A COMBINATORIAL METHOD FOR COMPUTING STEENROD SQUARES

Rocío González-Díaz and Pedro Real
Dpto. de Matemática Aplicada I
Facultad de Informática y Estadística
Universidad de Sevilla
rogodi@us.es, real@us.es

Abstract

We present here a combinatorial method for computing cup-i products and Steenrod squares of a simplicial set X. This method is essentially based on the determination of explicit formulae for the component morphisms of a higher diagonal approximation (i.e., a family of morphisms measuring the lack of commutativity of the cup product on the cochain level) in terms of face operators of X. A generalization of this method to Steenrod reduced powers is sketched.

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1 Introduction

Cohomology operations are algebraic operations on the cohomology groups of spaces commuting with the homomorphisms induced by continuous mappings. This machinery is useful when the graded vector space structure and the cup product on cohomology fail to distinguish two spaces by their cohomology. Steenrod squares [20] constitute an extremely important class of cohomology operations not only in Algebraic Topology but also in the area of simplicial methods in Homological Algebra (cohomology of groups, Hochschild cohomology of algebras, ...).

We here study in detail the underlying combinatorial structures of the definition of these cohomology operations. More concretely, we will move in the framework of Simplicial Topology [12] in which the basic objects are simplicial sets, that is, graded sets endowed with face (∂_i) and degeneracy (s_i) operators, satisfying several commutativity relations. Roughly speaking, a simplicial set can be considered as an algebraic generalization of the structure of a triangulated polyhedron although the former features a more rigid combinatorial structure than the latter. In this context, we develop a suitable setting in which all Steenrod cohomology operations can be studied simultaneously.

It is well-known that there are several methods for constructing Steenrod squares. One of them consists of making these operations using the cohomology of Eilenberg-Mac Lane spaces (see, for instance, [[12]; p. 107]). Another method is determined by the construction of a family of morphisms $\{D_i\}$ measuring the lack of commutativity of the cup product on the cochain level [20]. This sequence of morphisms is called *higher diagonal approximation*. Its existence is always guaranteed by the acyclic models method [5]. In this way, it is possible to derive a recursive procedure to obtain the explicit formula for any D_i (see [4], [[13]; Sect. 7]).

In this paper, we present an alternative method for obtaining the explicit formula for a higher diagonal approximation. In [15], the formula for D_i is established in terms of the component morphisms of a given Eilenberg-Zilber contraction (a special homotopy equivalence) from $C_*^N(X\times X)$ onto $C_*^N(X)\otimes C_*^N(X)$, where $C_*^N(X)$ denotes the normalized chain complex of a given simplicial set X.

Now, we have the following problems. On one hand, the component morphisms of the contraction above are defined in terms of face and degeneracy operators of the simplicial set X. On the other hand, the formula for a morphism D_i always involves the use of the homotopy operator of the Eilenberg-Zilber contraction and the explicit formula for this last morphism is determined by shuffles (a special type of permutation) of degeneracy operators. In consequence, if we try to express in this way the morphisms D_i in terms of face and degeneracy operators of X, the number of summands appearing in the formula for a morphism D_i evaluated over an element of degree n is, in general, at least 2^n. Therefore, an algorithm that would be designed starting from these formulae would be too slow for practical implementation.

Because of this, the idea of simplifying these formulae arises in a natural way. This simplification or normalization is based on the fact that any composition of face and degeneracy operators of the simplicial set X can be put in a "canonical" way. That is, a composition of this type can be expressed in the unique form:

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s},$$

where $j_t > \cdots > j_1 \geq 0$ and $i_s > \cdots > i_1 \geq 0$.

Moreover, taking into account that the image of a morphism D_i lies in $C_*^N(X) \otimes C_*^N(X)$, those summands of the simplified formula for D_i with a factor having a degeneracy operator in its expression, can be eliminated. The reason is that this factor applied to an element of the simplicial set X is zero in the normalized chain complex associated to X. In this way, we here obtain more simple formula for the morphism D_i.

In this way, the classical definition of Steenrod squares:

$$Sq^i(c)(x) \quad = \quad \begin{cases} \mu(< c \otimes c,\ D_{j-i}(x) >), & i \leq j \\ 0, & i > j \end{cases} \tag{1}$$

where $c \in \mathrm{Hom}(C_j^N(X), \mathbf{Z}_2)$, $x \in C_{i+j}^N(X)$ and μ is the homomorphism induced by the multiplication in $\mathbf{Z}_2$, is complemented by a manageable combinatorial formulation of the higher diagonal approximation. As we will remark later, this description can be considered as a direct translation of the most ancient definition of Steenrod squares (see [18]) to the general setting of the Simplicial Topology. We give a more detailed explanation in the third section.

Still, we think that this combinatorial machinery could be substantially improved in the future by exploiting the well-known properties of Steenrod squares [1, 20] and advanced techniques for calculating cocycles (see, for example, [9, 10]). In this way, a "reasonably efficient" algorithm computing the cohomology algebra of several important simplicial sets could be derived.

Finally, we start an analogous study to that given in [15] for Steenrod reduced powers. More precisely, we provide a simplicial description of these operations in terms of the component morphisms of an Eilenberg-Zilber contraction from

$C_*^N(X\times \overset{p\text{ times}}{\cdots} \times X)$ to $C_\star^N(X)\otimes \overset{p\text{ times}}{\cdots} \otimes C_*^N(X)$, and a 0-sequence $\{\gamma_i\}_{i\geq 0}$ in the symmetric group G_p.

Here is a summary of the present paper. Section 2 is dedicated to notation, terminology and a presentation of the problem. In Section 3, we show an explicit combinatorial definition of cup-i products and, consequently, of Steenrod squares. In Section 4, we study Steenrod reduced powers from this point of view. Finally, a proof of the main theorem enunciated in Section 3 is given in Section 5.

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2 The higher homotopy commutativity of the Alexander-Whitney operator

The aim of this section is to give some preliminaries and a brief account of the work done in [15] in order to facilitate the understanding of the rest of the paper. Most of the material given in this section can be found in [12], [11], [17] and [21].

A *simplicial set* X is a sequence of sets $X_0, X_1, \ldots$, together with *face operators* $\partial_i : X_n \to X_{n-1}$ and *degeneracy operators* $s_i : X_n \to X_{n+1}$ $(i = 0, 1, \ldots, n)$, which satisfy the following "simplicial" identities:

$$
\begin{array}{lll}
\textbf{(s1)} & \partial_i\partial_j = \partial_{j-1}\partial_i, & \text{if } i < j; \\
\textbf{(s2)} & s_i s_j = s_{j+1} s_i, & \text{if } i \leq j; \\
\textbf{(s3)} & \partial_i s_j = s_{j-1}\partial_i, & \text{if } i < j, \\
\textbf{(s4)} & \partial_i s_j = s_j \partial_{i-1}, & \text{if } i > j+1, \\
\textbf{(s5)} & \partial_j s_j = 1_X = \partial_{j+1} s_j. &
\end{array}
$$

The elements of X_n are called *n-simplices.* A simplex x is *degenerated* if $x = s_i y$ for some simplex y and degeneracy operator s_i; otherwise, x is non degenerated.

The following elementary lemma will be essential in the proof of the main theorem of this paper.

Lemma 2.1 *[12] Any composition $\mu : X_m \to X_n$ of face and degeneracy operators of a simplicial set X can be put in a unique "canonical" form:*

$$s_{j_t} \cdots s_{j_1} \partial_{i_1} \cdots \partial_{i_s},$$

where $n > j_t > \cdots > j_1 \geq 0$, $m \geq i_s > \cdots > i_1 \geq 0$ and $n - t + s = m$.

Let R be a ring which is commutative with unit. A *chain* (resp. *cochain*) *complex* $M = \{M_n, d_n\}$ (resp. $C = \{C^n, \delta^n\}$) is a graded (over the integers) R-module together with a R-module map d of degree -1 (resp. of degree $+1$), called the *differential*, such that $d_n\, d_{n+1} = 0$ (resp. $\delta^{n+1}\, \delta^n = 0$). An element c of C^n is called n-cochain. The *homology* $H_*(M)$ (resp. the *cohomology* $H^*(C)$ is the family of modules

$$H_n(M) \;=\; \text{Ker } d_n/\text{Im } d_{n+1} \qquad (\text{resp. } H^n(C) = \text{Ker } \delta^n/\text{Im } \delta^{n-1}).$$

If M is a chain complex over R and G is an R-module, there is a cochain complex $\text{Hom}(M, G) = \{\text{Hom}(M_n, G), \delta^n\}$, where, if $c \in \text{Hom}(M_n, G)$, then $\delta^n c \in \text{Hom}(M_{n+1}, G)$ is defined by

$$(\delta^n c)(x) = c(d_{n+1}(x)), \qquad x \in M_{n+1}.$$

We also write $< c, x >$ instead of $c(x)$ and set $< c, x >= 0$ if the degree of the cochain c is not equal to the degree of the element x. In this notation,

$$< \delta^n c, x > = < c, d_{n+1}(x) > .$$

Now, given a simplicial set X, let $C_*(X)$ denotes the chain complex $\{C_n(X), d_n\}$, in which $C_n(X)$ is the free R-module generated by X_n and $d_n : C_n(X) \to C_{n-1}(X)$ is defined by $d_n = \displaystyle\sum_{i=0}^{n} (-1)^i \partial_i$. Let us denote by $s(C_*(X))$ the graded R-module generated by all the degenerated simplices. In $C_*(X)$, we have that $d_n(s(C_{n-1}(X))) \subset s(C_{n-2}(X))$ and, then $C_*^N(X) = \{C_n(X)/s(C_{n-1}(X)), d_n\}$ is a chain complex called *the normalized chain complex associated to* X. Given a R-module G, let us define by $C^*(X; G)$ the cochain complex associated to $C_*^N(X)$. In this way, we define the homology and cohomology of X with coefficients in a

R-module G by $H_*(X;G) = H_*(C_*^N(X) \otimes G)$ and $H^*(X;G) = H^*(C^*(X;G))$, respectively.

Eilenberg and Mac Lane defined in [6] a *contraction* of chain complexes from N onto M, as a triple (f, g, ϕ) in which $f : N \to M$ (projection) and $g : M \to N$ (inclusion) are chain maps, and $\phi : N \to N$ (homotopy operator) is a map of R-module raising degree by 1. Moreover, it is required that

$$\textbf{(c1)} \quad fg = 1_M, \qquad \textbf{(c2)} \quad \phi d + d\phi + gf = 1_N,$$

$$\textbf{(c3)} \quad \phi g = 0, \qquad \textbf{(c4)} \quad f\phi = 0, \qquad \textbf{(c5)} \quad \phi\phi = 0.$$

Hence, this definition implies that the "big" complex N is homology equivalent to the "small" complex M in a strong way. In fact, a contraction is a special homotopy equivalence between chain complexes.

If we have two contractions (f_i, g_i, ϕ_i) from N_i to M_i, with $i = 1, 2$ then, the following contractions can be constructed (see [6]):

- The tensor product contraction $(f_1 \otimes f_2, g_1 \otimes g_2, \phi_1 \otimes g_2 f_2 + 1_{N_1} \otimes \phi_2)$ from $N_1 \otimes N_2$ to $M_1 \otimes M_2$.
- If $N_2 = M_1$, the composition contraction $(f_2 f_1, g_1 g_2, \phi_1 + g_1 \phi_2 f_1)$ from N_1 to M_2.

If p and q are non-negative integers, a (p, q)-*shuffle* (α, β) is a partition of the set $\{0, 1, \ldots, p + q - 1\}$ of integers into two disjoint subsets, $\alpha_1 < \cdots < \alpha_p$ and $\beta_1 < \cdots < \beta_q$, of p and q integers, respectively. The signature of the shuffle (α, β) is defined by $sig(\alpha, \beta) = \sum_{i=1}^{p} \alpha_i - (i - 1)$.

After these preliminaries, we are able to describe a very important homotopy equivalence in Algebraic Topology. This contraction tells us that the associated chain complex $C_*^N(X \times Y)$ reduces to the tensor product of chain complexes $C_*^N(X)$ and $C_*^N(Y)$.

An Eilenberg-Zilber contraction [8] from $C_*^N(X \times Y)$ to $C_*^N(X) \otimes C_*^N(Y)$, where X and Y are given simplicial sets, is defined by the triple $(AW,\ EML,\ SHI)$ where:

- The Alexander-Whitney operator $AW : C_*^N(X \times Y) \longrightarrow C_*^N(X) \otimes C_*^N(Y)$ is defined by:

$$AW(a_m \times b_m) = \sum_{i=0}^{m} \partial_{i+1} \cdots \partial_m a_m \otimes \partial_0 \cdots \partial_{i-1} b_m.$$

If $X = Y$, AW can be considered as a "simplicial approximation" to the diagonal and this operator provides a method for constructing the cup product in cohomology. If we interchange the factors a_m and b_m in the formula, we obtain a different approximation. Comparison of these two different approximations to the diagonal leads to the Steenrod squares.

- The Eilenberg-Mac Lane operator $EML : C_*^N(X) \otimes C_*^N(Y) \longrightarrow C_*^N(X \times Y)$ is defined by:

$$EML(a_p \otimes b_q) = \sum_{(\alpha,\beta)\in\{(p,q)-\text{shuffles}\}} (-1)^{sig(\alpha,\beta)} s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q.$$

This operator can be seen as a process of "triangulation" in the cartesian product $X \times Y$.

- And the Shih operator $SHI : C_*^N(X \times Y) \longrightarrow C_{*+1}^N(X \times Y)$ is defined by:

$$\begin{aligned} SHI(a_0 \times b_0) \quad &= 0; \\ SHI(a_m \times b_m) \quad &= \sum (-1)^{\bar{m}+sig(\alpha,\beta)+1} \quad s_{\beta_q+\bar{m}} \cdots s_{\beta_1+\bar{m}} s_{\bar{m}-1} \partial_{m-q+1} \cdots \partial_m a_m \\ &\qquad\qquad \times s_{\alpha_{p+1}+\bar{m}} \cdots s_{\alpha_1+\bar{m}} \partial_{\bar{m}} \cdots \partial_{m-q-1} b_m; \end{aligned}$$

where $\bar{m} = m - p - q$, $sig(\alpha, \beta) = \sum_{i=1}^{p+1} \alpha_i - (i-1)$, and the last sum is taken over all the indices $0 \leq q \leq m-1$, $0 \leq p \leq m-q-1$ and $(\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}$.

A recursive formula for the SHI operator has already been given by Eilenberg and Mac Lane in [7]. The explicit formula given here was stated by Rubio in [16] and proved by Morace in the appendix of [14]. In contrast to the deep studies found in the literature on the AW and EML operators, it turns to be quite surprising the lack of interest shown up to now in the study of the homotopy operator involved in an Eilenberg-Zilber contraction, not only from the point of view of getting its explicit formula but also of obtaining algebraic preservation results of this operator with regard to the underlying coalgebra structure on $C_*^N(X \times X)$.

Given a simplicial set X and a positive integer p, we can form a contraction (f, g, ϕ) from $C_*^N(X \times \overset{p \text{ times}}{\cdots} \times X)$ to $C_*^N(X) \otimes \overset{p \text{ times}}{\cdots} \otimes C_*^N(X)$, appropriately composing Eilenberg-Zilber contractions. If $p = 2$, then (f, g, ϕ) is the contraction (AW, EML, SHI). The contraction from $C_*^N(X \times X \times X)$ onto $C_*^N(X) \otimes C_*^N(X) \otimes C_*^N(X)$ is defined by the composition of the Eilenberg-Zilber contraction from $C_*^N(X \times X \times X)$ to $C_*^N(X) \otimes C_*^N(X \times X)$ and the tensor product contraction of the identity morphism $1_{C_*^N(X)}$ and the Eilenberg-Zilber contraction from $C_*^N(X \times X)$ onto $C_*^N(X) \otimes C_*^N(X)$. And so on.

From now on, the contractions obtained in this way will also be called Eilenberg-Zilber contractions.

Let p be a positive integer. Let us define several chain maps (or morphisms) we use in this paper. We will omit in the notation of these morphism its dependency on p. The *diagonal map*

$$\Delta : C_*^N(X) \to C_*^N(X \times \overset{p \text{ times}}{\cdots} \times X)$$

is defined by $\Delta(x) = (x, \overset{p \text{ times}}{\ldots}, x)$. The following automorphisms are also defined:

$$t : C_*^N(X \times \overset{p \text{ times}}{\cdots} \times X) \to C_*^N(X \times \overset{p \text{ times}}{\cdots} \times X),$$

such that $t(x_1, x_2, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$ and

$$T : C_*^N(X) \otimes \overset{p \text{ times}}{\cdots} \otimes C_*^N(X) \to C_*^N(X) \otimes \overset{p \text{ times}}{\cdots} \otimes C_*^N(X)$$

defined by $T(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = (-1)^{|x_1|(|x_2| + \cdots + |x_p|)} x_2 \otimes \cdots \otimes x_p \otimes x_1$.

We now outline the problem concerning Steenrod squares in which we are interested. It is well-known that the AW operator is not commutative, that is, assuming that $X = Y$, $AWt \neq AW$. On the other hand, this operator determines the cup product in cohomology. If G is a ring, given two cochains $c \in C^i(X; G)$ and $c' \in C^j(X; G)$, and $x \in C_{i+j}^N(X)$, the cup-product of c and c' is defined by:

$$\begin{aligned} c \smile c'(x) &= \mu(< c \otimes c', AW\ \Delta(x) >) \\ &= \mu(< c, \partial_{i+1} \cdots \partial_{i+j} x > \otimes < c', \partial_0 \cdots \partial_{i-1} x >), \end{aligned}$$

where μ is the homomorphism induced by the multiplication on G. Steenrod in [19] determined that there exists an infinite sequence of morphisms $\{D_i\}$, called *higher diagonal approximation*, which "measures" this lack of commutativity. More precisely, there is a sequence of graded homomorphisms $D_i : C_*^N(X) \to C_*^N(X) \otimes C_*^N(X)$ of degree i such that:

$$D_0 = AW\,\Delta$$
$$d_\otimes\, D_{i+1} + (-1)^i D_{i+1}\, d = T\, D_i + (-1)^{i+1} D_i,$$

where d and $d_\otimes$ are the differentials of $C_*^N(X)$ and $C_*^N(X) \otimes C_*^N(X)$, respectively.

Moreover, the morphism D_i can be expressed in the form $D_i = h_i\ \Delta$, where $h_i : C_*^N(X \times X) \to C_*^N(X) \otimes C_*^N(X)$ is a homomorphism of degree i. In the literature, the existence of the tower of iterated Alexander-Whitney operators $\{h_i\}$ is guaranteed by the acyclic models method (see [5]). This technique can be considered as a constructive method in the simplicial category and recursive formulae for $\{D_i\}$ can be established (see [13]).

In [15], a different approach is presented. The strong homotopy commutativity of AW is determined by making use of the explicit Eilenberg-Zilber contraction (AW, EML, SHI). Hence, the formula for a morphism D_i is given in terms of the morphisms AW, SHI, the diagonal Δ and the automorphism t. More precisely, the formula for h_i is $AW\ (t\ SHI)^i$, for all $i \in N$.

Now, the definition of the cohomology operation $Sq^i : H^j(X; \mathbf{Z}_2) \to H^{j+i}(X; \mathbf{Z}_2)$ (see (1)) takes the form:

$$Sq^i(c)(x) = \begin{cases} \mu(< c \otimes c, AW\ (t\ SHI)^{j-i}(x, x) >), & i \leq j \\ 0, & i > j \end{cases} \tag{2}$$

where $c \in C^j(X; \mathbf{Z}_2)$ and $x \in C_{i+j}^N(X)$.

It is obvious that the formulae for the morphism h_i can be given in terms of face and degeneracy operators of X. Our objective in the next section is to show how to "simplify" these formulae and to obtain an explicit definition of Steenrod squares only in terms of face operators of X.

3 An explicit combinatorial description of the cup-i products

It is clear that the image of h_i lies in $C_*^N(X) \otimes C_*^N(X)$. Therefore, if we express the factors of the summands of the formula for D_i in a canonical way (see Lemma 2.1), those summands of the simplified formula for D_i having a factor with a degeneracy operator in its expression must be eliminated.

Working in this way, we can state the following theorem. Section 5 is entirely devoted to the proof of this result.

Theorem 3.1 *Let R be the ground ring and let X be a simplicial set. Let us consider the Eilenberg-Zilber contraction (AW, EML, SHI) from $C_*^N(X \times X)$ onto $C_*^N(X) \otimes C_*^N(X)$. Then, the morphism $h_n = AW(t\,SHI)^n : C_m^N(X \times X) \to (C_*^N(X) \otimes C_*^N(X))_{m+n}$ can be expressed in the form:*

- *if n is even, then:*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} (-1)^{A(n)+B(n,m,\bar\imath)+C(n,\bar\imath)+D(n,m,\bar\imath)}$$
$$\partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m$$
$$\otimes\partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdot\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1},$$

- *if n is odd, then:*

$$AW(t\,SHI)^n = \sum_{i_n=n}^{m} \sum_{i_{n-1}=n-1}^{i_n-1} \cdots \sum_{i_0=0}^{i_1-1} (-1)^{A(n)+B(n,m,\bar\imath)+C(n,\bar\imath)+D(n,m,\bar\imath)}$$
$$\partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdot\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1}$$
$$\otimes\partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m,$$

where

$$A(n) = \begin{cases} 1, & \text{if } n \equiv 3,4,5,6 \ mod\ 8 \\ 0, & \text{otherwise} \end{cases}$$

$$B(n,m,\bar\imath) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} i_{2j}, & \text{if } n \equiv 1,2 \ mod\ 4 \\ \displaystyle\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} i_{2j+1} + nm, & \text{if } n \equiv 0,3 \ mod\ 4 \end{cases}$$

$$C(n,\bar\imath) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (i_{2j} + i_{2j-1})(i_{2j-1} + \cdots + i_0)$$

and

$$D(n,m,\bar\imath) = \begin{cases} (m+i_n)(i_n + \cdots + i_0), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

where $\bar\imath = (i_0, i_1, \ldots, i_n)$.

A first consequence of the theorem above is that taking into account that the cup-i product (see, for example, [3]) of a p-cochain $c \in C^p(X;G)$ and a q-cochain $c' \in C^q(X;G)$ is defined by:

$$c \smile_i c'(x) \;=\; \mu(< c \otimes c', D_i(x) >),$$

where $i \in \mathbf{N}$ and $x \in C^N_{p+q-i}(X)$, one immediately gets a simplicial description of these operations. In an analogous way, a combinatorial definition of Steenrod squares (2) is given by the following corollary:

Corollary 3.2 *Let $\mathbf{Z}_2$ be the ground ring and let X be a simplicial set. If $c \in C^j(X;\mathbf{Z}_2)$ and $x \in C^N_{i+j}(X)$, then $Sq^i : H^j(X;\mathbf{Z}_2) \to H^{j+i}(X;\mathbf{Z}_2)$ is defined by:*

- *If $i \leq j$ and $i+j$ is even, then:*

$$\begin{aligned} Sq^i(c)(x) \;=\; & \sum_{i_n=S(n)}^{m} \; \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} \\ & \mu(< c, \partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdots\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m x > \\ & \otimes < c, \partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdots\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1} x >). \end{aligned}$$

- *If $i \leq j$ and $i+j$ is odd, then:*

$$\begin{aligned} Sq^i(c)(x) \;=\; & \sum_{i_n=S(n)}^{m} \; \sum_{i_{n-1}=S(n-1)}^{i_n-1} \cdots \sum_{i_1=S(1)}^{i_2-1} \\ & \mu(< c, \partial_{i_0+1}\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\cdots\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i_n-1} x > \\ & \otimes < c, \partial_0\cdots\partial_{i_0-1}\partial_{i_1+1}\cdots\cdots\partial_{i_{n-1}-1}\partial_{i_n+1}\cdots\partial_m x >). \end{aligned}$$

- *If $i > j$, then $Sq^i(c)(x) = 0$.*

In these formulae, $n = j - i$, $m = i + j$,

$$S(k) = i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n} \lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor,$$

for all $0 \leq k \leq n$ and $i_0 = S(0)$.

Proof

Let us start with $c \in C^j(X;\mathbf{Z}_2)$. If $i > j$ then $Sq^i(c) = 0$. So, let us suppose that $i \leq j$. Then,

$$Sq^i(c)(x) = \mu(< c \otimes c,\ AW(t\,SHI)^n(x,x) >)$$

where $x \in C_m(X)$. It is not hard to notice that we only have to consider the summands of the explicit formula for $AW\,(t\,SHI)^n$ (see Theorem 3.1), having the same number of face operators in both factors.

- If $n = 0$, then $m - i_0 = i_0$, so $i_0 = \frac{m}{2}$. The formula only has one summand.
- If $n = 1$, then $i_1 - 1 - i_0 = i_0 + m - i_1$. So, $i_0 = i_1 - \frac{m+1}{2}$ and $i_1 \geq \frac{m+1}{2}$.
- In general, if n is even (when n is odd the proof is analogous):

$$\begin{aligned} &m - i_n + \cdots + i_{2k+1} - 1 - i_{2k} + \cdots + i_1 - 1 - i_0 \\ &= i_n - 1 - i_{n-1} + \cdots + i_{2k} - 1 - i_{2k-1} + \cdots + i_2 - 1 - i_1 + i_0, \end{aligned}$$

  hence, we have

$$i_0 = i_1 - i_2 + i_3 - \cdots - i_n + \frac{m}{2}. \tag{3}$$

  Taking into account in (3) that $i_0 \geq 0$, we get

$$i_1 \geq i_2 - i_3 + \cdots + i_n - \frac{m}{2}.$$

  Using $i_0 \leq i_1 - 1$ in (3), we have

$$i_2 \geq i_3 - i_4 + \cdots + i_{n-1} - i_n + \frac{m}{2} + 1.$$

  In general, let us suppose that

$$i_k \geq i_{k+1} - i_{k+2} + \cdots + (-1)^{k+n-1} i_n + (-1)^{k+n}\frac{m}{2} + \lfloor\frac{k}{2}\rfloor$$

  for all $1 \leq k \leq \ell$ and let us prove that this expression is true in $\ell + 1$. In the case $k = \ell - 1$, since $i_\ell - 1 \geq i_{\ell-1}$, we have

$$i_\ell - 1 \geq i_\ell - i_{\ell+1} + \cdots + (-1)^{\ell+n-2} i_n + (-1)^{\ell+n-1}\frac{m}{2} + \lfloor\frac{\ell-1}{2}\rfloor$$

  and simplifying, we conclude

$$i_{\ell+1} \geq i_{\ell+2} - i_{\ell+3} + \cdots + (-1)^{\ell+n} i_n + (-1)^{\ell+n+1}\frac{m}{2} + \lfloor\frac{\ell+1}{2}\rfloor.$$

$\square$

It is necessary to say a few words in order to evaluate the novelty of all this combinatorial formulation. The following historical observations can be found in [[3]; Ch. VI, Sect. 1.B, page 511]. In 1947, Steenrod [18] generalized the Čech-Whitney definition of the cup-product for a finite simplicial complex K (in our context, we can consider K as a polyhedral simplicial set [12]).The idea was to keep several common vertices between both factors of the "decomposition" of a considered simplex, instead of one as in the cup-product. In this way, Steenrod established formulae for the cup-i products, which were "awkward to handle", in his own words. He showed that the cohomology operations induced from this unwieldy description were in fact independent of the order chosen on the vertices of K.

Having this in mind, we could dare to say that in this paper we are rediscovering this old description given by Steenrod and clarifying it in a general combinatorial framework. On the other hand, it is important to note that we here determine the signs involved in the formulae of the cup-i products.

Assuming that the face operators are evaluated in constant time, the following result gives us a first measure of the computational complexity of these formulae.

Proposition 3.3 *Let $\mathbf{Z}_2$ be the ground ring. Let X be a simplicial set and k a non-negative integer. If $c \in C^{i+k}(X;\mathbf{Z}_2)$, then the number of face operators taking part in the formula for $Sq^i(c)$ is $O(i^{k+1})$.*

Proof

Let $j = i + k$. Here, it is not necessary to distinguish the cases $i + j$ even and $i + j$ odd, since the proof is the same in both cases.

Firstly we count the number of summands of the formula for $Sq^i(c)$ given in Corollary 3.2.

The parameter i_n contributes with

$$m - S(n) + 1 = m - \lfloor \frac{m+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 \tag{4}$$

summands in the formula, where $n = j - i$ and $m = i + j$.

Let us see that (4) is equal to $i+1$. If n is even then, this expression is equal to

$$m - \frac{m}{2} - \frac{n}{2} + 1 = \frac{m-n}{2} + 1 = i + 1$$

and if n is odd,

$$m - \frac{m+1}{2} - \frac{n-1}{2} + 1 = \frac{m-n}{2} + 1 = i + 1.$$

The parameter i_{n-1} contributes with

$$i_n - S(n-1) = \lfloor \frac{m+1}{2} \rfloor - \lfloor \frac{n-1}{2} \rfloor \leq i + 1$$

summands.

In general, the parameter i_k, with $1 \leq k \leq n-1$ contributes with

$$i_{k+1} - S(k) = i_{k+2} - i_{k+3} + \cdots + (-1)^{k+n} i_n + (-1)^{k+n+1} \lfloor \frac{m+1}{2} \rfloor - \lfloor \frac{k}{2} \rfloor \tag{5}$$

summands, and using that $i_{\ell-1} - i_\ell \leq -1$ and $i_n \leq m$, it is not difficult to see that the expression (5) is less or equal to $i+1$ for each $1 \leq k \leq n-1$.

Hence, the number of summands appearing in the formulae of Corollary 3.2 is:

$$\begin{aligned} &\sum_{i_n=S(n)}^{m} \cdots \sum_{i_3=S(3)}^{i_4-1} \sum_{i_2=S(2)}^{i_3-1} i_2 - S(1) \leq \sum_{i_n=S(n)}^{m} \cdots \sum_{i_3=S(3)}^{i_4-1} \sum_{i_2=S(2)}^{i_3-1} i+1 \\ &\leq (i+1) \sum_{i_n=S(n)}^{m} \cdots \sum_{i_3=S(3)}^{i_4-1} i_3 - S(2) \leq \cdots \leq (i+1)^n. \end{aligned}$$

Since there are $m-n$ face operators in each summand and $n = k$, the number of face operators that the formula has, is less or equal to

$$(m-n)(i+1)^k = 2i(i+1)^k.$$

$\square$

We now discuss several facts about the computation with the formulae of the Corollary 3.2. First of all, it is clear that at least in the case in which X has a finite number of non-degenerated simplices in each degree, our method can be seen as an actual algorithm for calculating Steenrod squares. For example, if the number of non-degenerated simplices in every X_ℓ is $O(\ell^2)$, then, assuming that each face operator of X is an elementary operation, the complexity of our algorithm for calculating $Sq^i(c_{i+2})$ is $O(i^5)$. This complexity is obtained by the followings facts. On one hand, the number of face operators taking part in the formula of $Sq^i(c_{i+2})$

is $O(i^3)$. On the other hand, $Sq^i(c_{i+2})$ is a $(2i+2)$-cochain and then, this cochain is determined by knowing its image over all the non-degenerated simplices in X_{2i+2}.

Nevertheless, the most interesting examples appearing in Algebraic Topology show, in general, a high complexity in the number of non-degenerated simplices in each degree. For example, let us take the classifying space of a finite 2-group. In this case, the number of non-degenerated simplices in degree ℓ is $O(2^\ell)$. Hence, our method will only be useful here in low dimensions. As we have mentioned in the introduction, perhaps appropriately combining these combinatorial formulae with classical properties of Steenrod squares and well-known studies for calculating cocycles could allow us to make a substantial improvement in our method.

Finally, this technique can be useful when dealing with chain complexes arising from simplicial modules, like, for instance, the Hochschild complex of an R-algebra A is a simplicial R-module (see, for example, [2]). For these differential graded modules, the use of our simplicial method may be fruitful.

4 A generalization to Steenrod reduced powers

Real in [15] established formulae for the morphisms $\{D_i\}$ in terms of the component morphisms of a given Eilenberg-Zilber contraction. In an analogous way, we show here that this result can be generalized to Steenrod reduced powers (see [20, 3]). It seems clear that this study must lead in the very near future to an explicit simplicial description of these cohomological operations.

Let us consider the Eilenberg-Zilber contraction:

$$(f, g, \phi) : C_*^N(X \times \overset{p \text{ times}}{\cdots} \times X) \to C_*^N(X) \otimes \overset{p \text{ times}}{\cdots} \otimes C_*^N(X),$$

the diagonal Δ and the automorphisms t and T we have defined in Section 2.

Now, the equality $tg = gT$ holds due to the associativity of EML and to the good behavior of this morphism with regard to the automorphisms t and T.

Let us take a family of automorphisms $\{\gamma_i\}_{i\geq 0}$ defined by

$$\gamma_{2j-1} = t \qquad \text{and} \qquad \gamma_{2j} = t + t^2 + \cdots + t^{p-1}$$

and let $\gamma = \gamma_i \gamma_{i-1}$.

Before proving the main result of this section, we need the following propositions.

Proposition 4.1 *Let p be prime with $p \geq 2$ and i a non-negative integer, then*

$$\phi d\gamma_i \cdots \phi\gamma_1\phi \quad = \quad (-1)^{i-1}\phi\gamma_i \cdots \phi\gamma_1 d\phi + \sum_{k=1}^{i-1}(-1)^{i-k}\phi\gamma_i \cdots \phi\gamma_{k+2}\phi\gamma\phi\gamma_{k-1} \cdots \phi\gamma_1\phi.$$

Proof

We prove the proposition by induction on the parameter i.

- If $i = 1$ then, $\phi d\gamma_1\phi = \phi\gamma_1 d\phi$.
- If $i = 2$ then,

$$\begin{aligned} \phi d\gamma_2\phi\gamma_1\phi &= \phi\gamma_2 d\phi\gamma_1\phi = \phi\gamma_2(gf - 1 - \phi d)\gamma_1\phi \\ &= -\phi\gamma\phi - \phi\gamma_2\phi\gamma_1 d\phi. \end{aligned}$$

- In general,

$$\begin{aligned} \phi d\gamma_i\phi\gamma_{i-1} \cdots \phi\gamma_1\phi &= \phi\gamma_i d\phi\gamma_{i-1} \cdots \phi\gamma_1\phi \\ &= \phi\gamma_i(gf - 1 - \phi d)\gamma_{i-1} \cdots \phi\gamma_1\phi \\ &= -\phi\gamma\phi\gamma_{i-2} \cdots \phi\gamma_1\phi - \phi\gamma_i\phi d\gamma_{i-1} \cdots \phi\gamma_1\phi \end{aligned}$$

(by induction assumption)

$$\begin{aligned} &= -\phi\gamma\phi\gamma_{i-2} \cdots \phi\gamma_1\phi - (-1)^{i-2}\phi\gamma_i\phi\gamma_{i-1} \cdots \phi\gamma_1 d\phi \\ &\quad - \sum_{k=1}^{i-2}(-1)^{i-1-k}\phi\gamma_i\phi\gamma_{i-1} \cdots \phi\gamma_{k+2}\phi\gamma\phi\gamma_{k-1} \cdots \phi\gamma_1\phi \\ &= (-1)^{i-1}\phi\gamma_i \cdots \phi\gamma_1 d\phi \\ &\quad + \sum_{k=1}^{i-1}(-1)^{i-k}\phi\gamma_i \cdots \phi\gamma_{k+2}\phi\gamma\phi\gamma_{k-1} \cdots \phi\gamma_1\phi. \end{aligned}$$

□

Let $\Gamma_i(k) = f\gamma_i\phi\gamma_{i-1} \cdots \phi\gamma_{k+2}\phi\gamma\phi\gamma_{k-1}\phi \cdots \phi\gamma_1\phi\Delta$. We can prove

Proposition 4.2 *Let p be prime and i a non-negative integer, then*

$$f\gamma_i\phi\gamma_{i-1} \cdots \phi\gamma_2\phi\Delta = f\gamma_{i-1}\phi\gamma_{i-2} \cdots \phi\gamma_2\phi\gamma_1\phi\Delta + \sum_{k=1}^{i-1}(-1)^{k+1}\Gamma_i(k).$$

Proof

Using $\gamma_{k+1} = \gamma_k + (-1)^{k+1}\gamma$, for all $1 < k < i$, we have

$$\begin{aligned}&f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_{k+2}\phi\gamma_{k+1}\phi\gamma_{k-1}\cdots\phi\gamma_1\phi\Delta\\ &= f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_{k+2}\phi\gamma_k\phi\gamma_{k-1}\cdots\phi\gamma_1\phi\Delta + (-1)^{k+1}\Gamma_i(k).\end{aligned}$$

If we use this fact successively, we obtain the following identity:

$$f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta = f\gamma_{i-1}\phi\gamma_{i-2}\cdots\phi\gamma_2\phi\gamma_1\phi\Delta + \sum_{k=1}^{i-1}(-1)^{k+1}\Gamma_i(k)$$

$\square$

The main result of this section is the following one.

Theorem 4.3 *Let p be a prime number, $p \geq 2$, and i a non-negative integer. Then, there exists a sequence of morphisms $\{D_i\}$ defined by*

$$D_i = f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta,$$

verifying that

$$dD_i + (-1)^{i+1}D_id = \alpha_iD_{i-1},$$

where $\alpha_{2j-1} = T - 1$ and $\alpha_{2j} = 1 + T + T^2 + \cdots + T^{p-1}$.

Proof

Let us begin with the first term of the identity:

$$\begin{aligned} dD_i + (-1)^{i+1}D_id &= df\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta + (-1)^{i+1}f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta d\\ &= f\gamma_id\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta\\ &\quad +(-1)^{i+1}f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_1(gf-1-d\phi)\Delta\\ &= f\gamma_id\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta\\ &\quad +(-1)^if\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta + (-1)^if\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_1d\phi\Delta\end{aligned}$$

(by Proposition 4.1, we get)

$$\begin{aligned} &= f\gamma_id\phi\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta + (-1)^if\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta\\ &\quad +(-1)^i(-1)^{i-2}f\gamma_i\phi d\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta\\ &\quad +\sum_{k=1}^{i-2}(-1)^i(-1)^kf\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_{k+2}\phi\gamma\phi\gamma_{k-1}\phi\cdots\phi\gamma_1\phi\Delta\end{aligned}$$

$$
\begin{aligned}
&= f\gamma_i(d\phi+\phi d)\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta \\
&\quad +(-1)^i f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta+\sum_{k=1}^{i-2}(-1)^{i-k}\Gamma_i(k) \\
&= f\gamma_i(gf-1)\gamma_{i-1}\cdots\phi\gamma_1\phi\Delta \\
&\quad +(-1)^i f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta+\sum_{k=1}^{i-2}(-1)^{i-k}\Gamma_i(k)
\end{aligned}
$$

(let $\beta_{2j}=T+T^2+\cdots+T^{p-1}$ and $\beta_{2j-1}=T$, then we have)

$$
\begin{aligned}
&= \beta_i f\gamma_{i-1}\phi\gamma_{i-2}\cdots\phi\gamma_1\phi\Delta \\
&\quad +(-1)^i f\gamma_i\phi\gamma_{i-1}\cdots\phi\gamma_2\phi\Delta+\sum_{k=1}^{i-1}(-1)^{i-k}\Gamma_i(k)
\end{aligned}
$$

(Proposition 4.2 implies)

$$
\begin{aligned}
&= \beta_i f\gamma_{i-1}\phi\gamma_{i-2}\cdots\phi\gamma_1\phi\Delta+(-1)^i f\gamma_{i-1}\phi\gamma_{i-2}\cdots\phi\gamma_1\phi\Delta \\
&\quad +\sum_{k=1}^{i-1}(-1)^{i+k+1}\Gamma_i(k)+\sum_{k=1}^{i-1}(-1)^{i-k}\Gamma_i(k) \\
&= \alpha_i f\gamma_{i-1}\phi\gamma_{i-2}\cdots\phi\gamma_1\phi\Delta.
\end{aligned}
$$

□

5 Proof of the main theorem

The proof consists in finding out the factors of the formula (written in the canonical way) that are degenerated and in eliminating the summands having these factors.

First of all, notice that using the commutativity properties of the operators of a simplicial set (essentially, **(s3)**), it is easy to see that a factor of the formula whose expression begins (on the left) by

$$\partial_{j_1}\cdots\partial_{j_t}s_k\cdots \tag{6}$$

such that $0\le j_1<\cdots<j_t<k$, is degenerated in its simplified form.

Having said that, let us begin with the proof of the theorem.

For $n=0$, we obtain the explicit formula for the Alexander-Whitney operator.

Let us assume that the formula is true for $k\le n$ so, let us prove that the formula is true for the case $n+1$.

Let us consider that n is even (when n is odd, the proof is similar). In this case, by induction assumption, the formula over an element of degree m is as follows:

$$
\begin{aligned}
AW\ (t\ SHI)^{n+1} &= AW\ (t\ SHI)^n\ (t\ SHI) \\
= \sum_{i_n=n}^{m+1} \cdots \sum_{i_0=0}^{i_1-1} \quad \sum \quad & (-1)^{A(n)+B(n,m+1,\bar{\imath})+C(n,m+1,\bar{\imath})+D(n,\bar{\imath})}(-1)^{\bar{m}+sig(\alpha,\beta)+1} \\
& \partial_{i_0+1}\cdots\cdot\partial_{m+1}s_{\alpha_{p+1}+\bar{m}}\cdots s_{\alpha_1+\bar{m}}\partial_{\bar{m}}\cdots\partial_{m-q-1} \\
& \otimes\partial_0\cdots\cdot\partial_{i_n-1}s_{\beta_q+\bar{m}}\cdots s_{\beta_1+\bar{m}}s_{\bar{m}-1}\partial_{m-q+1}\cdots\partial_m
\end{aligned}
$$

where $\bar{\imath} = (i_0, i_1, \ldots, i_n)$, $\bar{m} = m - p - q$, $sig(\alpha, \beta) = \sum_{i=1}^{p+1} \alpha_i - (i-1)$, and the last sum is taken over all the indices $0 \leq q \leq m-1$, $0 \leq p \leq m-q-1$ and $(\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}$.

Let us recall that if the formula is in the normalized form, the summands which have a degenerated factor must be eliminated. The following cases are considered:

If $i_n > m - p$, we have to consider the following cases:

- If $i_n = m - p + t$ and $\beta_q < q - 1 + t$, with $1 \leq t \leq p$ then,

  $$\alpha_{p+1} = p + q > \cdots > \alpha_{t+1} = q + t > \alpha_t = q + t - 1 > \beta_q.$$

  So, the first factor of these summands is:

  $$\partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{m-p+t+1}\cdots\partial_{m+1}s_m\cdots s_{m-p+t-1}s_{\alpha_{t-1}}\cdots s_{\alpha_1}\partial_{\bar{m}}\cdots\partial_{m-q-1}$$
  $$= \partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}s_{m-p+t-1}s_{\alpha_{t-1}}\cdots s_{\alpha_1}\partial_{\bar{m}}\cdots\partial_{m-q-1}.$$

  Since $i_{n-1} - 1 < i_n - 1 = m - p + t - 1$, this factor is degenerated by (6).

- If $i_n = m - p + t$ and $\beta_q = q - 1 + t$, with $1 \leq t \leq p$ then,

  $$\alpha_{p+1} = p + q > \cdots > \alpha_{t+1} = q + t > \beta_q = q + t - 1.$$

  And hence, in this case, the expression of the first factor begins in the form:

  $$\partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{m-p+t+1}\cdots\partial_{m+1}s_m\cdots s_{m-p+t}s_{\alpha_t+\bar{m}}\cdots$$
  $$= \partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}s_{\alpha_t+\bar{m}}\cdots.$$

Now, we have to consider two different cases:

- If $i_{n-1} - 1 < \alpha_t + \bar{m}$ then, this factor is degenerated.
- If $i_{n-1} - 1 \geq \alpha_t + \bar{m}$, let us denote $\alpha_t = q+t-1-j$, where $1 \leq j \leq q+t-1$. Then,
$$\alpha_{p+1} = p + q > \cdots > \alpha_{t+1} = q + t,$$
and
$$\beta_q = q + t - 1 > \cdots > \beta_{q-j+1} = q + t - j > \alpha_t = q + t - 1 - j.$$
Hence, the second factor of these summands is:
$$\partial_0 \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_{m-p+t-1} s_{m-p+t-1} \cdots s_{m-p+t-j} s_{\beta_{q-j}+\bar{m}} \cdots$$
$$= \partial_0 \cdots \cdot \partial_{i_{n-2}-1} s_{i_{n-1}} \cdots s_{m-p+t-j} s_{\beta_{q-j}+\bar{m}} \cdots$$
Since $i_{n-2} - 1 < i_{n-1}$, this factor is degenerated.

- If $i_n = m-p+t$ and $\beta_q > q-1+t$, with $1 \leq t \leq p$ then $i_n - 1 < \beta_q + \bar{m}$. So, the second factor of the summands has the form (6) and hence, these summands must be eliminated.

- If $i_n = m+1$ and $\beta_q < p+q$ then $\alpha_{p+1} = p+q$ and since $i_{n-1} - 1 < i_n - 1 = m$ then, the first factor of the summands is degenerated.

- If $i_n = m + 1$ and $\beta_q = p + q$ then
$$\beta_q = p + q > \cdots > \beta_{j+1} = p + j + 1 > \alpha_{p+1} = p + j$$
with $0 \leq j \leq q - 1$ and the first factor of the summands is:
$$\partial_{i_0+1} \cdots \cdot \partial_{i_{n-1}-1} s_{m-q+j} s_{\alpha_p+\bar{m}} \cdots s_{\alpha_1+\bar{m}} \partial_{\bar{m}} \cdots \partial_{m-q-1}.$$
We have to consider two different cases:
  - If $i_{n-1} - 1 < m - q + j$ then, this factor is degenerated.
  - If $i_{n-1} - 1 \geq m - q + j$ then the second factor of the summands is:
$$\partial_0 \cdots \cdot \partial_{i_{n-2}-1} \partial_{i_{n-1}+1} \cdots \partial_m s_m \cdots s_{m-q+j+1} s_{\beta_j+\bar{m}} \cdots$$
$$= \partial_0 \cdots \cdot \partial_{i_{n-2}-1} s_{i_{n-1}} \cdots s_{m-q+j+1} s_{\beta_j+\bar{m}} \cdots,$$
which is degenerated due to the fact that $i_{n-2} - 1 < i_{n-1}$.

If $i_n < m-p$, then $i_n - 1 < \beta_q + \bar{m}$. So, these summands have the second factor in the form (6) and hence, must be eliminated.

If $i_n = m-p$, two cases hold:

- If $\beta_q > q-1$, then the second factor of these summands is degenerated as above.
- If $\beta_q = q-1$ and $i_{n-1} > \bar{m}-2$ then

  $$\alpha_{p+1} = p+q > \cdots > \alpha_1 = q > \beta_q = q-1 > \cdots > \beta_1 = 0,$$

  and the second factor of the tensor product is

  $$\begin{aligned} &\partial_0 \cdots \cdot \partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{m-p-1}s_{m-p-1}\cdots s_{m-p-q-1}\partial_{m-q+1}\cdots\partial_m \\ &= \partial_0 \cdots \cdot \partial_{i_{n-2}-1}s_{i_{n-1}}\cdots s_{m-p-q-1}\partial_{m-q+1}\cdots\partial_m; \end{aligned}$$

  and since $i_{n-2}-1 < i_{n-1}$, this factor is degenerated.

  Finally, if $\beta_q = q-1$ and $i_{n-1} \leq \bar{m}-2$ then the formula (save for the signs) corresponding to $AW\ (t\ SHI)^{n+1}$ is:

  $$\sum_{i_{n-1}=n-1}^{\bar{m}-2} \cdots \sum_{i_0=0}^{i_1-1} \sum_{q=0}^{m-1} \sum_{p=0}^{m-q-1}$$

  $$\begin{aligned} &\partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{m-p+1}\cdots\partial_{m+1}s_m\cdots s_{m-p}\partial_{\bar{m}}\cdots\partial_{m-q-1} \\ &\otimes\partial_0\cdots\cdot\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{m-p-1}s_{m-p-1}\cdots s_{\bar{m}-1}\partial_{m-q+1}\cdots\partial_m \end{aligned}$$

  $$= \sum_{i'_{n+1}=n+1}^{m} \sum_{i'_n=n}^{i'_{n+1}-1} \sum_{i_{n-1}=n-1}^{i'_n-1} \cdots \sum_{i_0=0}^{i_1-1}$$

  $$\begin{aligned} &\partial_{i_0+1}\cdots\cdot\partial_{i_{n-1}-1}\partial_{i'_n+1}\cdots\partial_{i'_{n+1}-1} \\ &\otimes\partial_0\cdots\cdot\partial_{i_{n-2}-1}\partial_{i_{n-1}+1}\cdots\partial_{i'_n-1}\partial_{i'_{n+1}+1}\cdots\partial_m; \end{aligned}$$

  where $i'_n = \bar{m}-1$ and $i'_{n+1} = m-q$.

Now, let us study the signs of the formulae in this last case. Keeping in mind that we are working with the exponent of (-1), all the identities are mod 2.

This proof is based on the fact that if and only if $i_n = m-p$ and $\beta_q = q-1$, the summands of $AW\ (t\ SHI)^n$ are non-degenerated.

We first verify the formula in the case $n = 1$. The exponent of (-1) associated to each summand is:

$$\begin{aligned}\bar{m} + sig(\alpha, \beta) + 1 &= \bar{m} + \sum_{i=1}^{p+1} \alpha_i - (i-1) + 1 \\ &= \bar{m} + \sum_{i=1}^{p+1} (q - 1 + i - (i-1)) + 1 = \bar{m} + q(p+1) + 1\end{aligned}$$

(note $i_0 = \bar{m} - 1$ and $i_1 = m - q$)

$$= i_0 + (m + i_1)(i_0 + i_1) = A(1) + B(1, m, \bar{\imath}) + C(1, \bar{\imath}) + D(1, m, \bar{\imath})$$

where $\bar{\imath} = (i_0, i_1)$, as asserted.

In general, we have to prove that

$$A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath}) + D(n, m+1, \bar{\imath}) + \bar{m} + sig(\alpha, \beta) + 1$$

$$= A(n+1) + B(n+1, m, \bar{\imath}) + C(n+1, \bar{\imath}) + D(n+1, m, \bar{\imath}).$$

- If n is even then $D(n, m+1, \bar{\imath}) = 0$ and the exponent of (-1) in each summand is:

$$\begin{aligned}&A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath}) + \bar{m} + sig(\alpha, \beta) + 1 \\ &= A(n) + B(n, m+1, \bar{\imath}) + C(n-1, \bar{\imath}) \\ &\quad + (m + p + i_{n-1})(i_{n-1} + \cdots + i_0) + \bar{m} + q(p+1) + 1\end{aligned}$$

(since $i'_n = \bar{m} - 1$, $i'_{n+1} = m - q$ and these identities are mod 2 then)

$$\begin{aligned}&= A(n) + B(n, m+1, \bar{\imath}) + C(n-1, \bar{\imath}) \\ &\quad + (m + 1 + i'_{n+1} + i'_n + i_{n-1})(i_{n-1} + \cdots + i_0) \\ &\quad + i'_n + (m + i'_{n+1})(i'_{n+1} + i'_n) \\ &= A(n) + B(n, m+1, \bar{\imath}) + C(n-1, \bar{\imath}) + i_{n-1} + \cdots + i_0 \\ &\quad + (i'_n + i_{n-1})(i_{n-1} + \cdots + i_0) + (m + i'_{n+1})(i_{n-1} + \cdots + i_0) \\ &\quad + i'_n + (m + i'_{n+1})(i'_{n+1} + i'_n) \\ &= A(n) + B(n, m+1, \bar{\imath}) + i'_n + i_{n-1} + \cdots + i_0 \\ &\quad + C(n+1, \bar{\imath}) + D(n+1, m, \bar{\imath}).\end{aligned}$$

We have to distinguish two cases:

- If $n \equiv 0 \bmod 4$ then $A(n) = A(n+1)$ and

$$A(n) + B(n, m+1, \bar{\imath}) + i'_n + i_{n-1} + \cdots + i_0$$

$$= A(n+1) + \sum_{j=0}^{\frac{n-2}{2}} i_{2j+1} + i'_n + i_{n-1} + \cdots + i_0$$

$$= A(n+1) + \sum_{j=0}^{\frac{n-2}{2}} i_{2j} + i'_n = A(n+1) + B(n+1, m, \bar{\imath}).$$

- If $n \equiv 2 \bmod 4$ then $A(n) = A(n+1) + 1$ and

$$A(n) + B(n, m+1, \bar{\imath}) + i'_n + i_{n-1} + \cdots + i_0$$

$$= A(n+1) + 1 + \sum_{j=0}^{\frac{n-2}{2}} i_{2j} + m + p + i'_n + i_{n-1} + \cdots + i_0$$

$$= A(n+1) + 1 + \sum_{j=0}^{\frac{n-2}{2}} i_{2j} + m + i'_{n+1} + i'_n + 1$$

$$+ i'_n + i_{n-1} + \cdots + i_0$$

$$= A(n+1) + \sum_{j=0}^{\frac{n-2}{2}} i_{2j+1} + i'_{n+1} + m = A(n+1) + B(n+1, m, \bar{\imath}).$$

- If n is odd then the exponent of (-1) is:

$$A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath}) + D(n, m+1, \bar{\imath}) + \bar{m} + sig(\alpha, \beta) + 1$$
$$= A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath})$$
$$+ (m+1+m+p)(m+p+i_{n-1}+\cdots+i_0) + \bar{m} + q(p+1) + 1$$

(since $i'_n = \bar{m} - 1$, $i'_{n+1} = m - q$ and these identities are mod 2 then)

$$= A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath})$$
$$+ (i'_{n+1} + i'_n)(m + 1 + i'_{n+1} + i'_n + i_{n-1} + \cdots + i_0)$$
$$+ i'_n + (m + i'_{n+1})(i'_{n+1} + i'_n)$$
$$= A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath}) + (i'_{n+1} + i'_n)(i'_n + i_{n-1} + \cdots + i_0)$$
$$+ i'_{n+1} + i'_n + (i'_{n+1} + i'_n)(m + i'_{n+1}) + i'_n + (m + i'_{n+1})(i'_{n+1} + i'_n)$$
$$= A(n) + B(n, m+1, \bar{\imath}) + C(n, \bar{\imath})$$
$$+ (i'_{n+1} + i'_n)(i'_n + i_{n-1} + \cdots + i_0) + i'_{n+1}$$
$$= A(n) + B(n, m+1, \bar{\imath}) + i'_{n+1} + C(n+1, \bar{\imath}) + D(n+1, m, \bar{\imath}).$$

Since n is odd then $n \equiv 1, 3, 5, 7 \bmod 8$ and, in these cases, $A(n) = A(n+1)$. Now, we have to distinguish two cases:

– If $n \equiv 1 \bmod 4$ then

$$B(n, m+1, \bar{\imath}) + i'_{n+1} = \sum_{j=0}^{\frac{n-1}{2}} i_{2j} + i'_{n+1} = \sum_{j=0}^{\frac{n+1}{2}} i_{2j} = B(n+1, m, \bar{\imath}).$$

– If $n \equiv 3 \bmod 4$ then

$$\begin{aligned} B(n, m+1, \bar{\imath}) + i'_{n+1} &= \sum_{j=0}^{\frac{n-3}{2}} i_{2j+1} + i_n + m + 1 + i'_{n+1} \\ &= \sum_{j=0}^{\frac{n-3}{2}} i_{2j+1} + i'_n + m + 1 + m + 1 \\ &= \sum_{j=0}^{\frac{n-1}{2}} i_{2j+1} = B(n+1, m, \bar{\imath}). \end{aligned}$$

□

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