Steenrod reduced powers and computability

R. González-Díaz and P. Real

Dpto. Matemática Aplicada I (Universidad de Sevilla)

Avda. Reina Mercedes, s/n. CP: 41012 Sevilla (Spain)

E-mails: rogodi@cica.es, real@cica.es

Abstract

Steenrod cohomology operations are algebraic tools for distinguishing non–homeomorphic topological spaces. In this paper, starting off from the general method developed in [6] for cup–i products and Steenrod squares, we present an explicit combinatorial formulation for the particular Steenrod cohomology operation $\mathfrak{P}_1^p : H^q(X;\mathbb{F}_p) \to H^{qp-1}(X;\mathbb{F}_p)$, where p is an odd prime, q a non–negative integer, X a simplicial set and $\mathbb{F}_p$ the finite field with p elements. As an example, we design an algorithm for computing $\mathfrak{P}_1^p$ on the cohomology of a simplicial complex and we determine its complexity.

Extended Abstract

Algebraic Topology studies problems of purely qualitative nature making use of algebraic structures. An evident example of this "algebrization" of Topology is the theory of algebraic invariants associated to topological spaces. The most intuitive algebraic invariant is the number of connected components of a topological space. Appropriate generalizations of this notion are homology, cohomology and homotopy groups. Homology (or its dual, cohomology) groups are, in general, easier to compute than homotopy groups. Moreover, the cohomology of a topological space has an additional ring structure determined by the *cup product* of two cochains. In particular, this product allows us to distinguish topological spaces having isomorphic homology groups. A finer class of invariants are cohomology operations, that is, algebraic maps on the cohomology groups of spaces. Within these tools, an extremely important type are *Steenrod squares and Steenrod reduced powers* [15, 13]. The importance of these Steenrod cohomology operations is twofold: on the one hand, they have a deep geometrical meaning; on the other hand, they can be used in areas close to Homological Algebra: group cohomology, Hochschild cohomology of algebras, ... These algebraic operations are extremely well–studied objects from a topological viewpoint (see [17] and [1] for a non–exhaustive account of results). The fact that Steenrod cohomology operations are completely determined by combinatorial data is also well–known ([11],[2], [12], ...). However, the recursive algorithms derived from these studies are too slow for practical implementation.

Passing on now to other matters, the problem of computability in Algebraic Topology is a "delicate one", as George Whitehead stated in 1983. With regard to this question, we limit ourselves to say that it is necessary to distinguish the problem of recognizing parts of classical Algebraic Topology as effective from that of providing practical solutions for these parts. This assertion is motivated by the fact that the majority of algorithms (spectral sequences, ...) in this field carry very high computational costs.

In combinatorial design theory, there is a recent interest by cohomology as it is indicated in [8]. Then the obtention of efficient algorithms for computing n–cocycles can be useful in that field. With regard to 2–cocycles, several methods for finding cocycles representing 2–dimensional cohomology classes of finite groups have been designed recently [5, 8, 9]. Working within the framework of Simplicial Topology [11], an explicit description of certain simplicial cocycles on simplicial sets (that is, a combinatorial model of a topological space) is given in [6]. To be more precise, in that paper, an explicit formulation for cup–i products and Steenrod squares is presented. An algorithm for computing these invariants is derived in a natural way. For example, the data for computing Steenrod squares are a simplicial set X, a non–negative integer i and a n–cocycle c of X, giving back as answer a $(n+i)$–cocycle $Sq^i(c)$. As an application, algorithms for computing these invariants in simplicial complexes are given in [7].

In this paper, we analyze in detail the underlying combinatorial structure of the Steenrod reduced power $\mathfrak{P}_1^p : H^q(X;\mathbb{F}_p) \to H^{qp-1}(X;\mathbb{F}_p)$, p being an odd prime, q a positive integer, X a simplicial set and $\mathbb{F}_p$ the finite field with p elements. This study is based on the results obtained in [6] for Steenrod reduced powers. Regarding these results, our motivation in the very near future is to give an explicit combinatorial picture of all Steenrod cohomology operations.

We now give some simplicial and algebraic preliminaries in order to facilitate the understanding of the extended abstract. This material can be found, for example, in [11], [10] and [16].

Let R be a ring and X a *simplicial set* (that is, a graded set, $X_0, X_1, \ldots$, endowed with two kinds of operators: face operators $\partial_j : X_n \to X_{n-1}$ and degeneracy operators $s_i : X_n \to X_{n-1}$, $1 \leq j \leq n$, verifying several "commutativity" relations). Let us denote $C_*(X)$ by the chain complex $\{C_n(X), d_n\}$ in which $C_n(X)$ is the graded R-module generated by X_n and $d_n : C_n(X) \to C_{n-1}(X)$ is $\sum_{i=0}^{n}(-1)^i\partial_i$. Let $C_*^N(X)$ be the *normalized chain complex* $\{C_n^N(X), d_n\}$ where $C_n^N(X) = C_n(X)/s(C_{n-1}(X))$ and $s(C_{n-1}(X))$ is the graded R–module generated by all the non–degenerate simplices y of $C_n(X)$ (that is, $y = s_i(x)$ for some simplex x and degeneracy operator s_i).

Now, the *cochain complex* associated to $C_*^N(X)$, denoted by $C^*(X; R)$, is the free R–module generated by all the R-module maps from $C_n^N(X)$ to R together with a map δ defined by $\delta(c)(x) = c(d(x))$. In this way, we define the *cohomology* of X with coefficients in R by $H^*(X) =$Ker $\delta_n/$Im δ_{n-1}. Note that a cocycle c (that is, $c \in C^*(X; R)$ such that $\delta(c) = 0$) represents a class of cohomology.

We are able to enunciate the main result of this paper.

Theorem 0.1. *Let p be an odd prime. Let $\mathbb{F}_p$ be the ground ring and X a simplicial set. If $c \in C^q(X; \mathbb{F}_p)$ and $x \in C^N_{pq-1}(X)$ then $\mathfrak{P}_1^p : H^q(X; \mathbb{F}_p) \to H^{qp-1}(X; \mathbb{F}_p)$ is defined by:*

$$
\begin{aligned}
\mathfrak{P}_1^p(c)(x) = \sum_{0\leq j\leq p-1} \sum_{S(j)} \quad & (-1)^{i_{j-1}+(i_{j-1}+i_j)(i_j+m)} \\
& c(\partial_{i_0+1}\cdots\partial_m x) \\
& \bullet c(\partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\partial_m x) \\
& \vdots \\
& \bullet c(\partial_0\cdots\partial_{i_{j-3}-1}\partial_{i_{j-2}+1}\cdots\partial_m x) \\
& \bullet c(\partial_0\cdots\partial_{i_{j-2}-1}\partial_{i_{j-1}+1}\cdots\partial_{i_j-1}\partial_{i_{j+1}+1}\cdots\partial_m x) \\
& \bullet c(\partial_0\cdots\partial_{i_{j+1}-1}\partial_{i_{j+2}+1}\cdots\partial_m x) \\
& \vdots \\
& \bullet c(\partial_0\cdots\partial_{i_{p-2}-1}\partial_{i_{p-1}+1}\cdots\partial_m x) \\
& \bullet c(\partial_0\cdots\partial_{i_{p-1}-1} x) \\
& \bullet c(\partial_0\cdots\partial_{i_{j-1}-1}\partial_{i_j-1}\cdots\partial_m x)
\end{aligned}
$$

where $m = pq - 1$, $\bullet$ *is the natural product in* $\mathbb{F}_p$ *and*

$$S(j) = \{0 \le i_0 \le \cdots \le i_{j-2} \le i_{j-1} < i_j \le i_{j+1} \le \cdots \le i_{p-1} \le m\}.$$

The key of our combinatorial approach is the description given in [6] for Steenrod reduced powers in terms of the component morphisms of a given Eilenberg–Zilber contraction [4]. The proof of the previous theorem is a simplification of that combinatorial formulation based on the fact that any composition of face and degeneracy operators can be expressed in a unique form:

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s}$$

where $j_t > \cdots > j_1 \ge 0$ and $i_s > \cdots > i_1 \ge 0$.

Moreover, bearing in mind that c is a q–cochain, we only have to consider those summands with exactly $pq - q - 1$ face operators in each factor. Then, we can simplify the formula even more.

Proposition 0.2. *Let p be an odd prime. Let $\mathbb{F}_p$ be the ground ring and X a simplicial set. If $c \in C^q(X;\mathbb{F}_p)$ and $x \in C^N_{pq-1}(X)$ then $\mathfrak{P}^p_1 : H^q(X;\mathbb{F}_p) \to H^{qp-1}(X;\mathbb{F}_p)$ is defined by:*

$$\begin{aligned}\mathfrak{P}^p_1(c)(x) = \sum_{1 \le j \le p-1} \sum_{jq \le i \le (j+1)q-1} & (-1)^{i(q+1)+q(m+1)} \\ & c(\partial_{q+1} \cdots \partial_{pq-1} x) \\ & \bullet c(\partial_0 \cdots \partial_{q-1} \partial_{2q+1} \cdots \partial_{pq-1} x) \\ & \vdots \\ & \bullet c(\partial_0 \cdots \partial_{(j-2)q-1} \partial_{(j-1)q+1} \cdots \partial_{pq-1} x) \\ & \bullet c(\partial_0 \cdots \partial_{(j-1)q-1} \partial_{i-q+1} \cdots \partial_{i-1} \partial_{(j+1)q} \cdots \partial_{pq-1} x) \\ & \bullet c(\partial_0 \cdots \partial_{(j+1)q-2} \partial_{(j+2)q} \cdots \partial_{pq-1} x) \\ & \vdots \\ & \bullet c(\partial_0 \cdots \partial_{(p-2)q-2} \partial_{(p-1)q} \cdots \partial_{pq-1} x) \\ & \bullet c(\partial_0 \cdots \partial_{(p-1)q-2} x) \\ & \bullet c(\partial_0 \cdots \partial_{i-q-1} \partial_{i+1} \cdots \partial_{pq-1} x)\end{aligned}$$

where $\bullet$ is the natural product in $\mathbb{F}_p$.

Assuming that face operators are evaluated in constant time, the following result gives us a first measure of the computational complexity of these formulae.

Proposition 0.3. *Let p be an odd prime. Let $\mathbb{F}_p$ be the ground ring and X a simplicial set. If $c \in H^q(X;\mathbb{F}_p)$, then the number of face operators taking part in the formula for $\mathfrak{P}_1^p(c)$ of the proposition above is*

$$p(p-1)q[(p-1)q-1].$$

It is clear that at least in the case in which X has a finite number of non–degenerate simplices in each degree, our method can be seen as an actual algorithm for calculating $\mathfrak{P}_1^p$ for an odd prime p. For example, if the number of non–degenerate simplices in each X_ℓ is $O(\ell)$ and each face operator of X is evaluated in constant time, the complexity of our algorithm for calculating $\mathfrak{P}_1^p(c_q)$ is $O(p^4q^3)$. As an example, an algorithm for computing $\mathfrak{P}_1^p$ on the cohomology of a simplicial complex [14, 16] can be easily described and implemented. This example in particular shows that it is not difficult to integrate Computational Algebra tools in the framework of Algebraic Topology.

References

[1] J. Dieudonné. *A history of Algebraic and Differential Topology 1990-1960.* Birkhäuser, Boston (1989).

[2] A. Dold. *Uber die Steenrodschen Kohomologieoperationen.* Annals of Math., **v. 73** (1961), 258-294.

[3] S. Eilenberg and S. Mac Lane. *On the groups $H(\pi, n)$, I.* Annals of Math., **v. 58** (1953), pp. 55-106 .

[4] S. Eilenberg and J. A. Zilber, *On products of complexes.* Am. J. Math., **v. 75** (1959), pp. 200-204.

[5] T. Ekedahl , J. Grabmeier and L. Lambe. *Algorithms for algebraic computations with applications to the cohomology of finite p-groups.* Preprint of Department of Mathematics and Centre for Innovative Computation, University of Wales (1997).

[6] R. González-Díaz and P. Real. *A combinatorial method for computing Steenrod Squares.* To appear in an special volume of the Journal of Pure and Applied Algebra (June 1999).

[7] R. González-Díaz and P. Real. *Computing cocycles on simplicial complexes.* Sent to the Second Workshop on Computer Algebra and Scientific Computing CASC'99.

[8] K. J. Horadam and W. De Launey. *Cocyclic development of designs.* J. Algebraic Combin., **v. 2(3)** (1993), pp. 267-290. Erratum: **v.1** (1994), page 129.

[9] L. Lambe. *An algorithm for calculating cocycles.* Preprint of Department of Mathematics and Centre for Innovative Computation, University of Wales (1997).

[10] S. Mac Lane. *Homology.* Classics in Mathematics. Springer-Verlag, Berlin (1995). Reprint of the 1975 edition.

[11] P. May. *Simplicial objects in Algebraic Topology.* Van Nostrand, Princeton (1967).

[12] P. May. *A general algebraic approach to Steenrod operations.* Lect. Notes in Math., Springer, **v. 156** (1970), 153-231.

[13] J. Milnor. *The Steenrod algebra and its dual.* Ann. of Math. **v. 67** (1958) pp 150-171.

[14] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley Publishing Company (1984).

[15] N. E. Steenrod and D. B. A. Epstein. *Cohomology operations.* Ann. of Math. Studies **v. 50**, Princeton University Press (1962).

[16] C. A. Weibel. *An introduction to Homological Algebra.* Cambridge studies in advanced mathematics, **38**, Cambridge University Press (1994).

[17] R. M. W. Wood. *Problems in the Steenrod algebra.* Bull. London Math. Soc., **v. 30** (1998), pp. 449-517.