Space-Variant Gabor Decomposition for Filtering 3D Medical Images

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Abstract. This is an experimental paper in which we introduce the possibility to analyze and to synthesize 3D medical images by using multivariate Gabor frames with Gaussian windows. Our purpose is to apply a space-variant filter-like operation in the space-frequency domain to correct medical images corrupted by different types of acquisitions errors. The Gabor frames are constructed with Gaussian windows sampled on non-separable lattices for a better packing of the space-frequency plane. An implementable solution for 3D-Gabor frames with non-separable lattice is given and numerical tests on simulated data are presented.

1 Introduction

The study of noise removal in medical images was approached in different ways by numerous authors. From the first studies involving only convolutional filters [12], to the rank algorithms [13,14] and most recently to methods to average pixels depending on their neighborhood statistics [10,11], the search was both for maximizing the filter capabilities and to propose fast algorithms. While very successful for various signal analysis applications in medicine or telecommunications, time-frequency analysis and especially the Gabor frames expansions for image processing in 2D or 3D was of limited interest due to the large indexing problems i.e. a 3D image is analysed with a 6D lattice, and to the problems related to time consuming implementation.

In a fairly recent paper [15], the first author introduced a procedure to efficiently perform an nD Gabor frames decomposition. Based on that result, we propose in this paper a method to decompose, to filter and to reconstruct 3D medical images that overcomes both problems. We are using a tensor product decompositions [4] to reduce the decomposition to only 2D lattice case and also fast algorithms to implement this case. The result is an experimental framework for fast Gabor frames construction used to analyze 3D-data images sampled on quincunx-type lattices. The research is in the simulation stage and we present in here only tests done on the MATLAB MRI data set.

Therefore the main contribution of this conference paper is to introduce a completely invertible 3D Gabor transform applied to filter 3D medical images,

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different from existing examples in literature that present only 3D Gabor filters [7,9].

This paper is structured as follows: In the second section we recall the necessary results from Gabor analysis, while in the third section we present the 3D Gabor frames construction. The application to 3D medical imaging is given in the fourth section. In the last section, we present timings for the 3D Gabor construction and the conclusions are drawn.

2 Theoretical Preliminaries

Frames $(g_i)_{i\in I}$ generalize the idea of a basis in a Hilbert space $\boldsymbol{H}$ and they are formally defined as:

Definition 1. *A family $(g_i)_{i\in I}$ in a Hilbert space $\boldsymbol{H}$ is called a frame if there exist constants $A, B > 0$ such that for all $g \in \boldsymbol{H}$*

$$A\|g\|^2 \leq \sum_{i\in I} |\langle g, g_i\rangle|^2 \leq B\|g\|^2 \tag{1}$$

Every element $f \in \boldsymbol{H}$ has an expansion of the form:

$$f = SS^{-1}f = \sum_{i\in I} \langle S^{-1}f, g_i\rangle g_i = \sum_{i\in I} \langle f, S^{-1}g_i\rangle g_i$$

where S denotes the **invertible frame operator** [3]: $Sf = \sum_{i\in I} \langle f, g_i\rangle g_i$. The family $(\gamma_i)_{i\in I} = (S^{-1}g_i)_{i\in I}$ is again a frame with frame bounds B^{-1} and A^{-1} and is called a **canonical dual** frame. The main tool for time-frequency analysis is the **Short-Time Fourier Transform** in short STFT, defined for functions in $\boldsymbol{L}^2(\mathbb{R})$ as

$$V_g f(\lambda) = V_g f(a, b) = \langle f, M_b T_a g\rangle = \langle f, \pi(\lambda)g\rangle \tag{2}$$

where $T_a f(t) = f(t - a)$ is the translation (time shift) and $M_b f(t) = e^{2\pi i b\cdot t} f(t)$ is the modulation (frequency shift), for $\lambda = (a, b) in \mathbb{R}^2$. The operators $M_b T_a$ are called **time-frequency shifts**. Their composition is denoted by $\pi(\lambda)$. In order to obtain Gabor frames, the STFT is sampled over a time-frequency lattice. In the standard 1D case the regular lattice is of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ with the condition $ab < 1$. Therefore a **Gabor frame** is defined as:

$$\mathcal{G}(g, a, b) := \{g_{k,l} = M_{bl} T_{ak} g, \quad k, l \in \mathbb{Z}\}, \tag{3}$$

i.e. the elements of the Gabor frame are translated and modulated versions of one atom g.

Theorem 1 (Dual Gabor Frames [5]). *If $\mathcal{G}(g, a, b)$ is a frame for $\boldsymbol{L}^2(\mathbb{R})$, then the canonical dual frame takes the form $\mathcal{G}(\gamma, a, b)$ for $\gamma = S^{-1}g$. where the Gabor frame operator S is defined as:*

$$S := \sum_{k,\, l \in \mathbb{Z}} \langle f, M_{bl} T_{ak} g \rangle M_{bl} T_{ak} g \tag{4}$$

We introduce also the result of Bourouihiya [2], which extends the classical results of Lyubarski and Seip to higher dimensions [6] for $g_0(x) = 2^{-1/4} e^{-\pi x^2}$ with $x \in \mathbb{R}$:

Lemma 1. *Let $g = g_0 \otimes \cdots \otimes g_0$ (n factors) and $\Lambda_{ab} = (a_1\mathbb{Z} \times \cdots \times a_n\mathbb{Z}) \times (b_1\mathbb{Z} \times \cdots \times b_n\mathbb{Z})$. Then $\mathcal{G}(g, \Lambda_{ab})$ is a frame if and only if $a_j b_j < 1$ for $1 \leq j \leq n$.*

Based on this result, another extension to non-separable lattices of the form $N\Lambda_{a,b}$ is possible, which besides a multi-variate sampling in each dimension, give us a better packing of the time-frequency plane by using a (non-separable) hexagonal lattice that match with the circular contour lines of the Gaussian.

The representation of the non-separable lattice is based on the rectangular lattice via a shear operation. Therefore, we can give the following lemma:

Lemma 2. *Given a window $g = g_0 \otimes \cdots \otimes g_0$ (n factors) and a lattice of the form $N\Lambda_{a,b}$, where Λ is a rectangular lattice and N is a shear lattice, the system $\mathcal{G}(g, \Lambda_{ab})$ is a frame if and only if $a_j b_j < 1$ for $1 \leq j \leq n$.*

Proof. The shear matrix N is a symplectic matrix hence the determinant $det(N) = 1$. Therefore the volumes $vol(N\Lambda_{a,b}) = vol(\Lambda_{a,b})$ are equal and the Lemma 1 extends to non-separable lattices.

In applications sampled data of finite lengths are analysed; the process of sampling and periodization are employed [16]. In this way also the number of shifts in time and frequency becomes finite. The redundancy of a discrete system, not necessarily a Gabor system, is defined as the fraction of the number of used discrete function over the length of the domain, $\frac{\#shift}{L}$.

3 3D Gabor Frame Construction

In this section we present how to construct numerically a 3D Gabor transform for 3D image filtering. We will exploit the possibilities given by the Lemma 2 for choosing a non-separable lattice of quincunx-type. This situation can be easily expanded to more dimensions or can be reduced to less dimensions following the same procedure. We write n for data cube length in one direction and the modulation and translation operator are defined on $\mathbb{Z}_n$. For further discretization details [8] is a comprehensive source.

For the 3D case, we will use a multi-variate generalized Gaussian window, obtained as a tensor product of 1D windows in the form:

$$G_3 = g_1 \otimes\ g_2 \otimes g_3$$

We consider our 3D Gaussian window of size $n_1 \times n_2 \times n_3$ as a complex-valued function on the additive Abelian group $\mathcal{G} = n_1 \times n_2 \times n_3$. The joint position-frequency space is

$$\mathcal{G} \times \widehat{\mathcal{G}} = n_1 \times n_2 \times n_3 \times n_1 \widehat{\times n_2 \times} n_3.$$

Now, let's pay attention to the lattice. We would like to use a non-separable lattice obtained as in the hypothesis of the Lemma 2 by applying a shear matrix to the matrix generating the regular standard lattice. There are other symplectic matrices like the rotation matrix using the fractional Fourier transform [1] that can be used to transform a rectangular lattice into a quincunx-like lattice.

For our 3D case using a non-separable lattice, we will consider the following quincunx-like lattice:

$$\Lambda_3 = \begin{pmatrix} I_{3\times 3} & Q_3 \\ O_3 & I_{3\times 3} \end{pmatrix} A_6 \cdot \mathbb{Z}^6$$

where

$$Q_3 = \begin{pmatrix} a_1/2b_1 & 0 & 0 \\ 0 & a_2/2b_2 & 0 \\ 0 & 0 & a_3/2b_3 \end{pmatrix}$$

and

$$A_6 = diag(a_1, a_2, a_3, b_1, b_2, b_3)$$

This 6D non-separable lattice can be reduced to the case of 2D non-separability in time and frequency by the expansion in the canonical basis. Moreover, we can write the following tensor product:

$$\begin{pmatrix} 1 & a_1/2b_1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & a_2/2b_2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & a_3/2b_3 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this conditions it is feasible to perform a Gabor analysis followed by a Gabor synthesis at level of the constituting vectors, and therefore reducing the case of applying 3D matrices to the faster case of applying the transform over each dimension. Therefore, we can compute the coefficients of the expansion (e.g. the 3D Gabor filter) by using any fast algorithms developed for the 1D Gabor transform [8]. For a complete Gabor transform, we use the Theorem 1 and we invert the frame operator in order to obtain the dual frame. Using the computed dual frame, we can recover the 3D data cube.

4 Applications to 3D Medical Image Processing

We have tested our implementation for the MRI data set that comes with MATLAB2017. Loading mri.mat adds two variables to the workspace: D (128-by-128-by-1-by-27, class uint 8) and a grayscale colormap, map (89-by-3, class double). D comprises 27 128-by-128 horizontal slices from an MRI data scan of a human cranium. Values in D range from 0 through 88, so the colormap is needed to generate a figure with a useful visual range.

The procedure takes as inputs D = 3D medical image and the G = the Gaussian analysing atoms. The parameters a = time shift and b = frequency shift are defined depending of the application and under the liniar independence contraints. The output is DG = data cube recovered after Gabor analysis. The complete algorithm is summarized below:

Algorithm 1. 3D Gabor transform filtering with quincunx lattice

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Input: G - Gaussian atom, D- 3D medical image
Output: DG - 3D Gabor filtered medical image
1: Generate Gabor matrix G for a quincux sampling lattice;
2: Compute the Gabor coefficients GC over each dimension
3: Remove the low energy Gabor coefficients
4: Recover DG over each dimension with the use of the dual atom

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In the Figs. 1 and 2, we present the results of the applying the Gabor filtering.

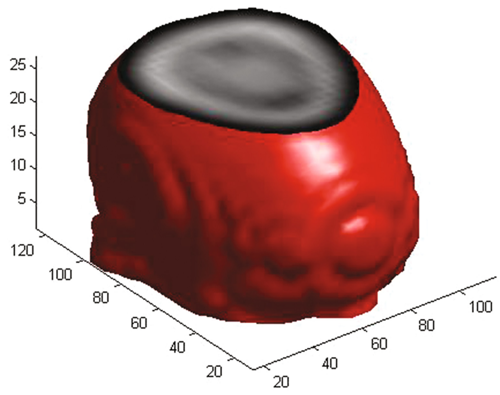

Fig. 1. Unfiltered 3D image

We observed that due to the reconstruction after the removal of the low energy coefficients the image becomes much smoother. The next step is to corroborate this results with the needs of medical practitioners or radiologists.

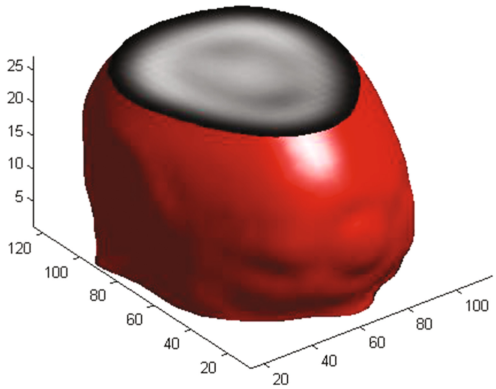

Fig. 2. Gabor filtered 3D image

5 Conclusions

In this paper, we described the construction of Gabor frames beyond the standard Gabor processing on regular lattices for signal and image processing. In this way, one has the freedom to choose a non-separable lattice and to sample the nD-Gaussian window in a multi-variate way, while still being assured that the result will be a Gabor frame. This added flexibility will be useful for application like 3D space-variant filtering or decomposition in the 3D Gabor domain for different features identifications (e.g. 3D plane waves).

The matrix we used for generating the lattice in our example, verifies the conditions under which the hypothesis of the Lemma 2 are true (i.e. shear matrix). The lattice parameters a_j, b_j should be chosen according to the time-frequency concentration of the corresponding windows g_j to obtain well concentrated systems. Therefore, under the conditions $a_j b_j < 1$ for $j = 1, 2, 3$, we obtain Gabor frames for decompositions and analysis. The timings for the implementation in $Matlab^{TM}$ on a notebook with Intel core i5 at 2.3 GHz and with 8.00 GB of RAM memory are given in Table 1.

Table 1. Timings comparison. The case 1. With redundancy 2, $ab = \frac{1}{2}$ the case 2., with redundancy 4, $ab = \frac{1}{4}$.

		Timings for 3D MRI data	
		1. n = 128	2. n = 128
Quincunx lattice	Coefficients	12.438 s	72.103 s
	Synthesis	13.134 s	78.363 s

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