“Coalgebra” Structures on 1 –Homological Models for Commutative Differential Graded Algebras

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Abstract. In [3] “small” 1–homological model H of a commutative differential graded algebra is described. Homological Perturbation Theory (HPT) [7–9] provides an explicit description of an A_∞–coalgebra structure $(\Delta_1, \Delta_2, \Delta_3, \ldots)$ of H. In this paper, we are mainly interested in the determination of the map $\Delta_2 : H \to H \otimes H$ as a first step in the study of this structure. Developing the techniques given in [20] (inversion theory), we get an important improvement in the computation of Δ_2 with regard to the first formula given by HPT. In the case of purely quadratic algebras, we sketch a procedure for giving the complete Hopf algebra structure of its 1–homology.

1 Introduction

In recent years, the relevance of Homological Algebra in the field of Theoretical Physics becomes more and more apparent. New emergent areas, such as Cohomological Physics [23] and Secondary Calculus [25], make use of notions from Homological Algebra to clearly describe a series of interesting physical problems. In particular, the role of A_∞-structures [22, 19] in mathematical physics has enormously increased at the beginning of the nineties [6, 17, 24]. Example of this was M. Kontsevich’s talk [15] at the International Congress of Mathematicians in 1994, in which he gave a conjectural interpretation of mirror symmetry as the “shadow” of an equivalence between two triangulated categories associated with A_∞-categories. His conjecture was proved in the case of elliptic curves by A. Polishchuk and E. Zaslow [18].

An A_∞–algebra $(A, m_1, m_2, m_3, \ldots)$ (see, for example, [14]) is a graded module $A = \oplus_{i=0}^{n} A_i$ endowed with graded maps $m_i : A^{\otimes n} \to A$, $n \geq 1$ of degree $n-2$ satisfying for $n \geq 1$:

$$\sum \pm m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$. Therefore, an A_∞-algebra is a differential (m_1) graded algebra with

* Authors are partially supported by the DGES–SEUID research project PB98–1621–C02–02 from Education and Science Ministry (Spain).

multiplication m_2, strongly homotopy associative (this lack of associativity is measured by a series of morphisms $(m_3, m_4, \ldots)$). In an analogous way, it is possible to define an A_∞-coalgebra $(C, \Delta_1, \Delta_2, \Delta_3, \ldots)$.

We are interested in the computation of the A_∞–algebra and A_∞–coalgebra structures of the 1–homology of a commutative differential graded algebra (briefly called CDGA) working with coefficients in a commutative ring Λ with $1 \neq 0$ (usually, the ground ring will be $\mathbf{Z}$). Let us recall that the 1-homology of a CDGA A is the homology of the reduced bar construction $\bar{B}(A)$ of A. The complex $\bar{B}(A)$ is a Hopf algebra, that is, it has both algebra and coalgebra structures such that they are compatible in some sense. Consequently, its homology carries A_∞–algebra and A_∞–coalgebra structures (unique, up to isomorphism [13]), both transferred from respective structures on $\bar{B}(A)$.

The problem of computing the structure of the A_∞-algebra of a "small" 1-homological model HBA (that means that there exists a homotopy equivalence between $\bar{B}(A)$ and HBA, such that HBA has less algebra generators than $\bar{B}(A)$), using homological perturbation tools [10, 20] is attacked in [3]. There, it was realized that this A_∞-structure reduces to a simple algebra structure $(HBA, m_1, m_2, 0, 0, \ldots)$, where m_1 represents the differential of the complex and m_2 is the associative product. The determination of m_2 is immediate and the attention is focused on the consecution of an "economical" formulation of m_1.

Here, we are interested in the dual, but extremely complicated, problem of calculating the A_∞-coalgebra structure of HBA. In this structure, $(HBA, \Delta_1, \Delta_2, \Delta_3, \ldots)$, Δ_1 coincides with the differential m_1, and the first step in solving this question consists of getting an "efficient" description of $\Delta_2 : HBA \to HBA \otimes HBA$. More concretely, in the present paper we obtain an important improvement in the computation of Δ_2 with regard to the initial formula provided by Homological Perturbation Theory [10] using the techniques which are comprised under the name of inversion theory [20, 3]. In some particular cases, such as purely quadratic algebras, the model HBA represents the actual 1-homology of A (that is, $m_1 = \Delta_1 = 0$). In that context, we sketch a "reasonable" algorithm for giving the complete Hopf algebra structure of HBA.

Finally, let us emphasize that our approach could be useful in providing new insights on the difficult problem of defining the A_∞-Hopf algebra structure [21].

2 Preliminaries

The algebraic setting and notation we need in this paper is conveniently described in [3]. In order to put into context the problem we deal with, most relevant notions of our framework are reviewed.

Let Λ be a commutative ring with non zero unit which is considered as ground ring. $(A, d_A, *_A, \xi_A, \eta_A)$ denotes a commutative differential graded algebra, endowed with an augmentation ξ_A and a cougmentation η_A. We will respect Koszul conventions.

Examples of CDGAs are the monogenic algebras: exterior algebra $E(x, 2n+1)$, polynomial algebra $P(y, 2n)$ and divided power algebra $\Gamma(y, 2n)$; all of them with trivial differential.

The *reduced bar construction* [16] associated to a CDGA A is defined as the differential graded bimodule $\bar{B}(A)$:

$$\bar{B}(A) = \Lambda \oplus \mathrm{Ker}\ \xi_A \oplus (\mathrm{Ker}\ \xi_A \otimes \mathrm{Ker}\ \xi_A) \oplus \cdots \oplus (\mathrm{Ker}\ \xi_A \otimes \cdots \otimes \mathrm{Ker}\ \xi_A) \oplus \cdots .$$

An element from $\bar{B}(A)$ is denoted by $\bar{a} = [a_1|\cdots|a_n]$. There is a *tensor graduation* given by $|\bar{a}|_t = \sum_{i=1}^{n} |a_i|$; as well as a *simplicial graduation* ($|\bar{a}|_s = |[a_1|\cdots|a_n]\,|_s = n$). The total differential is given by the sum of the *tensor* one, which depends on the differential of A, and the *simplicial* differential, which acts by using the product of A.

When the algebra A is commutative, it is possible to define a multiplicative structure upon $\bar{B}(A)$ (via an operator called *shuffle product*), so that the reduced bar construction also becomes a CDGA.

Given two non–negative integers p and q , a (p, q)–*shuffle* is defined as a permutation π of the set $\{0, \ldots, p+q-1\}$, such that $\pi(i) < \pi(j)$ when $0 \leq i < j \leq p-1$ or $p \leq i < j \leq p+q-1$.

Let us observe that there are $\binom{p+q}{p}$ different (p, q)–shuffles.

So, given a CDGA A, the *shuffle product* $\star : \bar{B}(A) \otimes \bar{B}(A) \longrightarrow \bar{B}(A)$, is defined, up to sign, by:

$$[a_1|\cdots|a_p] \star [b_1|\cdots|b_q] = \sum_{\pi \in \{(p,q)\text{–shuffles}\}} \pm [c_{\pi(0)}|\cdots|c_{\pi(p+q-1)}]\,;$$

where $(c_0, \ldots, c_{p-1}, c_p, \ldots, c_{p+q-1}) = (a_1, \ldots, a_p, b_1, \ldots b_q)$.

On the other hand, a coproduct can be define on $\bar{B}(A)$ which provides it a coalgebra structure,

$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n] \tag{1}$$

Both structures of algebra and coalgebra are compatible in the sense that $\bar{B}(A)$ is a Hopf algebra, that is, $\Delta\star = (\star\otimes\star)\,(1\otimes T\otimes 1)\,(\Delta\otimes\Delta)$, where $T : \bar{B}(A) \otimes \bar{B}(A) \to \bar{B}(A) \otimes \bar{B}(A)$ is the morphism that interchanges the factors.

A *contraction* $C : \{N, M, f, g, \phi\}$ [4,11], also denoted by $(f, g, \phi) : N \overset{C}{\Rightarrow} M$, from a differential graded module (N, d_N) to a differential graded module (M, d_M) consists in a homotopy equivalence determined by three morphisms f, g and ϕ; $f : N_* \to M_*$ (projection) and $g : M_* \to N_*$ (inclusion) being two differential graded module morphisms and $\phi : N_* \to N_{*+1}$ a homotopy operator. Moreover, these data are required to satisfy the following rules:

$$(c1)\ fg = 1_M,\ \ (c2)\ \phi d_N + d_N\phi + gf = 1_N,\ \ (c3)\ f\phi = 0,\ \ (c4)\ \phi g = 0,\ \ (c5)\ \phi\phi = 0\,.$$

Therefore, the homology groups of M and N coincide.

The Basic Perturbation Lemma (BPL) [7] states that given a contraction $C : \{N, M, f, g, \phi\}$ of chain complexes and a perturbation δ of d_N (that is, $(d_N + \delta)^2 = 0$), then there exists a new contraction $C_\delta = (f_\delta, g_\delta, \phi_\delta)$ from $(N, d_N + \delta)$ to $(M, d_M + d_\delta)$, verifying that

$$f_\delta = f(1 - \delta \Sigma_c^\delta \phi)\,, \quad g_\delta = \Sigma_c^\delta g\,, \quad \phi_\delta = \Sigma_c^\delta \phi\,, \quad d_\delta = f \delta \Sigma_c^\delta g\,, \qquad (2)$$

where $\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i \, (\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i (\phi\delta)^i + \cdots$.

It is commonly known that every CDGA A "factors", up to homotopy equivalence, into a *twisted tensor product* (or TTP) of exterior and polynomial algebras $\tilde{\otimes}^\rho_{i \in I} A_i$, that is, a tensor product of these algebras whose differential structure is enriched with a differential–derivation, ρ (see, for example, [16]). We will always assume that any CDGA A considered in this paper, is factored as a twisted tensor product $\tilde{\otimes}^\rho_{i \in I} A_i$ (being I a finite set of indexes, $I = \{1, 2, \ldots, n\}$) of exterior and polynomial algebras, A_i, of generators x_i, such that $|x_i| \leq |x_{i+1}|$. Notice that, consequently, an order is fixed on the factors.

Given two CDGAs A and A', a *semi–full algebra contraction* $(f, g, \phi) : A \Rightarrow A'$ [20, 2] consists of

- an inclusion, g, which is a morphism of DGAs (i.e., a multiplicative morphism);
- a *quasi–algebra projection* f, that is, $f *_A (\phi \otimes \phi) = 0$, $f *_A (\phi \otimes g) = 0$, $f *_A (g \otimes \phi) = 0$.
- and a *quasi–algebra homotopy* ϕ, that is, $\phi *_A (\phi \otimes \phi) = 0$, $\phi *_A (\phi \otimes g) = 0$, $\phi *_A (g \otimes \phi) = 0$.

The class of all semi–full algebra contractions is closed under composition, tensor product of contractions and perturbation.

Theorem 1. *[20]*
Let $C : \{N, M, f, g, \phi\}$ be a semi–full algebra contraction and $\delta : N \to N$ be a perturbation–derivation of d_N. Then, the perturbed contraction C_δ, is a new semi–full algebra contraction.

To obtain a 1–*homological model* for a CDGA A consists in establishing **a "chain" of *semi–full algebra contractions*** starting at the reduced bar construction $\bar{B}(A)$ and ending up at a CDGA HBA that is free and of finite type as graded module. An algorithm for computing 1–homological models for CDGAs was given in [1]. Now we recall the main steps on this algorithm which are essential in our work.

Three *almost–full* algebra contractions (that is, semi–full algebra contractions endowed with multiplicative projections) are used for this purpose:

- The contraction defined in [5] from $\bar{B}(A \otimes A')$ to $\bar{B}(A) \otimes \bar{B}(A')$, where A and A' are two CDGAs;

$$C_{\bar{B}\otimes} : \{\bar{B}(A \otimes A'), \bar{B}(A) \otimes \bar{B}(A'), f_{\bar{B}\otimes}, g_{\bar{B}\otimes}, \phi_{\bar{B}\otimes}\}\,;$$

- $f_{\bar{B}\otimes}([a_1 \otimes a'_1|\cdots|a_n \otimes a'_n])$

$$= \sum_{i=0}^{n} \xi_A(a_{i+1} *_A \cdots a_n)\xi_{A'}(a'_1 *_{A'} \cdots a'_i)[a_1|\cdots|a_i] \otimes [a'_{i+1}|\cdots|a'_n] \,. \quad (3)$$

- $g_{\bar{B}\otimes}([a_1|\cdots|a_n] \otimes [a'_1|\cdots|a'_m]) = [a_1|\cdots|a_n] \star [a'_1|\cdots|a'_m]$

  with $[a_1|\cdots|a_n] \in \bar{B}(A)\,, \quad [a'_1|\cdots|a'_m] \in \bar{B}(A')\,;$

  $[a_1|\cdots|a_n] = [a_1 \otimes \theta'|\cdots|a_n \otimes \theta']\,, \quad [a'_1|\cdots|a'_n] = [\theta \otimes a'_1|\cdots|\theta \otimes a'_n]\,,$

  where θ and θ' are units on A and A', respectively.
- up to sign, $\phi_{\bar{B}\otimes}([a_1 \otimes a'_1|\cdots|a_n \otimes a'_n])$

$$= \sum_{0\le p\le n-q-1\le n-1} \begin{array}{l} \pm\xi_A(a_{n-q+1} *_A \cdots a_n)[a_1 \otimes a'_1|\cdots|a_{\bar{n}-1} \otimes a'_{\bar{n}-1} \\ |(a'_{\bar{n}} *_{A'} \cdots *_{A'} a'_{n-q})|c_0|\cdots|c_{p+q}]\,, \end{array} \quad (4)$$

  where $\bar{n} = n-p-q$; $(c_{\pi(0)},\ldots,c_{\pi(p+q)}) = (a_{\bar{n}},\ldots,a_{n-q},a'_{n-q+1},\ldots a'_n)$; π is a $(p+1,q)$–shuffle and $0 \le p \le n-q-1 \le n-1$.

Given a tensor product $\otimes_{i\in I}A_i$ of CDGAs, a contraction from $\bar{B}(\otimes_{i\in I}A_i)$ to $\otimes_{i\in I}\bar{B}(A_i)$ is easily determined, by applying $C_{\bar{B}\otimes}$ several times in a suitable way. This new contraction is also denoted by $C_{\bar{B}\otimes}$.

– The isomorphism of DGAs described in [5]

$$C_{\bar{B}E} : \{\bar{B}(E(u,2n+1)), \Gamma(\underline{u},2n+2), f_{\bar{B}E}, g_{\bar{B}E}, 0\}\,,$$

where

$$f_{\bar{B}E}([u|\overset{m \text{ times}}{\cdots}|u]) = \underline{u}^{(m)};\;\; g_{\bar{B}E}(\underline{u}^{(m)}) = [u|\overset{m \text{ times}}{\cdots}|u]\,.$$

– The contraction also stated in [5]

$$C_{\bar{B}P} : \{\bar{B}(P(v,2n)), E(\underline{v},2n+1), f_{\bar{B}P}, g_{\bar{B}P}, \phi_{\bar{B}P}\}\,,$$

where

$$f_{\bar{B}P}([v^r]) = \begin{cases} 0 \text{ if } r \neq 1 \\ \underline{v} \text{ if } r = 1 \end{cases};\;\; f_{\bar{B}P}([v^{r_1}|\cdots|v^{r_m}]) = 0$$

$$g_{\bar{B}P}(\underline{v}) = [v];\;\; \phi_{\bar{B}P}([v^{r_1}|\cdots|v^{r_m}]) = [v|v^{r_1-1}|\cdots|v^{r_m}]\,.$$

If $\tilde{\otimes}^{\rho}_{i\in I}A_i$ is a twisted tensor product of exterior and polynomial algebras, the perturbation ρ produces a perturbation–derivation δ on the tensor differential of $\bar{B}(\otimes_{i\in I}A_i)$. Thanks to the contractions above, it is possible to establish, by composition and tensor product of contractions in a recurring way, a new semi–full algebra contraction $C^n : (f^n, g^n, \phi^n)$ from $\bar{B}(\otimes_{i=1}^n A_i)$ to $\otimes_{i=1}^n HBA_i$,

$$\bar{B}((\otimes_{i=1}^{n-1}A_i)\otimes A_n) \Rightarrow (\otimes_{i=1}^{n-1}\bar{B}(A_i))\otimes\bar{B}(A_n) \Rightarrow (\otimes_{i=1}^{n-1}HBA_i)\otimes HBA_n\,, \quad (5)$$

where

$$f^n = (f^{n-1} \otimes f_{BA_n}) f_{B\otimes}$$
$$g^n = g_{B\otimes}(g^{n-1} \otimes g_{BA_n})$$
$$\phi^n = \phi_{B\otimes} + g_{B\otimes}(\phi^{n-1} \otimes g_{BA_n} f_{BA_n} + 1 \otimes \phi_{BA_n}) f_{B\otimes} ,$$

HBA_i are exterior or divided power algebras and $(f^{n-1}, g^{n-1}, \phi^{n-1}) : \bar{B}(\otimes_{i=1}^{n-1} A_i) \stackrel{C_{n-1}}{\Longrightarrow} \otimes_{i=1}^{n-1} HBA_i$. A 1–homological model, HBA, of $A = \tilde{\otimes}_{i\in I}^{\rho} A_i$ is then obtained by perturbing this contraction.

$$C_\delta^n : (f_\delta^n, g_\delta^n, \phi_\delta^n) \; : \; \bar{B}(\tilde{\otimes}_{i\in I}^{\rho} A_i) \Rightarrow (\otimes_{i=1}^{n} HBA_i, d_\delta) , \qquad (6)$$

The differential d_δ as well as the morphims which compose the new contraction, are determined by the Basic Perturbation Lemma.

3 Inversion Theory

This section is devoted to the theory that initially appeared in [20] and which was later used in [3] for the simplification in the computation of d_δ on the 1–homological model (6). We further develop inversion theory, though the proof of almost all the results we state here, are only briefly sketched. Complete proofs will be widely showed in [12]. These techniques will allows to prove that the projection f_δ in (6) is multiplicative, what have important repercussion on the computation of the A_∞–structure of the 1–homological model.

We consider a commutative differential graded algebra under the conditions already described, $\tilde{\otimes}_{i\in I}^{\rho} A_i$, with A_i exterior or polynomial algebras of generators x_i. In order to shorten notation, we will consider $A = \otimes_{i=1}^{n-1} A_i$ and $A' = A_n$. The following definition complement the one given in [3] for inversions.

Definition 1. Let $A \otimes A'$ be a CDGA under conditions described above and let us consider a homogeneous element $[a_1 \otimes a'_1 | a_2 \otimes a'_2 | \cdots | a_n \otimes a'_n]$ from $\bar{B}(A \otimes A')$. We say that a component $a_i \otimes a'_i$ from that element is responsible for an *inversion*, if some of these cases takes place:

- $a_i = \theta$ and there exists an index $j > i$ with $a_j \neq \theta$. Then a'_i is responsible for a $\otimes$*–inversion*.
- whenever A is a polynomial algebra, $a_i \neq \theta$ and there exists an index $j > i$ such that $a_j \neq \theta$. Then a_i is responsible for a $p1$*–inversion*.
- whenever A' is a polynomial algebra, $a'_i \neq \theta'$ and $a_{i-1} = \theta$, $a_i = \theta, \ldots a_n = \theta$. Then a'_i is responsible for a p*–inversion*.

Notice that $\bar{a} \in \bar{B}(A \otimes A')$ has inversions caused by the "highest" algebra, $A' = A_n$, that is, p–inversions in the case that A_n is a polynomial algebra; inversions caused by the tensor product, that is, $\otimes$–inversions; and inversions caused by the "first" algebra $A = \otimes_{i=1}^{n-1} A_i$, that is, $p1$–inversions, if $n = 2$ and A is a polynomial algebra, or , in the case that $n > 2$, inversions of the first factor of $\lambda(\bar{a})$ in $\bar{B}(\otimes_{i=1}^{n-1} A_i)$, with (up to sign)

$$\lambda : \bar{B}(A \otimes A') \to \bar{B}(A) \times \bar{B}(A') : \lambda[a_1 \otimes a'_1 | \cdots | a_n \otimes a'_n] = [a_1 | \cdots | a_n] \times [a'_1 | \cdots | a'_n] .$$

So, an element has k inversions if there exist k components responsible for an inversion. We say an element from $\bar{B}(A\otimes A')$ has k inversions, if it is the sum of homogeneous elements with, at least, k inversions each one of them.

Let us consider the contraction $C_{\bar{B}\otimes}$,

$$(f_{\bar{B}\otimes}, g_{\bar{B}\otimes}, \phi_{\bar{B}\otimes}) : \bar{B}(A \otimes A') \Rightarrow \bar{B}(A) \otimes \bar{B}(A')$$

We analyze the behavior of the component morphisms with respect to the different types of inversions. For this purpose, we won't take into account signs in the referred formulas.

- **Lemma 1.** *$f_{\bar{B}\otimes}$ preserves the $p1$–inversions and p–inversions and is null if the element had one $\otimes$–inversion.*
  Recalling the formulation for this morphism (3), we realize that only the elements with the structure $[a_1|\cdots|a_k|a'_{k+1}|\cdots|a'_n]$, with $a_i \in A$ and $a'_i \in A'$, don't belong to the kernel of this morphism.
- **Lemma 2.** *$g_{\bar{B}\otimes}$ preserves the number of inversions of the factors (canonically included in $\bar{B}(A\otimes A')$).*
  Shuffle product doesn't change either the number of components from each algebra nor the relative position in between them, so the number of inversions keep the same or increases.
- **Lemma 3.** *$\phi_{\bar{B}\otimes}$ produces elements with, at least one more inversion than the original one.*
  After evaluating $\phi_{\bar{B}\otimes}$ over an element, $[a_1 \otimes a'_1|a_2 \otimes a'_2|\cdots|a_n \otimes a'_n]$, one obtains a sum of elements, that can be sketched as follows:
  $$\pm\xi_A(a_{n-q+1} *_A \cdots *_A a_n)[a_1 \otimes a'_1|\cdots|a_{n-p-q-1} \otimes a'_{n-p-q-1}|(a'_{\bar{n}} *_{A'} \cdots *_{A'} a'_{n-q})|$$
  $$[a_{n-p-q}|\cdots|a_{n-q}] \star [a'_{n-q+1}|\cdots|a'_n]] \,, \qquad (7)$$
  We can check that the number of non degenerate components from the first algebra stay the same at each term, so the number of $p1$–inversions doesn't come affected.
  In the case that A' is a polynomial algebra, $\phi_{\bar{B}\otimes}$ leaves each p-inversion the same or changes it into a $\otimes$-inversion. If the original element has k p-inversions, each term of the resultant sum will have $k-i$ p-inversions and, at least, i $\otimes$-inversions, where $0 \le i \le k$. So the k initial inversions are preserved.
  Concerning $\otimes$–inversions, the component $(a'_{\bar{n}} *_{A'} \cdots *_{A'} a'_{n-q})$ is always responsible for a new inversion of this kind, as well as those of the shuffle product indicated in (7).

Now we consider the already described contraction (5):

$$(f, g, \phi) : \bar{B}(A \otimes A') \Rightarrow \bar{B}(A) \otimes \bar{B}(A') \Rightarrow HBA \otimes HBA'$$

where

$$f = f^n = (f^{n-1} \otimes f_{\bar{B}A'})f_{\bar{B}\otimes},$$
$$g = g^n = g_{\bar{B}\otimes}(g^{n-1} \otimes g_{\bar{B}A'})$$
$$\phi = \phi^n = \phi_{\bar{B}\otimes} + g_{\bar{B}\otimes}(\phi^{n-1} \otimes g_{\bar{B}A'}f_{\bar{B}A'} + 1_{\bar{B}A} \otimes \phi_{\bar{B}A'})f_{\bar{B}\otimes}$$

Now, we will study the behavior of f and ϕ with respect to inversions.

- **Lemma 4.** *The evaluation of f over an element from $\bar{B}(A \otimes A')$ is null whenever such an element has, at least, one inversion.*
  This is clear since the formula for f is: $f = (f_{\bar{B}A} \otimes f_{\bar{B}A'})f_{\bar{B}\otimes}$, so the lemma follows from lemma 1 and the fact that $f_{\bar{B}P}([v^{r_1}|\cdots|v^{r_k}]) = 0$ whenever $k > 1$.
- **Lemma 5.** *ϕ increases the number of inversions, at least, by one.*
  The case $n = 2$ can be proved taking into account that both $\phi_{\bar{B}P}$ and $\phi_{\bar{B}\otimes}$ (lemma 3) satisfy this condition (recall that $\phi_{\bar{B}E} = 0$) and that shuffle product preserves inversions (lemma 2); case $n > 2$ is proved, then, by induction.

Now we assume there is a perturbation ρ for the tensor product of A and A', $A \otimes A'$, and that the perturbation–derivation induced on $\bar{B}(A \otimes A')$ is $\delta = 1 \otimes \rho + \rho \otimes 1$. We analyze also, the behavior of the latter with respect to inversions, coming to the following conclusion.

Lemma 6. *The evaluation of δ over a homogeneous element from $\bar{B}(A \otimes A')$ with k inversions, produces a sum of elements with at least $k-1$ inversions each one of them.*

Let us notice that each component of an homogeneous element from $\bar{B}(A \otimes A')$ is responsible for, at most, one inversion. Since the action of δ is reduced to the application of ρ to each component $a_i \otimes a'_i$ of the element $[a_1 \otimes a'_1|\cdots|a_n \otimes a'_n]$, then, only one inversion can be anihilated, if any.

4 An A_∞–Coalgebra Structure

We recall the definition of A_∞–coalgebra given in [19].

An A_∞–coalgebra is a graded Λ-module C endowed with a locally finite family of morphisms $\Delta_i : C \longrightarrow C^{\otimes i}$, $i \geq 1$, such that the degree of Δ_i is $i-2$ and

$$\sum_{k=1}^{n}\sum_{\lambda=0}^{n-k}(-1)^{k+\lambda+\lambda k}(1^{\otimes(n-\lambda-k)} \otimes \Delta_k \otimes 1^{\otimes\lambda})\Delta_{n+k+1} = 0.$$

Recall that, given a morphism of DG–modules $h : M \to N$, the notation $h^{\otimes i}$ is used for the morphism $h \otimes \overset{\text{i times}}{\cdots} \otimes h$.

Let us observe that in the case $n = 3$, the following expression is obtained:

$$(1\otimes\Delta_2)\Delta_2-(\Delta_2\otimes1)\Delta_2 = \Delta_3\Delta_1+(1^{\otimes2}\otimes\Delta_1)\Delta_3+(1\otimes\Delta_1\otimes1)\Delta_3+(\Delta_1\otimes1^{\otimes2})\Delta_3$$

That is, the morphism Δ_3 *measures* the coassociativity of Δ_2.

In the case of the reduced bar construction of a CDGA, the structure of coalgebra can be trivially considered as an A_∞–coalgebra with the differential $d = \Delta_1$, the coproduct $\Delta = \Delta_2$ and $\Delta_i = 0$, $i \geq 3$.

We are interested in the transference of this coalgebra structure from the reduced bar construction, $\bar{B}(A)$, of a CDGA, A, to the 1–homological model, HBA, described in section 2. Using HPT (see [10]), it is clear how to get an A_∞–coalgebra structure $(\Delta_1, \Delta_2, \Delta_3, \ldots)$ on the 1–homological model, where $\Delta_1 = d_\delta$ and whose formulation for Δ_2 and Δ_3 is:

$$\Delta_2 = (f_\delta \otimes f_\delta)\, \Delta\, g_\delta \tag{8}$$

$$\Delta_3 = f_\delta^{\otimes 3}\, (-\Delta \otimes 1 + 1 \otimes \Delta)\, (\phi_\delta \otimes g_\delta f_\delta + 1 \otimes \phi_\delta)\, \Delta\, g_\delta \tag{9}$$

We focus on the computation of Δ_2. At first, it would be necessary to evaluate this morphism over all the module generators of HBA. Nevertheless, the following proposition implies the compatibility of Δ_2 with the product, what makes possible to do this calculation only for the algebra generators. This proposition also allows to construct a **test of coassociativity** for Δ_2.

Proposition 1. *Let us consider an almost–full algebra contraction, $C : \{A, A', f, g, \phi\}$ and $\delta : A \to A$ a data perturbation for C. Then, the perturbed contraction*

$$C_\delta : \{(A, d_A + \delta, \xi_A, \eta_A),\ (A', d_{A'} + d_\delta, \xi_{A'}, \eta_{A'}),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is also an almost–full algebra contraction.

Proof. For proving that f_δ is a morphism of DGAs, let us consider the contraction

$$C_\delta \otimes C_\delta : \{\, A \otimes A,\, A' \otimes A',\, f_\delta \otimes f_\delta,\, g_\delta \otimes g_\delta,\, \phi_\delta^{[\otimes 2]} = \phi_\delta \otimes g_\delta f_\delta + 1 \otimes \phi_\delta \,\}\,.$$

Taking into account condition $(c2)$ from the definition of contraction,

$$\phi_\delta^{[\otimes 2]} \delta^{[2]} + \delta^{[2]} \phi_\delta^{[\otimes 2]} = 1^{\otimes 2} - g_\delta^{\otimes 2} f_\delta^{\otimes 2}$$

Recall that the $h^{[i]}$ denotates the morphism $\sum_k (1^{\otimes k} \otimes h \otimes 1^{\otimes i-k-1})$.

By composing to the left with the shuffle product $\star$ and then with the morphism f_δ, we obtain the expression:

$$f_\delta \star \phi_\delta^{[\otimes 2]} \delta^{[2]} + f_\delta \star \delta^{[2]} \phi_\delta^{[\otimes 2]} = f_\delta \star - f_\delta \star g_\delta^{\otimes 2} f_\delta^{\otimes 2} \tag{10}$$

Taking into account that g_δ is a morphism of DGAs and condition $(c1)$, the second member on the right hand side satisfies

$$f_\delta \star g_\delta^{\otimes 2} f_\delta^{\otimes 2} = f_\delta\, g_\delta \star f_\delta^{\otimes 2} = \star f_\delta^{\otimes 2}$$

So, if we prove that the first member of equality (10) is null, f_δ will be a morphism of DGAs. But the first term in the sum can be written

$$f_\delta \star (\phi_\delta \otimes g_\delta f_\delta + 1 \otimes \phi_\delta)\delta^{[2]} = f_\delta \star (\phi_\delta \otimes g_\delta f_\delta)\delta^{[2]} + f_\delta \star (1 \otimes \phi_\delta)\delta^{[2]}$$

where the first term on the right hand side is null because of the fact that f_δ is a quasi–algebra projection, in particular, that $f_\delta \star (\phi_\delta \otimes g_\delta) = 0$. On the other hand, the evaluation of $f_\delta \star (1 \otimes \phi_\delta)\delta^{[2]}$ is always null since ϕ, and hence ϕ_δ, produces elements with one inversion (lemma 5) that is preserved by the shuffle product (lemma 2).

Finally, the second summand in the first member of (10) can be expressed by:

$$f_\delta \star \delta^{[2]} \phi_\delta^{[\otimes 2]} = f_\delta \delta \star \phi_\delta^{[\otimes 2]} = d_\delta f_\delta \star \phi_\delta^{[\otimes 2]}$$

(since $\delta^{[2]}$ is a derivation and f_δ is a morphism of DG-modules). And now we reason in the same way as before to conclude that the evaluation of $f_\delta \star \phi_\delta^{[\otimes 2]}$ is null.

Taking into account this proposition, we can state the following one:

Proposition 2. *Let HBA be the 1–homological model of a CDGA A and $\Delta_2 = (f_\delta \otimes f_\delta)\, \Delta\, g_\delta$. The following property is satisfied:*

$$\Delta_2 \bullet = (\bullet \otimes \bullet)\, (1 \otimes T \otimes 1)\, (\Delta_2 \otimes \Delta_2)\,,$$

where $\bullet : HBA \otimes HBA \to HBA$ is the product of the CDGA HBA.

Proof. For the proof, it is only necessary to notice that g_δ and f_δ are both morphisms of DGAs and the fact that $\bar{B}(A)$ is a Hopf algebra. So the following chain of equalities can be established:

$$\begin{aligned}
\Delta_2 \bullet &= (f_\delta \otimes f_\delta)\, \Delta\, g_\delta \bullet \;=\; (f_\delta \otimes f_\delta)\, \Delta \,\star\, g_\delta^{\otimes 2} \\
&= (f_\delta \otimes f_\delta)\, (\star \otimes \star)\, (1 \otimes T \otimes 1)\, (\Delta \otimes \Delta)\, g_\delta^{\otimes 2} \\
&= (\star \otimes \star)\, (1 \otimes T \otimes 1)\, f_\delta^{\otimes 4}\, \Delta^{\otimes 2}\, g_\delta^{\otimes 2} \\
&= (\star \otimes \star)\, (1 \otimes T \otimes 1)\, ((f_\delta \otimes f_\delta)\, \Delta\, g_\delta) \otimes ((f_\delta \otimes f_\delta)\, \Delta\, g_\delta) \\
&= (\star \otimes \star)\, (1 \otimes T \otimes 1)\, (\Delta_2 \otimes \Delta_2)
\end{aligned}$$

We will denote $(\star \otimes \star)\, (1 \otimes T \otimes 1)$ by $\star_\otimes$.

In particular, this relationship means that, for determining the morphism Δ_2, we only need to evaluate it over the generators of HBA as an algebra (a finite number of elements!).

On the other hand, coassociativity of Δ_2 would guarantee the coalgebra structure on HBA. Proposition 1 allows to claim the following one:

Proposition 3. *Under conditions described above, if the morphism Δ_2 is coassociative for the algebra generators, then so it is for the rest of elements of HBA.*

Proof. It is sufficient to prove

$$(\Delta_2 \otimes 1)\Delta_2 \bullet = (1 \otimes \Delta_2)\Delta_2 \ \bullet \,. \tag{11}$$

Starting from the first term

$$\begin{aligned}(\Delta_2 \otimes 1)\,\Delta_2 \bullet &= (\Delta_2 \otimes 1)\,(\star \otimes \star)\,(1 \otimes T \otimes 1)\,(\Delta_2 \otimes \Delta_2)\\ &= ((\star \otimes \star)\,(1 \otimes T \otimes 1)\,(\Delta_2 \otimes \Delta_2) \otimes \star)(1 \otimes T \otimes 1)\,(\Delta_2 \otimes \Delta_2)\\ &= ((\star \otimes \star)\,(1 \otimes T \otimes 1)\ \otimes \star)((\Delta_2 \otimes \Delta_2) \otimes 1)(1 \otimes T \otimes 1)\,(\Delta_2 \otimes \Delta_2)\\ &= (\star_{\otimes}\ \otimes \star)\,(1^{\otimes 2} \otimes (1 \otimes T)(T \otimes 1) \otimes 1)\,((\Delta_2 \otimes 1)^{\otimes 2})\,\Delta_2^{\otimes 2}\\ &= (\star_{\otimes}\ \otimes \star)\,(1^{\otimes 2} \otimes (1 \otimes T)(T \otimes 1) \otimes 1)\,((\Delta_2 \otimes 1)\,\Delta_2)^{\otimes 2}\\ &= (\star_{\otimes}\ \otimes \star)\,(1^{\otimes 2} \otimes (1 \otimes T)(T \otimes 1) \otimes 1)\,((1 \otimes \Delta_2)\,\Delta_2)^{\otimes 2}\,.\end{aligned}$$

An analogous treatment can be done for the right hand side of (11) in order to get the same result.

Now, we attack the problem of the complexity of the associated algorithm to the explicit formula of Δ_2 and we considerably reduce the amount of elementary operations (in comparison with the algorithm derived from the initial formulation (8)) that are necessary in order to compute this morphism over an algebra generator.

On the other hand, the responsible for the high complexity in the evaluation of Δ_2 is the homotopy operator, ϕ, due, essentially, to the shuffles that are involved in the formulas of $\phi_{B\otimes}$ and $g_{B\otimes}$. We intend to reduce this complexity by eliminating unnecessary terms, and, for this aim, we use the inversion theory given in section 3.

We recall the formula for Δ_2:

$$\Delta_2 = ((f - f\delta\phi + f\delta\phi\delta\phi - \cdots) \otimes (f - f\delta\phi + f\delta\phi\delta\phi - \cdots))\,\Delta\,(g - \phi\delta g + \phi\delta\phi\delta g - \cdots)$$

We can observe that f is the last morphism applied. If, at the last stage, after applying Δ, the element y obtained by applying ϕ, has more than one inversion, then $\delta(y)$ will have at least one inversion. This way, each time we go on applying $\delta \circ \phi$, we obtain an element with at least one inversion (lemmas 5 and 6), and, therefore, the final evaluation by f is null (lemma 4). This means that, for the application of $f_\delta \otimes f_\delta$, we only have to consider summands of ϕ which produce elements with, at most, one inversion.

In consequence, we can establish the following theorem by which we considerably reduce the complexity in the computation of Δ_2.

Theorem 2. *When Δ_2 is applied to an algebra generator element from the $1-$homological model for a CDGA $A \otimes A'$, the formula for ϕ that is involved in the definition of $f_\delta \otimes f_\delta$, can be reduced to the following one:*

$$\phi = \bar{\phi}_{\bar{B}\otimes} + \bar{g}_{\bar{B}\otimes}(\phi_{\bar{B}A} \otimes g_{\bar{B}A'} f_{\bar{B}A'} + 1 \otimes \phi_{\bar{B}A'}) f_{\bar{B}\otimes},$$

where

- $\bar{\phi}_{\bar{B}\otimes}$ *is the one given in [3] for the simplification in the calculation of* d_δ.
- $\bar{g}_{\bar{B}\otimes}([a_1|\cdots|a_n]\otimes[a'_1|\cdots|a'_m]) = [a_1|\cdots|a_n|a'_1|\cdots|a'_m]$.

When Δ_2 *is applied to a product of two generators,* $z = x \bullet y$*, then,* $\Delta_2(z) = (\bullet\otimes\bullet)\,(1\otimes T\otimes 1)\,(\Delta_2(x)\otimes\Delta_2(y))$.

Recall that, the number of terms in the formula for $\bar{\phi}_{\bar{B}\otimes}$ was

$$\sum_{q=0}^{n-1}\sum_{p=0}^{n-q-1} 1 = \frac{n^2+n}{2}\,,$$

in contrast to the original number of terms:

$$\sum_{q=0}^{n-1}\sum_{p=0}^{n-q-1}\binom{p+q+1}{q} \;=\; 2^{n+1}-n-2\,.$$

Besides, the formula for $g_{\bar{B}\otimes}$ is reduced now to 1 term in the sum instead of $\binom{m+n}{n}$ (the reduction for this morphism in [3] was of n terms).

Even though the reduction in complexity is considerable, the algorithm for computing Δ_2 is still an extremely expensive procedure, due, mainly, to the morphism ϕ in g_δ. This fact forced us to look for cases of CDGAs in which this calculation becomes more reasonable.

5 The Case of Purely Quadratic CDGAs

We consider here an important subset of CDGAs for which $\Delta_1 = 0$. Therefore, the 1–homological model for these algebras coincides with its 1–homology, so if we study the A_∞–coalgebra structure on the model, we are actually considering such a structure on the 1–homology of a CDGA.

This is the case of the commutative differential algebras whose differential–perturbation, ρ, does'nt have any *linear* summand, that is, if $x, x_{i_1}, \ldots x_{i_k}$ are algebra generators, then $\rho(x) = \sum_i \lambda_i\, x_{i_1}^{r_{i_1}}\otimes\cdots\otimes x_{i_k}^{r_{i_k}}$, with $k\geq 2$. We denote this set of algebras by CDGA_0, for which we can state the following proposition.

Proposition 4. *If* A *is a* $CDGA_0$ *under the conditions described in section 2, the formula for* Δ_2 *in the* 1*–homological model,* HBA*, is*

$$\Delta_2 \;=\; (f\otimes f)\,\Delta\, g_\delta\,.$$

Proof. We recall that $f_\delta = \sum(-1)^i f\,(\delta\phi)^i$. Notice that the morphism $(f\,\delta)$ is applied at each summand, except for the first one. Such a composition is null since δ produces elements with, at least, one non-linear component and, hence, $f_{\bar{B}\otimes}$ becomes zero.

Definition 2. Let (A, ρ) be a CDGA, we say A is *quadratic*, if the action of the differential–perturbation on every algebra generator x is

$$\rho(x) = \sum_i \lambda_i \, x_{i_1}^{r_{i_1}} \otimes x_{i_2}^{r_{i_2}} \,,$$

where x_{i_1} , x_{i_2} are also algebra generators and $\lambda_i \in \Lambda$.

We say that A is *purely quadratic* if $r_{i_1} = 1$, $r_{i_2} = 1$ for all the indexes i.

Theorem 3. *Let A be a purely quadratic CDGA, the A_∞–coalgebra structure of its 1–homology, HBA, reduces to that of coalgebra.*

Notice that, in that case, HBA is a Hopf algebra.

Proof. It is sufficient to show that $\phi_\delta^{[\otimes 2]} \Delta \, g_\delta \;=\; 0$, what implies that $\Delta_3 = 0 = \Delta_4 = \cdots$ and, in particular, since Δ_3 is null, Δ_2 is coassociative and, hence, a real coproduct on HBA.

We say a homogeneous element from $\bar{B}(A)$ is *simple* if all the components of such an element are linear. We extent this concept in the natural way to any element from $\bar{B}(A)$.

We show that we always obtain simple elements by applying g_δ to the algebra generators. As a consequence, this property will be true for all the elements g_δ is applied to, since this morphism is multiplicative (theorem 1).

$$g_\delta(\underline{x}) = \sum_{i \geq 0} (-1)^i \, (\phi\delta)^i \, g \, (\underline{x})$$

We will prove, by induction on i, that $(\phi\delta)^i g(\underline{x}) \;=\; (\phi\delta)^i [x]$ are simple.

- $\phi\delta g(\underline{x}) \;=\; \phi\delta[x] \;=\; \sum_i \lambda_i \, \phi[x_{i_1} \otimes x_{i_2}] \;=\; \sum_i \lambda_i \, [x_{i_2} | x_{i_1}]$, which are simple.
- Let us assume that $(\phi\delta)^{i-1}[x]$ is simple. Then, each term of $\delta(\phi\delta)^{i-1}[x]$ is 1–*simple*, that is, has a unique quadratic component, $x_{k_1} \otimes x_{k_2}$,

  $$[x_1 | \cdots | x_{k-1} | x_{k_1} \otimes x_{k_2} | x_{k+1} | \cdots | x_m] \,. \qquad (12)$$

  Recall that A is factored as $A = \tilde{\otimes}_{i \in I}^{\rho} A_i$, with $I = \{1, 2, \ldots, n\}$ and, therefore, the homotopy operator ϕ can be written as follows,

  $$\phi \;=\; \phi^n \;=\; \phi_{B\otimes} \;+\; g_{B\otimes}(\phi^{n-1} \otimes g_{BA_n} f_{BA_n} + 1 \otimes \phi_{BA_n}) f_{B\otimes}$$

  Let us show, by induction on n, that ϕ turns that quadratic component into two linear components.

  - Case $n = 2$, A_1 and A_2 are a polynomial or an exterior algebra each one of them.

    It is clear that $f_{B\otimes}$ is null when applied to elements like (12). Concerning $\phi_{B\otimes}$, if we pay attention to the formula (4),we can check that the resultant element is is a sum of homogeneous elements with the structure

$$[x_1|\cdots|x_{k-j}|x_{k_2}|\,[x_{k-j+1}|\cdots|x_{k_1}|\cdots x_{k+i}]\star[x_{k+i+1}|\cdots|x_m]\,],$$

which is simple.

- Case $n>2$,
  1. If $x_{k_2}\in A_n$, $\phi_{\bar{B}\otimes}$ acts in the same way as before, as well as $f_{\bar{B}\otimes}$, which is null.
  2. If x_{k_2} is the algebra generator of A_j with $j<n$, $\phi_{\bar{B}\otimes}$ will be null. On the other hand, $f_{\bar{B}\otimes}$ will only be no null if the components corresponding to A_n are just the last, for example, i components, $x_{m-i+1},\dots,x_m$. In this case,

     $$f_{\bar{B}\otimes}([x_1|\cdots|x_{k-1}|x_{k_1}\otimes x_{k_2}|\cdots|x_{m-i}|\cdots|x_m]) = [x_1|\cdots|x_{k_1}\otimes x_{k_2}|\cdots|x_{m-i}]\otimes[x_{m-i+1}|\cdots|x_m] \quad (13)$$

     Now, it is easy to verify that

     $$(1\otimes\phi_{\bar{B}A_n})([x_1|\cdots|x_{k_1}\otimes x_{k_2}|\cdots|x_{m-i}]\otimes[x_{m-i+1}|\cdots|x_m])=0\,,$$

     since, on one hand, $\phi_{\bar{B}P}$ is null when applied to simple elements, and on the other hand, $\phi_{\bar{B}E}=0$.
     As for $g_{\bar{B}\otimes}(\phi^{n-1}\otimes g_{\bar{B}A_n}f_{\bar{B}A_n})$, its application to the element (13) gives place to simple elements, since if $g_{\bar{B}A_n}f_{\bar{B}A_n}[x_{m-i+1}|\cdots|x_m]$ is not null, it is clear that is simple; on the other hand, $\phi^{n-1}([x_1|\cdots|x_{k_1}\otimes x_{k_2}|\cdots|x_{m-i}]$ is simple by induction hypotheses, and finally, by shuffle product we obtain simple elements.

After g_δ, the morphism Δ, factors these simple elements as tensor product of pairs of simple factors.

Finally, we must show that ϕ is null when applied to simple elements and so will be $\phi_\delta^{[\otimes 2]}$. But, considering, again, the formula of ϕ, we realize that it is easy to prove it by induction. The key is that both $\phi_{\bar{B}\otimes}$ and $\phi_{\bar{B}A_n}$ are null when applied to simple elements.

Therefore, it is possible to derive an algorithm for computing the complete Hopf algebra structure of the 1–homology of a purely quadratic algebra. We intend to implement this algorithm in the near future starting from the program used in [3].

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