

Gradation in Greyscales of Graphs

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Abstract

In this work we present the notion of greyscale of a graph as a colouring of its vertices that uses colours from the real interval $[0,1]$. Any greyscale induces another colouring by assigning to each edge the non-negative difference between the colours of its vertices. These edge colours are ordered in lexicographical decreasing ordering and give rise to a new element of the graph: the gradation vector. We introduce the notion of minimum gradation vector as a new invariant for the graph and give polynomial algorithms to obtain it. These algorithms also output all greyscales that produce the minimum gradation vector. This way we tackle and solve a novel vectorial optimization problem in graphs that may produce more satisfactory solutions than those ones generated by known scalar optimization approaches. The interest of these new concepts lies in their possible applications for solving problems of engineering, physics and applied mathematics which are modeled according to a network whose nodes have assigned numerical values of a certain parameter delimited by a range of real numbers. The objective is to minimize the differences between each node and its neighbors, ensuring that the extreme values of the interval are assigned.

Keywords: graph colouring, greyscale, minimum gradation, graph algorithms.

MSC 2010 (Primary): 05C, 68R. **MSC 2010 (Secondary):** 05C15, 05C85, 90C47.

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1 Introduction

Graph colouring problems are among the most important combinatorial optimization problems in graph theory because of their wide applicability in areas such as wiring of printed circuits [4], resource allocation [22], frequency assignment problem [1, 10, 18], a wide variety of scheduling problems [17] or computer register allocation [3].

A variety of combinatorial optimization problems on graphs can be formulated similarly in the following way. Given a graph $G(V, E)$ and a mapping $f : V \rightarrow \mathbb{Z}$, a new mapping $\hat{f} : E \rightarrow \mathbb{Z}$ is induced by f such that $\hat{f}(e) = |f(u) - f(v)|$ for every $e = \{u, v\} \in E$. Then an optimization problem is formulated from several key elements: mappings f belonging to a mapping subset S , the image of V by f and the image of E by \hat{f} . In particular, the classic graph colouring problem, that is, colouring the vertices of G with as few colours as possible so that adjacent vertices always have different colours, can be stated in these terms as follows:

$$\chi(G) = \min_{f \in S} |f(V)| \quad \text{where } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } 0 \notin \hat{f}(E)\}.$$

It is well known that this minimum number $\chi(G)$ is called the chromatic number of the graph G and that its computing is an NP-hard problem [13].

It must be emphasized that the classic graph colouring problem bears in mind the number of colours used but not what they are. However, there are some works related to map colouring for which the nature of the colours is essential, whereas the number of them is fixed. The *maximum differential graph colouring problem* [12], or equivalently the *antibandwidth problem* [14], colours the vertices of the graph in order to maximize the smallest colour difference between adjacent vertices and using all the colours $1, 2, \dots, |V|$. Under the above formulation, these problems are posed as follows:

$$\max_{f \in S} \min \hat{f}(E) \quad \text{for } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } f(V) = \{1, 2, \dots, |V|\}\},$$

and therefore the complementary optimization case, the *bandwidth problem*, is given by

$$\min_{f \in S} \max \hat{f}(E) \quad \text{for } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } f(V) = \{1, 2, \dots, |V|\}\}.$$

Note that these problems are concerned with mappings that take values within a discrete spectrum and with the optimization of a scalar function. Dillencourt et

al. [6] studied a variation of the differential graph colouring problem under the assumption that all colours in the colour spectrum are available. This makes the problem continuous rather than discrete. The well-known frequency assignment problem is continuous by nature, although its treatment has traditionally been discrete. However, recent works [16] propose to shift the paradigm from discrete channel allocation to continuous frequency allocation. On the other hand, the key issue for process scheduling problems concerns the time representation, and in order to address real limitations, methods based on continuous-time representations have attracted a great amount of attention and provide great potential for the development of more accurate and efficient modeling and solution approaches [8, 20].

In this line, this paper deals with mappings taking values within the continuous spectrum $[0, 1]$, where 0 and 1 correspond to white and black colours, respectively, and the rest of the intermediate values are grey tones. Formally, given a graph $G(V, E)$, a *greyscale* f of G is a mapping $f : V \rightarrow [0, 1]$ such that the white and black colours are reached by f . Every greyscale induces a mapping \widehat{f} on E by assigning to each edge the non-negative difference between the values of f on its vertices. Whether the values of \widehat{f} are sorted by decreasing order, the *gradation vector* $grad(G, f)$ is obtained and the optimization problem of finding the minimum one, following the lexicographical order, among all the gradation vectors of greyscales of G arises in a natural way. Analogously, the notion of contrast is associated to increasing order and the maximum vector, and it is widely studied in work [2] by the same authors of this paper.

The bandwidth and antibandwidth problems are interested in optimizing the extreme colours of the edges of the graph, whereas Dillencourt et al. [6] focus on maximizing the sum of the colours of all the edges. Under this last approach, other papers work with different sum functions (for instance, see [15]). Nonetheless, both cases, extreme values and sum functions, deal with scalar objective functions. The notion of gradation vector leads us to a vectorial objective function which allocates grey tones in a manner which is both local and global: local due to the fact that the colour of every particular edge belongs to the gradation vector, and global because all edges of the graph participate in the vectorial objective function. Figure 1 visually displays an example of the goodness of minimum gradation vectors versus scalar optimization. Every vertex has been associated to a *big pixel* which is coloured with its grey tone and these *big pixels* are next to each other according to the adjacencies between vertices (in this construction, the notion of dual graph is underlying but without considering the external face). The idea of gradation is clearly better illustrated in Figure 1(a).

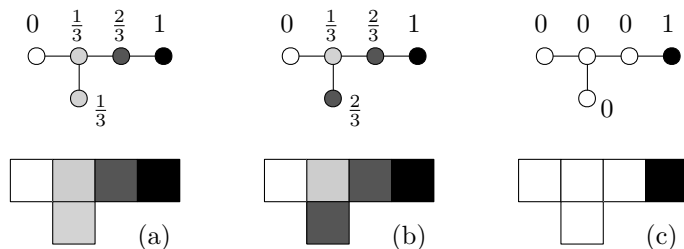


Figure 1: Comparison of minimum greyscales according to different criteria about the colours of edges: (a) minimum gradation vector, (b) minimax and (c) minisum.

Thus, under the formulation given above and following the lexicographical order, the minimum gradation problem on graphs is stated in the following terms:

$$\min_{f \in S} \text{grad}(G, f) \quad \text{for } S = \{f : V \rightarrow [0, 1] \text{ such that } \{0, 1\} \subset f(V)\}.$$

At present, although this problem seems to be a natural colouring question, we have not found the minimum gradation problem studied in these terms in the literature.

Even though it is not the main goal of this paper to deeply focus on possible applications of gradation in graphs, we think that the minimum gradation greyscale could contribute to interesting progresses on several problems concerning real networks. Next we present two of them.

For a wide variety of systems in different areas such as biological, social, technological, and information networks the community detection problem has become extremely useful. This problem consists of identifying special groups of vertices in a graph with high concentrations of edges within such vertices and low concentrations between these groups. This feature of real networks is called community structure [9], or clustering in graphs. For an extensive report on this topic see [19]. The minimum gradation greyscale solution could be interpreted as a new similarity measure quantifying some type of affinity between node pairs. This mapping locally minimizes the differences between each vertex and its neighbours and takes into account the global distances in the network. Besides, extremal values would be assigned to antipodal vertices.

It is well known that graph theory is used to modelize many kinds of networks services. The minimum gradation greyscale solution would model a possible almost uniform distribution of a service through a network, from the sources to

the sinks. For water supply networks, problems such as minimizing the amount of dissipated power in the water network and establishing pressure control techniques, among others, are studied. In [5] the method of graph partitioning is proposed to solve them. We guess that for a given water network graph model, the minimum gradation greyscale solution would help and complement the design of a good distribution of the water through the network in such a manner that the water or pressure losses between contiguous pipes would be minimized at the same time. A source or sink vertex can be modeled with extreme values prefixed in the greyscale. Thus, some of the vertices have preassigned grey tones and the aim is to obtain the minimum gradation vector preserving these fixed grey tones.

The outline of the paper is as follows: Section 2 formally introduces the notion of gradation on graphs through concepts such as greyscale and gradation vector, and formulates the two problems for study: *minimum gradation* and *restricted minimum gradation on graphs*. In Section 3, several results about the nature of the gradation problems are first established, and then the polynomial nature of both problems is proved by designing algorithms that provide minimum gradation vectors and all greyscales that give rise to them. Finally, in Section 4 we conclude with some remarks and highlight some open problems.

2 Preliminaries

This section is devoted to establishing the necessary definitions about gradation on graphs and to formulating the problems to be studied in this paper. Since the gradation and contrast notions together arise in greyscales of graphs, the basic concepts about contrast are also presented. Given a graph¹ $G(V, E)$, a *greyscale* f of G is a mapping on V to the interval $[0, 1]$ such that $f^{-1}(0) \neq \emptyset$ and $f^{-1}(1) \neq \emptyset$. For each vertex v of G , we call $f(v)$ the *grey tone* of v , or more generally, the *colour* of v , and notice that two adjacent vertices can have mapped the same grey tone. In particular, values 0 and 1 are called *the extreme tones*, that is, *white and black colours*, respectively. In a natural way, the notion of *complementary greyscale* arises for each greyscale f such that it maps every vertex v of G to $1 - f(v)$.

Associated to each greyscale f of the graph $G(V, E)$, the mapping $\hat{f} : E \rightarrow [0, 1]$ is defined as $\hat{f}(e) = |f(u) - f(v)|$ for every $e = \{u, v\} \in E$ and represents

¹Graphs in this paper are finite, undirected and simple and are denoted by $G(V, E)$, where V and E are its vertex-set and edge-set, respectively. The number of elements of V and E are denoted by n and m , respectively. For further terminology we follow [11].

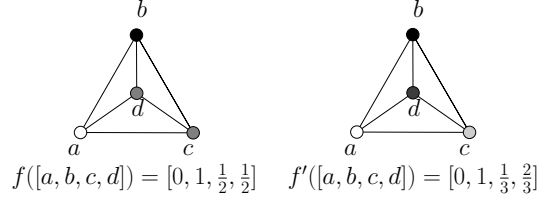


Figure 2: Two greyscales f and f' of the graph K_4 .

the gap or increase between the grey tones of vertices u and v . The value $\widehat{f}(e)$ is also said to be the *grey tone* of edge e . Thus, we deal with *coloured vertices and edges* by f and \widehat{f} , respectively. Note that the same mapping \widehat{f} associated to the greyscale f and its complementary one is obtained.

The *gradation vector* and the *contrast vector* associated to the greyscale f of G are vectors $\text{grad}(G, f) = (\widehat{f}(e_m), \widehat{f}(e_{m-1}), \dots, \widehat{f}(e_1))$ and $\text{cont}(G, f) = (\widehat{f}(e_1), \widehat{f}(e_2), \dots, \widehat{f}(e_m))$, respectively, where the edges of G are indexed such that $\widehat{f}(e_i) \leq \widehat{f}(e_j)$ whether $i < j$, that is, in ascending order of their grey tones. Thus, it can be noticed that the components of any contrast vector are ordered in ascending order and those of any gradation vector in decreasing order. For the sake of clarity and when the graph is fixed, the gradation and contrast vectors associated to a greyscale f will be denoted by \mathcal{G}_f and \mathcal{C}_f , respectively. Figure 2 shows two greyscales of the graph K_4 , f and f' , whose corresponding gradation vectors are $\mathcal{G}_f = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $\mathcal{G}_{f'} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, respectively.

Given two greyscales f and f' of a graph G , we say that f has *better gradation* than f' if the gradation vector \mathcal{G}_f is smaller than $\mathcal{G}_{f'}$ following the lexicographical order, that is, $\mathcal{G}_f < \mathcal{G}_{f'}$. Thus, the descending order of gradation vectors determines the goodness in terms of gradation. Then, f is said to be *smaller or greater by gradation* than f' if $\mathcal{G}_f < \mathcal{G}_{f'}$ or $\mathcal{G}_f > \mathcal{G}_{f'}$, respectively. In a similar way, we say that f has *better contrast* than f' if the contrast vector \mathcal{C}_f is greater than $\mathcal{C}_{f'}$ following the lexicographical order, that is, $\mathcal{C}_f > \mathcal{C}_{f'}$. Thus, the ascending order of contrast vectors determines the goodness in terms of contrast. Then, f is said to be *smaller or greater by contrast* than f' if $\mathcal{C}_f < \mathcal{C}_{f'}$ or $\mathcal{C}_f > \mathcal{C}_{f'}$, respectively. The greyscale f of Figure 2 is greater than f' by both contrast and gradation and so f has better contrast than f' but f' has better gradation than f .

Thus, the problems of finding the minimum gradation vector and the maximum contrast vector naturally arise in the contexts of gradation and contrast on graphs. From now on, we focus on gradation problem and point out [2] for the

contrast problem.

Then, a greyscale of a graph G whose gradation vector is minimum is called a *minimum gradation greyscale* of G and the following problem is formulated:

Minimum gradation on graphs (MIGG): given a connected graph $G(V, E)$, finding the minimum gradation vector and all the minimum gradation greyscales.

$$\min_{f \in S} \text{grad}(G, f) \quad \text{for } S = \{f : V \rightarrow [0, 1] \text{ such that } \{0, 1\} \subset f(V)\}.$$

We deal with the restricted version of this problem when the grey tones of some vertices are a priori known and the aim is to obtain the minimum gradation vector preserving the fixed grey tones. This situation leads to the concept of incomplete greyscale. Given a graph $G(V, E)$ and a nonempty proper subset V_c of V , an *incomplete V_c -greyscale* of G is a mapping on V_c to the interval $[0, 1]$. Note that this incompleteness means both that the mapping is not defined on all the vertices of G and that the extreme tones do not necessarily belong to the range of the incomplete mapping. A greyscale f is *compatible* with an incomplete V_c -greyscale g if $f(u) = g(u)$ for all $u \in V_c$.

Restricted minimum gradation on graphs (RMIGG): given a connected graph $G(V, E)$ and an incomplete V_c -greyscale g of G , finding the gradation vector that is minimum among all the gradation vectors of greyscales compatible with g , as well as determining all these greyscales.

$$\min_{f \in S} \text{grad}(G, f) \quad \text{for } S = S_1 \cap S_2$$

where $S_1 = \{f : V \rightarrow [0, 1] / \{0, 1\} \subset f(V)\}$ and

$S_2 = \{f : V \rightarrow [0, 1] / f \text{ is compatible with an incomplete greyscale given of } G\}$.

Resolving each of these problems means finding the appropriate minimum gradation vector and all their minimum gradation greyscales except the complementary ones. Note that the MIGG and RMIGG problems are posed for connected graphs but general graphs can be also considered, and in this case each connected component has to be considered separately.

3 Minimum gradation problem

In this section several results about the nature of the gradation problems are first established, which let us prove the correctness of our polynomial procedures.

These algorithms provide minimum gradation vectors and all greyscales that give rise to them, all minimum gradation greyscales. Some cases are distinguished in the RMIGG problem due to the special role that the extreme tones 0 and 1 play in the concept of greyscale and, from a common subroutine, several polynomial algorithms are designed according to the possible existence of 0's or 1's as prefixed colours. Finally, the MIGG problem is also solved in polynomial time.

Before giving our results, we make some interesting observations about the MIGG problem. Given a greyscale of a connected graph, the components of the gradation vector are sorted by decreasing order and then, our purpose is to obtain the minimum gradation vector. Therefore, our question can be considered as a minimax problem. By other hand, throughout this section we deal with gradation vectors and the greyscales which are associated to. Notice that, given a graph, the minimum gradation vector is unique but there can exist different minimum gradation greyscales which give rise to the same minimum gradation vector (see an example in Figure 3).

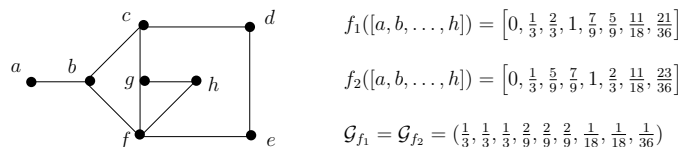


Figure 3: The minimum gradation vector can be achieved from different greyscales.

About paths and distance, we follow the terminology of [11]. Given a connected graph G , the *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining them; a $u - v$ *path* is a path joining the u and v vertices of G and a $u - v$ *geodesic* is a shortest $u - v$ path. The *diameter* $d(G)$ is the length of any longest geodesic, which is called a *diameter geodesic*, and two vertices u and v are *antipodal* if $d(u, v) = d(G)$.

The two following definitions, support greyscale and edge-colour-increase mapping, provide the key tools to solve the MIGG problem for connected graphs.

Let $G = (V, E)$ be a connected graph of diameter $d(G)$ and let u and v be two antipodal vertices of G . Then, the *support greyscale* for u and v is the mapping on $V(G)$ given by

$$f\langle u, v \rangle(w) = \frac{d(w, u) - d(w, v) + d(G)}{2d(G)}.$$

It is straightforward to check that $f\langle u, v \rangle$ is a greyscale of G , that is, it takes values from the interval $[0, 1]$ and, in particular, $f\langle u, v \rangle(u) = 0$ and $f\langle u, v \rangle(v) = 1$. The following lemma establishes the values of $\widehat{f}\langle u, v \rangle$.

Lemma 3.1. *Let $f\langle u, v \rangle$ be the support greyscale associated to the pair of antipodal vertices u and v of a connected graph $G(V, E)$ of diameter $d(G)$. Then, the only components of the gradation vector $\mathcal{G}_{f\langle u, v \rangle}$ are $\frac{1}{d(G)}$, $\frac{1}{2d(G)}$ and 0.*

Proof. Let $e = \{w_1, w_2\}$ be an edge of G . Due to the adjacency between w_1 and w_2 , it holds that $d(w_2, u) = d(w_1, u) + k$, with $k \in \{-1, 0, 1\}$. These three values of k give rise to three cases to analyze associated to u (index i) and, analogously, three cases for v (index j). That is, from the definition of the support greyscale and distinguishing these nine Cases i, j , with $i, j = 1, 2, 3$, the following values for $|f\langle u, v \rangle(w_1) - f\langle u, v \rangle(w_2)|$ are reached:

- Cases 1.1, 2.2 and 3.3: $|f\langle u, v \rangle(w_1) - f\langle u, v \rangle(w_2)| = 0$.

□

In particular, the values of $\widehat{f}\langle u, v \rangle(e)$ on the edges of any $u - v$ geodesic are $\frac{1}{d(G)}$ because of the diameter notion, that is, the longest geodesic in G . So, at least, the first $d(G)$ components of $\mathcal{G}_{f\langle u, v \rangle}$ are equal to $\frac{1}{d(G)}$.

However, let us remark that support greyscales do not lead to minimum gradation vectors in general. The support greyscale for u and v of the graph in Figure 4 is $f\langle u, v \rangle([u, a, b \dots h, i, v]) = [0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, 1]$, and so its associated gradation vector is given by $\mathcal{G}_{f\langle u, v \rangle} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0, 0)$. Nevertheless, the minimum gradation vector of this graph of diameter 4 is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, 0, 0)$, that can be obtained from the greyscale

$$f([u, a, b \dots h, i, v]) = [0, \frac{1}{4}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, 1].$$

Then, given two antipodal vertices, a special greyscale has been defined which maps them to the values 0 and 1. In some sense, Corollary 3.7 will state the reciprocal of this fact.

On the other hand, given a greyscale f of a connected graph $G(V, E)$, the *edge-colour-increase mapping* F is defined as the mapping $F : V \times V \longrightarrow [0, 1]$ such that

$$F(u, v) = \begin{cases} \frac{|f(u) - f(v)|}{d(u, v)} & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

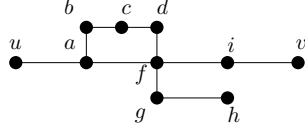


Figure 4: The support greyscale for u and v of this graph of diameter 4 is not a minimum gradation greyscale.

The value of $F(u, v)$ can be viewed as ‘the amount of colour’ that every edge of any $u - v$ geodesic would be given whether the colour increase between u and v were fairly distributed along the $u - v$ geodesic. Throughout this section, the relationship between \hat{f} and $F(u, v)$ is studied.

Geodesics play an essential role in the MIGG problem. Our next results are established on the set of geodesics of the given graph and they state several links between the values of f , \hat{f} and F , according to the position of the vertices into the geodesics.

Lemma 3.2. *Let f be a greyscale of a graph $G(V, E)$ and let u and v be a pair of vertices of G . For each vertex w of each $u - v$ geodesic and different to u and v , it holds that,*

$$F(u, v) < \max\{F(u, w), F(w, v)\} \quad \text{or} \quad F(u, v) = F(u, w) = F(w, v).$$

Moreover, the above equalities only hold whenever $f(w)$ belongs to the interval of extremes $f(u)$ and $f(v)$.

Proof. The relative position of $f(w)$ with respect to $f(u)$ and $f(v)$ determines three cases. For the sake of clarity and without loss of generality we may suppose that $f(u) \leq f(v)$. On the other hand, it is clear that $d(u, v) = d(u, w) + d(w, v)$, and since w is different to u and v , then $d(u, w) < d(u, v)$ and $d(w, v) < d(u, v)$.

1. If $f(w) \leq f(u) \leq f(v)$, then

$$\frac{f(v) - f(u)}{d(u, v)} \leq \frac{f(v) - f(w)}{d(u, v)} < \frac{f(v) - f(w)}{d(w, v)}.$$

That is, $F(u, v) < F(w, v)$ and $F(u, v) < \max\{F(u, w), F(w, v)\}$ trivially follows.

2. If $f(u) \leq f(v) \leq f(w)$, it proceeds as in the previous case obtaining the same inequality.

3. If $f(u) \leq f(w) \leq f(v)$, we prove that it is not possible $F(u, v) > \max\{F(u, w), F(w, v)\}$ and if $F(u, v) = \max\{F(u, w), F(w, v)\}$, then the three values of F are equal.

Assume to the contrary that $F(u, v) > \max\{F(u, w), F(w, v)\}$. Hence,

$$F(u, w) < F(u, v) \Rightarrow f(w) - f(u) < d(u, w)F(u, v) \quad (1)$$

and

$$F(w, v) < F(u, v) \Rightarrow f(v) - f(w) < d(v, w)F(u, v) \quad (2)$$

Adding (1) and (2), we obtain

$$\begin{aligned} f(v) - f(u) &< (d(u, w) + d(w, v))F(u, v) \Rightarrow f(v) - f(u) < d(u, v)F(u, v) \\ &\Rightarrow F(u, v) < F(u, v), \end{aligned}$$

which is a contradiction.

Next, if $F(u, v) = \max\{F(u, w), F(w, v)\}$ we assume $F(u, v) = F(u, w)$.

$$F(u, v) = F(u, w) \Rightarrow f(w) - f(u) = (f(v) - f(u)) \frac{d(u, w)}{d(u, v)} \quad (3)$$

Now,

$$F(w, v) = \frac{(f(v) - f(u)) - (f(w) - f(u))}{d(w, v)}$$

and by (3) and since $d(u, v) - d(u, w) = d(w, v)$,

$$F(w, v) = \frac{(f(v) - f(u))d(w, v)}{d(u, v)d(w, v)} = F(u, v).$$

So, $F(u, v) = F(u, w) = F(w, v)$.

Under the assumption that $F(u, v) = F(w, v)$ and by similar arguments, the same result is obtained.

□

The following two results establish connections between the mappings \hat{f} and F on geodesics.

Corollary 3.3. *Let f be a greyscale of a graph $G(V, E)$ and let u and v be a pair of vertices of G . For each $u - v$ geodesic P_{u-v} , it holds that*

$$F(u, v) \leq \max_{e \in P_{u-v}} \widehat{f}(e).$$

Proof. We denote P_{u-v} by $\{u = w_0, e_1, w_1, e_2, w_2, \dots, w_{l-1}, e_l, v = w_l\}$ as alternating sequence of vertices and edges and hence $l = d(u, v)$. In order to prove the result, a stronger assertion will be stated, that is, $F(u, v) \leq \max\{\widehat{f}(e_1), \dots, \widehat{f}(e_i), F(w_i, v)\}$ for $i = 1, \dots, l - 1$.

For $i = 1$, Lemma 3.2 applied to w_1 of P_{u-v} and the fact that $\widehat{f}(e_1) = F(u, w_1)$ lead trivially to $F(u, v) \leq \max\{\widehat{f}(e_1), F(w_1, v)\}$.

Inductively, suppose $F(u, v) \leq \max\{\widehat{f}(e_1), \dots, \widehat{f}(e_i), F(w_i, v)\}$. Lemma 3.2 is again applied, in this case to w_{i+1} as vertex of the path $\{w_i, e_{i+1}, w_{i+1}, \dots, v\}$, obtaining that

$$F(w_i, v) \leq \max\{F(w_i, w_{i+1}), F(w_{i+1}, v)\} = \max\{\widehat{f}(e_{i+1}), F(w_{i+1}, v)\}.$$

Thus, the induction hypothesis and this inequality about $F(w_i, v)$ give rise to the result for $i + 1$. \square

Lemma 3.4. *Let f be a greyscale of a graph $G(V, E)$, let u and v be a pair of vertices of G and let P_{u-v} be a $u - v$ geodesic. If $F(u, v) = \max_{e \in P_{u-v}} \widehat{f}(e)$ then $F(u, v) = \widehat{f}(e)$ for every edge $e \in P_{u-v}$.*

Proof. For the sake of simplicity and without loss of generality, let the alternating sequence of vertices and edges $\{u = w_0, e_1, w_1, e_2, w_2, \dots, w_{l-1}, e_l, v = w_l\}$ be the $u - v$ geodesic P_{u-v} where $f(u) \leq f(v)$. Assume to the contrary that there exists e_j an edge of P_{u-v} such that $\widehat{f}(e_j) < F(u, v)$.

The following intervals are considered for $1 \leq i \leq l$:

$$I_i = \begin{cases} [f(w_{i-1}), f(w_i)] & \text{if } f(w_{i-1}) < f(w_i) \\ [f(w_i), f(w_{i-1})] & \text{if } f(w_i) < f(w_{i-1}) \\ \emptyset & \text{if } f(w_{i-1}) = f(w_i) \end{cases}$$

Thus, the union of these intervals is a cover of $[f(u), f(v)]$ and therefore the following contradiction is achieved:

$$f(v) - f(u) \leq \sum_{i=1}^l |f(w_{i-1}) - f(w_i)| = \sum_{i=1}^l \widehat{f}(e_i) = \widehat{f}(e_j) + \sum_{i=1, i \neq j}^l \widehat{f}(e_i) \leq$$

$$\begin{aligned}
&\leq \widehat{f}(e_j) + (l-1) \max_{1 \leq i \leq l} \widehat{f}(e_i) = \widehat{f}(e_j) + (l-1)F(u, v) < \\
&< F(u, v) + (l-1)F(u, v) = lF(u, v) = f(v) - f(u).
\end{aligned}$$

□

The two following results highlight the key role that the maximum value of the edge-colour-increase mapping F plays in the MIGG problem.

Corollary 3.5. *Let f be a greyscale of a connected graph $G(V, E)$ and let u and v be a pair of vertices of G such that $F(u, v) = \max_{a, b \in V} F(a, b)$ and $f(u) \leq f(v)$.*

For each vertex w of a $u - v$ geodesic the following holds:

$$f(w) = f(u) + d(u, w)F(u, v).$$

Proof. If w is u or v , then the result holds trivially. Otherwise, since $F(u, v) = \max_{a, b \in V} F(a, b)$, in particular, it holds that

$$F(u, v) \geq \max\{F(u, w), F(w, v)\}.$$

Then, by Lemma 3.2, $F(u, v) = F(u, w) = F(w, v)$ and $f(w)$ belongs to the interval $[f(u), f(v)]$. Now, from $F(u, v) = F(u, w)$, it is trivial to obtain the required statement for $f(w)$ in the following way:

$$F(u, v) = F(u, w) = \frac{f(w) - f(u)}{d(u, w)} \Rightarrow f(w) = f(u) + d(u, w)F(u, v).$$

□

Note that, if $f(v) \leq f(u)$, the result states by a similar proof using $F(u, v) = F(w, v)$ that $f(w) = f(v) + d(w, v)F(u, v)$.

Corollary 3.6. *Let f be a greyscale of a connected graph $G(V, E)$ and let u and v be a pair of vertices of G such that $F(u, v) = \max_{a, b \in V} F(a, b)$. For each edge e of a $u - v$ geodesic the following holds:*

$$\widehat{f}(e) = F(u, v).$$

Proof. Without loss of generality, $f(u) \leq f(v)$ can be assumed. Let a and b be the vertices of the edge e . Corollary 3.5 is applied to a and b , and since $d(u, b) = d(u, a) \pm 1$, it holds that $f(a) = f(u) + d(u, a)F(u, v)$ and $f(b) = f(u) + (d(u, a) \pm 1)F(u, v)$.

Then, $\widehat{f}(e) = |f(b) - f(a)| = |\pm F(u, v)| = F(u, v)$. \square

Next, a property for minimum gradation greyscales related to antipodal vertices which are white- and black-coloured is established.

Corollary 3.7. *If f is a minimum gradation greyscale of a graph G and u and v are vertices of G such that $f(u) = 0$ and $f(v) = 1$, then u and v are antipodal vertices.*

Proof. By Lemma 3.1, the values of the colour of the edges by any support greyscale are $\frac{1}{d(G)}$, $\frac{1}{2d(G)}$ and 0, sorted by decreasing order. Hence, $\widehat{f}(e) \leq \frac{1}{d(G)}$ for all $e \in E$ and so $\max_{e \in E} \widehat{f}(e) \leq \frac{1}{d(G)}$. Furthermore, since $f(u) = 0$ and $f(v) = 1$,

$F(u, v) = \frac{1}{d(u, v)}$ and by Corollary 3.3 $F(u, v) = \frac{1}{d(u, v)} \leq \max_{e \in E} \widehat{f}(e) \leq \frac{1}{d(G)}$. Then $d(u, v) \geq d(G)$, that is, $d(u, v) = d(G)$. \square

The characterization of the minimum gradation vector for paths is a consequence of some of the above-proved results.

Corollary 3.8. *The minimum gradation vector of the path of length n is the vector whose all components are equal to $\frac{1}{n}$.*

Proof. Let $\{u = w_0, e_1, w_1, e_2, w_2, \dots, w_{n-1}, e_n, v = w_n\}$ be the path of length n and let f be a greyscale such that \mathcal{G}_f is the minimum gradation vector of the path. There must exist two vertices mapped to 0 and 1 by f , and by Corollary 3.7 the only option is that they are u and v . Thus, $F(u, v) = \frac{1}{n}$.

By other hand, it is easy to check that $\widehat{f}\langle u, v \rangle(e_i) = \frac{1}{n}$ for $i = 1 \dots n$, being $f\langle u, v \rangle$ the support greyscale for u and v . So, $\max_{1 \leq i \leq n} \widehat{f}(e_i) \leq \frac{1}{n}$ follows. Moreover, Corollary 3.3 guarantees that $F(u, v)$ is a lower bound of $\max_{1 \leq i \leq n} \widehat{f}(e_i)$.

Hence $\frac{1}{n} = F(u, v) \leq \max_{1 \leq i \leq n} \widehat{f}(e_i) \leq \frac{1}{n}$, and by Lemma 3.4 we conclude that $\widehat{f}(e_i) = \frac{1}{n}$ for $i = 1 \dots n$. \square

Our next aim is to design algorithms which provide all the minimum gradation greyscales of a connected graph, for both MIGG and RMIGG problems. These greyscales are obtained in a stepwise manner by incomplete greyscales such that each of these is compatible with the previous one. Thus, an iterative procedure is carried out that is based on the operation of deleting coloured edges and isolated coloured vertices. We proceed first to devise the V_c -COMPATIBLE-COMPLETE-MAPPING common subroutine which is applied to solve both the different RMIGG problems according to the possible existence of the extreme tones as prefixed colours, and the MIGG problem.

PROCEDURE: V_c -COMPATIBLE-COMPLETE-MAPPING

Input: An incomplete V_c -greyscale g of a connected graph $G(V, E)$.

Output: A mapping f on V compatible with g .

1. Initialize $G_1(V_1^1, E_1) \leftarrow G(V, E)$
2. Initialize $i \leftarrow 1$
3. Initialize $l(1) \leftarrow 1$
4. **If** $u \in V_c$ **do** $f(u) = g(u)$;
5. **While** $|V_c| < |V|$ **do**
 - (a) Compute the distance matrix D_i of G_i ;
 - (b) Compute the finite value $M_i = \max_{1 \leq j \leq l(i)} \{F(a, b) : \{a, b\} \subseteq V_i^j \cap V_c\}$ and the set $S_i = \{\{u, v\} \subseteq V_i \cap V_c : F(u, v) = M_i\}$, where distances are taken from D_i ;
 - (c) **For** each $\{u, v\} \in S_i$ and considering distances from D_i **do**
 - i. Compute $A = \{w \in V_i : w \in u - v \text{ geodesic of } G_i\}$;
 - ii. **For** each $w \in A$ **do**

$$f(w) = \begin{cases} f(u) + d(w, u) M_i & \text{if } f(u) \leq f(v) \\ f(v) + d(w, v) M_i & \text{if } f(u) > f(v) \end{cases}$$
 - iii. $V_c \leftarrow V_c \cup A$
 - (d) Let $G_{i+1}(V_{i+1}, E_{i+1})$ be the subgraph of $G_i(V_i, E_i)$ obtained by deleting all the edges $w_1 w_2$ with $w_1, w_2 \in A$ and removing the resulting isolated vertices. Let V_{i+1}^j be the vertex-sets of the connected components of G_{i+1} for $j = 1 \dots l(i+1)$;

- (e) **If** $S_i = \emptyset$, each set V_i^j contains exactly one vertex w^j in V_c **then**
- i. **For** $j := 1$ to $l(i)$ **do** $f(u) = f(w^j)$ with $u \in V_i^j$;
 - ii. $V_c \leftarrow V_c \cup V_i^1 \cup \dots \cup V_i^{l(i)}$
- (f) $i \leftarrow i + 1$;

Note that the mapping generated by V_c -COMPATIBLE-COMPLETE-MAPPING procedure and the input incomplete V_c -greyscale have the same range, and therefore that mapping is not necessarily a greyscale due to the possible nonexistence of the extreme tones as values reached by the mapping. It is explanatory to point out that the grey tone assigned by Step 5(c)ii can be also obtained as follows:

$$f(w) = \begin{cases} f(v) - d(w, v) M_i & \text{if } f(u) \leq f(v) \\ f(u) - d(w, u) M_i & \text{if } f(u) > f(v) \end{cases}$$

Next, the RMIGG problems are resolved according to the nature of the prefixed colours, that is, distinguishing whether or not the extreme tones are prefixed values.

Theorem 3.9. *Let g be an incomplete V_c -greyscale of a connected graph $G(V, E)$ such that the extreme tones 0 and 1 are reached by g . Then it is possible to obtain, in polynomial time, a greyscale of G compatible with g whose gradation vector is the minimum one among all gradation vectors of greyscales compatible with g .*

Moreover, such greyscale is unique and it is provided by the V_c -COMPATIBLE-COMPLETE-MAPPING algorithm.

Proof. In order to state the result, the V_c -COMPATIBLE-COMPLETE-MAPPING algorithm is first proved to be finite and polynomial. Then, it is shown to provide the unique greyscale compatible with g such that its gradation vector is minimum among all gradation vectors of greyscales compatible with g .

At least one vertex is coloured at every iteration of the while-loop in Step 5, either by Substeps (e) or (f), and hence it ends after at most $|V| - |V_c|$ iterations. The time complexity of computing distance matrices (Step 3(c)) dominates the time complexity of the rest of the steps and that can be done in $O(n^3)$ time applying the Floyd-Warshall algorithm [7, 21]. So, the while-loop and the Step 3(c) determine the polynomial time of the V_c -COMPATIBLE-COMPLETE-MAPPING algorithm, that is, $O(n^4)$ time.

Now, the algorithm output f is proved to be a well-defined greyscale compatible with g . Its values are either the values of g (Step 4) or are assigned

by Step 5(c)ii to vertices belonging to $u - v$ geodesics such that $M_i = F(u, v)$. Owing to Step 4, f is compatible with g and since both extreme tones are prefixed colours, its range is the interval $[0, 1]$. It is necessary to check that the colour assignment by Step 5(c)ii is consistent, that is, both when a vertex in V_c is again coloured by Step 5(c)ii in the i -iteration, and in the case of a vertex belonging to different such geodesics. First, let w be a vertex with colour $f(w)$ belonging to a $u - v$ geodesic such that $M_i = F(u, v)$ (maximum value of Step 5b); in particular, $F(u, v)$ is greater than $F(u, w)$ and $F(w, v)$. This fact along with Lemma 3.2 for f, u, v and the subgraph induced by $V_{i-1} \cap V_c$ lead to $F(u, v) = F(u, w) = F(w, v)$ and $f(w)$ belonging to the interval of extremes $f(u)$ and $f(v)$. Whether $f(u) \leq f(v)$ (the reasoning is similar when $f(u) > f(v)$ and taking into account $F(u, v) = F(w, v)$), it holds that

$$F(u, v) = F(u, w) \Rightarrow M_i = \frac{f(w) - f(u)}{d(w, u)} \Rightarrow f(w) = f(u) + d(w, u) M_i.$$

In other words, the value assigned to w by Step 5(c)ii and its previous colour are the same.

On the other hand, let $P_{u_1-v_1}$ and $P_{u_2-v_2}$ be two geodesics such that $M_i = F(u_1, v_1) = F(u_2, v_2)$ (we assume, without loss of generality, $f(u_1) \leq f(v_1)$ and $f(u_2) \leq f(v_2)$) and let w be a vertex in $V - V_c$ belonging to both u_1v_1 and u_2v_2 . Suppose on the contrary that $f(u_1) + d(u_1, w) M_i > f(u_2) + d(u_2, w) M_i$ (the arguments are similar if the other inequality is assuming), and therefore

$$f(u_1) > f(u_2) + (d(u_2, w) - d(u_1, w)) M_i, \quad (4)$$

From $M_i = F(u_1, v_1)$ it holds that

$$f(v_1) = f(u_1) + (d(u_1, w) + d(w, v_1)) M_i. \quad (5)$$

Taking into account (4) and (5),

$$\begin{aligned} f(v_1) &> f(u_2) + (d(u_2, w) - d(u_1, w)) M_i + (d(u_1, w) + d(w, v_1)) M_i = \\ &= f(u_2) + (d(u_2, w) + d(w, v_1)) M_i \geq f(u_2) + d(u_2, v_1) M_i \Rightarrow \\ &\Rightarrow \frac{f(v_1) - f(u_2)}{d(u_2, v_1)} \geq M_i, \end{aligned}$$

which is a contradiction.

The gradation vector of f is now proved to be

$$(M_1, \dots, M_1, M_2, \dots, M_2, \dots, M_r, \dots, M_r, 0, \dots, 0),$$

where r is the number of executions of the while-loop, and the number of zeros can be null. For every edge of G either its vertices belong to the set A at only one execution of Step 5c or its colour is white (extreme tone 0) due to the situation described in Step 5e. In the first case, its colour is M_i by Corollary 3.6 applied to each pair of vertices in S_i and every connected component of G_i at which the maximum value M_i is reached. Furthermore, it is necessary to guarantee that the sequence of maximum values computed in Step 5b is strictly decreasing in i . The value d_i denotes the distance measured in G_i and is listed in the matrix D_i ; it is clear that $d_i(u, v) \leq d_{i+1}(u, v)$. Let M_i and M_{i+1} be the maximum values of the edge-colour-increase mapping F on G_i and G_{i+1} and computed by the iterations i and $i + 1$ of Step 5b, respectively. Let also u_{i+1} and v_{i+1} be two vertices such that $M_{i+1} = F(u_{i+1}, v_{i+1})$ on G_{i+1} . The executions at which the vertices u_{i+1} and v_{i+1} are coloured determine three cases:

1. Both vertices u_{i+1} and v_{i+1} are coloured before the i -iteration takes place. Then, if $\{u_{i+1}, v_{i+1}\} \notin S_i$, it follows that

$$M_i > \frac{|f(u_{i+1}) - f(v_{i+1})|}{d_i(u_{i+1}, v_{i+1})} \geq \frac{|f(u_{i+1}) - f(v_{i+1})|}{d_{i+1}(u_{i+1}, v_{i+1})} = M_{i+1}.$$

Otherwise, the fact that $\{u_{i+1}, v_{i+1}\} \in S_i$ leads to $d_i(u_{i+1}, v_{i+1}) < d_{i+1}(u_{i+1}, v_{i+1})$ and then

$$M_i = \frac{|f(u_{i+1}) - f(v_{i+1})|}{d_i(u_{i+1}, v_{i+1})} > \frac{|f(u_{i+1}) - f(v_{i+1})|}{d_{i+1}(u_{i+1}, v_{i+1})} = M_{i+1}.$$

2. One of the vertices u_{i+1} and v_{i+1} is coloured at the i -iteration but the other one is previously. Without loss of generality, $f(u_{i+1}) \leq f(v_{i+1})$ can be assumed and then we distinguish two possibilities depending on whether either u_{i+1} receives its grey tone at the i -iteration or v_{i+1} does.

- (a) If the vertex u_{i+1} is coloured at the i -iteration, it belongs to some $u_i - v_i$ geodesic such that $M_i = F(u_i, v_i)$ on G_i (the inequality $f(u_i) \leq f(v_i)$ can be assumed) and so it holds that

$$f(u_{i+1}) = f(u_i) + d_i(u_i, u_{i+1}) M_i \Rightarrow f(u_i) = f(u_{i+1}) - d_i(u_i, u_{i+1}) M_i. \quad (6)$$

On the other hand, v_{i+1} is coloured previously to the i -iteration and the inequality between M_i and M_{i+1} is achieved by distinguishing if $\{u_i, v_{i+1}\}$ belongs or not to S_i . In case that $\{u_i, v_{i+1}\} \notin S_i$, the equality in (6) leads to:

$$\begin{aligned} M_i &> \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, v_{i+1})} \geq \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, u_{i+1}) + d_i(u_{i+1}, v_{i+1})} \geq \\ &\geq \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})} = \\ &= \frac{f(v_{i+1}) - f(u_{i+1}) + d_i(u_i, u_{i+1}) M_i}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})} = \\ &= \frac{d_{i+1}(u_{i+1}, v_{i+1}) M_{i+1} + d_i(u_i, u_{i+1}) M_i}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})}. \end{aligned}$$

If $M_i \leq M_{i+1}$, then

$$M_i > \frac{[d_{i+1}(u_{i+1}, v_{i+1}) + d_i(u_i, u_{i+1})] M_i}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})} = M_i,$$

which is a contradiction, and therefore $M_i > M_{i+1}$.

In case that $\{u_i, v_{i+1}\} \in S_i$ and since $d_i(u_i, v_{i+1}) < d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})$ due to the connection of u_{i+1} and v_{i+1} in G_{i+1} , it follows that

$$M_i = \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, v_{i+1})} > \frac{f(v_{i+1}) - f(u_i)}{d_i(u_i, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1})},$$

and the reasoning goes on as in the lines above.

- (b) If the vertex v_{i+1} is coloured at the i -iteration, similar arguments lead to the result taking into account two facts: v_{i+1} belongs to some $u_i - v_i$ geodesic such that $M_i = F(u_i, v_i)$ on G_i and $f(u_i) \leq f(v_i)$, which implies that $f(v_{i+1}) = f(v_i) - d_i(v_i, v_{i+1}) M_i$, and the membership or not of $\{u_{i+1}, v_i\}$ in S_i .
3. Both vertices u_{i+1} and v_{i+1} are coloured by the i -iteration. Therefore, there exist two pairs of vertices $\{u_i^1, v_i^1\}$ and $\{u_i^2, v_i^2\}$ of G_i such that $M_i = F(u_i^1, v_i^1) = F(u_i^2, v_i^2)$ and the vertices u_{i+1} and v_{i+1} belong to some

$u_i^1 - v_i^1$ and $u_i^2 - v_i^2$ geodesic, respectively (without loss of generality we may suppose that $f(u_{i+1}) \leq f(v_{i+1})$, $f(u_i^1) \leq f(v_i^1)$ and $f(u_i^2) \leq f(v_i^2)$). Then,

$$f(u_{i+1}) = f(u_i^1) + d_i(u_i^1, u_{i+1}) M_i \Rightarrow f(u_i^1) = f(u_{i+1}) - d_i(u_i^1, u_{i+1}) M_i \quad (7)$$

$$f(v_{i+1}) = f(v_i^2) - d_i(v_i^2, v_{i+1}) M_i \Rightarrow f(v_i^2) = f(v_{i+1}) + d_i(v_i^2, v_{i+1}) M_i \quad (8)$$

Whether $\{u_i^1, v_i^2\} \notin S_i$ and from (7) and (8), it holds that

$$\begin{aligned} M_i &> \frac{f(v_i^2) - f(u_i^1)}{d_i(u_i^1, v_i^2)} \geq \frac{f(v_i^2) - f(u_i^1)}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} = \\ &= \frac{f(v_{i+1}) + d_i(v_i^2, v_{i+1}) M_i - f(u_{i+1}) + d_i(u_i^1, u_{i+1}) M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} = \\ &= \frac{d_{i+1}(u_{i+1}, v_{i+1}) M_{i+1} + d_i(v_i^2, v_{i+1}) M_i + d_i(u_i^1, u_{i+1}) M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} \end{aligned}$$

If $M_i \leq M_{i+1}$, then

$$M_i > \frac{[d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)] M_i}{d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)} = M_i,$$

which is a contradiction, and hence $M_i > M_{i+1}$.

Whether $\{u_i^1, v_i^2\} \in S_i$, the inequality between M_i and M_{i+1} is achieved by applying that $d_i(u_i^1, v_i^2) < d_i(u_i^1, u_{i+1}) + d_{i+1}(u_{i+1}, v_{i+1}) + d_i(v_{i+1}, v_i^2)$, which follows from the connection of u_{i+1} and v_{i+1} in G_{i+1} .

Our next and final aim is to prove that f is the only greyscale compatible with g such that its gradation vector

$$\text{grad}(G, f) = (M_1, \dots, M_1, M_2, \dots, M_2, \dots, M_r, \dots, M_r, 0, \dots, 0)$$

is minimum among all gradation vectors of such greyscales. For this purpose, let C_k be the vertex-set of G containing the vertices that have been coloured at any of the first k executions of the while-loop in Step 5, for $k = 1 \dots r$, being r the total number of executions of Step 5. Given a minimum gradation greyscale h compatible with g we prove by induction on k that $h(w) = f(w)$ for all $w \in V$.

For $k = 1$, every vertex of C_1 belongs to some $u - v$ geodesic $P_{u-v} = \{u = w_0, e_1, w_1, e_2, w_2, \dots, w_{l-1}, e_l, v = w_l\}$ (alternating sequence of vertices and edges) such that $g(u) = f(u) = h(u)$ and $g(v) = f(v) = h(v)$. There does not exist

an edge e_i such that $\widehat{h}(e_i) > M_1$ due to the minimality of \mathcal{G}_h and the existence of \mathcal{G}_f , and the next argument guarantees the non-existence of an edge e_i such that $\widehat{h}(e_i) < M_1$, therefore $\widehat{h}(e_i) = \widehat{f}(e_i) = M_1$ for all edge of P_{u-v} . Both values of any greyscale on consecutive vertices of P_{u-v} define intervals whose union is a cover of the interval $[g(u), g(v)]$ or $[g(v), g(u)]$. The lengths of these intervals, that is, the grey tones of the corresponding edges, are all M_1 for f , and so, also for the greyscale h . Thus, if there exists one of them less than M_1 there must exist another one greater than M_1 , but this fact is not possible owing to the minimality of \mathcal{G}_h and the existence of \mathcal{G}_f .

Then, $f(w_i) = h(w_i)$ for all vertices of P_{u-v} since $\widehat{f}(e_i) = \widehat{h}(e_i) = M_1$ for all edges of P_{u-v} and $f(u) = h(u) = g(u)$ and $f(v) = h(v) = g(v)$.

For the induction step, the same previous reasoning is applied to the elements of the geodesics taking part in the execution $k + 1$ of Step 5, since the extreme vertices of such geodesics belong to C_k and therefore their grey tones assigned by f and h are equal. \square

The next result solves the RMIGG problem in the case of only one type of extreme colour, either white or black, among the prefixed values.

Theorem 3.10. *Let g be an incomplete V_c -greyscale of a connected graph $G(V, E)$ such that only one extreme tone, either 0 or 1, is reached by g . Then it is possible to obtain, in polynomial time, all the greyscales of G compatible with g whose gradation vector is the minimum one among all gradation vectors of greyscales compatible with g .*

Proof. Without loss of generality, we may suppose that g reaches the white colour 0 but not the black colour 1. There exists a vertex $w \in V - V_c$ such that $f(w) = 1$ for any greyscale f compatible with g , in particular, any minimum gradation greyscale. Then, for every vertex $w \in V - V_c$ a new incomplete greyscale g_w is defined such that $g_w(u) = g(u)$ whether $u \in V_c$ and $g_w(w) = 1$. In accordance with Theorem 3.9 for g_w , there exists only one greyscale f_w compatible with g_w whose gradation vector is minimum among all gradation vectors of greyscales compatible with g_w .

Among these $|V - V_c|$ greyscales f_w , those whose gradation vector is minimum are the solutions of the RMIGG problem for only one extreme colour, and they have been obtained by running the COMPATIBLE-COMPLETE-MAPPING polynomial procedure $|V - V_c|$ times. Hence $O(n^5)$ is achieved for this problem. \square

Now, the following result resolves the RMIGG problem in case of neither the black colour nor the white one appears among the prefixed values.

Theorem 3.11. *Let g be an incomplete V_c -greyscale of a connected graph $G(V, E)$ such that no extreme tone is reached by g . Then it is possible to obtain, in polynomial time, a greyscale of G compatible with g whose gradation vector is the minimum one among all gradation vectors of greyscales compatible with g .*

Proof. This proof is similar to the proof of Theorem 3.10, but in this case a new incomplete $V_c \cup \{w_1, w_2\}$ -greyscale is defined for every pair of vertices w_1 and w_2 of $|V - V_c|$ such that $g_{\{w_1, w_2\}}(u) = g(u)$ whether $u \in V_c$, $g_{\{w_1, w_2\}}(w_1) = 0$ and $g_{\{w_1, w_2\}}(w_2) = 1$. Since there are $\binom{|V - V_c|}{2}$ of these incomplete greyscales and the COMPATIBLE-COMPLETE-MAPPING polynomial procedure provides only one greyscale for each one of them, this problem can be solved in $O(n^6)$ time. \square

Finally, the following and last theorem solves the MIGG problem.

Theorem 3.12. *The MIGG problem can be solved in polynomial time, that is, the minimum gradation vector and all their minimum gradation greyscales are obtained in polynomial time.*

Proof. All possible greyscales can be considered taking all possible pairs of vertices to be coloured with the extreme tones, that is, $\binom{|V|}{2}$ incomplete greyscales. Theorem 3.9 is applied to each of these incomplete greyscales and hence, the best greyscales in the sense of gradation have to be selected among a set of $\binom{|V|}{2}$ greyscales, each of them computed in polynomial time. Thus, the minimum gradation vector of the given graph, as well as all their minimum gradation greyscales are achieved in polynomial time. \square

In accordance with Corollary 3.7 it is possible to reduce the actual time of the V_c -COMPATIBLE-COMPLETE-MAPPING procedure applied to the MIGG problem ($V_c = \emptyset$) only taking into account the pairs of antipodal vertices to be coloured with the extreme tones, instead of all the pairs of vertices of the graph. By other hand, the following observation also reduces the actual time of the algorithms that solve the MIGG problem and the RMIGG problems when both extreme tones are not reached by the incomplete greyscale. The V_c -COMPATIBLE-COMPLETE-MAPPING procedure has to be applied a quadratic number of times in the worst case, once for each pair of vertices coloured with black and white. These executions can be performed in parallel and since we are dealing with minimax problems, after each iteration of the while-loop, it suffices to continue with the executions that lead to the minimum value for M_i and, moreover, appearing the minimum number of times in the gradation vector. The rest of these executions of the V_c -COMPATIBLE-COMPLETE-MAPPING procedure can be discarded.

Finally, it is easy to check the following result.

Corollary 3.13. *At least, the first $d(G)$ components of the minimum gradation vector of the MIGG problem for a graph G with diameter $d(G)$ are $\frac{1}{d(G)}$.*

4 Open problems

The new concept of gradation of a graph, related to vertex and edge colourings, has been introduced, and polynomial algorithms have been designed to solve the problem of determining the minimum gradation vector depending on whether or not there exist prefixed colours. Nevertheless, the algorithms developed in this paper have high time complexity so that the main open problem that immediately raises by our work is to improve the computational time required to solve gradation problems. Our time complexities are determined by the computation of the distance matrix of the graph so different resolution techniques would have to be investigated in order to reduce the computational times.

It would be also interesting to pose gradation in digraphs, studying the more suitable way of assigning colours to the directed edges.

Acknowledgements

The authors gratefully acknowledge financial support by the Spanish Ministerio de Economía, Industria y Competitividad and Junta de Andalucía via grants, MTM2015-65397-P (M.T. Villar-Liñán) and PAI FQM-164, respectively.

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