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Average connectivity of minimally 2-connected graphs and average edge-connectivity of minimally 2-edge-connected graphs

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ABSTRACT

Let G be a (multi)graph of order n and let u, v be vertices of G. The maximum number of internally disjoint u–v paths in G is denoted by $\kappa_G(u, v)$, and the maximum number of edge-disjoint u–v paths in G is denoted by $\lambda_G(u, v)$. The *average connectivity* of G is defined by $\overline{\kappa}(G) = \sum \kappa_G(u, v)/\binom{n}{2}$, and the *average edge-connectivity* of G is defined by $\overline{\lambda}(G) = \sum \lambda_G(u, v)/\binom{n}{2}$, where both sums run over all unordered pairs of vertices $\{u, v\} \subseteq V(G)$. A graph G is called *ideally connected* if $\kappa_G(u, v) = \min\{\deg(u), \deg(v)\}$ for all unordered pairs of vertices $\{u, v\}$ of G.

We prove that every minimally 2-connected graph of order n with largest average connectivity is bipartite, with the set of vertices of degree 2 and the set of vertices of degree at least 3 being the partite sets. We use this structure to prove that $\overline{\kappa}(G) < \frac{9}{4}$ for any minimally 2-connected graph G. This bound is asymptotically tight, and we prove that every extremal graph of order n is obtained from some ideally connected nearly regular graph on roughly $n/4$ vertices and $3n/4$ edges by subdividing every edge. We also prove that $\overline{\lambda}(G) < \frac{9}{4}$ for any minimally 2-edge-connected graph G, and provide a similar characterization of the extremal graphs.

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1. Introduction

Throughout, we allow graphs to have multiple edges. A graph with no multiple edges is called a *simple graph*. A u–v path in a graph G is an alternating sequence of vertices and edges

$$v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$$

in which all vertices and edges are distinct, $v_0 = u$, $v_k = v$, and edge e_i has endvertices v_{i-1} and v_i for all $i \in \{1, \dots, k\}$. If G is a simple graph, then a path can be described by listing only its vertices. A set of u–v paths $\mathcal{P}$ is called *internally disjoint* if no two paths in $\mathcal{P}$ have an internal vertex (i.e., a vertex other than u or v) or an edge in common (which can only occur if u and v are adjacent), and is called *edge-disjoint* if no two paths in $\mathcal{P}$ have an edge in common. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$ or $d(u, v)$ if G is understood, is the length of a shortest u–v path.

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Let G be a non-trivial graph. The *connectivity* of G, denoted by $\kappa(G)$, is the smallest number of vertices whose removal disconnects G or produces the trivial graph. The *edge-connectivity* of G, denoted by $\lambda(G)$, is the smallest number of edges whose removal disconnects G. For $k \geq 1$, a graph G is *k-connected* (or *k-edge-connected*) if it has connectivity (edge-connectivity, respectively) at least k.

Following [2], for a pair u, v of distinct vertices of G, the *connectivity* between u and v in G, denoted by $\kappa_G(u, v)$, is defined as the maximum number of internally disjoint paths between u and v. By a well-known theorem of Menger [12], when u and v are non-adjacent, this matches the familiar alternate definition of the connectivity between u and v as the minimum number of vertices whose removal separates u and v. The definition from [2] used here is also well-defined if u and v are adjacent. Analogously, the *edge-connectivity* between u and v in G, denoted by $\lambda_G(u, v)$, is the maximum number of edge-disjoint u–v paths in G. Again, by an alternate version of Menger's theorem, this matches the familiar alternate definition of the edge-connectivity between u and v as the minimum number of edges whose removal separates u and v. When G is clear from context, we use $\kappa(u, v)$ and $\lambda(u, v)$ instead of $\kappa_G(u, v)$ and $\lambda_G(u, v)$, respectively. Whitney [16] showed that if G is a graph, then $\kappa(G) = \min\{\kappa(u, v) \mid u, v \in V(G)\}$. Similarly $\lambda(G) = \min\{\lambda(u, v) \mid u, v \in V(G)\}$. Thus, the connectivity and edge-connectivity of a graph are worst-case measures.

A more refined measure of the overall level of connectedness of a graph, introduced in [2], is based on the average values of the 'local connectivities' between all pairs of vertices. The *average connectivity* of a graph G of order n, denoted by $\overline{\kappa}(G)$, is the average of the connectivities over all pairs of distinct vertices of G. That is,

$$\overline{\kappa}(G) = \sum_{\{u,v\} \subseteq V(G)} \kappa(u, v) / \binom{n}{2}.$$

The *total connectivity* of G, denoted by $K(G)$, is the sum of the connectivities over all pairs of distinct vertices of G, i.e., $K(G) = \binom{n}{2}\overline{\kappa}(G)$.

Analogously, the *average edge-connectivity* of G, denoted by $\overline{\lambda}(G)$, is the average of the edge-connectivities over all pairs of distinct vertices of G. That is,

$$\overline{\lambda}(G) = \sum_{\{u,v\} \subseteq V(G)} \lambda_G(u, v) / \binom{n}{2}.$$

The *total edge-connectivity* of G, denoted by $\Lambda(G)$, is the sum of the edge-connectivities over all pairs of distinct vertices of G, i.e., $\Lambda(G) = \binom{n}{2}\overline{\lambda}(G)$.

Let u and v be distinct vertices of a graph G. It is well-known (see [13, Section 5]) that

$$\kappa(u, v) \leq \lambda(u, v) \leq \min\{\deg(u), \deg(v)\}.$$

If $\kappa(u, v) = \min\{\deg(u), \deg(v)\}$ for all pairs of distinct vertices u and v in G, then we say that G is *ideally connected*. If $\lambda(u, v) = \min\{\deg(u), \deg(v)\}$ for all pairs of distinct vertices u and v in G, then we say that G is *ideally edge-connected*. Evidently, if G is ideally connected, then it must also be ideally edge-connected. A graph G is called *nearly regular* if the difference between its maximum degree and its minimum degree is at most 1. Ideally connected nearly regular graphs will play an important role in our work.

Much work has been done on bounding the average connectivity in terms of various graph parameters, including order and size [2], average degree [6], and matching number [10]. Bounds have also been achieved on the average connectivity of graphs belonging to particular families, including planar and outerplanar graphs [6], Cartesian product graphs [6], strong product graphs [1], and regular graphs [10]. Average connectivity has also proven to be a useful measure for real-world networks, including street networks [3] and communication networks [15].

In this article, we demonstrate sharp bounds on the average connectivity of minimally 2-connected graphs and the average edge-connectivity of minimally 2-edge-connected graphs. For $k \geq 1$, a graph G is called *minimally k-connected* if $\kappa(G) = k$ and for every edge e of G, $\kappa(G - e) < k$. Analogously, G is called *minimally k-edge-connected* if $\lambda(G) = k$ and for every edge e of G, $\lambda(G - e) < k$. A graph G with $\kappa(G) = \overline{\kappa}(G) = k$ is called a *uniformly k-connected graph*. It was observed in [2] that uniformly k-connected graphs are minimally k-connected. It is obvious that every minimally 1-connected graph (i.e., tree) is uniformly 1-connected. However, for $k \geq 2$, minimally k-connected graphs need not be uniformly k-connected, as can be seen by considering the graphs $K_{k,n-k}$ for $n > 2k \geq 4$. So if $k \geq 2$, it is natural to ask by how much the average connectivity of a minimally k-connected graph can exceed k. Similarly, by how much can the average edge-connectivity of a minimally k-edge-connected graph exceed k? In this article, we answer both of these questions in the case where $k = 2$.

We show that

$$2 \leq \overline{\kappa}(G) < \tfrac{9}{4}$$

for every minimally 2-connected graph G. The lower bound is readily seen to be attained if and only if G is a cycle. We prove the upper bound in Section 2. We say that G is an *optimal* minimally 2-connected graph of order n if G has maximum average connectivity among all such graphs. We prove that any optimal minimally 2-connected graph of order at least 5 must be bipartite, with the set of vertices of degree 2 and the set of vertices of degree at least 3 being the partite sets. More specifically it is shown that every minimally 2-connected graph of order n having maximum average connectivity

is obtained from some ideally connected nearly regular graph on roughly $n/4$ vertices and $3n/4$ edges by subdividing every edge. This result demonstrates that the above bound of $9/4$ on $\overline{\kappa}(G)$ is asymptotically tight. It can be deduced, from this characterization, that the optimal minimally 2-connected graphs are ideally connected but not all ideally minimally 2-connected graphs are optimal.

We also show that

$$2 \leq \overline{\lambda}(G) < \frac{9}{4}$$

for any minimally 2-edge-connected graph G. The lower bound is readily seen to be attained if and only if every block of G is a cycle. We prove the upper bound in Section 3, where we study the structure of minimally 2-edge-connected graphs of order n with maximum average edge-connectivity (which we call *edge-optimal* minimally 2-edge-connected graphs). We obtain structural results on edge-optimal minimally 2-edge-connected graphs similar to those obtained for optimal minimally 2-connected graphs, though the proofs are quite different. This culminates in the same upper bound as for the vertex version, and an analogous characterization of the edge-optimal minimally 2-edge-connected graphs.

2. Average connectivity of minimally 2-connected graphs

In this section, we obtain results about the structure of optimal minimally 2-connected graphs, and use this to prove a sharp upper bound on the average connectivity of minimally 2-connected graphs. It is easy to see that minimally 2-connected graphs must be simple graphs. So throughout this section, we denote paths by listing only the vertices.

Minimally 2-connected graphs were characterized independently in [8,14]. A cycle C of a graph G is said to have a *chord* if there is an edge of G that joins a pair of non-adjacent vertices from C. The following characterization of Plummer [14] is used frequently throughout this section.

Theorem 2.1 ([14, Corollary 1a])**.** *A 2-connected graph G is minimally 2-connected if and only if no cycle of G has a chord.*

We state next another useful characterization of minimally 2-connected graphs from [14].

Theorem 2.2 ([14, Theorem 5])**.** *Let G be a 2-connected graph that is not a cycle. Let S be the vertices of degree 2, and let $T_1, \ldots, T_k$ be the components of $G - S$. Then G is minimally 2-connected if and only if $k \geq 2$, and for every $1 \leq i \leq k$, T_i is a tree, and if C is any cycle of G, then either $V(C) \cap V(T_i) = \emptyset$ or the subgraph induced by $V(C) \cap V(T_i)$ is connected.*

Remark 2.3. Let G be a minimally 2-connected graph and let C and $T_1, \ldots, T_k$ be as described in Theorem 2.2. If $V(C) \cap V(T_i)$ is not empty, then the vertices in $V(C) \cap V(T_i)$ induce a path in both C and T_i; otherwise C has a chord, contrary to Theorem 2.1.

In particular, Theorem 2.2 says that in a minimally 2-connected graph that is not a cycle, the vertices of degree exceeding 2 induce an acyclic graph. We remark that Mader extended this result in [11] by showing that in a minimally k-connected graph, for $k \geq 2$, the subgraph induced by the vertices of degree exceeding k is a forest.

2.1. Structural properties of optimal minimally 2-connected graphs

Let G be a minimally 2-connected graph, and let F be the subgraph of G induced by the set of vertices of degree exceeding 2. By Theorem 2.2, F is a forest. We begin by proving that if u and v are in the same component of F, then $\kappa_G(u, v) = 2$. This explains why we might expect the set of vertices of degree exceeding 2 to be independent in an optimal minimally 2-connected graph.

Theorem 2.4. *Let G be a minimally 2-connected graph that is not a cycle, and let S be the set of vertices of degree 2 in G. Suppose T is a non-trivial component of $G - S$. If u, v are vertices of T, then $\kappa(u, v) = 2$.*

Proof. By Theorem 2.2, T is a tree. Since G is 2-connected, $\kappa(u, v) \geq 2$. If u and v are adjacent, then $\kappa(u, v) = 2$, by [4, Lemma 4.2]. So we assume $uv \notin E(G)$. Assume, to the contrary, that $\kappa(u, v) \geq 3$. Then there exist three internally disjoint u–v paths P_1, P_2, and P_3 in G. At most one of these paths is contained in T. We may assume both P_2 and P_3 contain vertices not in T. Let C be the cycle induced by the vertices of P_2 and P_3. Then C contains both u and v and the subgraph of C induced by $V(C) \cap V(T)$ is not connected, contrary to Theorem 2.2. □

Corollary 2.5. *Let G, S, T and u, v be as in the statement of Theorem 2.4. If P_1 and P_2 are two internally disjoint paths in G, then one of P_1 and P_2 is the u–v path in T.*

Proof. Let C be the cycle formed from P_1 and P_2. Then $V(C) \cap V(T)$ is non-empty. So, by Remark 2.3, $V(C) \cap V(T)$ induces a path in T and in C that contains u and v. Since T has a unique u–v path, one of P_1 and P_2 must be the u–v path in T. □

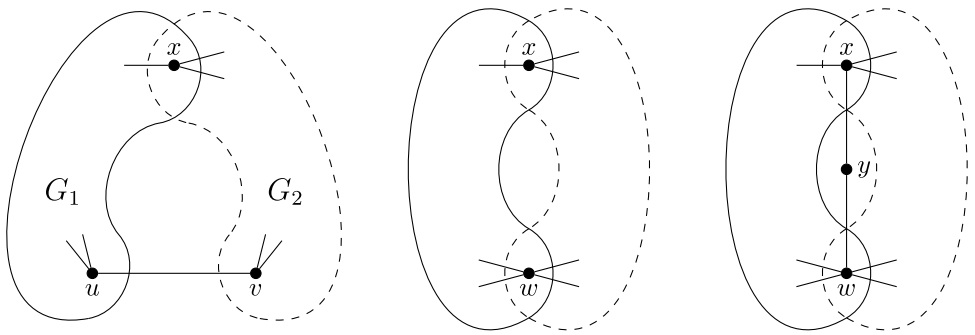

Fig. 1. A sketch of G (left), H (middle), and G' (right). Note that u, v, and x do not necessarily have degree exactly 3 as drawn.

We now show that if G is an optimal minimally 2-connected graph of order at least 5, then the set of vertices of degree 2 is independent, and so is the set of vertices of degree exceeding 2. This is the key structural result used in the sequel to obtain an upper bound on the average connectivity of minimally 2-connected graphs.

Theorem 2.6. *Let G be an optimal minimally 2-connected graph of order $n \geq 5$. Then the vertices of degree 2 in G form an independent set.*

Proof. Since $n \geq 5$ and $K_{2,n-2}$ is a minimally 2-connected graph with average connectivity exceeding 2, G is not a cycle. So, by Theorem 2.2, G has at least two vertices of degree exceeding 2.

If the vertices of degree 2 do not form an independent set, there exist vertices u and v of degree exceeding 2 and a u–v path $P : (u =)u_0u_1 \dots u_k(= v)$, such that $k \geq 3$ and $\deg_G(u_i) = 2$ for $1 \leq i \leq k-1$. Delete the edges of P from G and add the edges uu_i and u_iv for $1 \leq i \leq k-1$. Let G' be the resulting graph. Then G' has order n and it is readily checked that G' is minimally 2-connected. Moreover, the total connectivity of G' exceeds the total connectivity of G by $k-2$ since $\kappa_{G'}(u, v) = \kappa_G(u, v) + k - 2$, and for all pairs x, y of vertices of G where $\{x, y\} \neq \{u, v\}$ we have $\kappa_{G'}(x, y) = \kappa_G(x, y)$. □

In the next result we use the following notation. For vertices u and v in a graph G we use $u \sim_G v$ to indicate that u is adjacent with v and $u \not\sim_G v$ to indicate that u is not adjacent with v. The subscript is omitted if G is clear from context.

Theorem 2.7. *Let G be an optimal minimally 2-connected graph of order $n \geq 5$. Then the vertices of degree exceeding 2 in G form an independent set.*

Proof. As in the proof of the previous result we see that G has at least two vertices of degree exceeding 2.

Suppose, towards a contradiction, that u and v are adjacent vertices of degree at least 3 in G. Since G is minimally 2-connected, $G - uv$ has a cut-vertex, say x. Since $G - x$ is connected, it follows that uv is a bridge of $G - x$. So $G - uv - x$ has exactly two components A_1 and A_2, say, where A_1 contains u and A_2 contains v. Let G_i be the subgraph of G induced by $V(A_i) \cup \{x\}$ for $i = 1, 2$ (see Fig. 1).

Let H be the graph obtained from G by contracting the edge uv to a new vertex labelled w. Let G' be the graph obtained from H by adding vertex y and the edges xy and yw (see Fig. 1). We prove that G' is a minimally 2-connected graph of order n (Fact 3) with $\overline{\kappa}(G') > \overline{\kappa}(G)$ (Fact 4), contradicting the optimality of G. We begin by proving two useful facts.

Fact 1. $u \not\sim_G x$ and $v \not\sim_G x$.

Since $\deg_G(u) \geq 3$, u has a neighbour a in the set $V(G) - \{v, x\}$. Since G is 2-connected, there is an a–x path Q_1 that does not contain u, and it must lie in G_1. Similarly, v has a neighbour b in the set $V(G) - \{u, x\}$, and there is a b–x path Q_2 in G_2 that does not contain v. So the paths Q_1, Q_2, and $auvb$ produce a cycle, C say, in which neither u nor v is adjacent with x. Since G is minimally 2-connected, it follows from Theorem 2.1 that C has no chords. So $u \not\sim_G x$ and $v \not\sim_G x$. This completes the proof of Fact 1.

Thus H and hence G' contains no multiple edges.

Fact 2. G' is 2-connected.

Since G' is obtained from H by joining the new vertex y to the vertices x and w, it follows, by a straightforward argument, that G' is 2-connected if H is 2-connected. We will show that H is 2-connected by showing that every pair of distinct vertices of H lies on a cycle. First let $a, b \in V(H) - \{w\} = V(G) - \{u, v\}$. Since G is 2-connected, there is a cycle C of G containing a and b. If C does not contain u or v, then C is a cycle of H. If C contains exactly one of u (or v), then the cycle obtained from C by replacing u (or v, resp.) with w is a cycle of H that contains a and b. So we may assume that C contains both u and v. In this case, C must contain x as well. By Fact 1, $u \not\sim_G x$ and $v \not\sim_G x$. So by contracting the edge uv

of C to w, we obtain a cycle of H containing a and b. Finally, for any $a \in V(H) - \{w\}$, there is a cycle C of G containing a and the edge uv. Contracting the edge uv of C to w gives a cycle in H containing a and w. So we conclude that H, and hence G', is 2-connected. This completes the proof of Fact 2.

Fact 3. G' is minimally 2-connected.

Assume, to the contrary, that G' is not minimally 2-connected. Then, by Theorem 2.1, G' has a cycle C with a chord. Since y has degree 2, it is not incident with a chord of C. So the ends of the chord are either in G_1 or G_2, say in G_1. If w is not on C, then C is contained in G_1. However, then G has a cycle with a chord, contrary to Theorem 2.1. So we may assume that C contains w. Assume next that C contains the path wyx. Let P be any v–x path in G_2. Then the graph obtained from C by deleting the subpath wyx and adding P and the edge uv produces a cycle in G that has a chord, contrary to Theorem 2.1. So we may assume that C does not contain y. Also wx is not an edge of C since, by Fact 1, $w \not\sim_G x$. If C contains no vertices of $G_2 - \{x, v\}$, then the cycle obtained by replacing w in C with u is a cycle in G_1 and thus a cycle in G that has a chord, contrary to Theorem 2.1. So C contains a vertex of $G_2 - \{x, v\}$. By replacing w in C with uv, we obtain a cycle of G that has a chord, a contradiction. Thus G' is minimally 2-connected. This completes the proof of Fact 3.

Fact 4. $\overline{\kappa}(G) < \overline{\kappa}(G')$.

We show that $K(G) < K(G')$, from which the statement readily follows. We demonstrate the following:

(i) $\kappa_G(u, v) = \kappa_{G'}(w, y)$;
(ii) $\kappa_G(a, b) \leq \kappa_{G'}(a, b)$ for all $a, b \in V(G) - \{u, v\}$;
(iii) $\kappa_G(u, z) + \kappa_G(v, z) \leq \kappa_{G'}(w, z) + \kappa_{G'}(y, z)$ for all $z \in V(G) - \{u, v, x\}$; and
(iv) $\kappa_G(u, x) + \kappa_G(v, x) < \kappa_{G'}(w, x) + \kappa_{G'}(y, x)$.

Summing the left-hand side of (i), (ii), (iii), and (iv) over all possibilities gives $K(G)$, and summing the right-hand side of (i), (ii), (iii), and (iv) over all possibilities gives $K(G')$, so the desired result follows immediately.

For (i), $\kappa_G(u, v) = 2$ by Theorem 2.4, and since $\deg_{G'}(y) = 2$, we have $\kappa_{G'}(w, y) = 2$, by Fact 2.

For (ii), let $a, b \in V(G) - \{u, v\} = V(G') - \{w, y\}$. If one of a and b belongs to G_1 and the other to G_2, then $\kappa_{G'}(a, b) = 2 = \kappa_G(a, b)$. So assume, without loss of generality, that $a, b \in V(G_1) - u$. If $\mathcal{P}_{a,b}$ is a collection of $\kappa_G(a, b)$ pairwise internally disjoint a–b paths in G, then at most one of these paths contains the vertex u. If no member of $\mathcal{P}_{a,b}$ contains u, then $\mathcal{P}_{a,b}$ is a collection of $\kappa_G(a, b)$ internally disjoint a–b paths in G'. Otherwise, let P be the unique path in $\mathcal{P}_{a,b}$ containing u. If v is also on P, then let P' be the path obtained from P by contracting uv to w. Otherwise, if v is not on P, then let P' be the path obtained from P by replacing u with w. Then $(\mathcal{P}_{a,b} - P) \cup \{P'\}$ is a collection of $\kappa_G(a, b)$ internally disjoint a–b paths in G'. Either way, we conclude that $\kappa_G(a, b) \leq \kappa_{G'}(a, b)$.

For (iii), let $z \in V(G) - \{u, v, x\}$. Assume without loss of generality that $z \in V(G_1)$. Let $\mathcal{P}_{u,z}$ be a family of $\kappa_G(u, z)$ pairwise internally disjoint u–z paths in G. Any path between u and z that also contains at least one vertex of G_2 must necessarily contain both uv and x. Thus at most one of the paths in $\mathcal{P}_{u,z}$ contains v. If such a u–z path P exists, then the path obtained from P by contracting the edge uv to w is a w–z path in G'. If we replace u by w on all the remaining paths in $\mathcal{P}_{u,z}$, then we obtain a family of $\kappa_G(u, z)$ pairwise internally disjoint w–z paths in G'. So $\kappa_G(u, z) \leq \kappa_{G'}(w, z)$. Since the edge uv and the vertex x separate z and v in G, it follows that $\kappa_G(v, z) = 2$. Since $\deg_{G'}(y) = 2$, we have $\kappa_{G'}(y, z) = 2$, by Fact 2. So $\kappa_G(u, z) + \kappa_G(v, z) \leq \kappa_{G'}(w, z) + \kappa_{G'}(y, z)$.

For (iv), let $\mathcal{P}_{u,x}$ be a collection of $\kappa_G(u, x)$ pairwise internally disjoint u–x paths in G. Exactly one of these paths contains vertices of A_2, since such a path necessarily contains the edge uv, and there is a v–x path in G_2. Let $\mathcal{P}'_{u,x}$ be the collection of all paths in $\mathcal{P}_{u,x}$ whose internal vertices belong to G_1. So $|\mathcal{P}'_{u,x}| = \kappa_G(u, x) - 1$. By replacing u with w on every path of $\mathcal{P}'_{u,x}$, we obtain a family $\mathcal{P}''_{u,x}$ of $\kappa_G(u, x) - 1$ internally disjoint w–x paths of G' whose internal vertices all belong to G_1. By a similar argument, we obtain a family $\mathcal{P}''_{v,x}$ of $\kappa_G(v, x) - 1$ internally disjoint w–x paths of G' whose internal vertices all belong to G_2. The path wyx is a w–x path that is internally disjoint from the paths in $\mathcal{P}''_{u,x} \cup \mathcal{P}''_{v,x}$. So $\kappa_{G'}(w, x) \geq \kappa_G(u, x) + \kappa_G(v, x) - 1$. Finally, since $\deg_{G'}(y) = 2$, we have $\kappa_{G'}(y, x) = 2$, by Fact 2. Therefore,

$$\kappa_{G'}(w, x) + \kappa_{G'}(y, x) \geq \kappa_G(u, x) + \kappa_G(v, x) + 1 > \kappa_G(u, x) + \kappa_G(v, x).$$

This completes the proof of Fact 4 and the theorem. □

Theorem 2.8. *Let G be an optimal minimally 2-connected graph of order $n \geq 5$. Then G is bipartite with partite sets the set of vertices of degree 2 and the set of vertices of degree exceeding 2.*

Proof. This follows immediately from Theorems 2.6 and 2.7. □

We conclude this section by noting that, given a minimally 2-connected graph G of order $n \geq 5$, for which either the vertices of degree 2 or the vertices of degree exceeding 2 are not independent, the proofs of Theorems 2.6 and 2.7 implicitly describe an algorithm for constructing a minimally 2-connected graph G' of the same order n with higher average connectivity than G. By repeated application of this algorithm we obtain a minimally 2-connected graph of order n in which the vertices of degree 2 and those of degree exceeding 2 are independent. Moreover, the average connectivity of this graph exceeds that of the other graphs that preceded it in the process.

2.2. An upper bound on the average connectivity of minimally 2-connected graphs

Using the structural results on optimal minimally 2-connected graphs obtained in the previous section, we now demonstrate a sharp upper bound on the average connectivity of a minimally 2-connected graph of order n, and characterize the optimal minimally 2-connected graphs of order n, for all n sufficiently large.

Recall that a graph G is nearly regular if the difference between its maximum degree and its minimum degree is at most 1. If G is a nearly regular graph of order n and size m, then G has degree sequence

$$\underbrace{d+1,\ldots,d+1}_{r \text{ terms}},\underbrace{d,\ldots,d}_{n-r \text{ terms}}$$

where $d, r \in \mathbb{Z}$ are the unique integers satisfying $2m = dn + r$ and $0 \leq r < n$. We call this sequence a *nearly regular* sequence.

Let G be a graph. We know that $\kappa(u, v) \leq \min\{\deg(u), \deg(v)\}$ for all pairs of distinct vertices u and v of G. This motivates the following definition.

Definition 2.9. The *potential* of a sequence of positive integers $d_1, d_2, \ldots, d_n$ is defined by

$$P(d_1, d_2, \ldots, d_n) = \sum_{1 \leq i < j \leq n} \min\{d_i, d_j\}.$$

For a graph G on n vertices $v_1, v_2, \ldots, v_n$, the *potential* of G, denoted by $P(G)$, is the potential of the degree sequence of G; that is,

$$P(G) = P(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)) = \sum_{1 \leq i < j \leq n} \min\{\deg(v_i), \deg(v_j)\}.$$

We begin by stating two results from Beineke, Oellermann, and Pippert [2] that we will use to obtain the main result of this section. The first of these follows from the proof of Corollary 2.4 in [2], where sharpness was also demonstrated.

Theorem 2.10 (*[2, Corollary 2.4]*). *Let G be a graph of order n, size m, and with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $d \geq 0$ and $0 \leq r < n$ be integers such that $2m = \sum_{i=1}^{n} d_i = dn + r$. Then*

$$P(d_1, d_2, \ldots, d_n) \leq P(\underbrace{d+1,\ldots,d+1}_{r \text{ terms}},\underbrace{d,\ldots,d}_{n-r \text{ terms}}).$$

Recall that if $\kappa(u, v) = \min\{\deg(u), \deg(v)\}$ for all pairs of distinct vertices u and v of G, then we say that G is ideally connected. Since $\kappa(u, v) \leq \min\{\deg(u), \deg(v)\}$ for all u, v, we have $K(G) \leq P(G)$, with equality if and only if G is ideally connected.

Theorem 2.11 (*[2, Section 2]*). *Let n and m be integers such that $3 \leq n \leq m \leq \binom{n}{2}$. Then there is an ideally connected nearly regular simple graph of order n and size m.*

In fact, we note that most ideally connected nearly regular (multi)graphs are simple. More precisely we make the following straightforward observation.

Observation 2.12. *Let G be a nearly regular ideally connected graph of order $n \geq 3$ and size $m \geq n$. Then either G is simple, or G has exactly two vertices of maximum degree, this pair of vertices is joined by exactly two edges, and this is the only multiple edge.*

We observe that if G, as described in Observation 2.12, has a multiple edge, then the two edges joining the pair u, v of vertices of maximum degree, constitute two of the internally disjoint paths in a maximum collection of internally disjoint u–v paths in G.

We are now ready to prove the main result of this section.

Theorem 2.13. *Let G be a minimally 2-connected graph of order n. Then*

$$\overline{\kappa}(G) \leq 2 + \tfrac{(n-2)^2}{4n(n-1)} < \tfrac{9}{4}.$$

Moreover, let $n = 4k + \ell$, where $k, \ell \in \mathbb{Z}$ and $0 \leq \ell < 4$.

(a) *If $k \geq 8$ and $\ell = 0$, then*

$$\overline{\kappa}(G) \leq 2 + \tfrac{n^2-4n}{4n(n-1)} = 2 + \tfrac{n-4}{4(n-1)},$$

with equality if and only if G is obtained from an ideally connected 6-regular graph of order k by subdividing every edge.

(b) *If $k \geq 30$ and $\ell = 1$, then*

$$\overline{\kappa}(G) \leq 2 + \tfrac{n^2-6n+13}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally connected nearly regular (multi)graph of order k and size $n-k = 3k+1$ by subdividing every edge.

(c) *If $k \geq 68$ and $\ell = 2$, then*

$$\overline{\kappa}(G) \leq 2 + \tfrac{n^2-8n+60}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally connected nearly regular graph of either order k and size $n-k = 3k+2$, or order $k+1$ and size $n-k-1 = 3k+1$, by subdividing every edge.

(d) *If $k \geq 30$ and $\ell = 3$, then*

$$\overline{\kappa}(G) \leq 2 + \tfrac{n^2-6n+17}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally connected nearly regular graph of order $k+1$ and size $n-k-1 = 3k+2$ by subdividing every edge.

Proof. Let G be an optimal minimally 2-connected graph of order n. By Theorem 2.8, G is a bipartite graph, with the set of vertices of degree 2 and the set of vertices of degree exceeding 2 being independent sets. Let H be the (multi)graph obtained from G by replacing every vertex of degree 2 with an edge between its neighbours, and note that G can be recovered from H by subdividing each of its edges. Suppose that G has s vertices of degree at least 3, and hence $n-s$ vertices of degree 2. Then H has s vertices and $n-s$ edges. Note that $s \leq \frac{2}{5}n$, as the sum of the degrees of the s vertices of degree at least 3 must be equal to $2(n-s)$. By a straightforward argument, we have

$$\begin{aligned} K(G) &= 2\left[\tbinom{n}{2} - \tbinom{s}{2}\right] + K(H) \\ &\leq 2\left[\tbinom{n}{2} - \tbinom{s}{2}\right] + P(H), \end{aligned}$$

with equality if and only if H is ideally connected. Let $2(n-s) = ds + r$ for $d, r \in \mathbb{Z}$ and $0 \leq r < s$. Then, by Theorem 2.10,

$$\begin{aligned} 2\left[\tbinom{n}{2} - \tbinom{s}{2}\right] + P(H) &\leq 2\tbinom{n}{2} - 2\tbinom{s}{2} + d\tbinom{s}{2} + \tbinom{r}{2} \\ &= 2\tbinom{n}{2} + (d-2)\tbinom{s}{2} + \tbinom{r}{2} \\ &= n(n-1) + \left[\tfrac{2(n-s)-r}{s} - 2\right]\tbinom{s}{2} + \tbinom{r}{2} \\ &= n(n-1) + \left[\tfrac{2n-4s-r}{s}\right]\tfrac{s(s-1)}{2} + \tfrac{r(r-1)}{2} \\ &= n(n-1) + (2n-4s)(s-1)/2 - r(s-1)/2 + r(r-1)/2 \\ &= n(n-1) + (n-2s)(s-1) - r(s-r)/2, \end{aligned}$$

with equality if and only if H is nearly regular (i.e., H has r vertices of degree $d+1$ and $s-r$ vertices of degree d). So far, the bound on $K(G)$ is tight if and only if H is ideally connected and nearly regular. By Theorem 2.11, there exists such a graph H (in fact, a simple graph) for any choice of n and s where $n-s \leq \binom{s}{2}$.

To prove the general bound given in the theorem statement, we first observe, using elementary calculus, that $(n-2s)(s-1)$ achieves a maximum of $\frac{(n-2)^2}{8}$ at $s = \frac{n+2}{4}$. Thus

$$K(G) \leq n(n-1) + (n-2s)(s-1) - r(s-r)/2 \leq n(n-1) + (n-2s)(s-1) \leq n(n-1) + \tfrac{(n-2)^2}{8},$$

Dividing through by $\binom{n}{2}$ gives the general upper bound on $\overline{\kappa}(G)$.

We now prove the exact upper bound given in part (a) of the theorem statement. We include a proof of part (b), which is slightly more technical, in the Appendix. The proofs of parts (c) and (d), which are very similar to the proof of part (b), are omitted. To prove each part, we first find the exact value(s) of s at which the quantity

$$(n-2s)(s-1) - r(s-r)/2$$

is maximized. We then show that $n-s \leq \binom{s}{2}$ at all such values, which guarantees that the maximum is actually attained by some graph.

For part (a), let $n = 4k$ with $k \geq 8$. We show that

$$g_k(s) = (4k-2s)(s-1) - r(s-r)/2 \leq 2k^2 - 2k = \tfrac{n^2-4n}{8},$$

with equality if and only if $s = k$. First, if $s = k$, then $d = 6$ and $r = 0$, and thus $g_k(k) = 2k^2 - 2k$. Next, if $s = k+1$, then $d = 5$ and $r = k-7 > 0$, so $g_k(k+1) = 2k^2 - 2k - 4(k-7) < 2k^2 - 2k$. Lastly, if $s \notin \{k, k+1\}$, let $f_k(s) = (4k-2s)(s-1)$. Clearly $g_k(s) \leq f_k(s)$, and we show that $f_k(s) < 2k^2 - 2k$ for all $s \notin \{k, k+1\}$. The function $f_k(s)$ is a quadratic in s which attains its maximum value at $s = k + \frac{1}{2}$. Thus, if $s < k$, then $f_k(s) < f_k(k) = 2k^2 - 2k$, and if $s > k+1$, then $f_k(s) < f_k(k+1) = 2k^2 - 2k$.

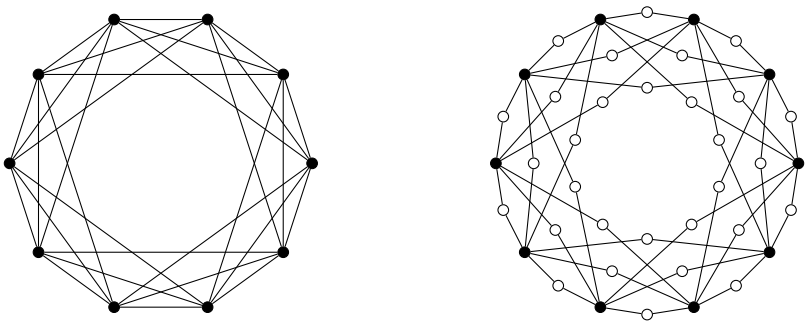

Fig. 2. The graph $C_{10}^{(3)}$ (left) and the graph S_{40} (right) obtained by subdividing every edge of $C_{10}^{(3)}$. The vertices resulting from subdivision are indicated by hollow circles.

In conclusion, we have

$$K(G) \leq n(n-1) + \tfrac{n^2-4n}{8},$$

with equality if and only if H is an ideally connected nearly regular (multi)graph on k vertices and $n-k = 3k$ edges (i.e., H is 6-regular). By Observation 2.12, H must be a simple graph. Since $k \geq 8$, we have $n-k = 3k \leq \binom{k}{2}$, so indeed, Theorem 2.11 guarantees sharpness. The bound on $\overline{\kappa}(G)$ follows by dividing through by $\binom{n}{2}$. This completes the proof of part (a). □

The ideally connected nearly regular graphs described in [2] can now be used to give explicit constructions of optimal minimally 2-connected graphs of order n in each of the parts of Theorem 2.13. In part (a), where $n = 4k$ with $k \geq 8$, the ideally connected nearly regular graph on k vertices and $3k$ edges (i.e., ideally connected 6-regular graph on k vertices) described in [2] is $C_k^{(3)}$ (the cube of the cycle C_k, obtained from C_k by joining all pairs of vertices at distance at most 3). Let S_n be the graph obtained by subdividing every edge of $C_k^{(3)}$. Then S_n is an optimal minimally 2-connected graph of order n. See Fig. 2 for a drawing of S_n in the case where $n = 40$. The other cases can be described in a similar manner.

We make particular mention of the fact that in case (b), where $n = 4k+1$ with $k \geq 30$, we can add any one edge to $C_k^{(3)}$ (we could even create one multiple edge) to produce an ideally connected nearly regular graph of order k and size $3k+1$. Subdividing every edge of such a graph gives an optimal minimally 2-connected graph of order n. So indeed, the ideally connected nearly regular graph in the statement of Theorem 2.13(b) may be a multigraph. In parts (a), (c), and (d), however, the ideally connected nearly regular graph will be simple.

Finally, if G is a minimally 2-connected graph of order n, where n is a small value not covered by Theorem 2.13, then with the notation used in the proof of Theorem 2.13, the bound

$$K(G) \leq n(n-1) + (n-2s)(s-1) - r(s-r)/2 \tag{1}$$

still holds, with equality if and only if H is an ideally connected nearly regular graph on s vertices and $n-s$ edges. The exact maximum value of the right-hand side of (1) can be determined by checking all possibilities for s. From the work of [2], we can guarantee that this bound will be sharp as long as some value of s at which the maximum occurs satisfies $n-s \leq \binom{n}{2}$.

3. Average edge-connectivity of minimally 2-edge-connected graphs

In this section, we obtain results about the structure of edge-optimal minimally 2-edge-connected graphs, and use this to prove a sharp upper bound on the average edge-connectivity of minimally 2-edge-connected graphs.

We first recall some elementary properties of minimally 2-edge-connected graphs, given by Chaty and Chein in [5]. A non-trivial graph having no cut vertices is called *nonseparable*, and the *blocks* of a non-trivial graph G are the maximal nonseparable subgraphs of G.

Lemma 3.1 (*[5]*)**.**

(a) *Every block of a minimally 2-edge-connected graph is minimally 2-edge-connected.*
(b) *If G and H are two minimally 2-edge-connected graphs, then the graph obtained from the disjoint union $G \cup H$ by identifying $u \in V(G)$ and $v \in V(H)$ is minimally 2-edge-connected.*
(c) *If G is a minimally 2-edge-connected graph, then G has no triple edges, and if G has a pair of parallel edges between vertices u and v, then the removal of these two edges separates u and v.*

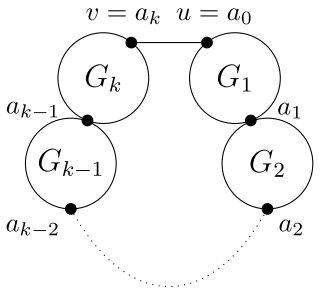

Fig. 3. A graph G with clasp uv. In particular, we have $k \geq 2$, the graph G_i is a necklace or an edge and is extensible between a_{i-1} and a_i for all i, and there exists some j such that G_j is an edge.

A *necklace* is a nonseparable minimally 2-edge-connected simple graph. A graph G is *extensible* between vertices x and y if the graph obtained from G by adding a new vertex z and the edges xz and yz is minimally 2-edge-connected. We refer to this operation as *extending* x and y through z.

Lemma 3.2 ([5, Corollary 2])**.** *Let G be a necklace. For distinct vertices x and y in G, if $\lambda(x, y) \geq 3$, then G is extensible between x and y.*

Following [5], a graph G is called an *E-chain* if it can be represented by an alternating sequence of graphs and vertices $G_1a_1G_2a_2 \cdots a_{k-1}G_k$ for some integer $k \geq 1$, where the following properties are satisfied:

(a) For all $i \in \{1, 2, \ldots, k\}$, either G_i is an edge or G_i is a necklace.
(b) For all $i \in \{1, 2, \ldots, k-1\}$, we have $V(G_i) \cap V(G_{i+1}) = \{a_i\}$.
(c) For all $i \in \{1, 2, \ldots, k-2\}$ and $j \in \{i+2, \ldots, k\}$, we have $V(G_i) \cap V(G_j) = \emptyset$.

We next introduce the terminology "clasp" and "claspable" to make Chaty and Chein's main result from [5] easier to state and work with. An E-chain $G_1a_1G_2a_2 \ldots G_{k-1}a_{k-1}G_k$ is *claspable* at vertices $u = a_0$ and $v = a_k$ if all of the following conditions are satisfied:

(a) $k \geq 2$, $a_0 \in V(G_1)$ with $a_0 \neq a_1$, and $a_k \in V(G_k)$ with $a_{k-1} \neq a_k$.
(b) G_i is extensible between a_{i-1} and a_i for all $i \in \{1, \ldots, k\}$.
(c) There exists some $j \in \{1, \ldots, k\}$ such that G_j is an edge.

We say that an edge uv is a *clasp* in G if the graph $G - uv$ is an E-chain $G_1a_1G_2a_2 \cdots G_{k-1}a_{k-1}G_k$ that is claspable at u and v (see Fig. 3).

Theorem 3.3 ([5, Theorem 2])**.** *Let G be a graph. The following are equivalent:*

(a) *G is a necklace.*
(b) *Every edge of G is a clasp.*
(c) *There exists an edge of G which is a clasp.*

3.1. Structural properties of edge-optimal minimally 2-edge-connected graphs

Recall that an edge-optimal minimally 2-edge-connected graph of order n is a minimally 2-edge-connected graph of order n having maximum average edge-connectivity. For $n \geq 5$, we prove that every edge-optimal minimally 2-edge-connected graph G is bipartite, with the set of vertices of degree 2 and the set vertices of degree at least 3 being the partite sets. We also demonstrate that G is 2-connected, i.e., G is a necklace. We begin with two short lemmas.

Lemma 3.4. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then G contains a pair of vertices x and y that lie in the same block of G and satisfy $\lambda(x, y) \geq 3$.*

Proof. Since there is a minimally 2-edge-connected graph on n vertices with average edge-connectivity strictly greater than 2 (take $K_{2,n-2}$, for example), and since G has maximum average edge-connectivity among all such graphs, there is at least one pair of vertices x, y in G such that $\lambda(x, y) \geq 3$. If x and y are in the same block, then we are done. If x and y are not in the same block, then let z be the first cut vertex that appears internally on every x–y path. Then x and z are in the same block and $\lambda(x, z) \geq 3$. □

Lemma 3.5. *Let G be a necklace, let u and v be adjacent vertices of degree 2 in G, and let G' be the graph obtained from G by contracting the edge uv. Then G' is a nonseparable minimally 2-edge-connected graph.*

Proof. First of all, if $G \cong C_3$, then $G' \cong C_2$, which is a nonseparable minimally 2-edge-connected graph.

So we may assume that G has order at least 4. Let u denote the contracted edge in G'. By Theorem 3.3, the edge uv is a clasp in G, so in particular, the graph $G - uv$ is an E-chain $G_1a_1G_2a_2 \dots G_{k-1}a_{k-1}G_k$ that is claspable at u and v. Since both u and v have degree 2 in G, it must be the case that both G_1 and G_k are edges, and hence that $k \geq 3$. Thus, we see that the E-chain $G_1a_1G_2a_2 \dots G_{k-1}$ is claspable at u and a_{k-1}, and hence the edge ua_{k-1} is a clasp in G'. By Theorem 3.3, it follows that G' is a necklace. □

We are now ready to prove that the vertices of degree 2 in an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$ form an independent set. We actually prove a slightly stronger result that has several useful corollaries.

Theorem 3.6. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$, and let B be a block of G. Then no two vertices of degree 2 in B are adjacent in B.*

Proof. Suppose otherwise that u and v are adjacent vertices of degree 2 in B. By Lemma 3.4, there is a pair of vertices x and y in G such that $\lambda_G(x, y) \geq 3$. Since $\lambda_G(u, v) = 2$, we have $\{u, v\} \neq \{x, y\}$. Let G' be the graph obtained from G by contracting the edge uv to a new vertex w and extending x and y through a new vertex z. Note that G' has the same order as G. We claim that G' is minimally 2-edge-connected, and that $\overline{\lambda}(G') > \overline{\lambda}(G)$, which contradicts the fact that G is edge-optimal.

For reference, let B' be the graph obtained from B by contracting the edge uv. First of all, if $B \cong C_2$, then B' is a single vertex, and by Lemma 3.1(a), Lemma 3.1(b), and Lemma 3.2, we see that G' is minimally 2-edge-connected. So we may assume that $|V(B)| \geq 3$, in which case B is a necklace. By Lemma 3.5, the graph B' is a nonseparable minimally 2-edge-connected graph. Again, it follows by Lemma 3.1(a), Lemma 3.1(b), and Lemma 3.2 that G' is minimally 2-edge-connected.

Now we show that $\overline{\lambda}(G') \geq \overline{\lambda}(G)$. Since G' is 2-edge-connected and has the same order as G, it suffices to show that for every pair of distinct vertices $a, b \in V(G)$ with $\lambda_G(a, b) \geq 3$, there is a corresponding pair of distinct vertices in G' whose edge-connectivity in G' is at least $\lambda_G(a, b)$. Since G is minimally 2-edge-connected, the graph $G - uv$ has a bridge e that separates u and v. Let G_1 and G_2 be the components of $G - uv - e$, with $u \in V(G_1)$ and $v \in V(G_2)$. Suppose that $\lambda_G(a, b) \geq 3$ for distinct vertices $a, b \in V(G)$. If $\{a, b\} \subseteq V(G_1)$, then we have $\lambda_{G'}(a, b) \geq \lambda_G(a, b)$, where we abuse notation slightly and replace u by w in G' if $u \in \{a, b\}$. Similarly, if $\{a, b\} \subseteq V(G_2)$, then we have $\lambda_{G'}(a, b) \geq \lambda_G(a, b)$, where we replace v by w in G' if $v \in \{a, b\}$. Thus, we have $\overline{\lambda}(G') \geq \overline{\lambda}(G)$.

Finally, note that $\lambda_{G'}(x, y) > \lambda_G(x, y)$ because of the new x–y path through z. (Again, we abuse notation slightly and must replace x or y with w in G' if $\{u, v\} \cap \{x, y\} \neq \emptyset$.) Therefore, we have $\overline{\lambda}(G') > \overline{\lambda}(G)$. □

The following corollaries of Theorem 3.6 are helpful in establishing that no pair of vertices of degree greater than 2 are adjacent in an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$.

Corollary 3.7. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then no block of G is a cycle, i.e., for every block B of G, we have $\overline{\lambda}(B) > 2$.*

Proof. Suppose otherwise that some block B of G is a cycle. But then B has at least two adjacent vertices of degree 2 in B, contradicting Theorem 3.6 □

Corollary 3.8. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then G is simple.*

Proof. Suppose, towards a contradiction, that G has a pair of parallel edges e_1 and e_2 between vertices u and v. Then by Lemma 3.1(c), the vertices u and v make up a block of G with average edge-connectivity 2. This contradicts Corollary 3.7. □

So in the remainder of this section, we describe paths by listing only the vertices. The next lemma describes a property of every cut vertex of an edge-optimal minimally 2-edge-connected graph.

Lemma 3.9. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. If G has a cut vertex v, then every block of G containing v, has some vertex $w \neq v$ such that $\lambda_G(v, w) \geq 3$.*

Proof. Let v be a cut vertex of G, and let $H_1, \dots, H_p$ be the components of $G - v$. For every $i \in \{1, \dots, p\}$, let H_i' be the subgraph of G induced by $V(H_i) \cup \{v\}$. By Lemma 3.1(a) and Lemma 3.1(b), we see that H_i' is a minimally 2-edge-connected graph. Note also that there are exactly p blocks of G containing v; let B_i be the block of G containing v that is a subgraph of H_i'. Suppose, towards a contradiction, that $\lambda_G(v, w) = 2$ for all $w \in V(B_i)$ for some i, $1 \leq i \leq p$. Without loss of generality, we assume that $i = 1$.

We now describe a construction of a graph G' that is minimally 2-connected with average connectivity exceeding that of G. Relabel the copy of v in H_i' with the label v_i. For every $i \in \{1, \dots, p\}$, if there is a vertex $w_i \in B_i$ such that $\lambda_G(v, w_i) \geq 3$, then define $u_i = v_i$. Otherwise, by Corollary 3.7, there is some pair of vertices in $B_i - v_i$, say x_i and y_i, such

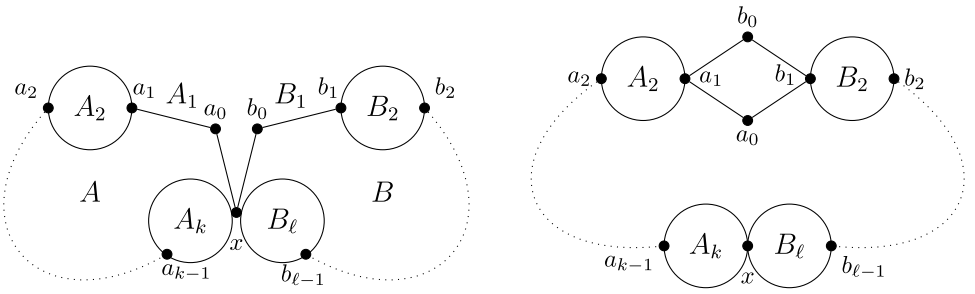

Fig. 4. The graph H (left) and the graph H' (right).

that $\lambda_G(x_i, y_i) \geq 3$. In this case, define $u_i = x_i$ (whether x_i or y_i is chosen does not matter). Since $\lambda_G(v, w) = 2$ for all $w \in B_1$, we see that $u_1 = x_1 \neq v_1$. Let G' be the graph obtained from the disjoint union $\bigcup_{i=1}^{k} H_i'$ by identifying all vertices in the set $\{u_1, \ldots, u_k\}$. By Lemma 3.1(b), G' is a minimally 2-edge-connected graph of order n, and it is straightforward to verify that $\overline{\lambda}(G') > \overline{\lambda}(G)$, which contradicts the fact that G is edge-optimal. □

We are now ready to prove that vertices of degree at least 3 are independent in every edge-optimal minimally 2-edge-connected graph of order $n \geq 5$.

Theorem 3.10. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then no two vertices of degree at least 3 are adjacent in G.*

Proof. Suppose otherwise that u and v are adjacent vertices of degree at least 3 in G. Let B be the block of G containing u and v. By Lemma 3.1(a) and Corollary 3.8, the block B is a necklace, and by Lemma 3.9, we have $\deg_B(u), \deg_B(v) \geq 3$. By Theorem 3.3, the edge uv is a clasp in B, i.e., the graph $B - uv$ is an E-chain $B_1a_1B_2a_2 \cdots B_{k-1}a_{k-1}B_k$ that is claspable at u and v. Since $\deg_B(u), \deg_B(v) \geq 3$, we see that neither B_1 nor B_k is an edge. By condition (c) of the definition of claspable E-chain, there exists some $j \in \{2, \ldots, k-1\}$ such that B_j is an edge, and thus $k \geq 3$. For ease of notation let $x = a_{j-1}$ and $y = a_j$. Let G' be the graph obtained from G by contracting the edge uv to a single vertex w and subdividing the edge xy, calling the new vertex z. For reference, let B' be the graph obtained from B by the same operations. We claim that G' is minimally 2-edge-connected and that $\overline{\lambda}(G') > \overline{\lambda}(G)$.

By construction, it is easy to see that xz (or zy) is a clasp in B', and hence by Theorem 3.3, we have that B' is a necklace. Thus, by Lemma 3.1(a) and Lemma 3.1(b), we see that G' is minimally 2-edge-connected. Therefore, to show that $\overline{\lambda}(G') \geq \overline{\lambda}(G)$, it suffices to show that for every pair of distinct vertices $a, b \in V(G)$ with $\lambda_G(a, b) \geq 3$, there is a corresponding pair of distinct vertices in G' whose edge-connectivity in G' is at least $\lambda_G(a, b)$. This follows by an argument similar to the one used in the proof of Theorem 3.6. Finally, to see that $\overline{\lambda}(G') > \overline{\lambda}(G)$, note that $\lambda_G(a_{k-1}, a_1) = 2$, while it is easy to see that $\lambda_{G'}(a_{k-1}, a_1) \geq 3$. □

As an immediate consequence of Theorem 3.6 and Theorem 3.10 we have the following structure result for edge-optimal minimally 2-edge-connected graphs.

Theorem 3.11. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then G is bipartite with partite sets the set of vertices of degree 2 and the set of vertices of degree exceeding 2.*

We close this section with a proof that every edge-optimal minimally 2-edge-connected graph of order $n \geq 5$ is 2-connected.

Theorem 3.12. *Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. Then G is 2-connected.*

Proof. Assume, to the contrary, that G is not 2-connected. Let x be a cut vertex of G, and let A and B be two distinct blocks of G that contain x. Let H be the subgraph of G induced by $V(A) \cup V(B)$. By Lemma 3.9, we have $\deg_A(x), \deg_B(x) \geq 3$. Let a_0 be a neighbour of x in A, and let b_0 be a neighbour of x in B. By Theorem 3.10, we have $\deg_G(a_0) = \deg_G(b_0) = 2$. Let $a_1 \neq x$ be the other neighbour of a_0, and let $b_1 \neq x$ be the other neighbour of b_0. By Theorem 3.6 and Lemma 3.9, we have $\deg_A(a_1), \deg_B(b_1) \geq 3$. Let G' be the graph obtained from G by deleting the edges xa_0 and xb_0, and adding the edges a_0b_1 and a_1b_0. For reference, let H' be the graph obtained from H by the same operations (see Fig. 4). We claim that G' is minimally 2-edge-connected and that $\overline{\lambda}(G') > \overline{\lambda}(G)$.

Since A is a necklace, the edge xa_0 is a clasp in A, meaning that the graph $A - xa_0$ is an E-chain $A_1a_1A_2a_2 \cdots A_{k-1}a_{k-1}A_k$ that is claspable at $a_0 \in A_1$ and $x \in A_k$. (Since $\deg_A(a_0) = 2$, it must be the case that A_1 contains only the edge a_0a_1.) Similarly, since B is a necklace, the graph $B - xb_0$ is an E-chain $B_1b_1B_2b_2 \cdots B_{\ell-1}b_{\ell-1}B_\ell$ that is claspable at $b_0 \in B_1$ and $x \in B_\ell$ (see Fig. 4). Now it is readily seen that H' is 2-connected, and since every edge of H (and hence H') is incident to a

vertex of degree 2, we conclude immediately that H' is minimally 2-edge-connected. By Lemma 3.1(a) and Lemma 3.1(b), we conclude that G' is minimally 2-edge-connected.

In order to show that $\overline{\lambda}(G') \geq \overline{\lambda}(G)$, it suffices to show that $\lambda_{H'}(u, v) \geq \lambda_H(u, v)$ for every pair of distinct vertices $u, v \in V(H)$. Since H' is minimally 2-edge-connected, we may assume that u and v both have degree at least 3, meaning that $\{u, v\} \cap \{a_0, b_0\} = \emptyset$. First suppose that u and v are in the same block of G, say A. Then there is a family of $\lambda_H(u, v)$ edge-disjoint paths between u and v in A. At most one of these paths contains the edge a_0x. If this is the case, then replace the edge a_0x with the edge a_0b_1 together with a path from b_1 to x in B. Thus, we obtain $\lambda_H(u, v)$ edge disjoint paths from u to v in H'. So we may assume that u and v are in different blocks of H, say $u \in A$ and $v \in B$. Let $X \subseteq E(H')$ be a minimum separating set for u and v in H'; so $|X| = \lambda_{H'}(u, v)$. We will find a subset of $E(H)$ of cardinality at most $|X|$ that separates u and v in H. Let $Z = \{a_0a_1, a_1b_0, b_0b_1, b_1a_0\}$. By the minimality of X, if $X \cap Z \neq \emptyset$, then $|X \cap Z| = 2$. But then $(X \cup \{a_0x, b_0x\})\backslash Z$ separates u and v in H, and has cardinality $|X|$. So we may assume that $|X \cap Z| = \emptyset$, hence in particular $X \subseteq E(H)$. Since X separates u and v in H', either u and x are separated by X in H', or v and x are separated by X in H'. Assume without loss of generality that u and x are separated by X in H'. If u and a_1 are also separated by X in H', then u and v are separated by X in H as well. So we may assume that u and a_1 are not separated by X in H'. But then X contains at least one edge of B, and hence $(X \cup \{a_0x\})\backslash E(B)$ separates u and v in H, and has cardinality at most $|X|$.

Finally, it is easy to see that $\lambda_{H'}(a_1, b_1) = \lambda_H(a_1, b_1) + 1$, from which it follows that $\overline{\lambda}(G') > \overline{\lambda}(G)$. □

We conclude this section by noting that, given a minimally 2-edge-connected graph G of order $n \geq 5$, for which either the vertices of degree 2 or the vertices of degree exceeding 2 are not independent or the graph is not 2-connected, the proofs of this section implicitly describe an algorithm for constructing a minimally 2-edge-connected graph G' of the same order n with higher average edge-connectivity than G. By repeated application of this algorithm we obtain a 2-connected minimally 2-edge-connected graph of order n in which the vertices of degree 2 and those of degree exceeding 2 are independent. Moreover, the average edge-connectivity of this graph exceeds that of the other graphs that preceded it in the process.

3.2. An upper bound on the average edge-connectivity of minimally 2-edge-connected graphs

The structural properties proven in Section 3.1 lead us to a tight upper bound on the average edge-connectivity of a minimally 2-edge-connected graph. Both the statement and the proof of this bound are very similar to those of Theorem 2.13. The proof of the edge-analogue of Theorem 2.13 uses the following two results of Hakimi [9], and Dankelmann and Oellermann [7].

Theorem 3.13 ([9]). *A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of non-negative integers is multigraphical if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq \sum_{i=2}^{n} d_i$.*

Theorem 3.14 ([7]). *Let $D : d_1 \geq d_2 \geq \cdots \geq d_n$, $n \geq 3$, be a multigraphical sequence with $d_n > 0$ and let n_1 denote the number of terms in D that equal 1. Then there is an ideally edge-connected multigraph with degree sequence D if and only if*

(a) $n_1 \leq d_1 - d_2$ *or*
(b) $D : n - 1, 1, 1, \ldots, 1$ *where D contains $n - 1$ terms equal to 1.*

In particular, we have the following:

Corollary 3.15. *Let m and n be integers such that $m \geq n \geq 3$. Then there is an ideally edge-connected nearly regular (multi)graph of order n and size m.*

Theorem 3.16. *Let G be a minimally 2-edge-connected graph of order n. Then*

$$\overline{\lambda}(G) \leq 2 + \tfrac{(n-2)^2}{4n(n-1)} < \tfrac{9}{4}.$$

Moreover, let $n = 4k + \ell$, where $k, \ell \in \mathbb{Z}$ and $0 \leq \ell < k$.

(a) *If $k \geq 8$ and $\ell = 0$, then*

$$\overline{\lambda}(G) \leq 2 + \tfrac{n^2-4n}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally edge-connected 6-regular (multi)graph of order k by subdividing every edge.

(b) *If $k \geq 30$ and $\ell = 1$, then*

$$\overline{\lambda}(G) \leq 2 + \tfrac{n^2-6n+13}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally edge-connected nearly regular (multi)graph of order k and size $n - k$ by subdividing every edge.

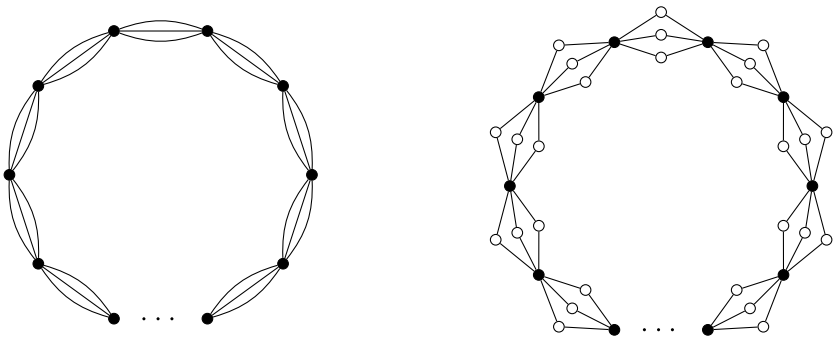

Fig. 5. The graph $C_k^{[3]}$ (left) and the graph G_{4k} (right) obtained by subdividing every edge of $C_k^{[3]}$. The vertices resulting from subdivision are indicated by hollow circles.

(c) *If* $k \geq 68$ *and* $\ell = 2$, *then*

$$\overline{\lambda}(G) \leq 2 + \tfrac{n^2-8n+60}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally edge-connected nearly regular (multi)graph of either order k and size $n-k$, *or order* $k+1$ *and size* $n-k-1$, *by subdividing every edge.*

(d) *If* $k \geq 30$ *and* $\ell = 3$, *then*

$$\overline{\lambda}(G) \leq 2 + \tfrac{n^2-6n+17}{4n(n-1)},$$

with equality if and only if G is obtained from an ideally edge-connected nearly regular (multi)graph of order $k+1$ *and size* $n-k-1$ *by subdividing every edge.*

Proof. Let G be an edge-optimal minimally 2-edge-connected graph of order $n \geq 5$. By Corollary 3.8 and Theorem 3.11, G is a simple bipartite graph, with the set of vertices of degree 2 and the set of vertices of degree exceeding 2 being independent sets. The remainder of the proof is analogous to that of Theorem 2.13, with the terminology and notation for connectivity changed to that of edge-connectivity, and Corollary 3.15 used in place of Theorem 2.11 to guarantee sharpness. □

The examples of optimal minimally 2-connected graphs described at the end of Section 2.2 are now easily seen to be edge-optimal minimally 2-edge-connected graphs as well. We can also provide examples of edge-optimal minimally 2-edge-connected graphs which are not optimal minimally 2-connected graphs. For example, for $n = 4k$ with $k \geq 8$, let $C_k^{[3]}$ be the graph obtained from C_k by replacing every edge with a bundle of three multiple edges, and let G_{4k} be the graph obtained from $C_k^{[3]}$ by subdividing every edge exactly once (see Fig. 5). Since $C_k^{[3]}$ is an ideally edge-connected 6 regular graph on k vertices, we conclude, by Theorem 3.16, that G_{4k} is an edge-optimal minimally 2-edge-connected graph. While G_{4k} is also a minimally 2-connected graph, note that $C_k^{[3]}$ is clearly not ideally (vertex-)connected, so G_{4k} is not an optimal minimally 2-connected graph.

4. Conclusion

In this paper we obtained sharp bounds for the average connectivity of minimally 2-connected graphs and the average edge-connectivity of minimally 2-edge-connected graphs, and we characterized the extremal structures. It remains an open problem to determine an upper bound for the average connectivity of minimally k-connected graphs and the average edge-connectivity of minimally k-edge-connected graphs for $k \geq 3$. What can be said about the structure of *optimal* minimally k-connected graphs (those with largest average connectivity among all minimally k-connected graphs of the same order)?

Conjecture 4.1. *Let* $k \geq 3$, *and let G be an optimal minimally k-connected graph of order n. Then for n sufficiently large, G is bipartite, with partite sets the set of vertices of degree k and the set of vertices of degree exceeding k.*

We also conjecture the analogous statement for the edge version.

Conjecture 4.2. *Let* $k \geq 3$, *and let G be an edge-optimal minimally k-edge-connected graph of order n. Then for n sufficiently large, G is bipartite, with partite sets the set of vertices of degree k and the set of vertices of degree exceeding k.*

These conjectures are supported by computational evidence for $k = 3$ and $k = 4$ and $n \leq 11$. If Conjecture 4.1 is true, then for every $k \geq 3$, the proof of the general upper bound of Theorem 2.13 generalizes easily to show that $\overline{\kappa}(G) < \frac{9}{8}k$ for any minimally k-connected graph G of sufficiently large order, depending on k. The edge version is analogous.

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Appendix

Proof of part (b) of Theorem 2.13. Let $n = 4k + 1$ with $k \geq 30$. We claim that

$$g_k(s) = (4k + 1 - 2s)(s - 1) - r(s - r)/2 \leq 2k^2 - 2k + 1 = \tfrac{n^2-6n+13}{8},$$

with equality if and only if $s = k$. First off, if $s = k$, then $d = 6$ and $r = 2$, and it follows that $g_k(k) = 2k^2 - 2k + 1$. It remains to show that $g_k(s) < g_k(k)$ for all $s \neq k$. We consider three cases.

Case 1: $s \in (k - \frac{k-2}{9}, k)$

Let $s = k - i$ for some integer $i \in [1, \frac{k-2}{9})$. It follows that $d = 6$ and $r = 2 + 8i < k - i = s$ (note that $2 + 8i < k - i$ since $i < \frac{k-2}{9}$). Now

$$g_k(k - i) = 34i^2 + (14 - 4k)i + 2k^2 - 2k + 1$$

is a quadratic in i with positive leading coefficient, so for $i \in [1, \frac{k-2}{9})$,

$$g_k(k - i) \leq \max\left\{g_k(k - 1), g_k\left(k - \tfrac{k-2}{9}\right)\right\}.$$

We verify that $g_k(k) > g_k(k - 1) = 2k^2 - 6k + 49$ for all $k \geq 13$, and that $g_k(k) > g_k\left(k - \frac{k-2}{9}\right) = \frac{160}{81}k^2 - \frac{100}{81}k - \frac{35}{81}$ for all $k \geq 30$. Therefore, for $s \in (k - \frac{k-2}{9}, k)$, we have $g_k(s) < g_k(k)$.

Case 2: $s \in (k, k + \frac{k+2}{7})$.

Let $s = k + i$ for some integer $i \in [1, \frac{k+2}{7})$. It follows that $d = 5$ and $r = k + 2 - 7i$ (note that $k + 2 - 7i < k + i$ since $i \geq 1$ and $k + 2 - 7i > 0$ since $i < \frac{k+2}{7}$). Now

$$g_k(k + i) = 26i^2 + (-12 - 4k)i + 2k^2 + 1$$

is a quadratic in i with positive leading coefficient, so for $i \in [1, \frac{k+2}{7})$,

$$g_k(k + i) \leq \max\left\{g_k(k + 1), g_k\left(k + \tfrac{k+2}{7}\right)\right\}.$$

We verify that $g_k(k) > g_k(k + 1) = 2k^2 - 4k + 15$ for all $k \geq 8$, and $g_k(k) > g_k\left(k + \frac{k+2}{7}\right) = \frac{96}{49}k^2 - \frac{36}{49}k + \frac{15}{49}$ for all $k \geq 30$. Therefore, for $s \in (k, k + \frac{k+2}{7})$, we have $g_k(s) < g_k(k)$.

Case 3: $s \leq k - \frac{k-2}{9}$ or $s \geq k + \frac{k+2}{7}$

Let $f_k(s) = (4k+1-2s)(s-1)$, and we certainly have $g_k(s) \leq f_k(s)$ (with equality if and only $r(s-r) = 0$). By elementary calculus, $f_k(s)$ is increasing when $s < k$ and decreasing when $s > k + 1$. So if $s \leq k - \frac{k-2}{9}$, then

$$g_k(s) \leq f_k(s) \leq f_k\left(k - \tfrac{k-2}{9}\right) = \tfrac{160}{81}k^2 - \tfrac{100}{81}k - \tfrac{35}{81},$$

which is strictly less than $g_k(k)$ for $k \geq 30$. Similarly, if $s \geq k + \frac{k+2}{7}$, then

$$g_k(s) \leq f_k(s) \leq f_k\left(k + \tfrac{k+2}{7}\right) = \tfrac{96}{49}k^2 - \tfrac{36}{49}k - \tfrac{15}{49},$$

which is strictly less than $g_k(k)$ for $k \geq 30$.

In conclusion, we have

$$K(G) \leq n(n - 1) + \tfrac{n^2-6n+13}{8},$$

with equality if and only if H is an ideally connected nearly regular (multi)graph on k vertices and $n - k = 3k + 1$ edges. One can verify that H has exactly two vertices of maximum degree 7, so by Observation 2.12, H may have a single multiple edge between these vertices, but has no other multiple edges. Since $k \geq 30$, we have $n - k = 3k + 1 \leq \binom{k}{2}$, so indeed, Theorem 2.11 guarantees sharpness. The bound on $\overline{\kappa}(G)$ follows by dividing through by $\binom{n}{2}$. □

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