Geometric Objects and Cohomology Operations *

R. González–Díaz, P. Real

Dept. of Applied Math.,
University of Seville, Spain
{rogodi, real}@us.es

Abstract. Cohomology operations (including the cohomology ring) of a geometric object are finer algebraic invariants than the homology of it. In the literature, there exist various algorithms for computing the homology groups of simplicial complexes ([Mun84], [DE95,ELZ00], [DG98]), but concerning the algorithmic treatment of cohomology operations, very little is known. In this paper, we establish a version of the incremental algorithm for computing homology given in [ELZ00], which saves algebraic information, allowing us the computation of the cup product and the effective evaluation of the primary and secondary cohomology operations on the cohomology of a finite simplicial complex. The efficient combinatorial descriptions at cochain level of cohomology operations developed in [GR99,GR99a] are essential ingredients in our method. We study the computational complexity of these processes and a program in Mathematica for cohomology computations is presented.

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1 Introduction

A simplicial complex is a well–known discrete model of a geometric object, which consists of a collection of simplices that fit together in a natural way to form the object. In order to classify simplicial complexes from a topological point of view, a first algebraic invariant that can be used is homology, which in some sense, counts the number of holes of the object.

We can cite two relevant algorithms for computing homology groups H_*K of a simplicial complex K in $\mathbf{R}^n$: (1) the classical algorithm based on reducing certain matrices to their Smith normal form [Mun84]; (2) the incremental algorithm [DE95,ELZ00,EZ01], avoiding the severe computational costs of the reduction to Smith normal form and consisting of assembling the complex simplex by simplex and at each step updates the Betti numbers of the current complex. Starting with the boundary of a negative simplex, this persistence process finds the cycle which is destroyed by this simplex through the search, computing in this way the geometric realization of a homology cycle. It runs in time at most $O(m^3)$, where m is the number of simplices of the complex. For simplicial complexes embedding in $\mathbf{R}^3$, this complexity is reduced to $O(m)$ in time and space [DE95]. The algorithm proposed in [DG98] is based on simulating a thickening of a given complex in $\mathbf{R}^3$ to a topological 3-manifold homotopic to it, and computing the homology groups of the last one using classical results. The time and space complexity is linear and this method also produces representations of generators of the homology groups.

In general, computing homology is not enough for determining whether two geometric objects are homeomorphic or not. Finer algebraic invariants such as the cohomology (an algebraic dual notion to homology), the cup product on cohomology or cohomology operations [Spa81], allow us to topologically distinguish two geometric objects having isomorphic homology groups. For example, a torus and the wedge product of a sphere and two circles have the same homology but the respective cup products on cohomology are "essentially" different. Using a field as the coefficient group, for example, $\mathbf{Z}_2$, the cohomology H^*K of a simplicial complex K gives us the same topological information as the homology of it. However, the additional ring structure on the cohomology determined by the cup product and cohomology operations cannot directly be produced from the algorithms previously mentioned for computing the homology. Roughly speaking, a cohomology operation $\theta : H^m(-;G) \to H^n(-;G')$ is a homomorphism that acts on cohomology (G and G' being groups); relevant examples of cohomology operations are Steenrod squares, Steenrod reduced powers and Adem secondary cohomology operations [MT68]. As an example of the strong constraints that these operations impose on the cohomology of spaces, we can cite that the use of this machinery is essential for showing that there do not exist spaces X having cohomology $H^*(X;\mathbf{Z})$ a polynomial ring $\mathbf{Z}[\alpha]$ unless α has dimension 2 or 4.

In this paper, we make use of an explicit chain contraction (a special chain equivalence) connecting the chain complex C_*K, canonically associated to a simplicial complex K and its homology H_*K. Moreover, from this datum we can derive a cochain contraction from the cochain complex $C^*K = Hom(C_*K;\mathbf{Z}_2)$, to the cohomology H^*K. Using this information, we can compute:

1. Geometric realizations of (co)homology generators.
2. The (co)homology class of a (co)cycle in terms of (co)homology generators.
3. The construction of a (co)boundary of a given (co)cycle.
4. The induced homomorphism at (co)homology level of a simplicial map between two complexes.
5. The cup product on cohomology and some primary and secondary cohomology operations.

The first problem is to construct such chain contractions from C_*K to H_*K. In [GR01], a translation of the classical matrix algorithm (1) in terms of chain contractions is designed. In this paper, we design a version of the incremental method described in [ELZ00] in terms of chain contractions. The complexity of our method is also $O(m^3)$ where m is the number of simplices of K, but our algorithm saves information which allows us, for example, to compute the following operations:

1. The cohomology ring of K in $O(m^5)$.
2. The Steenrod square operation $Sq^i\alpha_n$ of a cohomology class α_n of degree n in $O(i^{n-i+1}m)$ (see [GR99a])
3. The Adem secondary cohomology operation $\Psi_2\alpha_2$ of a cohomology class $\alpha_2 \in KerSq^2H^1(K;\mathbf{Z_2})$ in $O(m^3)$.

In fact, the modus operandi for evaluating a mod 2 cohomology operation $\bar{\mathcal{O}} : H^mK \to H^nK$ on a cohomology class α_m is the following:

1. First, given a finite simplicial complex K, construct the chain contraction from C^*K to H^*K (denoted $(f^*, g^*, \phi^*) : C^*K \Rightarrow H^*K$), using our version of the incremental technique.

2. Evaluate $\bar{\mathcal{O}}$ on the cohomology class α_m using the diagram

$$\begin{array}{ccc} C^*K & \overset{g^*}{\leftarrow} & H^*K \\ {\scriptstyle\mathcal{O}}\downarrow & & \downarrow{\scriptstyle\bar{\mathcal{O}}} \\ C^*K & \overset{f^*}{\rightarrow} & H^*K\,, \end{array}$$

   where $\mathcal{O} : C^*K \to C^*K$ is a cochain operation associated to $\bar{\mathcal{O}}$ whose formulation is explicitly given in simplicial terms. An efficient combinatorial description $\mathcal{O}$ for $\bar{\mathcal{O}}$ being a Steenrod square [GR99,GR99a], a Steenrod reduced power [GR99] or some Adem secondary cohomology operations [GR01] have already been done by the authors. We do not deal with this question in this paper, but it is necessary to say that the algorithmic approach we give here will only be valid if combinatorial pictures of cohomology operations at cochain level are determined.

Let us observe that in this paper we deal with the general case of $\mathbf{R}^n$. Versions in terms of chain contractions of the algorithms given in [DE95] and [DG98], designed for the special case of $\mathbf{R}^3$, would allow us to considerably reduce the computational costs of the processes.

2 Homology and Chain Contractions

In this section, we design a version of the incremental algorithm of [ELZ00] in terms of chain contractions. In this way, we construct a chain contraction from the chain complex canonically associated to a simplicial complex K, to its homology. Let us observe that passing to cohomology is not a problem if we use a field as the ground ring. The resulting cochain contraction from C^*K to H^*K will help us to compute the cup product on cohomology and cohomology operations.

Now, we give a brief summary of concepts and notations. The terminology follows Munkres [Mun84].

Throughout this paper, we consider $\mathbf{Z}_2$ is the ground ring and μ denotes the product on $\mathbf{Z}_2$. A *q–simplex* σ in $\mathbf{R}^n$(where $q \leq n$) is the convex hull of $q+1$ affinely independent points $\{v_0, ..., v_q\}$. We denote $\sigma = \langle v_0, ..., v_q\rangle$. The *dimension of* σ is $|\sigma| = q$. A 0–simplex is a vertex, a 1–simplex is an edge, a 2–simplex is a triangle, a 3–simplex is a tetrahedron, and so on. An *i–face* of $\sigma = \langle v_0, ..., v_q\rangle$ $(i < q)$ is an i–simplex whose vertices are in the set $\{v_0, ..., v_q\}$. The $(q-1)$–faces of σ are called the *facets* of σ. A simplex is *shared* if it is a face of more than one simplex. Otherwise, the simplex is *free* if it belongs to one higher–dimensional simplex, and *maximal* if it does not belong to any. A *simplicial complex* K is a collection of simplices such that:

- If τ is a face of $\sigma \in K$, then $\tau \in K$.
- If $\sigma', \sigma \in K$, then $\sigma' \cap \sigma \in K$ or $\sigma' \cap \sigma = \emptyset$.

Let us notice that K can be given by the set of its maximal simplices. The *dimension of* K is $dimK = max\{|\sigma| :\ \sigma \in K\}$. In this paper, all the simplices have finite dimension and all the simplicial complexes are finite collections. The set of all the q–simplices of K is denoted by $K^{(q)}$. If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a *subcomplex* of K. Let K and K' be two simplicial complexes. A map $f : K^{(0)} \to K'^{(0)}$ such that whenever $\langle v_0, ..., v_q\rangle \in K$ then $f(v_0), ..., f(v_n)$ are vertices of a simplex of K', is called a *vertex map*.

Algebraic Topology is the study of algebraic objects attached to topological spaces; the algebraic invariants reflect some of the topological structure of the spaces.

The *chain complex* C_*K associated to a simplicial complex K is a family $\{C_qK, \partial_q\}_{q\geq 0}$ defined in each dimension q by:

- C_qK is the free abelian group generated by the q–simplices of K. An element $a = \sigma_1 + \cdots + \sigma_m$ of C_qK $(\sigma_i \in K^{(q)})$ is called a *q–chain.*
- $\partial_q : C_qK \to C_{q-1}K$ called the *boundary operator* is given by

$$\partial_q\langle v_0, ..., v_q\rangle = \sum_{i=0}^{q} \langle v_0, ..., \hat{v}_i, ..., v_q\rangle$$

  where $\langle v_0, ..., v_q\rangle$ is a q–simplex of K and the hat means that v_i is omitted. By linearity, ∂_q can be extended to C_qK, where it is a homomorphism.

A q–chain a is called a *q–cycle* if $\partial a = 0$. If $a = \partial b$ for some $b \in C_{q+1}K$ then a is called a *q–boundary.* We denote the groups of q–cycles and q–boundaries by Z_qK and B_qK respectively, and define $Z_0K = C_0K$. Since $B_qK \subseteq Z_qK$, we can define the *qth homology group* to be the quotient group Z_qK/B_qK, denoted by H_qK. Given that elements of this group are cosets of the form $a + B_qK$, where $a \in Z_qK$, we say that the coset $a + B_qK$, denoted by $[a]$, is the *homology class* in H_qK determined by a or a is a *representative cycle* of $[a]$. Let K and L be two simplicial complexes. A *chain map* $f : C_*K \to C_*L$ is a family of homomorphisms

$$\{f_q : C_qK \to C_qL\}_{q \geq 0}$$

such that $\partial_q f_q = f_{q-1}\partial_q$ for all q. Observe that for every vertex map $f : K^{(0)} \to L^{(0)}$, we can obtain the corresponding chain map $f_\# : C_*K \to C_*L$ such that

$$f_\#\langle v_0, ..., v_q\rangle = \begin{cases} \langle f(v_0), ..., f(v_q)\rangle & \text{if } f(v_i) \text{ distinct} \\ 0 & \text{otherwise} \end{cases}$$

Let h and k be two chain maps from C_*K to C_*L. A *chain homotopy* from h to k is a family of homomorphisms

$$\{\phi_q : C_qK \to C_{q+1}L\}_{q \geq 0}$$

such that $\partial_{q+1}\phi_q + \phi_{q-1}\partial_q = h_q + k_q$. We write $h \sim k$ if a chain homotopy between h and k exists. Two chain complexes C_*K and C_*L are *chain equivalent* if there exist two chain maps $f : C_*K \to C_*L$ and $g : C_*L \to C_*K$ such that

$$fg \sim 1_{C_*L} \qquad \text{and} \qquad gf \sim 1_{C_*K}\,.$$

Observe that, in this case, $\phi_q : C_qK \to C_{q+1}K$ for all $q \geq 0$. A *chain contraction* [EM52] from C_*K to C_*L is a chain equivalence such that

$$fg = 1_{C_*L} \qquad \text{and} \qquad gf \sim 1_{C_*K} \quad (\text{that is, } 1_{C_*K} + gf = \partial\phi + \phi\partial)$$

and ϕ has the following "annihilation" properties: $f\phi = 0$, $\phi g = 0$ and $\phi\phi = 0$. We denote such chain contraction as $(f, g, \phi) : C_*K \Rightarrow C_*L$. Observe that if a chain contraction from C_*K to C_*L exists then L has fewer or the same number of simplices than K. Now, we show some examples of contractions.

(a) Edge Contractions.
Conditions under which edge contractions are homeomorphisms appear in [DEGN99]. Here, we show one condition under which edge contractions become, at algebraic level, chain contractions.
Let K be a simplicial complex and $\tau = \langle a, b\rangle$ an edge in K. An *edge contraction* is given by the vertex map $f : K^{(0)} \to L^{(0)} = K^{(0)} - \{a, b\} \cup \{c\}$ where $f(a) = f(b) = c$, and $f(v) = v$ for all $v \neq a, b$.
Let B be a subset of K that is not necessarily a subcomplex. Define

$$\overline{B} = \{\sigma' \in K : \ \sigma' \leq \sigma \in B\}, \qquad St\,B = \{\sigma \in K : \ \sigma \geq \sigma' \in B\} \quad \text{and} \quad Lk\,B = \overline{St\,B} - St\,\overline{B}\,,$$

where $\sigma' < \sigma$ means that σ' is a face of σ.
If $Lk\,a \cap Lk\,b = Lk\,\tau$, then a chain contraction $(f_\#, g, \phi)$ from C_*K to C_*L is defined as follows:

- $f_\#$ is the chain map induced by the vertex map f.
- $g : C_*L \to C_*K$ is such that

$$\begin{aligned} &g\tau = \tau \qquad \forall \tau \notin St\,c, \\ &g\langle c\rangle = \langle a\rangle, \\ &g(\omega \cup \langle c\rangle) = \begin{cases} \omega \cup \langle a\rangle & \text{if } \omega \in Lk\,a, \\ \omega \cup \langle b\rangle + \bar{\omega} \cup \langle a, b\rangle & \text{if } \omega \in Lk\,b - Lk\,\tau \text{ and } \bar{\omega} \in Lk\,\tau \text{ is a facet of } \omega, \\ \omega \cup \langle b\rangle & \text{if } \omega \in Lk\,b - Lk\,\tau \text{ and no facet of } \omega \text{ belongs to } Lk\,\tau. \end{cases} \end{aligned}$$

- $\phi : C_*K \to C_{*+1}K$ is given by

$$\phi\langle v_0, ..., v_q, b\rangle = \langle v_0, ..., v_q, a, b\rangle \ \text{ if } \langle v_0, ..., v_q\rangle \in Lk\,\tau$$

  and $\phi\tau = 0$ otherwise.

(b) Simplicial Collapses.
Suppose K is a simplicial complex, $\sigma \in K$ is a maximal q–simplex and σ' is a free $(q-1)$–face of σ. Then, K *simplicially collapses* onto $K - \{\sigma, \sigma'\}$. More generally, a *simplicial collapse* is any sequence of such operations. A *thinned* simplicial complex $M_{\text{scol}}(K)$ is a subcomplex of K with the condition that all the faces of the maximal simplices of $M_{\text{scol}}(K)$ are shared. Then, it is obvious that it is no longer possible to collapse. There is an explicit chain contraction from C_*K onto $C_*(M_{\text{scol}}K)$ [For99]. The following algorithm computes $M_{\text{scol}}K$ and the chain contraction from C_*K onto $C_*(M_{\text{scol}}K)$. Suppose that K is given by the set of its maximal simplices.

```
Initially, M_scol K = K,  φτ = 0,  fτ = gτ = τ for each τ ∈ K.
While there exists a maximal simplex σ with a free face σ' do
        M_scol K = M_scol K − {σ, σ'},
        φσ' = σ,  fσ' = σ' + ∂σ and  fσ = 0
End while
```

(c) Contraction to a Vertex.
Let $\sigma = \langle v_0, \ldots, v_q\rangle$ be a simplex and let $K[\sigma]$ be the simplicial complex whose maximal simplex is σ. It is obvious that we can obtain a chain contraction from $C_*K[\sigma]$ to $\langle v_0\rangle$ using simplicial collapses. But now, we show another contraction from $C_*K[\sigma]$ to $\langle v_0\rangle$ determining the acyclicity of the simplex σ. This last chain contraction is the key for constructing another one from any simplicial complex to its homology as we will see in the following section. We define $(f_\sigma, g_\sigma, \phi_\sigma) : C_*K[\sigma] \Rightarrow \langle v_0\rangle$ as follows:

$$f_\sigma\langle v_i\rangle = \langle v_0\rangle \qquad 0 \le i \le q\,, \qquad \text{and} \qquad f_\sigma(\tau) = 0 \qquad \text{otherwise,}$$

$$\phi_\sigma\langle v_0, v_{j_1}, \ldots, v_{j_n}\rangle = 0 \qquad \text{and} \qquad \phi_\sigma\langle v_{j_1}, \ldots, v_{j_n}\rangle = \langle v_0, v_{j_1}, \ldots, v_{j_n}\rangle \qquad 1 \le j_1 < \cdots < j_n \le q,$$

$$g_\sigma\langle v_0\rangle = \langle v_0\rangle.$$

Let us observe that in this case $\langle v_0\rangle$ represents the unique class of homology in $H_*K[\sigma]$.

2.1 Incremental Homology Algorithm and Chain Contractions

Our algorithm for computing a chain contraction from the chain complex of a simplicial complex K to its homology is based on the incremental algorithm for computing the persistence of the Betti numbers developed in [ELZ00].

The input of our algorithm implemented in Mathematica is the sorted set of all the simplices, $K = \{\sigma_1, \ldots, \sigma_m\}$, with the property that any subset of it, $\{\sigma_1, \ldots, \sigma_i\}$, $i \le m$, is a simplicial complex itself. The output $\ell =$`contraction[`K`]` is a list of sorted lists. Each sorted list has three elements. The first one is a simplex σ of K, the second one is the image of σ under f and the third one consists of the image of σ under ϕ. We omit in the list the simplices such that the image of them are null under f and ϕ. In general, a class of homology α is represented by a simplex τ, so in order to obtain the image of α under g, we only have to compute $a = \tau + \phi\partial\tau$. Moreover, a will be a representative cycle of α.

Now, let us suppose we have constructed the list $\ell =$`contraction[`L`]` for $L = \{\sigma_1, \ldots, \sigma_{i-1}\}$, $i \le m$ (if $L = \emptyset$, we assume $\ell = \emptyset$). We construct `contraction[`$\{\sigma_1, \ldots, \sigma_i\}$`]` as follows:

```
If     f[∂σ_i, ℓ]= 0 then,
       ℓ ∪ {(σ_i, σ_i, φσ_i)},
Else
      Replace [ Replace [ ℓ,
                          Solve[f[∂σ_i, ℓ]=0]
                        ],
                Solve[φ[∂σ_i, ℓ]=σ_i]
              ]
End if
```

where, for a simplex τ, $f[\tau, \ell]$ and $\phi[\tau, \ell]$ are, respectively, the second and the third element of the list of ℓ that has τ as the first element. If this list does not exist, then $f[\tau, \ell] = 0$ and $\phi[\tau, \ell] = 0$. Now, let us explain what `contraction[`$\{\sigma_1, \ldots, \sigma_i\}$`]` computes. If $f[\partial\sigma_i, \ell] = 0$ then σ_i “creates a cycle”, so in fact, σ_i is a new generator of homology. Otherwise, $f[\partial\sigma_i, \ell]$ is a sum of elements of the form $\sum_{\sigma_j \in N \subset L} \sigma_j$. The idea of this last case is that σ_i destroys the cycle generated by $\partial\sigma_i$ in L. Therefore, we impose $f[\partial\sigma_i, \ell] = 0$ and $\phi[\partial\sigma_i, \ell] = \sigma_i$. We replace these relations in ℓ with the commands Replace and Solve.

At the end of the algorithm, all the elements of the form $\phi\tau$ are replaced by zero. For obtaining the morphism g and the representative cycles of the homology classes of K, we compute $\tau + \phi\partial\tau$ for each simplex τ (the generators of homology) satisfying that $f[\tau, \ell] = \tau$ in the list $\ell =$`contraction`$[K]$. We create a new list of sorted lists, called `representativeCycles`$[K]$ such that in each sorted list the first element is a generator of homology, τ, and the second element is its image under g, $\tau + \phi\partial\tau$. Observe that this last chain is, in fact, a cycle:

$$\begin{aligned}\partial(\tau+\phi\partial\tau) &= \partial\tau+\partial\phi\partial\tau\\ &= \partial\tau+(gf-1-\phi\partial)\partial\tau\\ &= gf\partial\tau-\phi\partial\partial\tau \qquad [\text{ since } \partial\partial\tau=0\text{, then }]\\ &= gf\partial\tau \qquad [\text{ since, by construction, } f\partial\tau=0\text{, then }]\\ &= 0.\end{aligned}$$

It is easy to check that (f,g,ϕ) is, in fact, a chain contraction from C_*K to H_*K. Observe that given a cycle a, if $fa=0$ then a is also a boundary. In order to compute a chain a' such that $a=\partial a'$, we can use the relation

$$a-gfa=\phi\partial a+\partial\phi a\,.$$

Since $\partial a=0$ and $fa=0$, we have $a=\partial\phi a\,.$

Theorem 1. The complexity of our algorithm for computing the homology of a finite simplicial complex K and a chain contraction from C_*K on H_*K is $O(m^3)$, where m is the number of simplices of K.

Proof. Let $K=\{\sigma_1,\dots,\sigma_m\}$ and $d=dim\,K$. Suppose that we have computed $\ell=$`contraction[`$\{\sigma_1,\dots,\sigma_{i-1}\}$`]`. In the worst case, we have to solve f`[`$\partial\sigma_i,\ell$`]`$=0$ and ϕ`[`$\partial\sigma_i,\ell$`]`$=\sigma_i$. Observe that the number of simplices involved in $\partial\sigma_i$ is less or equal than the dimension of σ_i which is at most d and then, the number of simplices involved in the formulas of f`[`$\partial\sigma_i,\ell$`]` and ϕ`[`$\partial\sigma_i,\ell$`]` is $O(dm)=O(m)$. Since we have to solve the equations and replace the solution in ℓ, the total cost of these operations is $O(m^2)$. Moreover, for obtaining the representative cycles, we have to compute $\tau+\phi\partial\tau$ for every generator of homology. The cost of this is also $O(m^2)$. Therefore, the total algorithm runs in time at most $O(m^3)$. □

3 Cohomology and Cohomology Operations

One reason in order to use the cohomology for distinguishing spaces instead of homology, is that the cohomology has additional structures, such as the cup product and cohomology operations. If two spaces have isomorphic (co)homology groups but the behaviour of the ring structure or cohomology operations is different, then they are not homeomorphic. In this section we explain how we can compute the cup product and cohomology operations starting from a chain contraction from an algebraic object to its homology. We first need to define more concepts.

The *cochain complex* associated to K, denoted by C^*K, is the family

$$\{C^qK,\delta^q\}_{q\geq 0}\,,$$

defined in each dimension q by:

- The group $C^qK=Hom(C_qK;\mathbf{Z}_2)$=$\{c:\,C_qK\to\mathbf{Z}_2,\ c$ is a homomorphism$\}$.
- The homomorphism $\delta^q:C^qK\to C^{q+1}K$ called the *coboundary operator* given by

$$\delta^q c\,a=c\,\partial_{q+1}a$$

  where $c\in C^qK$ and $a\in C_{q+1}K$.

The elements of C^qK are called *q–cochains*. Observe that a q–cochain can be defined on $K^{(q)}$ and it is naturally extended by linearity on C_qK. Z^qK and B^qK are the kernel of δ^q and the image of δ^{q-1}, respectively. The elements in Z^qK are called *q–cocycles* and those in B^qK are called *q–coboundaries.* The *qth cohomology group*

$$H^qK=Z^qK/B^qK$$

can be defined for each integer q. Take into account that since the ground ring is a field, the homology and cohomology of K are isomorphic. Moreover, given a generator of homology, α, of dimension q, we can define the corresponding generator of cohomology $\alpha^*:H_qK\to\mathbf{Z}_2$ such as

$$\alpha^*\alpha=1 \qquad\text{and}\qquad \alpha^*\beta=0 \quad\text{for } \alpha\neq\beta\in H_qK\,.$$

One can also define the dual concept of chain maps and chain contractions, in the obvious way. Furthermore, starting from a chain contraction (f,g,ϕ) from C_*K to H_*K, we construct a cochain contraction

(f^*, g^*, ϕ^*) from C^*K to H^*K as follows. Let $c \in C^*K$ and $\alpha^* \in H^*K$. Define $f^*c = c\,g$, $g^*\alpha^* = \alpha^* f$ and $\phi^* c = c\,\phi$.

The cohomology of K is a ring with the *cup product*

$$\smile:\ H^iK \otimes H^jK \to H^{i+j}K$$

defined at a cocycle level by $(c \smile c')\sigma = \mu(c\langle v_0, \dots, v_i\rangle \otimes c'\langle v_i, \dots, v_{i+j}\rangle)$, where c and c' are an i–cocycle and a j–cocycle, respectively, and $\sigma = \langle v_0, \dots, v_{i+j}\rangle \in K^{(i+j)}$ is such that $v_0 < \cdots < v_{i+j}$. Using the chain contraction (f, g, ϕ) from C_*K to H_*K, we can compute the cohomology ring of K in the following way:

Take α^* and β^*, cohomology classes of K
For every $\gamma \in H_{i+j}K$
    compute $((\alpha^* f) \smile (\beta^* f))g\gamma$
End for

Notice that the resulting cohomology class is determined by the cocycle $c = (\alpha^* f) \smile (\beta^* f)$.

In order to compute a cohomology operation $\bar{\mathcal{O}} : H^*K \to H^{*+i}K$, on one hand, we need to compute `contraction[K]` in order to obtain a chain contraction (f, g, ϕ) from C_*K to its homology and, on the other hand, we need a simplicial version $\mathcal{O} : C^*K \to C^{*+i}K$ of $\bar{\mathcal{O}}$. Therefore, for obtaining $\bar{\mathcal{O}}(\alpha^*)$, where $\alpha^* \in H^*K$, we only need to compute $\mathcal{O}(\alpha^* f)g$ (for more details, see [GR01]). For example, from the combinatorial formulae of Steenrod squares given in [Ste47,SE62],

$$Sq^i : H^*K \to H^{*+i}K\,,$$

for calculating the cohomology class $Sq^i(\alpha^*)$ with α^* in H^qK, we only have to compute $Sq^i(\alpha^* f)g$. More concretely, at cochain level, $Sq^i c = c \smile_{q-i} c \bmod 2$. Moreover, given a p–cochain c and a q–cochain c', $c \smile_n c'$ is a $(p + q - n)$–cochain defined by

$$(c \smile_n c')\sigma = \sum_{0 \le i_0 < \cdots < i_n \le p+q-n} \mu(c(\cup_{j\ \mathrm{even}} z^j) \otimes c'(\cup_{j\ \mathrm{odd}} z^j))$$

where $\sigma = \langle v_0, \dots, v_{p+q-i}\rangle$, $v_0 < \cdots < v_{p+q-i}$; $z^0 = \langle v_0, \dots, v_{i_0}\rangle$, $z^j = \langle v_{i_{j-1}}, \dots, v_{i_j}\rangle$, for $1 \le j \le n$, and $z^{n+1} = \langle v_{i_n}, \dots, v_{p+q-n}\rangle$. Finally, we can express Steerond squares in a matrix form due to the fact that these cohomology operations are homomorphisms. The process of diagonalization of such matrices can give us detailed information about the kernel and image of these cohomology operations. This information will be very useful in the next section in order to compute Adem secondary cohomology operations.

4 Adem Secondary Cohomology Operations

For attacking the computation of secondary cohomology operations, we will see in this section that the homotopy operator ϕ of the chain contraction (f, g, ϕ) from C_*K to the homology of K, is essential.

First of all, we will need the following mod 2 relation [Ste47]:

$$\delta(c \smile_n c') = c \smile_{n-1} c' + c' \smile_{n-1} c + \delta c \smile_n c' + c \smile_n \delta c' \qquad (1)$$

where c and c' are two cochains. Now, we shall indicate how Adem secondary cohomology operations

$$\Psi_q : N^qK \to H^{q+3}(K; \mathbf{Z}_2)/Sq^2 H^{q+1}(K; \mathbf{Z}), \qquad q \ge 2$$

can be constructed (see [Ade52,Ade58]). N^qK denotes the kernel of $Sq^2 : H^q(K; \mathbf{Z}) \to H^{q+2}(K; \mathbf{Z}_2)$. These operations appear using the known relation:

$$Sq^2 Sq^2 \alpha + Sq^3 Sq^1 \alpha = 0$$

for any $\alpha \in H^*(K; \mathbf{Z})$. For this particular relation there exist cochain mappings

$$E_j : C^*(K \times K \times K \times K) \to C^{*-j}K$$

such that mod 2

$$(c \smile_{q-2} c) \smile_q (c \smile_{q-2} c) + (c \smile_{q-1} c) \smile_{q-2} (c \smile_{q-1} c) = \delta E_{3q-3} c^4\,,$$

where c is a q–cochain with integer coefficients. Making use of the relation (1) we have that mod 2

$$(c \smile_{q-2} c) \smile_q (c \smile_{q-2} c) = \delta(b \smile_q \delta b + b \smile_{q-1} b)$$

$$(c \smile_{q-1} c) \smile_{q-2} (c \smile_{q-1} c) = \delta(\eta \smile_{q-2} \delta\eta + \eta \smile_{q-3} \eta)$$

where b is a $(q+1)$–cochain such that $c \smile_{q-2} c = \delta b$ and $\eta = \frac{1}{2}(c \smile_q c + c)$. Therefore

$$w = \begin{cases} E_{3q-3}c^4 + b \smile_{q-1} b + b \smile_q \delta b + \eta \smile_{q-2} \delta\eta + \eta \smile_{q-3} \eta, & q > 2 \\ E_3 c^4 + b \smile_1 b + b \smile_2 \delta b + \eta \smile \delta\eta, & q = 2 \end{cases}$$

is a mod 2 cocycle. If c is a representative q–cocycle of a cohomology class $\alpha \in N^q K$ with integer coefficients then,

$$\Psi_q \alpha = [w] + Sq^2 H^{q+1} K\,.$$

Now, suppose $\mathbf{Z}_2$ is the ground ring and suppose we have computed the contraction (f, g, ϕ) from C_*K to H_*K, $\ell =$`contraction[K]`. Then, the cochain b is $\phi^*(c \smile_{q-2} c) = (c \smile_{q-2} c)\phi$. Observe that for computing $\Psi_q \alpha^*$, $\alpha^* \in H^*K$, we need to have a combinatorial expression of the morphism E_{3q-3}. A method for obtaining "economical" combinatorial formulae for E_{3q-3} is given in [Gon00]. For example,

$$\begin{aligned}(E_3c^4)\sigma = \mu(&c\langle v_0, v_2, v_3\rangle \otimes c\langle v_0, v_1, v_2\rangle \otimes c\langle v_3, v_4, v_5\rangle \otimes c\langle v_2, v_3, v_5\rangle \\ &+c\langle v_0, v_4, v_5\rangle \otimes c\langle v_3, v_4, v_5\rangle \otimes c\langle v_0, v_1, v_2\rangle \otimes c\langle v_0, v_1, v_2\rangle \\ &+c\langle v_0, v_1, v_5\rangle \otimes c\langle v_3, v_4, v_5\rangle \otimes c\langle v_1, v_2, v_3\rangle \otimes c\langle v_1, v_2, v_3\rangle \\ &+c\langle v_0, v_1, v_2\rangle \otimes c\langle v_2, v_4, v_5\rangle \otimes c\langle v_2, v_3, v_4\rangle \otimes c\langle v_2, v_3, v_4\rangle \\ &+c\langle v_0, v_1, v_2\rangle \otimes c\langle v_2, v_3, v_5\rangle \otimes c\langle v_3, v_4, v_5\rangle \otimes c\langle v_3, v_4, v_5\rangle)\,,\end{aligned}$$

where c is a 2–cochain and $\sigma = \langle v_0, v_1, v_2, v_3, v_4, v_5\rangle$ is a 5–simplex such that $v_0 < v_1 < v_2 < v_3 < v_4 < v_5$. Therefore, the steps for computing Ψ_q are the following:

1. Take $\alpha^* \in N^q K$ making use of the diagonalization of the matrix of $Sq^2 H^q K$.
2. Compute $c = \alpha^* f$.
3. Compute $b = (c \smile_{q-2} c)\phi$, $\eta = \frac{1}{2}(c \smile_q c + c)$, $b \smile_{q-1} b$, $b \smile_q \delta b$, $\eta \smile_{q-3} \eta$, $\eta \smile_{q-2} \delta\eta$ and $E_{3q-3}c^4$.
2. Compute wg.

Let us explain with more detail the first step. In our implementation in Mathematica, the command `hclass[`ℓ, q`]` computes the list of all the cohomology classes of K in dimension q. We compute $Sq^2\alpha^*$ for each $\alpha^* \in$ `hclass[`ℓ, q`]` and we write the result as a vector `sq2[`ℓ, α^*`]` of $0's$ and $1's$ such that $Sq^2\alpha^* =$`sq2[`ℓ, α^*`].hclass[`$\ell, q+2$`]`. Then, we construct the matrix corresponding to $Sq^2 H^q K$ with the command

```
matrixSq2[ℓ,q]=Table[sq2[ℓ, hclass[ℓ,q][[i]]],{i,1,Length[hclass[ℓ,q]]}
```

After this, we compute

```
NullSpace[matrixSq2[ℓ,q], Modulus→ 2].hclass[ℓ,q]
```

in order to obtain a base of $N^q K$.

An example of the computation of Adem secondary cohomology operation using our algorithm is the following. Let K be a simplicial complex whose set of maximal simplices is

$\{\langle 1,3,7\rangle, \langle 3,4,7\rangle, \langle 1,4,7\rangle, \langle 1,2,8\rangle, \langle 2,3,8\rangle, \langle 1,3,8\rangle, \langle 4,5,9\rangle, \langle 4,6,9\rangle, \langle 5,6,9\rangle, \langle 3,4,10\rangle,$
$\langle 3,6,10\rangle, \langle 4,6,10\rangle, \langle 1,2,3,4,5,6\rangle, \langle 1,2,3,4,5,11\rangle, \langle 1,2,3,4,6,11\rangle, \langle 1,2,3,5,6,11\rangle, \langle 1,2,4,5,6,11\rangle,$
$\langle 1,3,4,5,6,11\rangle, \langle 2,3,4,5,6,11\rangle\}$

We first compute the chain contraction to the homology:

$\{\{\langle 1\rangle, \langle 1\rangle, 0\}, \{\langle 2\rangle, \langle 1\rangle, \langle 1,2\rangle\}, \{\langle 3\rangle, \langle 1\rangle, \langle 1,3\rangle\}, \{\langle 4\rangle, \langle 1\rangle, \langle 1,3\rangle + \langle 3,4\rangle\}, \{\langle 5\rangle, \langle 1\rangle, \langle 1,3\rangle + \langle 3,4\rangle + \langle 4,5\rangle\},$
$\{\langle 6\rangle, \langle 1\rangle, \langle 1,3\rangle + \langle 3,4\rangle + \langle 4,6\rangle\}, \{\langle 7\rangle, \langle 1\rangle, \langle 1,7\rangle\}, \{\langle 8\rangle, \langle 1\rangle, \langle 1,8\rangle\}, \{\langle 9\rangle, \langle 1\rangle, \langle 1,3\rangle + \langle 3,4\rangle + \langle 4,9\rangle\},$
$\{\langle 10\rangle, \langle 1\rangle, \langle 1,3\rangle + \langle 3,10\rangle\}, \{\langle 11\rangle, \langle 1\rangle, \langle 1,11\rangle\}, \{\langle 1,4\rangle, 0, \langle 1,3,7\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle\},$

$\{\langle 1,5\rangle, 0, \langle 1,3,7\rangle + \langle 1,4,5\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle\}, \{\langle 1,6\rangle, 0, \langle 1,3,6\rangle + \langle 3,4,10\rangle + \langle 3,6,10\rangle + \langle 4,6,10\rangle\},$
$\{\langle 2,3\rangle, 0, \langle 1,2,8\rangle + \langle 1,3,8\rangle + \langle 2,3,8\rangle\}, \{\langle 2,4\rangle, 0, \langle 1,2,4\rangle + \langle 1,3,7\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle\},$
$\{\langle 2,5\rangle, 0, \langle 1,2,5\rangle + \langle 1,3,7\rangle + \langle 1,4,5\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle\},$
$\{\langle 2,6\rangle, 0, \langle 1,2,6\rangle + \langle 1,3,6\rangle + \langle 3,4,10\rangle + \langle 3,6,10\rangle + \langle 4,6,10\rangle\},$
$\{\langle 2,8\rangle, 0, \langle 1,2,8\rangle\}, \{\langle 2,11\rangle, 0, \langle 1,2,11\rangle\}, \{\langle 3,5\rangle, 0, \langle 1,3,5\rangle + \langle 1,3,7\rangle + \langle 1,4,5\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle\},$
$\{\langle 3,6\rangle, 0, \langle 3,4,10\rangle + \langle 3,6,10\rangle + \langle 4,6,10\rangle\}, \{\langle 3,7\rangle, 0, \langle 1,3,7\rangle\}, \{\langle 3,8\rangle, 0, \langle 1,3,8\rangle\},$
$\{\langle 3,11\rangle, 0, \langle 1,3,11\rangle\}, \{\langle 4,7\rangle, 0, \langle 1,3,7\rangle + \langle 3,4,7\rangle\}, \{\langle 4,10\rangle, 0, \langle 3,4,10\rangle\},$
$\{\langle 4,11\rangle, 0, \langle 1,3,7\rangle + \langle 1,4,7\rangle + \langle 1,4,11\rangle + \langle 3,4,7\rangle\}, \{\langle 5,6\rangle, 0, \langle 4,5,9\rangle + \langle 4,6,9\rangle + \langle 5,6,9\rangle\},$
$\{\langle 5,9\rangle, 0, \langle 4,5,9\rangle\}, \{\langle 5,11\rangle, 0, \langle 1,3,7\rangle + \langle 1,4,5\rangle + \langle 1,4,7\rangle + \langle 1,5,11\rangle + \langle 3,4,7\rangle\},$
$\{\langle 6,9\rangle, 0, \langle 4,6,9\rangle\}, \{\langle 6,10\rangle, 0, \langle 3,4,10\rangle + \langle 4,6,10\rangle\},$
$\{\langle 6,11\rangle, 0, \langle 1,3,6\rangle + \langle 1,6,11\rangle + \langle 3,4,10\rangle + \langle 3,6,10\rangle + \langle 4,6,10\rangle\},$
$\{\langle 1,2,3\rangle, \langle 1,2,3\rangle, 0\}, \{\langle 1,3,4\rangle, \langle 1,3,4\rangle, 0\}, \{\langle 1,4,6\rangle, \langle 1,4,6\rangle, 0\}, \{\langle 1,5,6\rangle, \langle 1,5,6\rangle, 0\},$
$\{\langle 2,3,4\rangle, \langle 1,2,3\rangle + \langle 1,3,4\rangle, \langle 1,2,3,4\rangle\}, \{\langle 2,3,5\rangle, \langle 1,2,3\rangle, \langle 1,2,3,5\rangle\},$
$\{\langle 2,3,6\rangle, \langle 1,2,3\rangle, \langle 1,2,3,6\rangle\}, \{\langle 2,3,11\rangle, \langle 1,2,3\rangle, \langle 1,2,3,11\rangle\}, \{\langle 2,4,5\rangle, 0, \langle 1,2,4,5\rangle\},$
$\{\langle 2,4,6\rangle, \langle 1,4,6\rangle, \langle 1,2,4,6\rangle\}, \{\langle 2,4,11\rangle, 0, \langle 1,2,4,11\rangle\}, \{\langle 2,5,6\rangle, \langle 1,5,6\rangle, \langle 1,2,5,6\rangle\},$
$\{\langle 2,5,11\rangle, 0, \langle 1,2,5,11\rangle\}, \{\langle 2,6,11\rangle, 0, \langle 1,2,6,11\rangle\}, \{\langle 3,4,5\rangle, \langle 1,3,4\rangle, \langle 1,3,4,5\rangle\},$
$\{\langle 3,4,6\rangle, \langle 1,3,4\rangle + \langle 1,4,6\rangle, \langle 1,3,4,6\rangle\}, \{\langle 3,4,11\rangle, \langle 1,3,4\rangle, \langle 1,3,4,11\rangle\}, \{\langle 3,5,6\rangle, \langle 1,5,6\rangle, \langle 1,3,5,6\rangle\},$
$\{\langle 3,5,11\rangle, 0, \langle 1,3,5,11\rangle\}, \{\langle 3,6,11\rangle, 0, \langle 1,3,6,11\rangle\}, \{\langle 4,5,6\rangle, \langle 1,4,6\rangle + \langle 1,5,6\rangle, \langle 1,4,5,6\rangle\},$
$\{\langle 4,5,11\rangle, 0, \langle 1,4,5,11\rangle\}, \{\langle 4,6,11\rangle, \langle 1,4,6\rangle, \langle 1,4,6,11\rangle\}, \{\langle 5,6,11\rangle, \langle 1,5,6\rangle, \langle 1,5,6,11\rangle\},$
$\{\langle 2,3,4,5\rangle, 0, \langle 1,2,3,4,5\rangle\}, \{\langle 2,3,4,6\rangle, 0, \langle 1,2,3,4,6\rangle\}, \{\langle 2,3,4,11\rangle, 0, \langle 1,2,3,4,11\rangle\},$
$\{\langle 2,3,5,6\rangle, 0, \langle 1,2,3,5,6\rangle\}, \{\langle 2,3,5,11\rangle, 0, \langle 1,2,3,5,11\rangle\}, \{\langle 2,3,6,11\rangle, 0, \langle 1,2,3,6,11\rangle\},$
$\{\langle 2,4,5,6\rangle, 0, \langle 1,2,4,5,6\rangle\}, \{\langle 2,4,5,11\rangle, 0, \langle 1,2,4,5,11\rangle\}, \{\langle 2,4,6,11\rangle, 0, \langle 1,2,4,6,11\rangle\},$
$\{\langle 2,5,6,11\rangle, 0, \langle 1,2,5,6,11\rangle\}, \{\langle 3,4,5,6\rangle, 0, \langle 1,3,4,5,6\rangle\}, \{\langle 3,4,5,11\rangle, 0, \langle 1,3,4,5,11\rangle\},$
$\{\langle 3,4,6,11\rangle, 0, \langle 1,3,4,6,11\rangle\}, \{\langle 3,5,6,11\rangle, 0, \langle 1,3,5,6,11\rangle\}, \{\langle 4,5,6,11\rangle, 0, \langle 1,4,5,6,11\rangle\},$
$\{\langle 2,3,4,5,6\rangle, 0, \langle 1,2,3,4,5,6\rangle\}, \{\langle 2,3,4,5,11\rangle, 0, \langle 1,2,3,4,5,11\rangle\}, \{\langle 2,3,4,6,11\rangle, 0, \langle 1,2,3,4,6,11\rangle\},$
$\{\langle 2,3,5,6,11\rangle, 0, \langle 1,2,3,5,6,11\rangle\}, \{\langle 2,4,5,6,11\rangle, 0, \langle 1,2,4,5,6,11\rangle\}, \{\langle 3,4,5,6,11\rangle, 0, \langle 1,3,4,5,6,11\rangle\},$
$\{\langle 2,3,4,5,6,11\rangle, \langle 2,3,4,5,6,11\rangle, 0\}\}\,.$

Notice that if a simplex of K doesn't appear in this list, it is because its image under f and ϕ is null. The representative cycle of every homology class is:

$$
\begin{aligned}
&g\langle 1\rangle = \langle 1\rangle \\
&g\langle 1,2,3\rangle = \langle 1,2,3\rangle + \langle 1,2,8\rangle + \langle 1,3,8] + \langle 2,3,8\rangle \\
&g\langle 1,3,4\rangle = \langle 1,3,4\rangle + \langle 1,3,7\rangle + \langle 1,4,7\rangle + \langle 3,4,7\rangle \\
&g\langle 1,4,6\rangle = \langle 1,3,4\rangle + \langle 1,3,6\rangle + \langle 1,4,6\rangle + \langle 3,4,10\rangle + \langle 3,6,10\rangle + \langle 4,6,10\rangle \\
&g\langle 1,5,6\rangle = \langle 1,4,5\rangle + \langle 1,4,6\rangle + \langle 1,5,6\rangle + \langle 4,5,9\rangle + \langle 4,6,9\rangle + \langle 5,6,9\rangle \\
&g\langle 2,3,4,5,6,11\rangle = \langle 1,2,3,4,5,6\rangle + \langle 1,2,3,4,5,11\rangle + \langle 1,2,3,4,6,11\rangle \\
&\qquad + \langle 1,2,3,5,6,11\rangle + \langle 1,2,4,5,6,11\rangle + \langle 1,3,4,5,6,11\rangle + \langle 2,3,4,5,6,11\rangle\,.
\end{aligned}
$$

A base of the kernel of $Sq^2 H^2 K$ is:

$$\{\langle 1,2,3\rangle^*, \langle 1,3,4\rangle^*, \langle 1,4,6\rangle^*, \langle 1,5,6\rangle^*\}\,.$$

Now, given an element α of this kernel, we first have to compute $c = g^*\alpha$. Let us study a concrete example with all the details. Let us take $\alpha = \langle 1,2,3\rangle^* + \langle 1,5,6\rangle^*$. Then

$$
\begin{aligned}
c = g^*\alpha = \alpha f = {} & \langle 1,2,3\rangle^* + \langle 1,5,6\rangle^* + \langle 2,3,4\rangle^* + \langle 2,3,5\rangle^* + \langle 2,3,6\rangle^* \\
& + \langle 2,3,11\rangle^* + \langle 2,5,6\rangle^* + \langle 3,5,6\rangle^* + \langle 4,5,6\rangle^* + \langle 5,6,11\rangle^*\,.
\end{aligned}
$$

We now compute the cochains of the 3rd step of the algorithm for computing Ψ_2.

$$\delta b = c \smile c = \langle 1,2,3,5,6\rangle^* + \langle 2,3,4,5,6\rangle^* + \langle 2,3,5,6,11\rangle^* \qquad b = (c \smile c)\phi = \langle 2,3,5,6\rangle^*$$

Then, we have that $b \smile_1 b = 0$ and $b \smile_2 \delta b = 0$. On the other hand, $\delta\eta = c \smile_1 c = 0$ therefore $\eta \smile \delta\eta = 0$. We thus get,

$$w = f^*(E_3 c^4) = (E_3 c^4)g = \langle 1,2,3,4,5,6\rangle^* g = \langle 2,3,4,5,6,11\rangle^* .$$

Therefore, $\Psi_2(\langle 1,2,3\rangle^* + \langle 1,5,6\rangle^*) = \langle 2,3,4,5,6,11\rangle^*$. Finally, observe that since there are no classes of cohomology of dimension 3, then $\langle 2,3,4,5,6,11\rangle^* \notin Im\, Sq^2 H^3 K$.

5 Some Comments

All these results can be given in a more general framework working not necessarily with finite simplicial complexes. Nevertheless, a contraction from the (co)chain complex associated to the simplicial complex to its (co)homology must exist in order to develop the method.

In this paper, the ground ring is $\mathbf{Z}_2$ for simplicity, but the same process can be done working with any field as the ground ring. For example, let $\mathbf{Z}_p$ (p being a prime) be the group of coefficients. From the combinatorial formulae for the reduced pth powers P_i [Ste47,SE62] at cochain level in terms of face operators established in [GR99,Gon00] and the algorithm for computing the chain contraction (f, g, ϕ) from $C_*(K;\mathbf{Z}_p)$ to $H_*(K;\mathbf{Z}_p)$, Steenrod cohomology operations can effectively be computed. Let $\alpha^* \in H^q(K;\mathbf{Z}_p)$, for calculating the cohomology class $\mathcal{P}_i(\alpha^*)$ with $\alpha^* \in H^q(K;\mathbf{Z}_p)$, we only have to compute $P_i(\alpha f)g$.

Finally, in order to obtain the image of any cohomology operation at cochain level over a representative cocycle using our formulae, we have to compute them on a base of $C_*(K)$ in the desired dimension. A way of decreasing the complexity of this is to do a "topological" thinning of the simplicial complex K in order to obtain a thinned simplicial subcomplex $M_{\mathrm{top}}K$ of K (such that there exists a chain contraction from C_*K to $C_*(M_{\mathrm{top}}K)$). Two examples of thinning in this way are edge contractions (example (a)) and simplicial collapses (example (b)). Therefore, we can apply our machinery to compute cohomology operations in the thinned simplicial complex $M_{\mathrm{top}}K$ and then, the results can be easily interpreted in the "big" simplicial complex K via composition of contractions.

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