Euler Well-Composedness*

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Abstract. In this paper, we define a new flavour of well-composedness, called Euler well-composedness, in the general setting of regular cell complexes: A regular cell complex is Euler well-composed if the Euler characteristic of the link of each boundary vertex is 1. A cell decomposition of a picture I is a pair of regular cell complexes $\big(K(I), K(\bar{I})\big)$ such that $K(I)$ (resp. $K(\bar{I})$) is a topological and geometrical model representing I (resp. its complementary, $\bar{I}$). Then, a cell decomposition of a picture I is self-dual Euler well-composed if both $K(I)$ and $K(\bar{I})$ are Euler well-composed. We prove in this paper that, first, self-dual Euler well-composedness is equivalent to digital well-composedness in dimension 2 and 3, and second, in dimension 4, self-dual Euler well-composedness implies digital well-composedness, though the converse is not true.

Keywords: Digital topology, discrete geometry, well-composedness, cubical complexes, cell complexes, manifolds, Euler characteristic.

1 Introduction

The concept of well-composedness of a picture was first introduced in [13] for 2D pictures and extended later to 3D in [14]: a well-composed picture satisfies that the continuous analog of the given picture has a boundary surface that is a manifold. The concept is described in terms of forbidden subsets for which the picture is not well-composed. In [8], the author defines a gap in a binary object in a digital space of arbitrary dimension, an analogous concept to that of forbidden subset of Latecki et al. and similar to the notion of tunnel that had been defined in [1] for digital hyperplanes. In [3], the concept of critical configurations (i.e., forbidden subsets) was extended to nD.

3D well-composed images may have some computational advantages regarding

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the application of several algorithms in computer vision, computer graphics and image processing. But in general, images are not a priori well-composed. There are several "repairing" methods for turning them into well-composed images (see, for example, [15, 12, 19, 18]). Besides, in [17], the authors extended the notion of "digital well-composedness" to nD sets.
Equivalences between different flavours of well-composedness have been studied in [4], namely: continuous well-composedness (CWCness), digital well-composedness (DWCness), well-composedness in the Alexandrov sense (AWCness), well-composedness based on the equivalence of connectivities (EWCness), and well-composedness on arbitrary grids (AGWCness). More specifically, as stated in [4], it is well-known that, in 2D, AWCness, CWCness, DWCness and EWCness are equivalent, so a 2D picture is well-composed if and only if it is XWC (X = A, C, D, E). In 3D, only AWCness, CWCness, DWCness are equivalent. Note that no link between AGWCness and the other flavours of well-composedness were known in nD, and for $n \geq 4$ the equivalences between the different flavours of well-composedness (AWCness, CWCness, DWCness and EWCness) have not been proved yet (except that AWCness implies DWCness, see [7] and that DWCness implies EWCness, see [3])

Recently, in [6], a counterexample has been given to prove that DWCness does not imply CWCness, what is an important result since it breaks with the idea that all the flavours of well-composedness are equivalent.

In the papers [9, 10, 5], the authors developed an nD topological method for repairing digital pictures (in the cubical grid) with "pinches", turning them into weakly well-composed complexes. More specifically, such a method constructs a "simplicial decomposition" $\big(P_S(I),\, P_S(\bar{I})\big)$ of a given n-dimensional (nD) picture I (initially represented by a cubical complex $Q(I)$) such that: (1) $P_S(I)$ is homotopy equivalent to $Q(I)$ and $P_S(\bar{I})$ is homotopy equivalent to $Q(\bar{I})$ being $\bar{I}$ the nD picture that is the "complementary" of I; (2) $\big(P_S(I), P_S(\bar{I})\big)$ is self-dual weakly well-composed, that is, for each vertex v on the boundary of $P_S(I)$, the set of n-simplices of $P_S(I)$ incident to v are "face-connected" (defined later), as well as those of $P_S(\bar{I})$ incident to v. As we will see later, in the setting of cubical complexes canonically associated to nD pictures, self-dual weak well-composedness is equivalent to digital well-composedness.

In fact, our ultimate goal is to prove that this method provides continuously well-composed complexes, that is, the boundary of their underlying polyhedron is an (n-1)D topological manifold. Since this goal is not reachable yet according to us, we propose an "intermediary" flavour of well-composedness, called Euler well-composedness, that is stronger than weak well-composedness but weaker than continuous well-composedness. The aim of the present paper is then to prove that Euler well-composedness implies weak well-composedness but that the converse is not true. The plan is the following: Section 2 and 3 recall the background relative to self-dual weak and digital well-composedness respectively. Section 4 introduces the definition of Euler well-composedness based on what we call "χ-critical vertices" and shows that: Euler well-composedness is equivalent to self-dual weak and digital well-composedness on 2D and 3D cubical grids; and

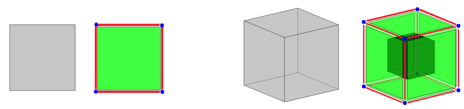


Fig. 1. Left: a 2-dimensional cube and its faces. Right: a 3-dimensional cube and its faces.

Euler well-composedness is stronger than weak and digital well-composedness on 4D cubical grids. Section 5 concludes the paper.

2 Background on regular cell complexes

Roughly speaking, a *regular cell complex* K is a collection of cells (where k-cells are homeomorphic to k-dimensional balls) glued together by their boundaries (faces), in such a way that a non-empty intersection of any two cells of K is a cell in K. Regular cell complexes have particularly nice properties, for example, their homology is effectively computable (see [16, p. 243]). When the k-cells in K are k-dimensional cubes, we refer to K as a *cubical complex* (see Figure 1). When they are k-dimensional simplices (points, edges, triangles, tetrahedra, etc.), we refer to K as a *simplicial complex.*

Let K be a regular cell complex. A k-cell μ is a *proper face of* an ℓ-cell $\sigma \in K$ if μ is a face of σ and $k < \ell$. A cell of K which is not a proper face of any other cell of K is said to be a *maximal* cell of K.

Let k, k' be integers such that $k < k'$. Then, the set $\{k, k+1, \dots, k'-1, k'\}$ will be denoted by $[\![k, k']\!]$.

Definition 1 (face-connectedness). *Let μ be a cell of a regular cell complex K. Let $\mathcal{A}_K^{(\ell)}(\mu)$ be a set of ℓ-cells of K sharing μ as a face. Let σ and σ' be two ℓ-cells of $\mathcal{A}_K^{(\ell)}(\mu)$. We say that σ and σ' are face-connected in $\mathcal{A}_K^{(\ell)}(\mu)$ if there exists a path $\pi(\sigma, \sigma') = (\sigma_1 = \sigma, \sigma_2 \dots, \sigma_{m-1}, \sigma_m = \sigma')$ of ℓ-cells of $\mathcal{A}_K^{(\ell)}(\mu)$ such that for any $i \in [\![1, m-1]\!]$, σ_i and σ_{i+1} share exactly one $(\ell-1)$-cell.*
We say that a set $\mathcal{A}_K^{(\ell)}(\mu)$ is face-connected if any two ℓ-cells σ and σ' in $\mathcal{A}_K^{(\ell)}(\mu)$ are face-connected in $\mathcal{A}_K^{(\ell)}(\mu)$.

An *external* cell of K is a proper face of exactly one maximal cell in K. A regular cell complex is *pure* if all its maximal cells have the same dimension. The *rank* of a cell complex K is the maximal dimension of its cells. The *boundary surface* of a pure regular cell complex K, denoted by ∂K, is the regular cell complex composed by the external cells of K together with all their faces. Observe that ∂K is also pure.

Definition 2 (nD cell complex). *An nD cell complex K is a pure regular cell complex of rank n embedded in $\mathbb{R}^n$. The underlying space (i.e., the union of the cells as subspaces of $\mathbb{R}^n$) will be denoted by $|K|$.*

An nD cell complex K is said to be *(continuously) well-composed* if $|\partial K|$ is an $(n-1)$-manifold, that is, each point of $|\partial K|$ has a neighborhood in $|\partial K|$ homeomorphic to $\mathbb{R}^{n-1}$.

Definition 3 (Weak well-composedness). *An nD cell complex K is weakly well-composed (wWC) if for any 0-cell (also called vertex) v in K, $\mathcal{A}_K^{(n)}(v)$ is face-connected.*

Definition 4 (Euler characteristic). *Let K be a finite regular cell complex. Let a_k denote the number of k-cells of K. The* Euler characteristic *of K is defined as*

$$\chi(K) = \sum_{k=0}^{\infty}(-1)^k a_k.$$

Recall that the Euler characteristic of a finite regular cell complex depends only on its homotopy type [11, p. 146].

Definition 5 (star, closed star and link). *Let v be a vertex of a given regular cell complex K.*

- *The star of v (denoted $\mathrm{St}_K(v)$) is the set of cells having v as a face. Note that the star of v is generally not a cell complex itself.*
- *The closed star of v (denoted $\mathrm{ClSt}_K(v)$) is the cell complex obtained by adding to $\mathrm{St}_K(v)$ all the faces of the cells in $\mathrm{St}_K(v)$.*
- *The link of v (denoted $\mathrm{Lk}_K(v)$) is the closed star of v minus the star of v, that is, $\mathrm{ClSt}_K(v) \setminus \mathrm{St}_K(v)$.*

3 Background on nD pictures

Now, let us formally introduce some concepts related to digital well-composedness of nD pictures.

Definition 6 (nD picture). *Let $n \geq 2$ be an integer and $\mathbb{Z}^n$ the set of points with integer coordinates in $\mathbb{R}^n$. An nD picture is a pair $I = (\mathbb{Z}^n, F_I)$, where F_I is a subset of $\mathbb{Z}^n$. The set F_I is called the foreground of I and the set $\mathbb{Z}^n \setminus F_I$ the background of I. The picture "complement" of I is defined as $\bar{I} = (\mathbb{Z}^n, \mathbb{Z}^n \setminus F_I)$.*

Definition 7 (cubical complex $Q(I)$). *The nD cubical complex $Q(I)$ canonically associated to an nD picture $I = (\mathbb{Z}^n,\, F_I)$ is composed by those n-dimensional unit cubes centered at each point in F_I, whose $(n-1)$-faces are parallel to the coordinate hyperplanes, together with all their faces.*

Figure 2 shows the geometric realization of cubical complexes representing a 2D binary picture of two pixels (left) and two 3D pictures of 2 voxels each.
Roughly speaking, two topological spaces are *homotopy equivalent* if one can be continuously deformed into the other. A specific example of homotopy equivalence is a *deformation retraction* of a space X onto a subspace A which is a

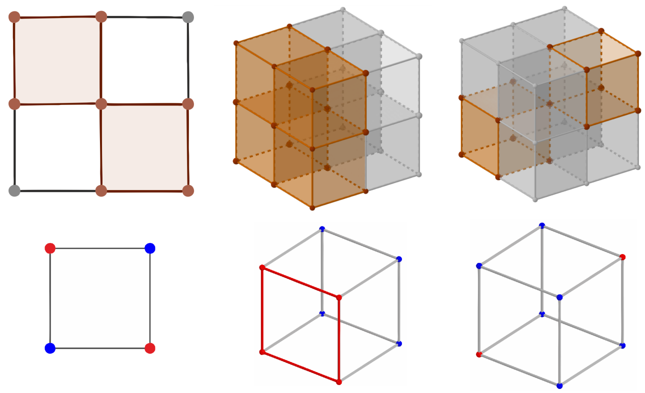


Fig. 2. Top figures: cubical complexes (in brown) of dimension 2 (left) and 3 (middle and right). Bottom figures: point representation of the cubical complexes on the top to help the intuition of Table 6 and 7. Points in red correspond to the maximal cubes, that are joined by a red edge if the corresponding cubes are face–connected.

family of maps $f_t : X \to X$, $t \in [0,1]$, such that: $f_0(x) = x$, $\forall x \in X$; $f_1(X) = A$; $f_t(a) = a$, $\forall a \in A$ and $t \in [0,1]$. The family $\{f_t : X \to X\}_{t\in[0,1]}$ should be continuous in the sense that the associated map $F : X \times I \to X$, where $F(x,t) = f_t(x)$, is continuous. See [11, p. 2].

Definition 8 (cell complex over an nD picture). *A cell complex over an nD picture I is an nD cell complex, denoted by $K(I)$, such that there exists a deformation retraction from $K(I)$ onto $Q(I)$.*

In [2], the concept of blocks was introduced. For two integers $k \leq k'$, let $\mathcal{E} = \{e^1, \dots, e^n\}$ be the canonical basis of $\mathbb{Z}^n$. Given a point $z \in \mathbb{Z}^n$ and a family of vectors $\mathcal{F} = \{f^1, \dots, f^k\} \subseteq \mathcal{E}$, the *block of dimension* k associated to the couple $(z, \mathcal{F})$ is the set defined as:

$$B(z,\mathcal{F}) = \left\{ z + \sum_{i\in[\![1,k]\!]} \lambda_i\, f^i \ :\ \lambda_i \in \{0,1\},\ \forall i \in [\![1,k]\!] \right\}.$$

This way, a 0-block is a point, a 1-block is a set of two points in $\mathbb{Z}^n$ on an unit edge, a 2-block is a set of four points on a unit square, and so on. A subset $B \subset \mathbb{Z}^n$ is called a *block* if there exists a couple $(z, \mathcal{F}) \in \mathbb{Z}^n \times \mathcal{P}(\mathcal{E})$ (where $\mathcal{P}(\mathcal{E})$ represents the set of all the subsets of $\mathcal{E}$), such that $B = B(z, \mathcal{F})$. We will denote the set of blocks of $\mathbb{Z}^n$ by $\mathcal{B}(\mathbb{Z}^n)$.

Definition 9 (antagonists). *Two points p, q belonging to a block $B \in \mathcal{B}(\mathbb{Z}^n)$ are said to be antagonists in B if their distance equals the maximum distance using the L^1-norm*[3] *between two points in B, that is, $\|p - q\|_1 = \max\big\{\|r - s\|_1 : r, s \in B\big\}$.*

[3] The L^1-norm of a vector $\alpha = (x_1, \dots, x_n)$ is $||\alpha||_1 = \sum_{i\in[\![1,n]\!]} |x_i|$.

Remark 1. The antagonist of a point p in a block $B \in \mathcal{B}(\mathbb{Z}^n)$ containing p exists and is unique. It is denoted by $\text{antag}_B(p)$.

Note that when two points $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are antagonists in a block of dimension $k \in [\![0, n]\!]$, then $|x_i - y_i| = 1$ for $i \in \{i_1, \dots, i_k\} \subseteq [\![1, n]\!]$ and $x_i = y_i$ otherwise.

Definition 10 (critical configuration). *Let $I = (\mathbb{Z}^n, F_I)$ be an nD picture and $B \in \mathcal{B}(\mathbb{Z}^n)$ a block of dimension $k \in [\![2, n]\!]$. We say that I contains a critical configuration in the block B if $F_I \cap B = \{p, p'\}$ or $F_I \cap B = B \setminus \{p, p'\}$, with p, p' being two antagonists in B.*

Definition 11 (digital well-composedness). *An nD picture is said to be digitally well-composed (DWC) if it does not contain any critical configuration in any block $B \in \mathcal{B}(\mathbb{Z}^n)$.*

We say that a property of an nD picture I is *self-dual*, if its complement $\bar{I}$ also satisfies the property. Hence, last definition of digital well-composedness is self-dual and based on local patterns.

4 Introducing the concept of Euler well-composedness

In this section, we introduce the new concept of Euler well-composedness for regular cell complexes and show that, in the cubical setting, digital well-composedness is equivalent to Euler well-composedness in 2D and 3D, but digital well-composedness is weaker than Euler well-composedness in 4D.

Definition 12 (χ-critical vertex). *Given an nD cell complex K, $n \geq 2$, a vertex $v \in K$ is χ-critical for K if:*

$$v \in \partial K \quad \textit{and} \quad \chi\big(\text{Lk}_K(v)\big) \neq \chi(\mathbb{B}^{n-1}) = 1,$$

where $\mathbb{B}^{n-1}$ is an $(n-1)$-dimensional ball.

In Figure 3, different cases of χ-critical and non-χ-critical vertices are shown.

Definition 13 (Euler well-composedness). *An nD cell complex is Euler well-composed if it has no χ-critical vertices.*

For example, in Figure 3, only case (c) represents a cubical complex that is Euler well-composed.

Definition 14 (cell decomposition of an nD picture). *A cell decomposition of an nD picture I consists of a pair of nD cell complexes, $\big(K(I), K(\bar{I})\big)$, such that:*

- *$K(I)$ is a cell complex over I and $K(\bar{I})$ is a cell complex over $\bar{I}$.*
- *$|K(I) \cup K(\bar{I})| = \mathbb{R}^n$.*
- *$K(I) \cap K(\bar{I}) = \partial K(I) = \partial K(\bar{I})$.*

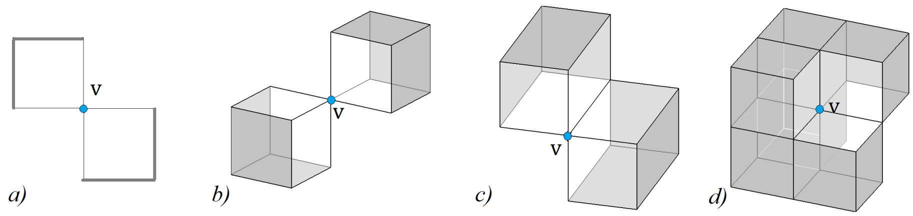
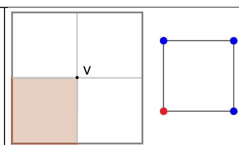
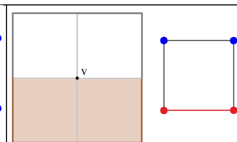
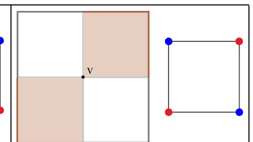


Fig. 3. Different cases of a vertex v on the boundary of a cubical complex $Q(I)$. $\mathrm{Lk}_Q(v)$ has been drawn in grey. a) A 2D case of a χ- critical vertex, with $\chi\big(\mathrm{Lk}_Q(v)\big) = 2$; b) a 3D case of a χ-critical vertex, with $\chi\big(\mathrm{Lk}_Q(v)\big) = 2$; c) a 3D case of a vertex on the boundary that is not a χ-critical vertex, since $\chi\big(\mathrm{Lk}_Q(v)\big) = 1$; d) complementary configuration of case (c) in which v is a χ-critical vertex, with $\chi\big(\mathrm{Lk}_Q(v)\big) = 0$.

Table 1. All the possible configurations (cubical and points representations) for $U = (\mathbb{Z}^2, F_U)$ in $B(o, \mathbb{Z}^2)$ satisfying that $o \in F_U$ and $\mathrm{card}(F_U) \leq 2$, up to rotations and reflections around $v = (1/2, 1/2)$.

$\mathrm{card}(F_U) = 1, 2$						
	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$
wWC	Yes	Yes	Yes	Yes	No	No
χWC	Yes	Yes	Yes	Yes	No	No
DWC	Yes		Yes		No	

Definition 15 (self-dual Euler well-composedness). *A cell decomposition* $\big(K(I), K(\bar{I})\big)$ *of an nD picture I is self-dual Euler well-composed (sχWC) if both* $K(I)$ *and* $K(\bar{I})$ *are Euler well-composed.*

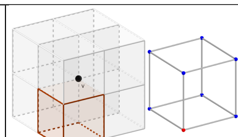
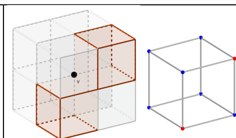
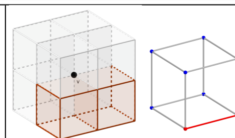
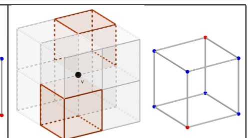
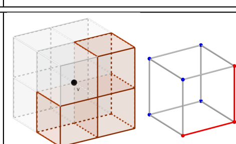
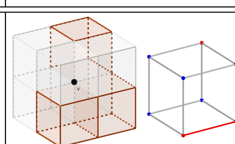
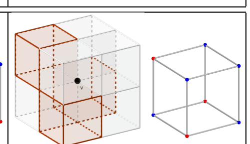
The definition of self-dual weak well-composedness was introduced in [5]. We recall it here.

Definition 16 (self-dual weak well-composedness). *A cell decomposition* $\big(K(I), K(\bar{I})\big)$ *of an nD picture I is self-dual weakly well-composed (swWC) if both* $K(I)$ *and* $K(\bar{I})$ *are weakly well-composed.*

We recall now that self-dual weak well-composedness is equivalent to digital well-composed in the cubical setting.

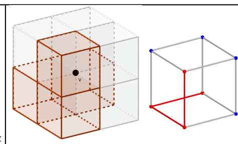
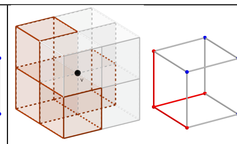
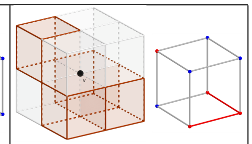
Theorem 1 ([3]). *Let* $I = (\mathbb{Z}^n, F_I)$ *be an nD picture and* $Q(I)$ *the cubical complex canonically associated to I. Then, I is digitally well-composed if and only if* $\big(Q(I), Q(\bar{I})\big)$ *is self-dual weakly well-composed.*

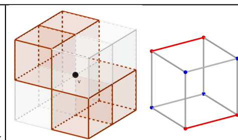
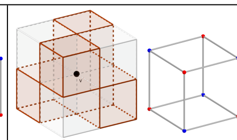
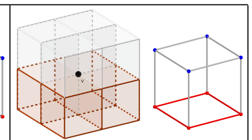
Table 2. All the possible configurations for $U = (\mathbb{Z}^3, F_U)$ in $B(o, \mathbb{Z}^3)$ satisfying that $o \in F_U$ and $\text{card}(F_U) \leq 3$, up to rotations and reflections around $v = (1/2, 1/2, 1/2)$.

$\text{card}(F_U) = 1$						
	$Q(I)$	$Q(\bar{I})$				
wWC	Yes	Yes				
χWC	Yes	Yes				
DWC	Yes					
$\text{card}(F_U) = 2$						
	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$
wWC	No	Yes	Yes	Yes	No	Yes
χWC	Yes	No	Yes	Yes	No	No
DWC	No		Yes		No	
$\text{card}(F_U) = 3$						
	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$
wWC	Yes	Yes	No	Yes	No	No
χWC	Yes	Yes	Yes	No	No	No
DWC	Yes		No		No	

We study now the possible equivalences between self-dual weak well-composedness and self-dual Euler well-composedness. We will prove that, as expected, self-dual Euler well-composedness is equivalent to digital well-composedness in 2D and 3D. Nevertheless, as we will see later, this equivalence is no longer true for 4D pictures. We will prove that self-dual Euler well-composedness implies digital well-composedness in 4D although the converse is not true. Observe that considering the definition of Euler well-composedness, we can study local patterns only. To prove such results, we should check the exhaustive lists of all the possible configurations $U = (\mathbb{Z}^n, F_U)$ in any block $B(z, \mathbb{Z}^n)$ for $z \in \mathbb{Z}^n$, for $n = 2, 3, 4$. To reduce the list and without loss of generality, we will only study configurations $U = (\mathbb{Z}^n, F_U)$ in the block $B(o, \mathbb{Z}^n)$ for o being the coordinates origin. Besides,

Table 3. All the possible configurations for $U = (\mathbb{Z}^3, F_U)$ in $B(o, \mathbb{Z}^3)$ satisfying that $o \in F_U$ and $\text{card}(F_U) = 4$, up to rotations and reflections around $v = (1/2, 1/2, 1/2)$.

$\text{card}(F_U) = 4$						
	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$
wWC	Yes	Yes	Yes	Yes	No	No
χWC	Yes	Yes	Yes	Yes	No	No
DWC	Yes		Yes		No	

$\text{card}(F_U) = 4$						
	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$	$Q(I)$	$Q(\bar{I})$
wWC	No	No	No	No	Yes	Yes
χWC	No	No	No	No	Yes	Yes
DWC	No		No		Yes	

we will also assume, again without loss of generality, that o is always in F_U. Let $\text{card}(F_U)$ denote the number of points in F_U. Since $\text{card}(F_{\bar{U}}) = 2^n - \text{card}(F_U)$, we will only study configurations U in $B(o, \mathbb{Z}^n)$ satisfying that $\text{card}(F_U) \leq 2^{n-1}$.

Fix a configuration U satisfying all the requirements listed above. Let $v \in \partial Q(U)$ be the vertex with coordinates $(1/2, \overset{n \text{ times}}{\ldots}, 1/2)$. Then, to see if such configuration is digitally well-composed, we will check if both $\mathcal{A}^{(n)}_{Q(U)}(v)$ and $\mathcal{A}^{(n)}_{Q(\bar{U})}(v)$ are face-connected or not. That is, we will check if the pair $(Q(U), Q(\bar{U}))$ is self-dual weakly well-composed. Similarly, to see if such configuration is self-dual Euler well-composed, we will check if vertex v is χ-critical in both $Q(U)$ and $Q(\bar{U})$.

Theorem 2. *Self-dual Euler well-composedness in the 2D and 3D cubical setting is equivalent to digital well-composedness.*

Proof. Table 1 shows all the possible configurations for $U = (\mathbb{Z}^2, F_U)$ in $B(o, \mathbb{Z}^2)$ satisfying that $o \in F_U$ and $\text{card}(F_U) \leq 2$, up to rotations and reflections around v. Looking at the table, we can check that DWCness $\Leftrightarrow$ sχWCness in 2D. Similarly, Tables 2 and 3 show that DWCness $\Leftrightarrow$ sχWCness in 3D. $\square$

Theorem 3. *Digital well-composedness does NOT imply self-dual Euler well-composedness in 4D.*

Table 4. Amount of configurations $U = (\mathbb{Z}^n, F_U)$ in $B(o, \mathbb{Z}^4)$ for the different cases in 4D clustered depending on the number of points in F_U and the property of being (or not) $Q(U)$ and $Q(\bar{U})$ weakly well-composed and/or Euler well-composed.

card(F_U)								$Q(U)$		$Q(\bar{U})$	
1	2	3	4	5	6	7	8	wWc	χWc	wWc	χWc
0	0	0	0	0	0	0	120	Yes	No	No	No
0	0	0	0	0	24	189	96	No	Yes	No	No
0	1	18	149	500	870	490	120	No	No	Yes	No
0	0	0	0	0	0	28	96	No	No	No	Yes
0	0	0	0	0	0	0	0	Yes	Yes	No	No
0	0	0	0	0	0	0	60	Yes	No	Yes	No
0	0	0	0	0	0	112	672	Yes	No	No	Yes
0	10	69	232	565	1074	1554	672	No	Yes	Yes	No
0	0	0	0	0	0	0	0	No	Yes	No	Yes
0	0	0	0	0	0	0	0	No	No	Yes	Yes
0	0	0	0	0	12	84	240	Yes	Yes	Yes	No
0	0	0	0	0	0	0	0	Yes	Yes	No	Yes
0	0	0	0	0	72	336	240	Yes	No	Yes	Yes
0	0	0	0	0	0	0	0	No	Yes	Yes	Yes
0	0	0	4	55	303	861	1811	No	No	No	No
1	4	18	70	245	648	1351	2308	Yes	Yes	Yes	Yes

Proof. An exhaustive list of configurations of hypercubes in $4D$ incident to a vertex that are digitally well-composed but not self-dual Euler well-composed is provided in Table 6. The complete list is summed up in Table 5 and can be found in the GitHub repository: https://github.com/Cimagroup/Euler-WCness. □

Theorem 4. *Self-dual Euler well-composedness in the 4D cubical setting implies digital well-composedness.*

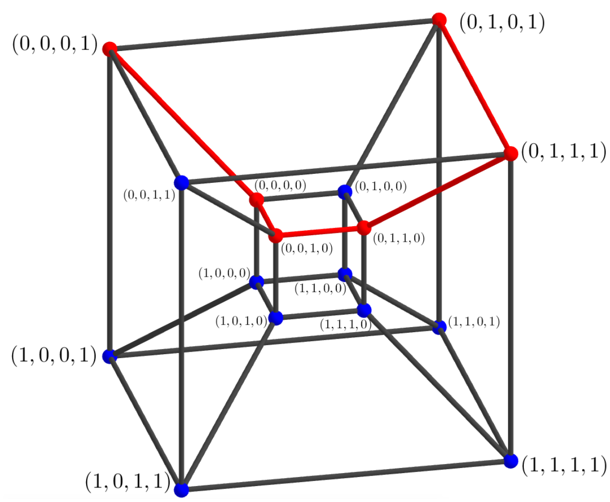
Proof. An exhaustive list of all the possible configurations $U = (\mathbb{Z}^n, F_U)$ in the block $B(o, \mathbb{Z}^4)$ satisfying that U is not digitally well-composed is given in Table 7. All such cases satisfy that they are not self-dual Euler well-composed either. □

Table 5. Exhaustive list in 4D of configurations for U in $B(o, \mathbb{Z}^4)$ clustered in the number of points in F_U and the property of being or not DWC and/or sχWC.

card(F_U)								U	$\big(Q(U), Q(\bar{U})\big)$
1	2	3	4	5	6	7	8	DWC	sχWC
1	4	18	70	245	648	1351	2308	Yes	Yes
0	0	0	0	0	84	420	540	Yes	No
0	0	0	0	0	0	0	0	No	Yes
0	11	87	385	1120	2271	3234	3587	No	No

Table 6: Exhaustive list of configurations for U in $B(o, \mathbb{Z}^4)$ satisfying that U is DWC but $(Q(U), Q(\bar{U}))$ is not sχWC. First column indicates the number of points of F_U; second column corresponds to the amount of configurations for U in $B(o, \mathbb{Z}^4)$ with the corresponding number of points in F_U and third column is an example of such kind of configuration.

card(F_U)	Amount	Example
6	84	{(0,0,0,0),(0,0,0,1),(0,0,1,0), (0,1,0,1),(0,1,1,0),(0,1,1,1)}



card(F_U)	Amount	Example
7	420	{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1), (0,1,0,0),(1,0,1,1),(1,1,0,0)}

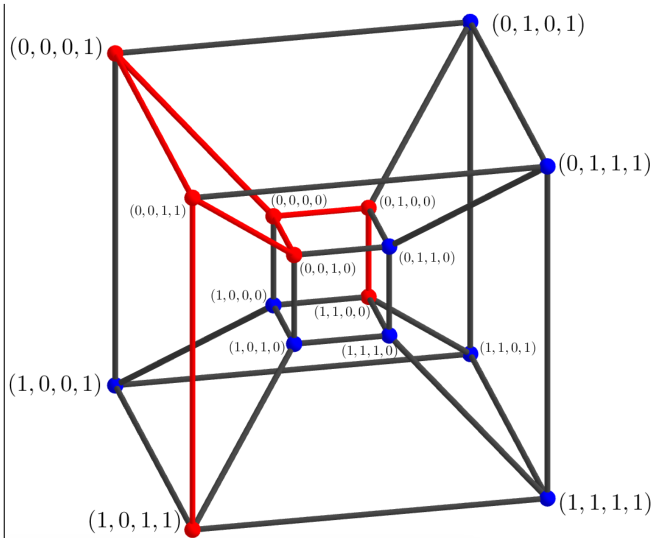
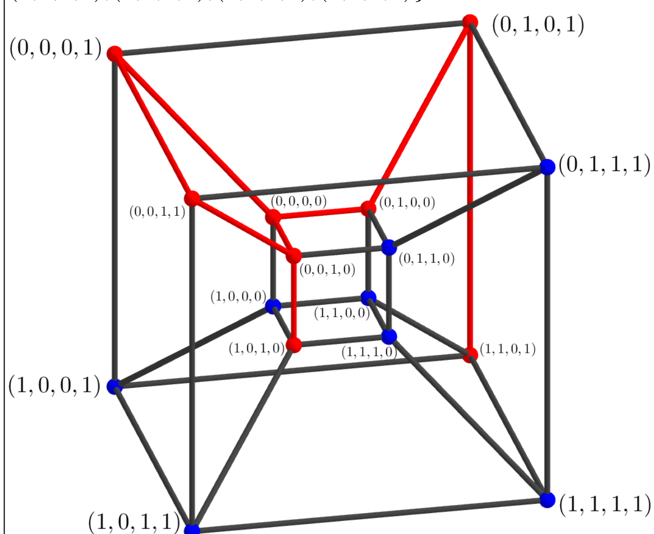
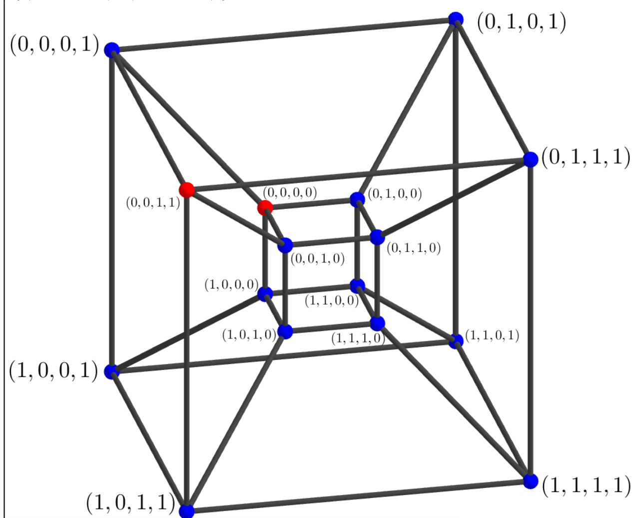
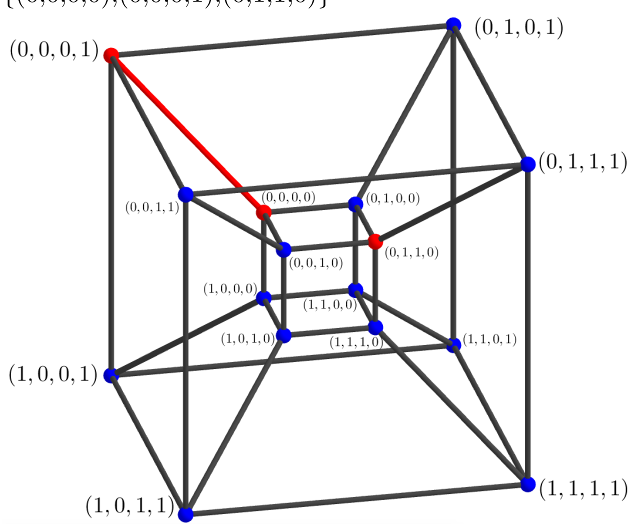
		
8	540	{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1), (0,1,0,0),(0,1,0,1),(1,0,1,0),(1,1,0,1)} 

Table 7: Exhaustive list of all the possible configurations $U = (\mathbb{Z}^n, F_U)$ in the block $B(o, \mathbb{Z}^4)$ satisfying that U is not digitally well-composed. All such cases satisfy that they are not self-dual Euler well-composed either. First column corresponds to the number of points in F_U, second column corresponds to the amount of configurations that there exist with such number of points in F_U. Third column shows an example of such configuration.

not DWC $\Rightarrow$ not sχWC		
card(F_U)	Amount	Example
1	0	-
2	11	{(0,0,0,0),(0,0,1,1)} 
3	87	{(0,0,0,0),(0,0,0,1),(0,1,1,0)} 
4	385	{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,1,1,1)}

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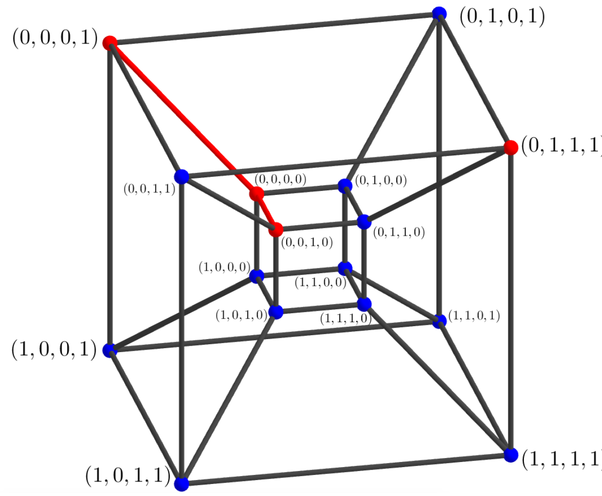
<tr><td></td><td></td><td>
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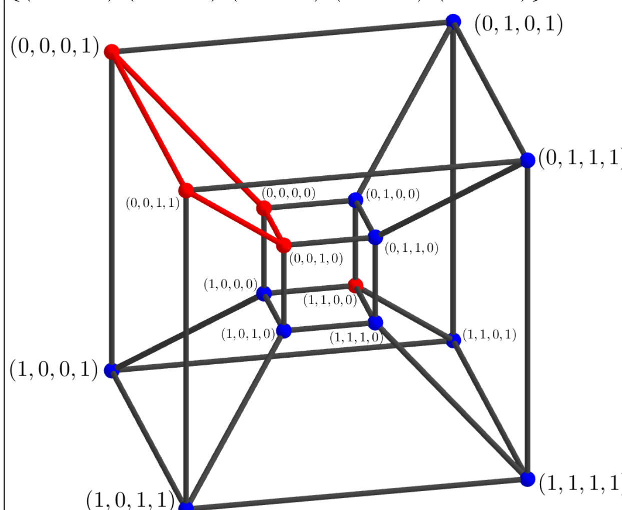
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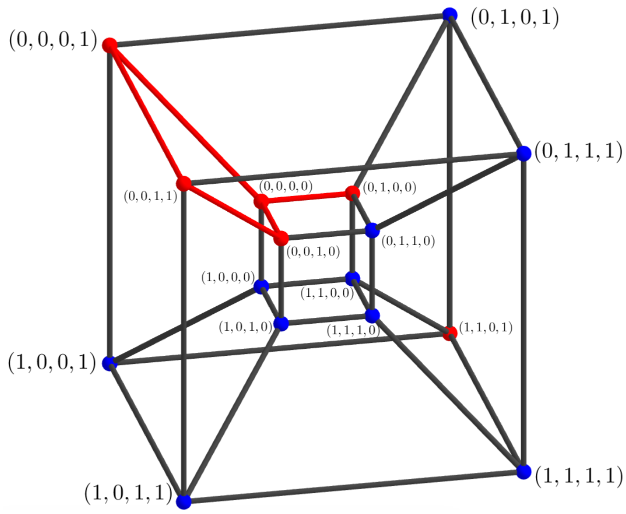
<tr><td></td><td></td><td>
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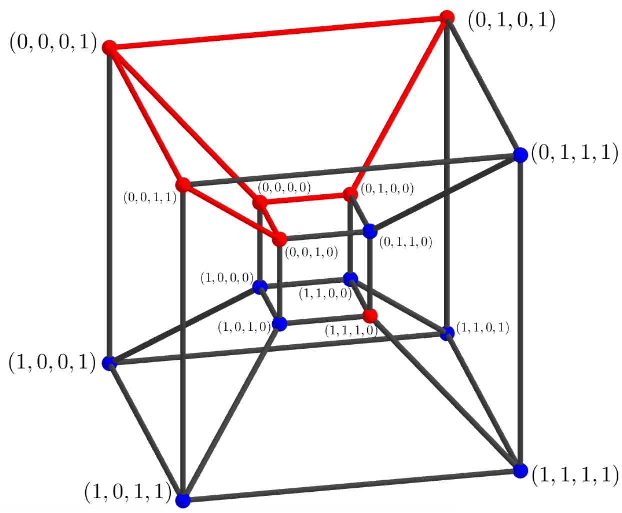
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<tr><td>8</td><td>3587</td><td>{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1),
(0,1,0,0),(0,1,0,1),(0,1,1,0),(1,1,1,1)}</td></tr>
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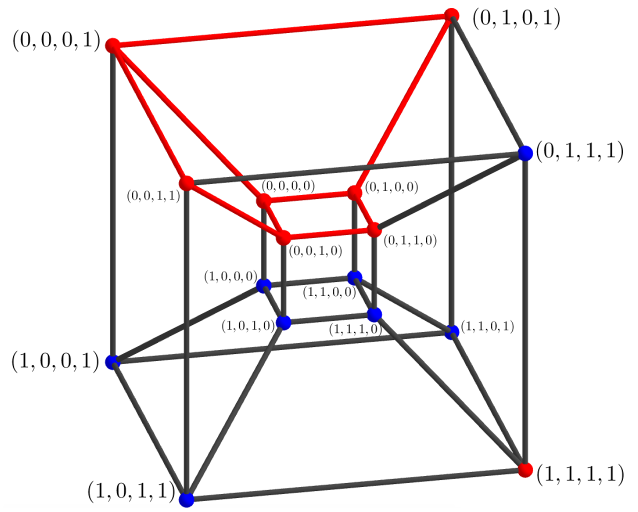

5 Conclusions and future works

We have proved via exhaustive lists of cases that self-dual weak well-composedness and digital well-composedness do not imply self-dual Euler well-composedness, but that the converse is true in 2D, 3D and 4D. In a future paper, we plan to cluster the 4D configurations obtained in equivalent classes up to rotations and reflections around the vertex v, similarly as what we have done to study the 2D and 3D cases. We also plan to prove the claim "self-dual Euler well-composedness implies digital well-composedness" in nD, $n \geq 2$ and study the existence of counter-examples for the converse in nD, for $n > 4$. Moreover, we plan to prove that the nD repairing method of [9, 10, 5] provides self-dual Euler well-composed simplicial complexes, providing a step forward to continuous well-composedness.

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