DieudonnÉ—kÖthe duality for vector—valued function spaces: Localization of bounded sets and barrelledness

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Dieudonné-Köthe duality for vector-valued function spaces: localization of bounded sets and barrelledness*

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Abstract

We study Dieudonné-Köthe spaces of Lusin-measurable functions with values in a locally convex space. Let Λ be a solid locally convex lattice of scalar-valued measurable functions defined on a measure space Ω. If E is a locally convex space, define $\Lambda\{E\}$ as the space of all Lusin-measurable functions $f : \Omega \to E$ such that $q(f(\cdot))$ is a function in Λ for every continuous seminorm q on E. The space $\Lambda\{E\}$ is topologized in a natural way and we study some aspects of the locally convex structure of $\Lambda\{E\}$; namely, bounded sets, completeness, duality and barrelledness. In particular, we focus the important case when Λ and E are both either metrizable or (DF)-spaces and derive good permanence results for reflexivity when the density condition holds.
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1 Introduction

The Köthe-Toeplitz theory of perfect sequence spaces (see [25, §30] or [44, Ch. 2]) has been one of the most influential in the study of the structure of locally convex spaces. It has provided the specialists with plenty of hints, examples and counterexamples. Dieudonné [9] extended this theory to spaces of measurable functions by replacing the $\sum$ in the definition of Köthe and Toeplitz with a suitable $\int$. More recent developments include the study of the vector-valued cases: Köthe spaces of sequences from a locally convex space [5], [14], [17], [18], [33], [35], L^p spaces with values in a Banach space [8] or a locally convex space [10], [11], [12], [13], [23], and Dieudonné-Köthe spaces of measurable functions with values in a normed space [15], [30], [32].

The aim of this paper is to study Dieudonné-Köthe spaces of measurable functions with values in a locally convex space and we shall be concerned mainly with two problems: localization of bounded sets and barrelledness. Of course, the idea is to extend to the case of functions the techniques used for vector-valued sequences. One of the first troubles that we face is that we do not have coordinates anymore, so we have to choose an appropriate definition of measurable function. The most suitable one for our purposes turns out to be Lusin measurability. In the case of sequences, one uses properties of the space of all absolutely convergent sequences from a locally convex space. Hence in our case we need a theory of absolute integration of functions with values in a locally convex space analogous to the theory of Bochner integral for Banach spaces. We shall use the theories of summable functions and absolutely p-integrable functions developed by Thomas [41], [43] and continued in our previous paper [12].

Roughly speaking, let Λ be a solid locally convex lattice of scalar-valued measurable functions defined on a measure space Ω. If E is a locally convex space, define $\Lambda\{E\}$ as the space of all Lusin-measurable functions $f : \Omega \to E$ such that $q(f(\cdot))$ is a function in Λ for every continuous seminorm q on E. The space $\Lambda\{E\}$ is topologized in a natural way and we study some aspects of the locally convex structure of $\Lambda\{E\}$; namely, bounded sets, completeness, duality and barrelledness. The organization of the paper is as follows: Section 2 contains the precise definitions and describes our framework for the rest of the paper. Section 3 is devoted to the problem of localization of bounded sets. We introduce an extension of the well-known property (B) of Pietsch [33, 1.5.5] and prove that if either both Λ and E are metrizable or both are (df)-spaces, then the bounded sets in $\Lambda\{E\}$ can be lifted from suitable bounded sets in Λ and E. In Section 4, we give an example showing that $L^1\{E\}$ need not to be complete even if E is complete. This forces us to introduce the additional hypothesis "$L^1\{E\}$ is complete" to derive completeness results for $\Lambda\{E\}$. Denote by $\Lambda^\times$ the Dieudonné-Köthe dual of Λ. The topological dual of Λ with the strong topology $\beta(\Lambda, \Lambda^\times)$ is not always $\Lambda^\times$ as the example $\Lambda = L^\infty$ shows. Even if the strong dual of Λ is $\Lambda^\times$, it may happen that the dual of $\Lambda\{E\}$ does not coincide with $\Lambda^\times\{E'_b\}$; e.g. for a Banach space E, the equality $(\Lambda\{E\})' = \Lambda^\times\{E'_b\}$ holds if and only if E' has the Radon-Nikodym property with respect to μ [15, Thm. 1]. When $\Lambda = L^1$, this was extended to quasi-barrelled spaces E such that E'_b has property (B) of Pietsch [12, 4.7]. Then, in order to ensure that $(\Lambda\{E\})' = \Lambda^{\times}\{E'_b\}$ for a general locally convex space E, we must impose some Radon-Nikodym type of condition on E. This problem will be studied in Section 5. In Section 6 we use the results about duality to obtain the barrelledness of $\Lambda\{E\}$ in some situations. The last section is devoted to the important case when Λ and E are both either metrizable or (DF)-spaces and derive good permanence results for reflexivity when the density condition holds.

2 Terminology and Notation

MEASURE SPACE. Throughout, Ω stands for a Hausdorff, locally compact and σ-compact topological space and μ for a Radon measure on Ω, i.e. μ is a (non-negative) measure defined on the σ-algebra of all Borel subsets of Ω such that

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}$$

for every Borel set $A \subset \Omega$. For the sake of notational simplicity, we shall work with the completion (Ω, Σ, μ) of the Radon measure space. For the same reason, we shall simply write a.e. or L^p instead of μ-a.e. or $L^p(\mu)$. We denote by p^* the conjugate number of $p \in [1, \infty]$. We assume, to avoid trivial cases, that our measure space does not reduce to a finite number of atoms. A particular instance of Radon measure space is the set $\mathbb{N}$ of natural numbers with the discrete topology and the counting measure. In this case, the spaces we deal with are vector-valued sequence spaces and, as we pointed out in the Introduction, they will be relevant because of the insights they provide to tackle the general case.

DIEUDONNÉ-KÖTHE DUALITY. Let L^1_{loc} be the space of all (classes of a.e. equal) measurable and locally integrable scalar-valued functions. The family of semi-norms

$$\phi \in L^1_{\text{loc}} \to \int_K |\phi|\, d\mu = \int_\Omega |\phi| \cdot \chi_K\, d\mu$$

obtained when K runs through the compact subsets of Ω defines a metrizable topology on L^1_{loc} because Ω is σ-compact. With this topology L^1_{loc} is a Fréchet space. The dual of L^1_{loc} is the space L^∞_c of all essentially bounded measurable functions having compact support. In what follows, Λ will stand for a solid subspace of L^1_{loc} containing L^∞_c; solid meaning that if $\phi \in \Lambda$ and $\psi : \Omega \to \mathbb{R}$ is a measurable function such that $|\psi| \leq |\phi|$ a.e., then ψ is also in Λ. The Dieudonné-Köthe dual of Λ is the space $\Lambda^\times$ defined by:

$$\Lambda^\times := \left\{\theta \in L^1_{\text{loc}} : \phi \cdot \theta \in L^1 \text{ for all } \phi \in \Lambda\right\},$$

and Λ is said to be perfect if $\Lambda = (\Lambda^\times)^\times$. The space $\Lambda^\times$ is perfect and solid. For instance, (L^p, L^{p^*}) and $(L^1_{\text{loc}}, L^\infty_c)$ are Dieudonné-Köthe dual pairs of perfect spaces. The spaces Λ and $\Lambda^\times$ are put into separate duality by means of the bilinear form

$$(\phi, \theta) \in \Lambda \times \Lambda^\times \to \int_\Omega \phi(t) \cdot \theta(t)\, d\mu(t).$$

This duality was defined and studied by Dieudonné [9]. This line of research was continued by several authors: [6], [16], [19], [24], [27], [28], [29], [34], [37], [38], [39], [40] and [45].

We recall here some relevant facts. The space $\Lambda^\times$ is sequentially complete for the weak topology $\sigma(\Lambda^\times, \Lambda)$. Therefore, by the Banach-Mackey Theorem, every weakly bounded set of $\Lambda^\times$ is bounded for the strong topology $\beta(\Lambda^\times, \Lambda)$. We shall consider on Λ solid topologies. These are polar topologies of uniform convergence on solid and bounded subsets of $\Lambda^\times$. If $\mathcal{M}$ is a saturated family of solid subsets of $\Lambda^\times$ that are $\sigma(\Lambda^\times, \Lambda)$-bounded, and such that $\mathcal{M}$ covers $\Lambda^\times$ and is stable by finite unions and positive multiples, then the solid topology of uniform convergence on $\mathcal{M}$ is called the $\mathcal{M}$-topology. Since every $M \in \mathcal{M}$ is solid, the corresponding polar seminorm on Λ is given by

$$\phi \in \Lambda \to q_M(\phi) := \sup \left\{ \int_\Omega |\phi(t) \cdot \theta(t)| \, d\mu(t) : \theta \in M \right\} \in \mathbb{R}.$$

The coarsest solid topology is the one obtained for the family of the solid hulls of singletons in $\Lambda^\times$. This topology is compatible with the dual pair $(\Lambda, \Lambda^\times)$. On the other extreme, the finest solid topology is obtained for the family of all $\sigma(\Lambda^\times, \Lambda)$-bounded and solid sets in $\Lambda^\times$. This topology coincides with the strong topology $\beta(\Lambda, \Lambda^\times)$. The Mackey topology $\tau(\Lambda, \Lambda^\times)$ is also a solid topology that not always coincides with the strong topology, as the case $\Lambda = L^\infty$ shows. Some necessary and sufficient conditions for the equality of the Mackey and strong topologies are given in [16]. As another consequence of the Banach-Mackey Theorem, let us mention that all solid topologies on Λ share with the weak topology $\sigma(\Lambda, \Lambda^\times)$ the same family of bounded sets. When Λ is perfect, all solid topologies are complete and this is also the case for solid topologies on $\Lambda^\times$ of uniform convergence on saturated families of solid bounded sets in Λ (not necessarily perfect, merely solid). A useful condition when dealing with sequence spaces is the convergence of the finite sections to the element. In our case, we have that if the $\mathcal{M}$-topology is coarser than the Mackey topology, then for every $\phi \in \Lambda$ the following hold: (a) $\lim_n \phi\chi_{K_n} = \phi$ for every increasing fundamental sequence of compact sets covering Ω a.e., and (b) $\lim_{\mu(A)\to 0} \phi\chi_A = 0$ [16, Thm. 1, (i) $\Rightarrow$ (ii)] (the statement of this theorem required Λ to be perfect, but the proof of the implication (i) $\Rightarrow$ (ii) works also for solid spaces).

Lusin-Measurable Functions. Let E be a locally convex space over the field of the real or complex numbers. We denote by $\mathcal{Q}(E)$ the class of all continuous seminorms on E, by E'_b the dual of E endowed with the strong topology $\beta(E', E)$, and by p_D the Minkowski functional of an absolutely convex set $D \subset E$. If D is a disc, i.e. an absolutely convex, closed and bounded set, then p_D is a norm on the linear span E_D of D. We shall work with Lusin-measurable functions. A function $f : \Omega \to E$ is said to be Lusin-measurable if for every compact set $K \subset \Omega$ and every $\varepsilon > 0$ there exists a compact set $K' \subset K$ such that $\mu(K \setminus K') < \varepsilon$ and the restriction $f : K' \to E$ is continuous. Equivalently, since our measure space is σ-finite, f is Lusin-measurable if there exists a sequence (K_n) of compact sets in Ω (that we may suppose either disjoint or increasing) such that $\mu(\Omega \setminus \bigcup_n K_n) = 0$ and each restriction $f : K_n \to E$ is continuous. For the sake of brevity, in what follows we shall simply say that f is measurable; if some other type of measurability is used, it will be mentioned explicitely. We think that measurability in the sense of Lusin, used e.g. in [3], [4], [10], [11], [12], [13], [41], [42], and [43], is the most appropriate for the case of a general locally convex space. If E is a Fréchet space, it coincides with the usual notion of strongly measurable function as the a.e. limit of a sequence of simple functions. A measurable function f is zero a.e. if and only if the composition with every $x' \in E'$ is also zero a.e.; in particular, two measurable functions $f, g : \Omega \to E$ are equal a.e. if and only if the scalar functions $\langle f(\cdot), x' \rangle$ and $\langle g(\cdot), x' \rangle$ coincide a.e. for each $x' \in E'$. In what follows, we shall identify, without further mention, measurable functions that are a.e. equal. We say that a function $f : \Omega \to E$ is localized in a set $B \subset E$ if $f(t) \in B$ a.e.; in particular, a function $f : \Omega \to E$ is localized in a bounded set if and only if its composition with every continuous seminorm on E, or only every scalar function $\langle f(\cdot), x' \rangle$ ($x' \in E'$), is essentially bounded. Another useful fact is that if a measurable function f is localized in E_D for some absolutely convex and closed set D, then the scalar function $t \to p_D(f(t))$ is also measurable. We shall make frequent use of the following result from [12].

Estimation Lemma [12, 3.7]. *Let K be a compact topological space, $f : K \to E$ a continuous function and $B \subset E$ an absolutely convex and closed set.*

1. *If $f(K)$ does not meet E_B then for each $n \in \mathbb{N}$ there exists a simple function $s_n : K \to B^\circ \subset E'$ such that $\operatorname{Re} \langle f(t), s_n(t) \rangle > n$ for all $t \in K$.*

2. *If $f(K) \subset E_B$ and $p_B(f) : K \to \mathbb{R}$ is continuous then for each $\varepsilon > 0$ there exists a simple function $s : K \to B^\circ \subset E'$ such that*

$$p_B(f(t)) < \operatorname{Re} \langle f(t), s(t) \rangle + \varepsilon \qquad \textit{for all } t \in K.$$

Let us remark that in the original statement B was a disc. However, the proof works for the more general case of an absolutely convex and closed set B because all you need is $B = B^{\circ\circ}$.

Summable Functions. A continuous function $f : K \to E$ defined on a compact topological space K and with values in a quasi-complete locally convex space E always has an integral $\int_K f \, d\mu \in E$, defined as the limit of Riemann sums [4, §6, Prop. 8]. This integral is characterized by the fact that

$$\left\langle \int_K f \, d\mu, x' \right\rangle = \int_K \langle f, x' \rangle \, d\mu \qquad \text{for all } x' \in E'.$$

This was used by Thomas [41] to define and study the integration of measurable functions. Let E be a locally convex space and $f : \Omega \to E$ be a measurable function. By analogy with the notion of summable families of numbers, and following [41, 1.2], we say that f is summable if the following net converges:

$$\left\{ \int_K f \, d\mu : K \in \mathcal{C}_f \right\},$$

where $\mathcal{C}_f$ is the family, ordered by inclusion, of all compact subsets $K \subset \Omega$ such that $f : K \to E$ is continuous. In that case, the integral of f is

$$\int_\Omega f \, d\mu := \lim_{K \in \mathcal{C}_f} \int_K f \, d\mu.$$

For $A \in \Sigma$ define $\int_A f\, d\mu := \int_\Omega f \cdot \chi_A\, d\mu$. Then $\langle \int_A f\, d\mu, x' \rangle = \int_A \langle f, x' \rangle\, d\mu$ for all $x' \in E'$ and $A \in \Sigma$.

Radon-Nikodym Property. A vector measure is a countably additive set function $m : \Sigma \to E$. A vector measure m is said to be μ-continuous if $m(A)$ converges to zero when $\mu(A)$ converges to zero and is said to have bounded variation if for each $q \in \mathcal{Q}(E)$ one has

$$|m|_q(\Omega) := \sup \left\{ \sum_{A \in \Pi} q(m(A)) : \Pi \text{ is a measurable partition of } \Omega \right\} < \infty.$$

It is easy to see that for a μ-continuous vector measure having bounded variation, each set function $A \in \Sigma \to |m|_q(A)$ is a μ-continuous positive measure. We say that a vector measure $m : \Sigma \to E$ has a density (with respect to μ) if there exists a summable function $g : \Omega \to E$, the density of m, such that the scalar function $t \to q(g(t))$ is in L^1 for every $q \in \mathcal{Q}(E)$ and $m(A) = \int_A g\, d\mu$ for all $A \in \Sigma$. A locally convex space E is said to have the Radon-Nikodym property (with respect to μ) if each μ-continuous vector measure $m : \Sigma \to E$ with bounded variation has a density. When E is a Banach space, this Radon-Nikodym property coincides with the usual one. Every reflexive strict (LF)-space has the Radon-Nikodym property [12, 4.10]. On the other hand, all locally convex spaces have the Radon-Nikodym property with respect to the counting measure on $\mathbb{N}$.

We refer the reader to the books by Jarchow [22], Köthe [25], [26] or Pérez Carreras and Bonet [31] for the terminology about locally convex spaces and to the monographs by Bourbaki [3], [4], Diestel and Uhl [8], Schwartz [36] or Thomas [41] for the properties of measurable functions and vector measures. Our paper [12] contains several Radon-Nikodym theorems for Lusin-measurable functions.

3 Localization of Bounded Sets

Let us give the natural extension to the function case of the vector-valued sequence spaces defined by Pietsch [33] and studied later by De Grande-De Kimpe [5] and Rosier [35].

Definition. A function $f : \Omega \to E$ is said to be Λ-integrable if it is measurable and for every $q \in \mathcal{Q}(E)$ the scalar function

$$q(f) : t \in \Omega \to q(f(t)) \in \mathbb{R}$$

is a function in Λ. We shall denote by $\Lambda\{E\}$ the vector space of all Λ-integrable functions. Note that $\Lambda\{E\}$ is solid in the sense that if $\theta \in L^\infty$ and $f \in \Lambda\{E\}$, then $\theta \cdot f \in \Lambda\{E\}$. Observe also that $\Lambda\{E\}$ contains the space $S_c(E)$ of all simple functions with compact support and also the space $L_c^\infty(E)$. If we have a solid $\mathcal{M}$-topology on Λ, we define on $\Lambda\{E\}$ a natural topology by means of the seminorms

$$f \in \Lambda\{E\} \to q_M(q(f)) := \sup \left\{ \int_\Omega |\theta(t)| \cdot q(f(t))\, d\mu(t) : \theta \in M \right\} \in \mathbb{R},$$

where $q \in \mathcal{Q}(E)$ and $M \in \mathcal{M}$. When E is a Banach space and $\Lambda = L^p$, the space $\Lambda\{E\}$ is the space $L^p\{E\}$ of Bochner p-integrable functions. When E and Λ are metrizable (resp. normable) then $\Lambda\{E\}$ also is.

It follows from the scalar case mentioned in the Introduction that if the $\mathcal{M}$-topology is coarser than the Mackey topology $\tau(\Lambda, \Lambda^\times)$, then for each $f \in \Lambda\{E\}$ we have (a) $\lim_n f\chi_{K_n} = f$ for every increasing fundamental sequence of compact sets and (b) $\lim_{\mu(A)\to 0} f\chi_A = 0$. The measurability of a function tells us that we can approximate it by simple functions uniformly on appropriate compact sets. Using this together with properties (a) and (b) above, a straightforward computation shows that $S_c(E)$ is dense in $\Lambda\{E\}$ in this case.

In this section, we shall be mainly concerned with the structure of the bounded subsets of $\Lambda\{E\}$ or, rather, with the possibility of finding them in a natural way from bounded sets in Λ and E. Since all solid topologies in Λ share the same families of bounded sets, it follows that a set $C \subset \Lambda\{E\}$ is bounded if and only if for each $q \in \mathcal{Q}(E)$ and $\theta \in \Lambda^\times$, the set $\{\int_\Omega |\theta|\, q(f)\, d\mu : f \in C\}$ is bounded in $\mathbb{R}$. For vector-valued sequence spaces, Rosier [35] (see [31, 4.9.7–11] as well) introduced the notion of fundamental λ-boundedness; a concept generalizing property (B) of Pietsch [33, 1.5.5] that translated into our context reads as follows.

Definition. We say that E is fundamentally Λ-bounded if each bounded subset of $\Lambda\{E\}$ is contained in a set of the form

$$[R, B] := \{f \in \Lambda\{E\} : f(t) \in E_B \text{ a.e. and } p_B(f) \in R\},$$

where B is a disc in E and R is a solid disc in Λ. (It is clear that such a set $[R, B]$ is bounded in $\Lambda\{E\}$.) Equivalently, E is fundamentally Λ-bounded if for each bounded set $C \subset \Lambda\{E\}$ there exists a disc $B \subset E$ such that each function $f \in C$ is localized in E_B and $\{p_B(f) : f \in C\}$ is a bounded subset of Λ. It is easy to see that fundamental Λ-boundedness is hereditary for subspaces but it is an open question whether it lifts to completions.

With this definition, fundamental l^1-boundedness is just property (B). For examples of fundamentally λ-bounded spaces, where λ is a Köthe sequence space, we refer the reader to [14] and [35]. Normed spaces are fundamentally Λ-bounded [15]. For L^p spaces we know that if E is quasi-complete then E is fundamentally L^p-bounded if and only if it is fundamentally ℓ^p-bounded [12, 3.8]. As a special case, we have that metrizable, strict (LF)-spaces and (df)-spaces are fundamentally L^p-bounded for every $p \in [1, \infty]$ [12, 3.10]. Every locally convex space is fundamentally L^∞-bounded (in particular, ℓ^∞-bounded) [10, Lem. 1].

Our first result extends to our case one of the main contributions of Rosier's [35, 6.(5)]. The root of this result can be traced back to the localization of bounded sets in $L^1\{E\}$ and $\ell^1\{E\}$ given by Grothendieck [20, pp. 68–69] for Fréchet spaces, and Pietsch [33, 1.5.8] for metrizable and (df)-spaces.

Theorem 1. *Let Λ be a perfect space that is metrizable for the strong topology $\beta(\Lambda, \Lambda^\times)$. The following hold:*

1. *If E is metrizable then E is fundamentally Λ-bounded.*

2. If E is a (df)-space then E is fundamentally $\Lambda^\times$-bounded.

Proof. Since $\Lambda(\beta(\Lambda,\Lambda^\times))$ is metrizable, $\Lambda^\times$ has an increasing fundamental sequence of $\sigma(\Lambda^\times,\Lambda)$-bounded solid discs (M_n).

Part (1): Let C be a bounded subset in $\Lambda\{E\}$. For each $q\in\mathcal{Q}$ and $n\in\mathbb{N}$, the following supremum is finite

$$\sup\left\{\int_\Omega |\theta|\, q(f)\, d\mu : \theta\in M_n, f\in C\right\},$$

hence $M_n\cdot C := \{\theta\cdot f : \theta\in M_n, f\in C\}$ is a bounded subset of $L^1\{E\}$. Since metrizable spaces are fundamentally L^1-bounded [12, 3.10], there exists a disc $B_n\subset E$ such that $M_n\cdot C\subset [V_1, B_n]$ where V_1 is the closed unit ball of L^1. This tells us that for each $f\in C$ and $\theta\in M_n$ we have

$$\text{(i) } \theta(t)f(t)\in E_{B_n} \text{ a.e.,}\quad\text{and}\quad \text{(ii) } p_{B_n}(\theta f)\in V_1.$$

Since E is metrizable, for the sequence (B_n) of bounded subsets in E, there is a sequence of positive numbers (ρ_n) such that $\bigcup_n \rho_n B_n$ is contained in a disc B. Use that $\Lambda^\times$ contains the characteristic functions of a fundamental sequence of compact subsets of Ω to deduce from (i) that every $f\in C$ is localized in E_B. The first part will be proved once we show that $\{p_B(f) : f\in C\}$ is a bounded subset of $\Lambda^{\times\times}=\Lambda$. Take $\theta\in\Lambda^\times$, then θ is in some M_n. Since $\rho_n B_n\subset B$, it follows from (ii) above that for every $f\in C$, the function $\theta p_B(f)$ is in L^1 and $\|\theta p_B(f)\|_1\le \rho_n^{-1}$, hence

$$\sup\left\{\int_\Omega |\theta|\, p_B(f)\, d\mu : f\in C\right\}\le \rho_n^{-1}.$$

Therefore, $\{p_B(f) : f\in C\}$ is a bounded subset of $\Lambda^{\times\times}=\Lambda$.

Part (2): If $C\subset\Lambda^\times\{E\}$ is bounded then for each $\phi\in\Lambda$ one has that $\{\phi f : f\in C\}$ is a bounded subset of $L^1\{E\}$. Since (df)-spaces are fundamentally L^1-bounded [12, 3.10], there is a disc $D(\phi)\subset E$ such that for each $f\in C$ we have $\phi(t)f(t)\in E_{D(\phi)}$ a.e. and $\left\|\phi p_{D(\phi)}(f)\right\|_1\le 1$. First let us see that there is a bounded set $D\subset E$ that does not depend on any ϕ and such that every $f\in C$ is localized in E_D. Proceed by contradiction: Let $B_1\subset B_2\subset\ldots$ be a fundamental sequence of discs in E and suppose that for each $n=1,2,\ldots$ there exists some $f_n\in C$ such that the set $A_n=\{t\in\Omega : f_n(t)\notin E_{B_n}\}$ has positive measure. Since f_n is measurable, we can obtain a compact set $K_n\subset A_n$ with positive measure and such that $f_n : K_n\to E$ is continuous. On the other hand, since M_n° is absorbent in $\Lambda=\Lambda^{\times\times}$ and $\chi_{K_n}\in\Lambda$, there exists $\rho_n>0$ such that $\rho_n\chi_{K_n}\in M_n^\circ$. Applying the Estimation Lemma in §2 to $\rho_n f_n : K_n\to E$, we deduce that there exists a simple function $s_n : K_n\to B_n^\circ$ such that

$$\operatorname{Re}\langle \rho_n f_n(t), s_n(t)\rangle > \frac{n}{\mu(K_n)}\qquad \text{for all } t\in K_n.$$

By integrating we have

$$\int_{K_n}\operatorname{Re}\langle \rho_n f_n(t), s_n(t)\rangle\, d\mu(t) > n.$$

Now, every $s_n(K_n)$ is a finite set. Since $s_n(K_n) \subset B_n^\circ$ and (B_n°) is a fundamental system of zero-neighbourhoods in E'_b, it follows that we can arrange the set $\bigcup_n s_n(K_n)$ as a null sequence in E'_b. This null sequence is equicontinuous because E is a (df)-space. Therefore, there exists an absolutely convex and closed zero-neighbourhood U in E such that $\bigcup_n s_n(K_n) \subset U^\circ$. Let $q \in \mathcal{Q}(E)$ be the Minkowski functional of U. The set $\{q(f) : f \in C\}$ is bounded in $\Lambda^\times$ by hypothesis, hence it is included in some M_j. For every $t \in K_j$ we have

$$q(f_j(t)) = \sup\{|\langle f_j(t), x'\rangle| : x' \in U^\circ\} \geq \operatorname{Re}\langle f_j(t), s_j(t)\rangle .$$

Bearing in mind that f_j is in C and that $\rho_j \chi_{K_j} \in M_j^\circ$ we have the contradiction

$$1 \geq \int_\Omega q(f_j)\rho_j \chi_{K_j}\, d\mu \geq \int_{K_j} \operatorname{Re}\langle \rho_j f_j(t), s_j(t)\rangle\, d\mu > j.$$

The proof will be finished if we show that there is a disc $B \subset E$ such that $D \subset B$ and $\{p_B(f) : f \in C\}$ is bounded in $\Lambda^\times$. Assume, without loss of generality, that D is the first set in a fundamental sequence $D \subset B_1 \subset B_2 \subset \ldots$ of bounded sets in E and suppose, on the contrary, that for each $n = 1, 2, \ldots$ there exists $f_n \in C$ such that $p_{B_n}(f_n) \notin M_n$. Then for some non-negative function $\phi_n \in M_n^\circ$ we have

$$\int_\Omega \phi_n p_{B_n}(f_n)\, d\mu > 1.$$

Since ϕ_n, $p_{B_n}(f_n)$ and f_n are measurable functions, there exists a compact set $K_n \subset \Omega$ with positive measure and such that the restriction of each of them to K_n is continuous and, moreover,

$$\int_{K_n} \phi_n p_{B_n} f_n\, d\mu > 1.$$

By applying the Estimation Lemma once again, given $\varepsilon_n > 0$ we get a simple function $s_n : K_n \to B_n^\circ \subset E'$ such that $p_{B_n}(\phi_n f_n) < \operatorname{Re}\langle \phi_n f_n, s_n\rangle + \varepsilon_n$ pointwisely in K_n. Moreover, by choosing an appropriate ε_n, we also have that $\int_{K_n} \operatorname{Re}\langle \phi_n f_n, s_n\rangle\, d\mu > 1$. Proceeding as before, one can check that the set $\bigcup_n s_n(K_n)$ is equicontinuous. Let U be a zero-neighbourhood in E, with Minkowski functional q, such that $\bigcup_n s_n(K_n) \subset U^\circ$. Since C is a bounded subset of $\Lambda^\times\{E\}$, we have that $\{q(f) : f \in C\}$ is a bounded subset of $\Lambda^\times$ and therefore it is included in some M_j. Using that $\phi_j \in M_j^\circ$ is non-negative, that the function s_j takes its values in U° and that $f_j \in C$, we have

$$1 < \int_{K_j} \operatorname{Re}\langle \phi_j f_j, s_j\rangle\, d\mu \leq \int_{K_j} \phi_j q(f_j)\, d\mu \leq 1,$$

a contradiction. ∎

The hypothesis that Λ is perfect is essential here; see [13, Remark after Lem. 2]. Since perfect spaces are sequentially complete for all solid topologies, the hypothesis of this theorem is, in other words, that Λ is a perfect Fréchet space for some solid topology. For $1 \leq p \leq \infty$ let L^p_{loc} be the Fréchet space of all measurable and locally p-integrable scalar functions and L^p_c be the (DF)-space of all measurable and p-integrable functions having compact support (see [21]). The couple $(L^p_{\mathrm{loc}}, L^{p^*}_c)$ is a Dieudonné-Köthe dual pair of perfect spaces.

Corollary 1. *For $1 \le p \le \infty$ the following hold:*

1. *If E is metrizable then E is fundamentally L^p_{loc}-bounded.*
2. *If E is a (df)-space then E is fundamentally L^p_c-bounded.*

Corollary 2. *Let $\Lambda(\beta(\Lambda, \Lambda^\times))$ be a perfect Banach space such that its dual is $\Lambda^\times$. Let $E = \mathrm{ind}_n\, E_n$ be a regular inductive limit of a sequence Banach spaces (E_n). If either the Banach spaces E_n are all separable or the inductive limit is strict, then $\Lambda\{E\} = \mathrm{ind}_n \Lambda\{E_n\}$. In particular, $L^p\{E\} = \mathrm{ind}_n L^p\{E_n\}$ for $1 \le p < \infty$.*

Proof. Under the hypotheses for Λ, we know that $\mathrm{ind}_n \Lambda\{E_n\}$ is a topological subspace of $\Lambda\{E\}$ [11, 3.3]. Since E is a (DF)-space, Theorem 1 tells us that for every $f \in \Lambda\{E\}$ there is a disc $B \subset E$ such that f is localized in E_B and $p_B(f) \in \Lambda$. The inductive limit is regular, therefore B is contained and bounded in some E_n. Then f is localized in E_n and so it is measurable for the inductive limit topology restricted to E_n. If the limit is strict, then f is measurable for the norm of E_n. If every step in the inductive sequence is separable, a theorem due to Meyer and Schwartz [36, Part I, II.3 Cor. 2 of Thm. 10 on pp. 122–124] [41, p. 51] ensures that f is measurable for the norm of E_n. In either case $f \in \Lambda\{E_n\}$ and this proves that $\Lambda\{E\} = \mathrm{ind}_n \Lambda\{E_n\}$. ■

4 Completeness

This section is devoted to study the completeness of the spaces $\Lambda\{E\}$. The starting point is that for a complete locally convex space E, the space $L^1\{E\}$ is not complete in general. This was already noted in Köthe's book [26, §41.7, p. 200], but no example was mentioned. Here we give one by using an example, due to Thomas [41, 6.11], of a vector measure with values in the dual of a non-separable Banach space that has bounded variation but no density with respect to the weak-star topology.

Example. Let Z be the unit ball of $L^\infty[0,1]$. For each $z \in Z$ we fix a representative such that $|z(t)| \le 1$ for every $t \in [0,1]$. Consider Z as an index set and let $F = \ell^\infty(Z)$ be the dual of the Banach space $E = \ell^1(Z)$ endowed with the topology of uniform convergence on the norm-compact subsets of E. Then F is a complete (gDF)-Schwartz space [22, 9.4.1–3, 11.1.4, 12.5.2 and 12.5.6]. (Moreover, by the Banach-Dieudonné Theorem, the topology on F equals the topology of uniform convergence on the norm-null sequences of E [22, 9.4.3].) We shall prove that if we take the unit interval with the Lebesgue measure as measure space, then $L^1\{F\}$ is not complete. For each finite set $Y \subset Z$, let $(\chi_Y(z))_{z\in Z}$ be the element of $\ell^\infty(Z)$ that takes the value 1 in the coordinates z such that $z \in Y$, and 0 otherwise, and define the simple function f_Y by

$$t \in [0,1] \to f_Y(t) = (z(t) \cdot \chi_Y(z))_{z\in Z} \in F.$$

Then $f_Y \in L^1\{F\}$. Let us see that, with the order induced by the inclusion relation, the net $\{f_Y : Y \subset Z, Y \text{ finite}\}$ is a Cauchy net in $L^1\{F\}$. Let q be a continuous seminorm on F; we may assume that q is the seminorm of uniform convergence on some norm-compact set $C \subset \ell^1(Z)$. A well-known property of $\ell^1(Z)$ is that the absolute convergence of the series that gives the norm of each element is uniform on every compact set. Therefore, given $\varepsilon > 0$ there is a finite set Y_0 such that

$$\sum_{z \notin Y_0} |\xi(z)| < \varepsilon \qquad \text{for all } \xi \in C.$$

Now, if Y is a finite set contained in $Z \setminus Y_0$, for $t \in [0,1]$ we have

$$\begin{aligned} q(f_Y(t)) &= \sup\{|\langle \xi, f_Y(t)\rangle| : \xi \in C\} = \sup\left\{\left|\sum_{z\in Z} \xi(z)\cdot \chi_Y(z)\cdot z(t)\right| : \xi \in C\right\} \\ &\leq \sup\left\{\sum_{z\in Y} |\xi(z)\cdot z(t)| : \xi \in C\right\} < \varepsilon. \end{aligned}$$

Then $\int_0^1 q(f_Y(t))\, d\mu(t) < \varepsilon$. This proves that $\{f_Y : Y \subset Z, Y \text{ finite}\}$ is a Cauchy net. The space $L^1\{F\}$ is isomorphically embedded in the space $cabv(\Sigma, \mu, F)$ of all μ-continuous vector measures $m : \Sigma \to F$ with bounded variation; to every $f \in L^1\{F\}$ we asign the vector measure

$$m_f : A \in \Sigma \to m_f(A) = \left(\int_A (f(z))(t)\, d\mu(t)\right)_{z\in Z} \in F.$$

Since $cabv(\Sigma, \mu, F)$ is complete (this can be proved as in the Banach space case), our net $\{f_Y : Y \subset Z, Y \text{ finite}\}$ converges to a vector measure $m : \Sigma \to F$. By looking at the coordinates, we obviously have

$$m(A) = \left(\int_A z(t)\, d\mu(t)\right)_{z\in Z} \in F.$$

But m does not have a density even for the weak topology $\sigma(F, E)$ [41, 6.11], hence $\{f_Y : Y \subset Z, Y \text{ finite}\}$ does not converges in $L^1\{F\}$. Since the net is bounded, it also follows that $L^1\{F\}$ is not quasi-complete.

The pathology exhibited by this example is the only one that can happen in the sense explained by our next result.

Theorem 2. *If $L^1\{E\}$ and Λ are complete (resp. sequentially complete or quasi-complete) then so is $\Lambda\{E\}$.*

Proof. We give the proof for the first case; the remaining two can be proved analogously. Let $(f_i)_{i\in I}$ be a Cauchy net in $\Lambda\{E\}$. Fix an a.e. partition (K_n) of Ω into pairwise disjoint compact subsets. For each $n \in \mathbb{N}$, the function χ_{K_n} is in $\Lambda^\times$, hence $(f_i \cdot \chi_{K_n})_{i\in I}$ is a Cauchy net in $L^1\{E\}$. By hypothesis $(f_i \cdot \chi_{K_n})_{i\in I}$ converges to some function g_n in $L^1\{E\}$. Since each g_n is measurable and the sequence (K_n) is pairwise disjoint, the function $g = \sum_{n=1}^\infty g_n \cdot \chi_{K_n}$ is also measurable. Let us first check that g is in $\Lambda\{E\}$. For each $q \in \mathcal{Q}(E)$, the net of scalar functions $(q(f_i))_{i\in I}$ is a Cauchy net in Λ so that it converges to some function $\phi \in \Lambda$. In particular, for every $n \in \mathbb{N}$ we have that $\phi \cdot \chi_{K_n}$ is the limit in L^1 of the net

$(q(f_i)\cdot\chi_{K_n})_{i\in I}$, hence $q(g)\cdot\chi_{K_n}=\phi\cdot\chi_{K_n}$ a.e. in Ω and it follows that $q(g)\in\Lambda$ because $q(g)=\phi$ a.e. in Ω. Since $(f_i\cdot\chi_{K_n})_{i\in I}$ converges to g_n in $L^1\{E\}$, we have that our Cauchy net $(f_i)_{i\in I}$ converges to $g\in\Lambda\{E\}$ for the topology induced by $L^1_{\text{loc}}\{E\}$. The desired conclusion will follow from the Bourbaki-Robertson lemma as soon as we prove that $\Lambda\{E\}$ has a basis of zero-neighbourhoods that are closed for the topology of $L^1_{\text{loc}}\{E\}$. Take $q\in\mathcal{Q}(E)$ and $M\in\mathcal{M}$. Bearing in mind that M is solid, we can write

$$\begin{aligned}\{f\in\Lambda\{E\}:q_M(q(f))\le 1\} &= \bigcap_{\theta\in M}\left\{f\in\Lambda\{E\}:\int_\Omega|\theta|\cdot q(f)\,d\mu\le 1\right\}\\ &= \bigcap_{\theta\in M\cap L^\infty_c}\left\{f\in\Lambda\{E\}:\int_\Omega|\theta|\cdot q(f)\,d\mu\le 1\right\}\end{aligned}$$

and this set is closed for the topology of $L^1_{\text{loc}}\{E\}$. ■

By looking at the proof, one can see that the hypothesis "$L^1\{E\}$ is complete" can be replaced by its equivalent "$L^1_{\text{loc}}\{E\}$ is complete."

Corollary 3. *If $L^1\{E\}$ is complete (resp. sequentially complete or quasi-complete), the following hold:*

1. *If Λ is perfect then $\Lambda\{E\}$ is complete (resp. sequentially complete or quasi-complete).*
2. *$\Lambda^\times\{E\}$ is complete (resp. sequentially complete or quasi-complete) for every solid topology on $\Lambda^\times$ of uniform convergence on a saturated family of solid bounded subsets of Λ.*

A locally convex space E is said to have the metrizable property (B) of Pietsch, property (BM) for short, if every bounded set $C\subset l^1\{E\}$ is bounded in $l^1\{E_B\}$ for some metrizable disc $B\subset E$. Note that if E has property (B) and every bounded subset of E is metrizable then E has property (BM). Metrizable spaces have property (BM). More generally, so does every strict (LF)-space. For (DF)-spaces, property (BM), or rather that bounded subsets are metrizable, equals the dual density condition introduced by Bierstedt and Bonet [1, 1.5]. We have proved in [12, Th. 4.12] that for a quasi-complete locally convex space E with property (BM) the space $L^1\{E\}$ is quasi-complete. Then we can give the following corollaries.

Corollary 4. *If E is a quasi-complete locally convex space having property (BM), the following hold:*

1. *If Λ is perfect then $\Lambda\{E\}$ is quasi-complete.*
2. *$\Lambda^\times\{E\}$ is quasi-complete.*

Corollary 5. *If E is a Fréchet (resp. Banach) space and Λ is a perfect Fréchet (resp. Banach) space, then $\Lambda\{E\}$ is also a Fréchet (resp. Banach) space.*

Since quasi-complete (LB)-spaces are complete [31, 8.3.18], we can use Corollary 2 above (or also [11, 3.6]) to deduce the following result.

Corollary 6. *Let $\Lambda(\beta(\Lambda, \Lambda^\times))$ be a perfect Banach space such that its dual is $\Lambda^\times$. Let $E = \mathrm{ind}_n E_n$ be a strict inductive limit of a sequence Banach spaces (E_n). Then $\Lambda\{E\} = \mathrm{ind}_n \Lambda\{E_n\}$ is a complete (LB)-space. In particular, $L^p\{E\}$ is a complete (LB)-space for $1 \leq p < \infty$.*

The situation for local completeness is different. We do not need the extra hypothesis that $L^1\{E\}$ is locally complete when E is fundamentally Λ-bounded.

Theorem 3. *If Λ is perfect and E is locally complete and fundamentally Λ-bounded, then $\Lambda\{E\}$ is locally complete.*

Proof. Let us see that every disc in $\Lambda\{E\}$ is a Banach disc. Since the sets $[R, B]$ form a fundamental family of bounded sets in $\Lambda\{E\}$ when R and B run, respectively, through the solid discs in Λ and discs in E, it is enough to prove that every $[R, B]$ is a Banach disc. By [31, p. 83] we must show that for each sequence (f_n) from $[R, B]$, the series $\sum_n 2^{-n} f_n$ converges in $\Lambda\{E\}$ to an element of $[R, B]$. Let R° be the polar of R in $\Lambda^\times$ and define $R^\circ \cdot [R, B] := \{\theta \cdot f : \theta \in R^\circ, f \in [R, B]\}$. Clearly, $R^\circ \cdot [R, B]$ is a subset of $[V_1, B]$, where V_1 is the unit ball of L^1. But $[V_1, B]$ is a Banach disc [12, 3.12], hence for each $\theta \in R^\circ$ the series $\sum_n 2^{-n} \theta f_n$ converges in $L^1\{E\}$ to an element of $[V_1, B]$ and the convergence is a.e. in (E_B, p_B) [12, Proof of 3.12]. Since R° is absorbent in $\Lambda^\times$, and this space contains the characteristic functions of compact subsets of Ω, it follows that the series $\sum_n 2^{-n} f_n$ defines a measurable function from Ω into E_B. Finally, for every $\theta \in R^\circ$ we have $p_B(\theta \cdot f) = |\theta|\, p_B(f) \in V_1$, hence $p_B(f) \in \Lambda^{\times\times} = \Lambda$ and $p_B(f) \in R^{\circ\circ} = R$. ■

5 Duality

We want to give conditions for $(\Lambda\{E\})' = \Lambda^\times\{E'_b\}$ to hold. It will be useful to consider the following generalized Dieudonné-Köthe dual of $\Lambda\{E\}$.

Definition. The Dieudonné-Köthe dual of a space $\Lambda\{E\}$ is defined as the set $(\Lambda\{E\})^\times$ of all measurable functions $g : \Omega \to E'_b$ such that

$$\int_\Omega |\langle f(t), g(t) \rangle|\, d\mu < \infty \qquad \text{for all } f \in \Lambda\{E\}.$$

For $g \in (\Lambda\{E\})^\times$ define the linear map

$$T_g : f \in \Lambda\{E\} \to T_g(f) := \int_\Omega \langle f(t), g(t) \rangle\, d\mu(t).$$

Remark. The absolute value inside the integral is not really needed in this definition; one may think of defining T_g as above for a function $g : \Omega \to E'_b$ by requiring merely that $\int_\Omega \langle f(t), g(t) \rangle\, d\mu(t)$ exists for all $f \in \Lambda\{E\}$. But, as a matter of fact, for such a function g and $f \in \Lambda\{E\}$ there exists a function ξ in the unit ball of

L^∞ such that $|\langle f(t), g(t)\rangle| = \xi(t)\langle f(t), g(t)\rangle$ a.e. in Ω. Now $\xi f \in \Lambda\{E\}$ because $\Lambda\{E\}$ is solid, hence

$$\int_\Omega |\langle f, g\rangle|\, d\mu = \int_\Omega \langle \xi f, g\rangle\, d\mu = T_g(\xi f).$$

We have left the absolute value inside the integral because that is the historical way of defining Köthe duals.

Lemma 1. *For $g \in (\Lambda\{E\})^\times$ the operator T_g is continuous on $\Lambda\{E\}$ if and only if there exists an equicontinuous disc $D \subset E'$ such that g is localized in E'_D and $p_D(g) \in \Lambda^\times$.*

Proof. If T_g is continuous then there is a seminorm $q \in \mathcal{Q}(E)$ and a solid disc $M \in \mathcal{M}$ such that

$$|T_g(f)| \le \sup\left\{\int_\Omega q(f)\,|\theta|\, d\mu : \theta \in M\right\} \qquad \text{for all } f \in \Lambda\{E\}.$$

Call U the closed unit ball of q and take $D = U^\circ$. Let us see first that g is localized in E'_D. If, on the contrary, $\mu\{t \in \Omega : g(t) \notin E'_D\} > 0$, there is a compact subset $K \subset \Omega$ with positive measure and such that g is continuous on K and $g(K)$ does not meet E'_D. Since $g : K \to E'$ is continuous for the strong topology, it is also continuous for the weak topology $\sigma(E', E)$. By the Estimation Lemma, there exists a simple function $s_n : K \to D^\circ = U$ such that $\operatorname{Re}\langle s_n(t), g(t)\rangle > n$ for all $t \in K$. Integrate to obtain

$$\begin{aligned} n \cdot \mu(K) &\le \int_\Omega \operatorname{Re}\langle s_n, g\rangle\, d\mu = \operatorname{Re} T_g(s_n) \le \sup\left\{\int_\Omega q(s_n)\,|\theta|\, d\mu : \theta \in M\right\} \\ &\le \sup\left\{\int_\Omega \chi_K\,|\theta|\, d\mu : \theta \in M\right\} = q_M(\chi_K), \end{aligned}$$

a contradiction for big enough values of n.

We have to prove now that $p_D(g) \in \Lambda^\times$. Fix a non-negative function $\phi \in \Lambda$. Let (K_n) be an increasing sequence of compact sets covering a.e. Ω and such that all the restrictions of $\phi \cdot g$ and $\phi \cdot p_D(g)$ to K_n are continuous. Using again the Estimation Lemma, we can find a simple function $s_n : K_n \to D^\circ = U$ such that

$$\phi(t) p_D(g(t)) \le \operatorname{Re}\langle s_n(t), \phi(t) g(t)\rangle + \mu(K_n)^{-1} \qquad \text{for all } t \in K_n.$$

Integration on K_n yields

$$\begin{aligned} \int_{K_n} \phi \cdot p_D(g)\, d\mu &\le \left|\int_{K_n} \langle \phi \cdot s_n, g\rangle\, d\mu\right| + 1 = |T_g(\phi \cdot s_n)| + 1 \\ &\le \sup\left\{\int_\Omega q(\phi s_n)\,|\theta|\, d\mu : \theta \in M\right\} + 1 \\ &\le \sup\left\{\int_\Omega \phi \cdot |\theta|\, d\mu : \theta \in M\right\} + 1 < \infty. \end{aligned}$$

By the Monotone Convergence Theorem, we have that $\phi \cdot p_D(g)$ is integrable. Since ϕ was arbitrary, it follows that $p_D(g) \in \Lambda^\times$.

Conversely, assume that there exists an equicontinuous disc $D \subset E'$ such that g is localized in E'_D and $p_D(g) \in \Lambda^\times$. Take any set $M \in \mathcal{M}$ such that

$p_D(g) \in M$. Let q be the polar seminorm associated to D. Since we have that $|\langle f(t), g(t)\rangle| \leq q(f(t)) \cdot p_D(g(t))$ for all $t \in \Omega$, it follows that

$$|T_g(f)| \leq \int_\Omega q(f) p_D(g)\, d\mu \leq \sup\left\{\int_\Omega q(f)\, |\theta|\, d\mu : \theta \in M\right\}.$$

Therefore, T_g is continuous. ■

What is the relation between $\Lambda^\times\{E'_b\}$ and $(\Lambda\{E\})^\times$?

Theorem 4. *The following hold:*

1. $(\Lambda\{E\})^\times \subset \Lambda^\times\{E'_b\}$.
2. *If E is fundamentally Λ-bounded then $(\Lambda\{E\})^\times = \Lambda^\times\{E'_b\}$.*
3. *If E is quasi-barrelled and E'_b is fundamentally $\Lambda^\times$-bounded, then we have that $(\Lambda\{E\})^\times = \Lambda^\times\{E'_b\}$. Moreover, in this case every $g \in \Lambda^\times\{E'_b\}$ defines, via T_g, an element of the topological dual of $\Lambda\{E\}$, in other words $\Lambda^\times\{E'_b\} \subset (\Lambda\{E\})'$.*

Proof. Part (1): Take $g \in (\Lambda\{E\})^\times$. We need to prove that $q(g)$ is in $\Lambda^\times$ for every continuous seminorm q on E'_b. So assume that q is the polar seminorm corresponding to a disc $B \subset E$ and fix $\phi \in \Lambda$. Let (K_n) be a disjoint sequence of compact subsets of Ω such that the restrictions of both g and $q(g)$ to each K_n are continuous and $\mu(\Omega \setminus \bigcup_n K_n) = 0$. For each $n \in \mathbb{N}$ define ε_n as $2^{-n}\|\phi\chi_{K_n}\|_1^{-1}$ if $\|\phi\chi_{K_n}\|_1 \neq 0$ and $\varepsilon_n = 1$ otherwise. Since $g : K_n \to E'_b$ is continuous, it will be also continuous for the weak topology $\sigma(E', E)$, so that we can apply the Estimation Lemma to deduce that there is a simple function $s_n : K_n \to B^{\circ\circ} = B$ such that $q(g(t)) < \operatorname{Re}\langle s_n(t), g(t)\rangle + \varepsilon_n$ for all $t \in K_n$. Define the measurable function $s = \sum_n s_n \chi_{K_n}$, then we have

$$|\phi(t)|\, q(g(t)) \leq \operatorname{Re}\langle |\phi(t)|\, s(t), g(t)\rangle + \sum_n \varepsilon_n \chi_{K_n}(t)\, |\phi(t)| \quad \text{a.e. in } \Omega.$$

The range of s is bounded, hence $|\phi|\, s \in \Lambda\{E\}$. It follows that $\langle |\phi|\, s, g\rangle \in L^1$ because $g \in (\Lambda\{E\})^\times$. On the other hand, by the choice of the numbers ε_n, it is clear that the function $\sum_n \varepsilon_n \chi_{K_n} |\phi|$ is also in L^1. Therefore, $|\phi|\, q(g) \in L^1$ and, since ϕ was arbitrary, $q(g) \in \Lambda^\times$ as desired.

Part (2): We have to prove the inclusion $\Lambda^\times\{E'_b\} \subset (\Lambda\{E\})^\times$. Take a function $g \in \Lambda^\times\{E'_b\}$. Given $f \in \Lambda\{E\}$, there is a disc $B \subset E$ such that $f(t) \in E_B$ a.e. and $p_B(f) \in \Lambda$. Let q be the continuous seminorm on E'_b of uniform convergence on B. Then $|\langle f(t), g(t)\rangle| \leq p_B(f(t)) \cdot q(g(t))$ holds pointwisely. But $p_B(f) \in \Lambda$ and $q(g) \in \Lambda^\times$, hence $p_B(f) \cdot q(g) \in L^1$, so that $\langle f, g\rangle$ is also in L^1. Since f was arbitrary in $\Lambda\{E\}$, we have that $g \in (\Lambda\{E\})^\times$.

Part (3): Take $g \in \Lambda^\times\{E'_b\}$. Since E'_b is fundamentally $\Lambda^\times$-bounded, there exists a disc $D \subset E'_b$ such that $g(t) \in E'_D$ a.e. and $p_D(g) \in \Lambda^\times$. Since E is quasi-barrelled, we have that D is equicontinuous so that the polar seminorm q of uniform convergence on D is continuous on E. Now, for each $f \in \Lambda\{E\}$ we have

$|\langle f(t), g(t)\rangle| \leq q(f(t))p_D(g(t))$ and this latter function is in L^1 because $q(f) \in \Lambda$ and $p_D(g) \in \Lambda^\times$. Since f was arbitrary in $\Lambda\{E\}$, we have that $g \in (\Lambda\{E\})^\times$. The inclusion $\Lambda^\times\{E'_b\} \subset (\Lambda\{E\})'$ follows from Lemma 1 and the fact that E is quasi-barrelled. ∎

As we mentioned in the Introduction, if we want $(\Lambda\{E\})' = \Lambda^\times\{E'_b\}$ to hold it will be necessary not only to require $\Lambda' = \Lambda^\times$, but also that E'_b satisfies the Radon-Nikodym property.

Theorem 5. *Assume that the dual of Λ is $\Lambda^\times$ and that E'_b is quasi-complete and has the Radon-Nikodym property. Then every element of the dual of $\Lambda\{E\}$ can be represented by means of a function $g \in \Lambda^\times\{E'_b\}$, i.e. $(\Lambda\{E\})' \subset \Lambda^\times\{E'_b\}$.*

Proof. Take $T \in (\Lambda\{E\})'$. We must find $g \in \Lambda^\times\{E'_b\}$ such that

$$T(f) = \int_\Omega \langle f(t), g(t)\rangle \, d\mu(t) \qquad \text{for all } f \in \Lambda\{E\}.$$

T is continuous, hence there is a seminorm $q \in \mathcal{Q}(E)$ and $M \in \mathcal{M}$ such that

$$|T(f)| \leq \sup\left\{\int_\Omega |\theta|\, q(f)\, d\mu : \theta \in M\right\} \qquad \text{for all } f \in \Lambda\{E\}.$$

Let $K \subset \Omega$ be compact. For a measurable set $A \subset K$, define an element $m_K(A) \in E'$ by $x \in E \to \langle x, m_K(A)\rangle := T(x\chi_A)$. Indeed, $x\chi_A \in S_c(E) \subset \Lambda\{E\}$, so that

$$\begin{aligned} |\langle x, m_K(A)\rangle| &= |T(x\chi_A)| \leq \sup\left\{\int_A |\theta|\, q(x)\, d\mu : \theta \in M\right\} \\ &= q(x) \cdot \sup\left\{\int_A |\theta| \; d\mu : \theta \in M\right\} = q(x) \cdot q_M(\chi_A). \end{aligned}$$

This defines a finitely additive set function $m_K : A \in \Sigma_K \to m_K(A) \in E'$, where Σ_K is the σ-algebra of all measurable sets contained in K. A straightforward computation shows that m_K has bounded variation; as a matter of fact, if we take a strong seminorm q_B on E', where B is a disc in E, we have

$$\sum_{j=1}^n q_B(m_K(A_j)) \leq \sup\{q(x) : x \in B\} \cdot q_M(\chi_A)$$

for every measurable partition $\{A_1, A_2, \ldots, A_n\}$ of a set $A \subset K$. Since $\Lambda' = \Lambda^\times$, it follows that the $\mathcal{M}$-topology is coarser than the Mackey topology, therefore $\lim_{\mu(A)\to 0} q_M(\chi_A) = 0$. This and the inequality above yield that m_K is μ-continuous. Let us see that m_K is countably additive. For a disjoint sequence $(A_n) \subset \Sigma_K$ and $A = \cup_n A_n$ we have $\lim_n \mu\left(A \setminus \cup_{j=1}^n A_j\right) = 0$. But

$$q_B\left(m_K(A) - m_K\left(\cup_{j=1}^n A_j\right)\right) \leq \sup\{q(x) : x \in B\} \cdot q_M\left(\chi_{(A\setminus\cup_{j=1}^n A_j)}\right),$$

therefore

$$\lim_{n\to\infty} q_B\left(m_K(A) - m_K\left(\cup_{j=1}^n A_j\right)\right) = 0.$$

By the Radon-Nikodym property, there is a summable function $g_K : K \to E'_b$ such that $g_K \in L^1\{E'_b\}$ and $\langle x, m_K(A)\rangle = \int_A \langle x, g_K(t)\rangle \, d\mu(t)$ for all $x \in E$ and

$A \in \Sigma_K$. Take different compact sets K_1 and K_2. It is clear that g_{K_1} and g_{K_2} coincide a.e. on $K_1 \cap K_2$, so we may define a measurable function $g : \Omega \to E'_b$ by requiring $g = g_K$ a.e. on every compact set K. Plainly, $g \in L^1_{\text{loc}}\{E'_b\}$ and for every simple function s with compact support we have $T(s) = \int_\Omega \langle s, g\rangle \, d\mu$. Since $S_c(E)$ is dense in $\Lambda\{E\}$ and T is continuous, it follows that the integral $T_g(f) = \int_\Omega \langle f, g\rangle \, d\mu$ exists and satisfies $T_g(f) = T(f)$ for every $f \in \Lambda\{E\}$. Finally, as we pointed out before Lemma 1, this implies that $g \in (\Lambda\{E\})^\times \subset \Lambda^\times\{E'_b\}$. ∎

Since the strong dual of a quasi-barrelled space is quasi-complete [22, 12.2.4], the previous theorems yield the following result.

Theorem 6. *Assume that the dual of Λ is $\Lambda^\times$, that E is quasi-barrelled and that E'_b is fundamentally $\Lambda^\times$-bounded and has the Radon-Nikodym property. Then $(\Lambda\{E\})' = \Lambda^\times\{E'_b\} = (\Lambda\{E\})^\times$.*

6 Barrelledness Conditions

We have just seen that under certain conditions the equality $\Lambda^\times\{E'_b\} = (\Lambda\{E\})'$ holds. In this case, the precise knowledge of the dual space enables us to study the strong topology and derive some barrelledness properties. When $(\Lambda\{E\})'$ equals $\Lambda^\times\{E'_b\}$, we have two topologies on this space. Namely, the strong topology $\beta(\Lambda^\times\{E'_b\}, \Lambda\{E\})$ as a dual space, and the natural topology as a space of vector-valued functions when $\Lambda^\times$ and E' are endowed with their respective strong topologies $\beta(\Lambda^\times, \Lambda)$ and $\beta(E', E)$. In this section, we shall speak about them as the strong and the natural topologies, respectively.

Theorem 7. *Assume that the dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$ is $\Lambda^\times$, that E is quasi-barrelled and fundamentally Λ-bounded, and that E'_b is fundamentally $\Lambda^\times$-bounded and has the Radon-Nikodym property. Then $(\Lambda\{E\})' = \Lambda^\times\{E'_b\}$, the strong topology on $\Lambda^\times\{E'_b\}$ coincides with its natural topology, and $\Lambda\{E\}$ is quasi-barrelled.*

Proof. The equality $(\Lambda\{E\})' = \Lambda^\times\{E'_b\}$ is Theorem 6 above applied to Λ endowed with the strong topology $\beta(\Lambda, \Lambda^\times)$.

Let us prove now that the strong topology on $\Lambda^\times\{E'_b\} = (\Lambda\{E\})'$ is coarser that the natural topology. Let C be a bounded set in $\Lambda\{E\}$ and consider the strong seminorm

$$q_C : g \in \Lambda^\times\{E'_b\} \to q_C(g) := \sup\left\{\left|\int_\Omega \langle f, g\rangle \, d\mu\right| : f \in C\right\}.$$

Since E is fundamentally Λ-bounded, there exists a disc $B \subset E$ and a solid disc $R \subset \Lambda$ such that for each $f \in C$ we have $f(t) \in E_B$ a.e. and $\{p_B(f) : f \in C\} \subset R$. Since $|\langle f(t), g(t)\rangle| \le p_B(f(t)) \cdot q_B(g(t))$, we have

$$q_C(g) \le \sup\left\{\int_\Omega |\langle f, g\rangle| \, d\mu : f \in C\right\} \le \sup\left\{\int_\Omega |\phi| \, q_B(g) \, d\mu : \phi \in R\right\}.$$

The last member in the preceding inequality defines a seminorm continuous for the natural topology, hence the strong topology is coarser than the natural topology.

On the other hand, the natural topology is coarser than the strong topology. To see this, let q_B be the strong seminorm on E'_b of uniform convergence on a disc $B \subset E$, and R a solid disc in Λ. Take the bounded set $C = [R, B] \subset \Lambda \{E\}$. It will be enough to prove that for every $g \in \Lambda^\times \{E'_b\}$, the following inequality holds: $q_R(q_B(g)) \le q_C(g)$. Since

$$q_R(q_B(g)) = \sup \left\{ \int_\Omega |\phi|\, q_B(g)\, d\mu : \phi \in R \right\},$$

given $\varepsilon > 0$ there exists a non-negative function $\phi \in R$ such that

$$q_R(q_B(g)) < \varepsilon + \int_\Omega \phi \cdot q_B(g)\, d\mu.$$

Take a sequence (K_n) of disjoint compact sets covering Ω a.e. and such that the restrictions of both g and $q_B(g)$ to each K_n are continuous. Choose a sequence of positive numbers (δ_n) such that the measurable function $\delta = \sum_n \delta_n \chi_{K_n}$ satisfies $\int_\Omega \phi \cdot \delta\, d\mu < \varepsilon$. By the Estimation Lemma in §2, for each δ_n there exists a simple function $s_n : K_n \to B$ such that $q_B(g) \le \operatorname{Re} \langle s_n, g\rangle + \delta_n$ pointwisely in K_n. The function $s = \sum_n s_n \chi_{K_n}$ is measurable and verifies $q_B(g) \le \operatorname{Re} \langle s, g\rangle + \delta$ a.e. in Ω. Multiply by ϕ, integrate and take into account that $\phi \cdot s$ is in C to deduce

$$\begin{aligned} q_R(q_B(g)) &\le \varepsilon + \int_\Omega \phi \cdot \operatorname{Re} \langle s, g\rangle\ d\mu + \int_\Omega \phi \cdot \delta\, d\mu \\ &\le \varepsilon + \int_\Omega \operatorname{Re} \langle \phi \cdot s, g\rangle\ d\mu + \varepsilon \\ &\le 2\varepsilon + \sup \left\{ \int_\Omega |\langle f, g\rangle|\ d\mu : f \in C \right\} = 2\varepsilon + q_C(g). \end{aligned}$$

Since ε was arbitrary, we have $q_R(q_B(g)) \le q_C(g)$.

Let us prove that $\Lambda \{E\}$ is quasi-barrelled. If $H \subset (\Lambda \{E\})' = \Lambda^\times \{E'_b\}$ is strongly bounded, then H is bounded for the natural topology. Since E'_b is fundamentally $\Lambda^\times$-bounded, there is a disc $D \subset E'_b$ and a solid disc $M \subset \Lambda^\times$ such that $H \subset [M, D]$; i.e. every $g \in H$ is localized in E'_D and $p_D(g) \in M$. Since E is quasi-barrelled, the polar seminorm q_D is continuous. For $f \in \Lambda \{E\}$ we have

$$\begin{aligned} q_H(f) &\le \sup \left\{ \int_\Omega |\langle f, g\rangle|\ d\mu : g \in H \right\} \le \sup \left\{ \int_\Omega q_D(f) p_D(g)\, d\mu : g \in H \right\} \\ &\le \sup \left\{ \int_\Omega q_D(f) \cdot \theta\, d\mu : \theta \in M \right\} = q_M(q_D(f)), \end{aligned}$$

and this last expression is a continuous seminorm on $\Lambda \{E\}$. ■

Theorems 3 and 7 yield the following corollary.

Corollary 7. *Under the hypothesis of Theorem 7, if Λ is perfect and E is locally complete, then $\Lambda \{E\}$ is barrelled.*

If we want to get that $\Lambda \{E\}$ is barrelled without requiring E to be locally complete, a different approach is to use the abstract result given in [7]. A family $P_\Sigma = \{P_A : A \in \Sigma\}$ of continuous linear projections on a locally convex space X is called an (Ω, Σ, μ)-Boolean algebra of projections if:

1. P_Ω is the identity on X.

2. $P_A = 0$ whenever $A \in \Sigma$ and $\mu(A) = 0$.

3. $P_{A_1 \cap A_2} = P_{A_1} \cdot P_{A_2}$ for all $A_1, A_2 \in \Sigma$.

4. $P_{A_1 \cup A_2} = P_{A_1} + P_{A_2}$ for all disjoint $A_1, A_2 \in \Sigma$.

The result mentioned above can be stated as follows.

Theorem A ([7, Cor. 1 and 2]) *Let X be a Hausdorff locally convex space with an (Ω, Σ, μ)-Boolean algebra of projections P_Σ. Assume that P_Σ is equicontinuous and that the following condition holds:*

($*$) *Whenever (A_n) is a decreasing sequence in Σ with $\mu(\bigcap_n A_n) = 0$, (x_n) is a bounded sequence in X such that x_n is supported in A_n (i.e. $P_{A_n}(x_n) = x_n$) for each $n \in \mathbb{N}$, and (α_n) is a sequence in ℓ^1, then the series $\sum_n \alpha_n x_n$ converges in X.*

Then we have:

1. *If X is quasi-barrelled and $P_A(X)$ is barrelled for each atom $A \in \Sigma$ then X is barrelled.*

2. *If X is quasi-barrelled and μ is atomless then X is barrelled.*

Let us see first that a space $\Lambda\{E\}$ satisfies condition ($*$) above provided that Λ is locally complete. In our case the equicontinuous (Ω, Σ, μ)-Boolean algebra of projections on $\Lambda\{E\}$ is given by

$$P_A : f \in \Lambda\{E\} \to P_A(f) := \chi_A \cdot f \in \Lambda\{E\}.$$

Lemma 2. *Assume that Λ is locally complete. Let (A_n) be a decreasing sequence of measurable sets with $\mu(\bigcap_n A_n) = 0$ and (f_n) be a bounded sequence in $\Lambda\{E\}$ such that f_n is supported in A_n for every $n \in \mathbb{N}$. If $(\alpha_n) \in \ell^1$ then the series $\sum_n \alpha_n f_n$ converges in $\Lambda\{E\}$.*

Proof. Since (A_n) is decreasing, if $t \notin A_n$, then $f_j(t) = 0$ for $j \geq n$. Therefore, except for t on a zero measure set, each of the series $\sum_n \alpha_n f_n(t)$ contains only a finite number of nonzero terms. This tells us that the series $\sum_n \alpha_n f_n$ is pointwise convergent a.e. in Ω to a function $f : \Omega \to E$. It is easy to see that this convergence is almost uniform on compact sets, so that f is measurable. Let us see that $f \in \Lambda\{E\}$ and that the series $\sum_n \alpha_n f_n$ converges to f for the topology of $\Lambda\{E\}$. Take $q \in \mathcal{Q}(E)$, then pointwisely we have $q(f) \leq \sum_n |\alpha_n|\, q(f_n)$. Since $(\alpha_n) \in \ell^1$ and the sequence $(q(f_n))$ is bounded in Λ, it follows that the locally Cauchy series $\sum_n |\alpha_n| q(f_n)$ is convergent to a function $\phi \in \Lambda$ because this space is locally complete. Now $\sum_n |\alpha_n| q(f_n)$ is an increasing series that converges pointwisely a.e. in Ω. By integrating on compact sets we easily deduce that its pointwise limit is also ϕ. Then $q(f(t)) \leq \phi(t)$ a.e. in Ω, so that $q(f) \in \Lambda$ because this space is solid. Finally, take $M \in \mathcal{M}$. Then, since (f_n) is bounded, we have

$$\begin{aligned}\lim_{n\to\infty} q_M\left(q\left(f - \sum_{k=1}^{n} \alpha_k f_k\right)\right) &= \lim_{n\to\infty} q_M\left(q\left(\sum_{k=n+1}^{\infty} \alpha_k f_k\right)\right) \\ &\leq \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |\alpha_k| \cdot q_M\left(q(f_k)\right) = 0,\end{aligned}$$

so that $\sum_n \alpha_n f_n$ converges to f in $\Lambda\{E\}$. ■

Theorem 8. *Under the hypothesis of Theorem 7, if Λ is locally complete, and if either the measure space is atomless or E is barrelled, then $\Lambda\{E\}$ is barrelled.*

Proof. We shall apply Theorem A. Since Λ is solid, it follows that the family $P_\Sigma = \{P_A : A \in \Sigma\}$ is an equicontinuous Boolean algebra of projections on $\Lambda\{E\}$, in the sense given above. Now, if A is an atom and E is barrelled, then $P_A(\Lambda\{E\})$ is barrelled because, in this case, $P_A(\Lambda\{E\})$ is isomorphic to E. Next, the previous lemma tells us that $\Lambda\{E\}$ satisfies condition $(*)$ of Theorem A. Theorem 7 says that $\Lambda\{E\}$ is quasi-barrelled. Finally, apply Theorem A. ■

7 The Special Case of Fréchet and (DF)-spaces

We treat here the special case of Fréchet and (DF)-spaces. As we saw in Theorem 1, if Λ and E are both metrizable or (DF) then they have good localization of bounded sets. Since these classes of spaces are, roughly speaking, dual to each other, one can localize the bounded sets also in $\Lambda^\times\{E'_b\}$. This enables us to make good use of the results of the last two sections.

We shall fix some terminology for this last section in order to avoid clumsy repetitions. First of all, we shall assume that Λ is perfect and that it is endowed with the strong topology $\beta(\Lambda, \Lambda^\times)$; in particular, Λ is complete. If Λ is a Fréchet space then $\Lambda^\times$ is a (DF)-space. Conversely, if Λ is (DF) then $\Lambda^\times$ is Fréchet. Echelon and coechelon spaces are important examples of (Fréchet,(DF)) dual pairs of perfect spaces, they have been studied in [6], [16], [27], [28] and [34].

We say that the dual pair $(\Lambda, \Lambda^\times)$ is reflexive if the topological dual of Λ is $\Lambda^\times$ and vice versa; equivalently, Λ with the Mackey topology $\tau(\Lambda, \Lambda^\times)$ is reflexive.

According to Theorem 1, if Λ is a Fréchet space and E is metrizable then E is fundamentally Λ-bounded and E'_b is fundamentally $\Lambda^\times$-bounded. Conversely, and also by Theorem 1, if Λ is a (DF)-space and E is a (DF)-space then E is fundamentally Λ-bounded and E'_b is fundamentally $\Lambda^\times$-bounded. We shall not repeat these two facts in the proofs below.

Theorem 9. *(1) Let Λ be a Fréchet space with topological dual $\Lambda^\times$. If E is a metrizable space such that E'_b has the Radon-Nikodym property, then we have $(\Lambda\{E\})' = \Lambda^\times\{E'_b\} = (\Lambda\{E\})^\times$. Moreover, if either the measure is atomless or E is barrelled, then $\Lambda\{E\}$ is barrelled.*

(2) Let Λ be a Fréchet space with topological dual $\Lambda^\times$. If E is a Fréchet space such that E'_b has the Radon-Nikodym property, then $\Lambda\{E\}$ is a Fréchet space with dual $(\Lambda\{E\})' = \Lambda^\times\{E'_b\} = (\Lambda\{E\})^\times$.

(3) Let Λ be a (DF)-space with topological dual $\Lambda^\times$. If E is a quasi-barrelled (DF)-space such that E'_b has the Radon-Nikodym property, then $\Lambda\{E\}$ is a quasi-barrelled (DF)-space with dual $(\Lambda\{E\})' = \Lambda^\times\{E'_b\} = (\Lambda\{E\})^\times$. Moreover, if either the measure is atomless or E is barrelled then $\Lambda\{E\}$ is barrelled.

Proof. Part (1): Apply Theorems 6 and 8 above. For part (2) we also need these two results together with Corollary 5. For part (3), simply note that if (B_n) and (R_n) are fundamental sequences of bounded sets in E and Λ, respectively, the sequence of bounded sets $([R_n, B_n])$ is fundamental in $\Lambda\{E\}$. The conclusion follows from Theorems 6 and 8 again. ■

Part (3) of this result should be compared with [13, Thm. 4]. Their conclusions are the same but the hypotheses are somewhat different. Namely, in [13, Thm. 4] it is required Λ to be a Banach space, but it is not required its strong dual to be $\Lambda^\times$; E is a (DF)-space that satisfies the dual density condition, but no mention of the Radon-Nikodym property is made (however, see Lemma 3 below); besides, we also assumed that the countably valued functions are dense in $\Lambda\{E\}$.

Corollary 8. *If E is metrizable and E'_b has the Radon-Nikodym property, then the dual of $L^p\{E\}$ is $L^{p^*}\{E'_b\}$ for all $p \in [1, \infty)$. Moreover, if either the measure is atomless or E is barrelled then $L^p\{E\}$ is barrelled*

Corollary 9. *If E a quasi-barrelled (DF)-space and E'_b has the Radon-Nikodym property then for all $p \in [1, \infty)$ $L^p\{E\}$ is a quasi-barrelled (DF)-space with dual $L^{p^*}\{E'_b\}$. Moreover, if either E is barrelled or the measure is atomless then $L^p\{E\}$ is barrelled.*

As we mentioned above, every reflexive Fréchet space has the Radon-Nikodym property. We give now a class of (DF)-spaces with the Radon-Nikodym property.

Lemma 3. *If E is a reflexive (DF)-space satisfying the dual density condition then E has the Radon-Nikodym property.*

Proof. Since E satisfies the dual density condition, we have that the bounded subsets of E are metrizable [1, 1.2.(d)], hence E has property (BM). To prove that E has the Radon-Nikodym property, let $m : \Sigma \to E$ be a μ-continuous vector measure with bounded variation, then m has locally bounded average range by [2, Lem. 3.3]. Therefore, m has locally relatively weakly compact average range because E is reflexive. Our Radon-Nikodym theorem [12, 4.9] allows us to deduce that m has a density because E is quasi-complete and has property (BM). ■

Theorem 10. *(1) Let Λ be a (DF)-space with topological dual $\Lambda^\times$. If E is a reflexive (DF)-space then $\Lambda\{E\}$ is a barrelled (DF)-space with strong dual $\Lambda^\times\{E'_b\}$.*

(2) Let Λ be a Fréchet space with topological dual $\Lambda^\times$. If E is a reflexive Fréchet space with the density condition then the strong dual of $\Lambda\{E\}$ is $\Lambda^\times\{E'_b\}$.

(3) Let Λ be a Fréchet space such that the dual pair $(\Lambda, \Lambda^\times)$ is reflexive. If E is a reflexive Fréchet space with the density condition, then $\Lambda\{E\}$ is a reflexive Fréchet space.

(4) Let Λ be a (DF)-space such that the dual pair $(\Lambda, \Lambda^\times)$ is reflexive. If E is a reflexive (DF)-space satisfying the dual density condition, then $\Lambda\{E\}$ is a reflexive (DF)-space.

(5) If E is either a reflexive Fréchet space with the density condition or a reflexive (DF)-space with the dual density condition, then the space $L^p\{E\}$ is reflexive for $1 < p < \infty$.

(6) If E is a reflexive Fréchet space with the density condition, then $L^1\{E\}$ is distinguished.

Proof. Part(1): Since E is reflexive, it is barrelled. Now apply Theorem 9 (3).

Part (2): Since E has the density condition, we have that E'_b satisfies the dual density condition [1, 1.2.(c)]. Lemma 3 above tells us that E'_b has the Radon-Nikodyn property. Finally, apply Theorem 9 (2).

Part (3): By (2), the strong dual of $\Lambda\{E\}$ is $\Lambda^\times\{E'_b\}$ and this space has $\Lambda\{E\}$ as strong dual by (1).

Part (4): By (1), the strong dual of $\Lambda\{E\}$ is $\Lambda^\times\{E'_b\}$. Since E'_b is a reflexive Fréchet space stisfying the density condition [1, 1.2.(c)], the strong dual of $\Lambda^\times\{E'_b\}$ is $\Lambda\{E\}$ by (2).

Part (5) follows from parts (3) and (4).

Part(6): We have to see that the strong dual of $L^1\{E\}$ is quasi-barrelled. The strong dual of $L^1\{E\}$ is $L^\infty\{E'_b\}$ by part (2) and this space is quasi-barrelled by [10, Thm. 2] or [13, Thm. 4] because E'_b is a (DF)-space satisfying the dual density condition. ■

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