

$$\mathcal{M}_f(I,J) = \mathcal{M}_{f_1}(I,J) + \mathcal{M}_{f_2}(I,J)$$

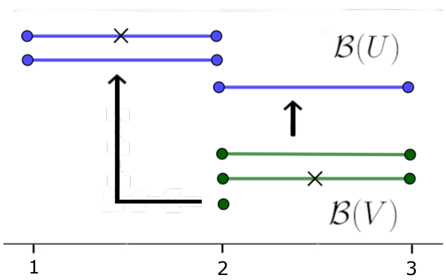
Proof. It follows directly from Proposition 4.4. □

Let us see with the following example that Theorem 4.5 makes computation much easier when a decomposition is available.

Example 4.6. *Recall that in Example 2.19 we show that the ladder module $L : V \xrightarrow{f} U$ of Example 2.10 can be decomposed as:*

$$\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0. \end{smallmatrix}$$

Then, the morphism f can be expressed as the direct sum of three "simpler morphisms", $f = f_1 \oplus f_2 \oplus f_3$. Recall that $\mathcal{M}_{f_2}(I,J) \neq 0$ if and only if $I = J = [2,3]$ and, in that case, $\mathcal{M}_{f_2}(I,J) = 1$, as explained in Example 3.10. Besides, it can be checked that $\mathcal{M}_{f_1}$ is always 0 and $\mathcal{M}_{f_3}(I,J) \neq 0$ if and only if $I = [2,2]$ and $J = [1,2]$ and, in that cases, $\mathcal{M}_{f_3}(I,J) = 1$ by Proposition 4.3. See the partial matching $\mathcal{M}_f$ represented in the following picture:



In general, indecomposable modules are not as simple as just a morphism between two interval modules. Actually, even when the domain or image of f is an interval module, X_{IJ} may not be $k_{I\cap J}$. In addition, as explained in the introduction, there does not exist a parametrization for all indecomposable modules. The following proposition explains how persistence partial matchings behave when the domain is an interval module.

Proposition 4.7. *Consider a morphism $f : k_I \to U$ and a persistence basis β for U. Let $\mathcal{J}$ be the finite set of intervals that appear in the expression of fk_I in terms of β. Then, X_{IJ} is not null only when $J = \tilde{J}$ where $\tilde{J}$ is the interval with the smallest length in $\mathcal{J}$. Besides, $X_{I\tilde{J}}$ is an interval submodule of $k_{I\cap\tilde{J}}$.*

Proof. If U is a direct sum of copies of an interval module, this proposition is a particular case of Proposition 4.3. If not, since $f \neq 0$, there exists a finite set $\{\beta_i\}_{i=1\ldots n}$ and $t \in I$ such that $f_t(1_k) = \sum_{i=1}^{n} \mu_i(\beta_i)_t(1_k)$, for some $\mu_1, \ldots, \mu_n \in k$.
Now, sort in decreasing order the set $\{b' : \exists i \text{ with } \beta_i \sim \langle\cdot, b'\rangle\}$ and define b, d as the first and second value respectively. Then, there is a set $\Lambda \subset \{1, \ldots n\}$, such that, for $i \in \Lambda$, $\beta_i \sim \langle\cdot, b\rangle$. We have that $f_s(1_k) = \sum_{i\in\Lambda} \mu_i(\beta_i)_s(1_k)$ for any $s \in \langle d, b\rangle$. Now, let $\tilde{J}$ be the interval with the smallest length in $\{J : \exists i \in \Lambda,\ \beta_i \sim J\}$. Then, by definition of $U^{\pm}_{\tilde{J}}$, we have that for any $s \in \langle d, b\rangle$, $f_s(1_k) \in U^{+}_{\tilde{J}s}$ but $f_s(1_k) \notin U^{-}_{\tilde{J}s}$. Then, there is K with $\langle d, b\rangle \subset K \subset I \cap \tilde{J}$ such that $k_K \simeq X_{I\tilde{J}}$. □

5 Relation between $\mathcal{M}_f$ and χ_f

In this section, we will compare some properties that satisfy the operations χ_f and $\mathcal{M}_f$. Note that, by construction, χ_f is completely determined by $\mathcal{B}(\operatorname{Im} f)$, $\mathcal{B}(V)$ and $\mathcal{B}(U)$. As a consequence, χ_f is not linear with respect to direct sums of ladder modules as $\mathcal{M}_f$ is. This way, χ_f has some counter-intuitive result: In Example 2.10, χ_f leaves $[2, 2]_1$ unmatched despite f has the indecomposable module $\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}$ in its decomposition, suggesting that an induced partial matching should match $[2, 2]_1$ and $[1, 2]_1$.

Now, let us see that although $\mathcal{M}_f$ and χ_f are not equivalent, they are related.

Theorem 5.1. *Consider a morphism $f : V \to U$ between pointwise finite dimensional persistence modules. Then, there exists $g : V \to U$ for which χ_f is a representation of the induced partial matching $\mathcal{M}_g$.*

Proof. Let α and β be persistence basis for V and U, respectively. Let us consider bijections $\sigma_1 : \{\alpha_i\} \to \mathit{Rep}(\mathcal{B}(V))$ and $\sigma_2 : \mathit{Rep}(\mathcal{B}(U)) \to \{\beta_i\}$ and define the function f_* between $\{\alpha_i\}$ and $\{\beta_i\}$ as follows:

$$f_* := \sigma_2 \circ \chi_f \circ \sigma_1.$$

In particular, if $\alpha_i \sim I$ and $\beta_j \sim J$ and $f_*(\alpha_i) = \beta_j$ then $J \leq I$ by Proposition 2.11. Then, we can define the morphism h_i between the persistence modules k_I and k_J being the identity in $I \cap J$ and 0 otherwise. Besides, we can define the projection π_i which sends each vector in $\operatorname{dom}\alpha = \oplus_i \operatorname{dom}\alpha_i$ to its component in $\operatorname{dom}\alpha_i$. We define the function $g_i = h_i \circ \pi_i \circ \alpha^{-1}$. If f_* does not match α_i with any β_j, we define $g_i = 0$. Since α is an isomorphism between V and its decomposition, we can also define

$$g := \beta \circ (\oplus_i g_i) \circ \Delta : V \longrightarrow U$$

being $\Delta : V \to \oplus_i V$ the diagonal map. Now, let

$$g_* := \sigma_2 \circ \chi_g \circ \sigma_1 : \{\alpha_i\} \to \{\beta_i\}.$$

Notice that g is defined in such a way that $\mathcal{B}(\operatorname{Im} g) = \{I \cap J : \chi_f(I) = J\}$, and that,by Proposition 2.12, $\mathcal{B}(\operatorname{Im} f) = \{I \cap J : \chi_f(I) = J\}$. By the same proposition, $\chi_f = \chi_g$ and $f_* = g_*$.
Let us prove now that if $t \in I \cap J$ then:

$$X_{IJ}[g](t) = \#\left\{\beta_j(t) \,\middle|\, \beta_j \sim J \text{ and } \exists \alpha_i \sim I \text{ with } g_*\alpha_i = \beta_j\right\}.$$

By Lemma 2.7, if $t \in I \cap J$ then:

$$U^+_{Jt} = \operatorname{span}\left\{\beta_j(t) : \beta_j \sim J', J' \leq J\right\}, \quad gV^+_{It} = \operatorname{span}\left\{g\alpha_i : \alpha_i \sim I', I' \leq I\right\}.$$

By the definition of g, we have that $g\,\alpha_i(t)$ is either equal to 0 or equal to $\beta_j(t)$ for some j. Then,

$$\begin{aligned} Y^+_{IJ}[g](t) = gV^+_{It} \cap U^+_{Jt} = \operatorname{span}\big\{\beta_j(t) :\ \beta_j \sim J', J' \leq J, \text{ and } \exists \alpha_i \sim I',\, I' \leq I, \\ \text{with } g\alpha_i(t) = \beta_j(t)\big\}. \end{aligned}$$

By the same reasoning, we have:

$$\begin{aligned} gV^+_{It} \cap U^-_{Jt} = \operatorname{span}\big\{\beta_j(t) : \beta_j \sim J', J' < J \text{ and } \exists \alpha_i \sim I',\, I' \leq I, \\ \text{with } g\alpha_i(t) = \beta_j(t)\big\} \end{aligned}$$

and

$$\begin{aligned} gV^-_{It} \cap U^+_{Jt} = \operatorname{span}\big\{\beta_j(t) : \beta_j \sim J', J' \leq J \text{ and } \exists \alpha_i \sim I',\, I' \leq I, \\ \text{with } g\alpha_i(t) = \beta_j(t)\big\}. \end{aligned}$$

Therefore, the space

$$Y^+_{IJ}[g](t) \cap Y^-_{IJ}[g](t) = gV^+_{It} \cap U^-_{Jt} + gV^+_{It} \cap U^-_{Jt}$$

can be seen as the subspace of $Y^+_{IJ}[g](t)$ generated by

$$\operatorname{span}\left\{\beta_j(t) : \beta_j \sim J', J' < J \text{ or } \exists \alpha_i \sim I',\, I' < I, \text{ with } g\alpha_i(t) = \beta_j(t)\right\}$$

and then,

$$X_{IJ}[g](t) = \operatorname{span}\left\{\beta_j(t) : \beta_j \sim J \text{ and } \exists \alpha_i \sim I, \text{ with } g\alpha_i(t) = \beta_j(t)\right\}$$

concluding the proof. □

Theorem 5.1 directly leads to a result related with Lemma 2.13.

Corollary 5.2. *Two pointwise finite dimensional persistence modules V and U are ε-interleaved if and only if there exists $f : V \to U_\varepsilon$ such that any representation of $\mathcal{M}_f$ is a 2ε-matching.*

Proof. By Lemma 2.13, if V and U are ε-interleaved then there exists $f : V \to U_\varepsilon$ such that χ_f is a 2ε-matching. Then, there exists a morphism $g : V \to U$ such that any representation of $\mathcal{M}_g$ also induces a 2ε-matching. On the other hand, consider a representation of $\mathcal{G}_f$ which is a 2ε-matching. In particular, it gives a ε-matching between $\mathcal{B}(V)$ and $\mathcal{B}(U)$. By the same argument appearing in the proof of Lemma 2.13, V and U are ε-interleaved. □

6 About the stability of X_{IJ}

In this section, we prove that X_{IJ} is "robust to" perturbations represented by the functor $\mathbf{1}_\varepsilon$ in the following sense: if two intervals $\langle a,b\rangle$ and $\langle c,d\rangle$ are matched by the partial matching $\mathcal{M}_f$ induced by a morphism f, then the intervals $\langle a,b-\epsilon\rangle$ and $\langle c,d-\epsilon\rangle$ are matched by the the partial matching $\mathcal{M}_{\mathbf{1}_\varepsilon(f)}$ induced by the morphism $\mathbf{1}_\varepsilon(f)$. To prove such result, we need first to introduce some preliminary lemmas.

Lemma 6.1. *Consider the vector spaces P', P, Q', Q, R such that $P' \subset P \subset R$ and $Q' \subset Q \subset R$. Then, for any linear map g with domain R, we have:*

$$g(P'\cap Q + P\cap Q') = \big(gP'\cap gQ + gP\cap gQ'\big)\cap g(P\cap Q).$$

Proof. Firstly, note that, by definition,$g(P'\cap Q+P\cap Q')\subset g(P\cap Q)$ and

$$g(P'\cap Q+P\cap Q') = g(P'\cap Q)+g(P\cap Q')\subset gP'\cap gQ+gP\cap gQ'.$$

Now, consider $x = gu+gw$ with $u\in gP'$, $w\in gQ'$ and $x=g(v)$ with $v\in P\cap Q$. Then, $0=gu+gw-gv$ which implies that $gu\in gQ$ and $gw\in gP$. Lastly, $x=gu+gw=g(u+w)$ and $x\in g(P'\cap Q+P\cap Q')$. Therefore,

$$(gP'\cap gQ+gP\cap gQ')\cap g(P\cap Q)\subset g(P'\cap Q+gP\cap gQ').$$

□

From now on, given $a,b\in\mathbb{E}$ and $\varepsilon\in\mathbb{R}$, the interval $\langle a,b+\varepsilon\rangle$ will be denoted by $\langle a,b\rangle^\varepsilon$.

Proposition 6.2. *Let $f: V\to U$ be a morphism between two persistence modules V and U, and let $0<\varepsilon\in\mathbb{R}$. Then, for any $t\in I\cap J$, we have:*

$$X_{IJ}[\mathbf{1}_\varepsilon(f)](t) = X_{I^\varepsilon J^\varepsilon}[f](t+\varepsilon).$$

Proof. Let us prove that $Y^+_{IJ}[\mathbf{1}_\varepsilon(f)](t) = Y^+_{I^\varepsilon J^\varepsilon}[f](t+\varepsilon)$. The case Y^-_{IJ} follows similarly. First, let us assume that $t\in I=\langle a,b\rangle$. Let ρ be the structure maps of V and ρ' the structure maps of $\mathbf{1}_\varepsilon(V)$. Let $\psi^V: V\to V_\varepsilon$ denote the set of linear isomorphisms $\{\psi_t: V(t)\to V_\varepsilon(t-\varepsilon)\}_{t\in\mathbb{R}}$ induced by the identity. Then:

$$\begin{aligned}
\mathrm{Im}^+_{at}(\mathbf{1}_\epsilon(V)) &= \bigcap_{s\in\langle a,t^+\rangle}\rho'_{st}(1^V_\varepsilon V_s) = \bigcap_{s\in\langle a,t^+\rangle}\rho'_{st}(\psi^V_{s+\varepsilon}(\rho_{s,s+\varepsilon}V_s))\\
&= \bigcap_{s\in\langle a,t^+\rangle}\psi^V_{t+\varepsilon}(\rho_{s+\varepsilon,t+\varepsilon}\rho_{s,s+\varepsilon}V_s)\\
&= \psi^V_{t+\varepsilon}\big(\bigcap_{s\in\langle a,t^+\rangle}\rho_{s,t+\varepsilon}V_s\big) = \psi^V_{t+\varepsilon}\,\mathrm{Im}^+_{a,t+\varepsilon}(V).
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathrm{Ker}^+_{bt}(\mathbf{1}_\epsilon(V)) &= \bigcap_{r\in\langle b,+\infty\rangle}\mathrm{Ker}\,\rho'_{st} = \bigcap_{r\in\langle b,+\infty\rangle}\mathrm{Ker}\,\psi^V_{t+\varepsilon}\rho_{t+\varepsilon,r+\varepsilon}|_{1^V_\varepsilon V_t}\\
&= \psi^V_{t+\varepsilon}\left((\rho_{t,t+\varepsilon}V_t)\cap\left(\bigcap_{r\in\langle b,+\infty\rangle}\mathrm{Ker}\,\rho_{t+\varepsilon,r+\varepsilon}\right)\right)\\
&= \psi^V_{t+\varepsilon}\left((\rho_{t,t+\varepsilon}V_t)\cap\mathrm{Ker}^+_{b+\varepsilon,t+\varepsilon}(V)\right).
\end{aligned}$$

Since $\mathrm{Im}^+_{a,t+\varepsilon}(V) \subset \rho_{t,t+\varepsilon} V_t$ and $\psi^V_{t+\varepsilon}$ is an isomorphism, we have:

$$(\mathbf{1}_\varepsilon(V))^+_{\langle a,b\rangle t} = \psi_{t+\varepsilon}\left(\mathrm{Im}^+_{a,t+\varepsilon}(V) \cap \mathrm{Ker}^+_{b+\varepsilon,t+\varepsilon}(V)\right) \simeq V^+_{\langle a,b+\varepsilon\rangle,t+\varepsilon}.$$

Then, for $t \in I \cap J$, we have that

$$\begin{aligned} Y^+_{IJ}[\mathbf{1}_\varepsilon(f)](t) &= (\mathbf{1}_\varepsilon(f))\big((\mathbf{1}_\varepsilon(V))^+_{It} \cap (\mathbf{1}_\varepsilon(U))^+_{Jt}\big) \\ &= (\mathbf{1}_\varepsilon(f))_t\big(\psi^V_{t+\varepsilon}(V^+_{I^\varepsilon,t+\varepsilon}) \cap \psi^U_{t+\varepsilon}(U^+_{J^\varepsilon,t+\varepsilon})\big) \\ &= \big(\psi^U_{t+\varepsilon}(fV^+_{I^\varepsilon,t+\varepsilon})\big) \cap \big(\psi^U_{t+\varepsilon}(U^+_{J^\varepsilon,t+\varepsilon})\big) \simeq Y^+_{I^\varepsilon J^\varepsilon}[f](t+\varepsilon). \end{aligned}$$

□

Corollary 6.3. *For any $0 < \varepsilon \in \mathbb{R}$ and any two intervals $I^\varepsilon, J^\varepsilon$, we have:*

$$\mathcal{M}_{\mathbf{1}_\varepsilon(f)}(I,J) = \mathcal{M}_f(I^\varepsilon, J^\varepsilon).$$

Proof. The results follows directly from Proposition 6.2. □

The following corollary is a consequence of Proposition 6.2 and the fact that $X_{IJ}[Id] = V_I$ if $I = J$ and 0 otherwise.

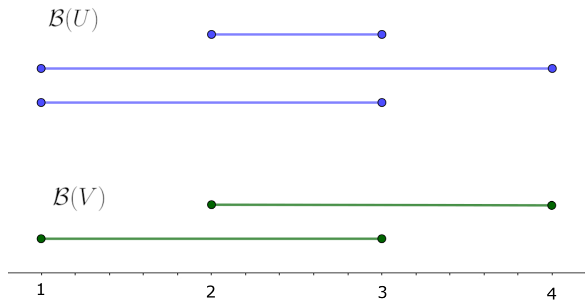
Corollary 6.4. *Given $0 < \varepsilon \in \mathbb{R}$ and a persistence module V indexed over $\mathbb{R}$, we have:*

$$X_{IJ}[1^V_\varepsilon](t) \simeq V_{I^\varepsilon,t+\varepsilon}$$

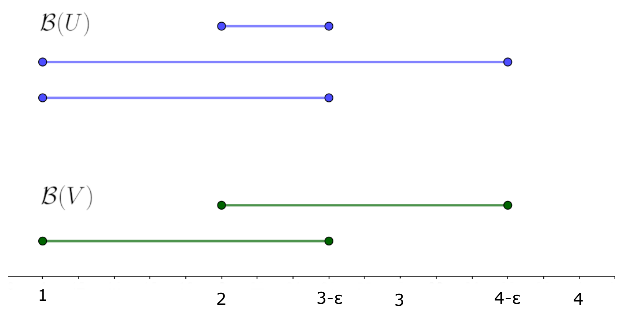
when $t \in I$ and $J = I$, and $X_{IJ}[1^V_\varepsilon](t) = 0$ otherwise.

Note that the effect of the functor $\mathbf{1}_\varepsilon$ on a barcode is to shift to the right by ε the left endpoint of each of its intervals. What Proposition 6.2 is saying is that $\#\mathcal{B}(X_{I^\varepsilon J^\varepsilon}[f]) = \#\mathcal{B}(X_{IJ}[\mathbf{1}_\varepsilon(f)])$ unless $I \cap J = \emptyset$.

Example 6.5. *Let us consider the morphisms f and $\mathbf{1}_\varepsilon(f)$ showed in Example 2.1. The barcodes of V and U are pictured below:*



and the ones from $\mathbf{1}_\varepsilon(V)$ and $\mathbf{1}_\varepsilon(U)$ are pictured below:



The direct sum of interval modules in the decomposition of V and U are given by the following submodules,

$$\begin{aligned}
V^+_{[1,3]} &= k \to \langle \begin{smallmatrix}1\\0\end{smallmatrix} \rangle \to \langle \begin{smallmatrix}1\\0\end{smallmatrix} \rangle \to 0,\\
V^-_{[1,3]} &= 0,\\
V^+_{[2,4]} &= 0 \to \langle \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \rangle \to \langle \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \rangle \to k,\\
V^-_{[2,4]} &= 0 \to \langle \begin{smallmatrix}1\\0\end{smallmatrix} \rangle \to \langle \begin{smallmatrix}1\\0\end{smallmatrix} \rangle \to 0,
\end{aligned}$$

and

$$\begin{aligned}
U^+_{[1,3]} &= \langle \begin{smallmatrix}1\\0\end{smallmatrix} \rangle \to \left\langle \begin{smallmatrix}1\\0\\0\end{smallmatrix} \right\rangle \to \left\langle \begin{smallmatrix}1\\0\\0\end{smallmatrix} \right\rangle \to 0,\\
U^-_{[1,3]} &= 0,\\
U^+_{[1,4]} &= \langle \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \rangle \to \left\langle \begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix} \right\rangle \to \left\langle \begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix} \right\rangle \xrightarrow{[0\ 1\ 0]} k,\\
U^-_{[1,4]} &= U^+_{[1,3]},\\
U^+_{[2,3]} &= 0 \to \left\langle \begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix} \right\rangle \to \left\langle \begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix} \right\rangle \to 0,\\
U^-_{[2,3]} &= 0 \to \left\langle \begin{smallmatrix}1\\0\\0\end{smallmatrix} \right\rangle \to \left\langle \begin{smallmatrix}1\\0\\0\end{smallmatrix} \right\rangle \to 0.
\end{aligned}$$

Observe that X_{IJ} is not zero only in the following cases:

$$X_{[1,3]\,[1,3]}[f] \simeq k_{[1,3]},$$

$$X_{[2,4]\,[1,4]}[f] \simeq k_{(3,4]}.$$

Then, by Proposition 6.2, we have:

$$X_{[1,3-\varepsilon]\,[1,3-\varepsilon]}[\mathbf{1}_\varepsilon(f)] \simeq k_{[1,3-\varepsilon]},$$

$$X_{[2,4-\varepsilon]\,[1,4-\varepsilon]}[\mathbf{1}_\varepsilon(f)] \simeq k_{(3-\varepsilon,4-\varepsilon]}.$$

7 Conclusions and future work

In this paper, we have defined a persistence partial matching $\mathcal{G}_f$ and a partial matching $\mathcal{M}_f$ between barcodes $\mathcal{B}(V)$ and $\mathcal{B}(U)$ induced by a morphism between two persistence modules V and U. We have proven that $\mathcal{G}_f$ and $\mathcal{M}_f$ are linear with respect to the direct sum of ladder modules. Besides, we have proven that $\mathcal{G}_f$ and $\mathcal{M}_f$ are robust to perturbations represented by the functor $\mathbf{1}_\varepsilon$ to f. The assumptions made in this paper are quite general and most of the results hold for any morphism between persistence modules indexed over $\mathbb{R}$ satisfying the descending chain condition for images and kernels. Only the results related to χ_f require the pointwise finite dimensional condition.

The following questions are left for future work. Firstly, we have observed that, in TDA, persistence modules are often obtained by applying the homology functor to a filtration of a topological space M. In particular, given a subspace $N \subset M$, the homology functor can be applied to filtrations of M/N. As a result, we could obtain a morphism between the persistence modules associated to M and M/N, respectively. We leave as a future work the study of how $\mathcal{G}_f$ and $\mathcal{M}_f$ can be applied to study the relation between $\mathcal{B}(M)$ and $\mathcal{B}(M/N)$.

Secondly, we also plan to develop algorithms that compute $\mathcal{G}_f$ and $\mathcal{M}_f$ specially when dealing with data modeled as simplical complexes.

Thirdly, from the theoretical point of view, we plan to study how $\mathcal{M}_f$ and $\mathcal{G}_f$ relate to the special cases of diagonal or block-decomposable modules. Theorem 4.5 and the proof of Theorem 5.1 might indicate that $\mathcal{M}_f$ defines a kind of projection from ladder modules to these decomposable modules.

Stability results in this paper are in terms of perturbations to morphisms between persistence modules, through the functor $\mathbf{1}_\varepsilon$. We also expect to obtain stability results in terms of ε-interleaved ladder modules. Besides, since the properties of χ_f when *(co)kerf* is ε-trivial are well-known [3], we would like to study if there are similar results for $\mathcal{M}_f$.

Finally, relations between persistence modules which come from dynamical systems are not usually given by morphisms but by another intermediate persistence modules through diagrams of the form $V \leftarrow W \rightarrow U$ [14, 15]. We leave as a future work how to define $\mathcal{G}_f$ in this case.

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References

[1] F. Chazal, V. de Silva, M. Glisse, and S. Oudot, *The Structure and Stability of Persistence Modules.* SpringerBriefs in Mathematics, Springer International Publishing, 2016.

[2] P. Bubenik and J. Scott, "Categorification of persistent homology," *Discrete and Computational Geometry*, vol. 51, pp. 600–627, 2014.

[3] U. Bauer and M. Lesnick, "Induced matchings and the algebraic stability of persistence barcodes," *Journal of Computational Geometry*, vol. 6, no. 2, pp. 162–191, 2015.

[4] W. Crawley-Boevey, "Decomposition of pointwise finite-dimensional persistence modules," *Journal of Algebra and Its Applications*, vol. 14, no. 5, 2015.

[5] E. Escolar and Y. Hiraoka, "Persistence modules on commutative ladders of finite type," *Discrete and Computational Geometry*, vol. 55, pp. 100–157, 2014.

[6] M. Botnan, V. Lebovici, and S. Oudot, "On rectangle-decomposable 2-parameter persistence modules," in *36th International Symposium on Computational Geometry, SoCG 2020* (S. Cabello and D. Chen, eds.), Leibniz International Proceedings in Informatics, LIPIcs, pp. 1–16, Schloss Dagstuhl- Leibniz-Zentrum fur Informatik GmbH, Dagstuhl Publishing, June 2020.

[7] H. Asashiba, E. Escolar, Y. Hiraoka, and H. Takeuchi, "Matrix method for persistence modules on commutative ladders of finite type," *Japan Journal of Industrial and Applied Mathematics volume*, vol. 36, p. 97–130, 2019.

[8] J. Cochoy and S. Oudot, "Decomposition of exact pfd persistence bimodules," *Discrete and Computational Geometry*, vol. 63, pp. 255–293, 2020.

[9] U. Bauer and M. Lesnick, "Persistence diagrams as diagrams: A categorification of the stability theorem," *Arxiv, unpublished*, 2016.

[10] A. T. Casas, "Distributing persistent homology via spectral sequences," 2020.

[11] C. Weibel, *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.

[12] A. Grothendieck, "Sur quelques points d'algèbre homologique, ii," *Tohoku Math. J. (2)*, vol. 9, no. 3, pp. 119–221, 1957.

[13] T. Leinster, *Basic Category Theory.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2014.

[14] H. Takeuchi, "The persistent homology of a sampled map: From a viewpoint of quiver representations," 2018.

[15] H. Edelsbrunner, G. Jabłoński, and M. Mrozek, "The persistent homology of a self-map," *Foundations of Computational Mathematics*, vol. 15, pp. 511–533, 2015.