Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **12** (2018), 297–317.
https://doi.org/10.2298/AADM160726003A

ON GENERALIZED 3-CONNECTIVITY OF THE STRONG PRODUCT OF GRAPHS

Encarnación Abajo, Rocío Casablanca, Ana Diánez and Pedro García-Vázquez[*]

Let G be a connected graph with n vertices and let k be an integer such that $2 \leq k \leq n$. The generalized connectivity $\kappa_k(G)$ of G is the greatest positive integer ℓ for which G contains at least ℓ internally disjoint trees connecting S for any set $S \subseteq V(G)$ of k vertices. We focus on the generalized connectivity of the strong product $G_1 \boxtimes G_2$ of connected graphs G_1 and G_2 with at least three vertices and girth at least five, and we prove the sharp bound $\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1$.

1. INTRODUCTION

Throughout this paper, all the graphs are simple, that is, with neither loops nor multiple edges. Notations and terminology not explicitly given here can be found in the books by Chartrand, Lesniak and Zhang [**3**] and by Hammack, Imrich and Klavžar [**6**].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For u, v two distinct vertices of $V(G)$, a path from u to v, also called an uv-path in G, is a subgraph P with vertex set $V(P) = \{u = x_0, x_1, \ldots, x_r = v\}$ and edge set $E(P) = \{x_0x_1, \ldots, x_{r-1}x_r\}$. This path is usually denoted by $P : x_0x_1 \ldots x_r$, where

[*]Corresponding author. Pedro García-Vázquez
2010 Mathematics Subject Classification. $05C40, 05C76$
Keywords and Phrases. Strong product of graphs, vertex connectivity, generalized connectivity, Menger's theorem, Whitney's theorem.

r is the length of the path.

Two uv-paths P and Q are said to be *internally disjoint* if $V(P) \cap V(Q) = \{u, v\}$. A cycle in G of length r is a path $C : x_0 x_1 \cdots x_r$ such that $x_0 = x_r$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G, and if G contains no cycles, then $g(G) = \infty$. The set of adjacent vertices to $v \in V(G)$ is denoted by $N_G(v)$. The *degree* of v is $d_G(v) = |N_G(v)|$, whereas $\delta(G) = \min_{v \in V(G)} d_G(v)$ is the *minimum degree* of G. A graph is said to be *connected* if there is a path from each vertex to any other vertex in the graph.

For connected graphs, Menger [**10, 11**] states that the maximum number of pairwise internally disjoint paths between a given pair of non adjacent vertices in a graph equals the minimum number of vertices whose deletion disconnects the pair. As a measure of the degree of connectedness of a graph, the *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices whose deletion produces a disconnected or trivial graph. A graph G for which $\kappa(G) \geq k$ is said to be k-connected. The first characterization of k-connected graphs was given by Whitney [**15**] in 1932, who states that a graph G is k-connected if and only if every pair of vertices in $V(G)$ is connected by k internally disjoint paths. Whitney in [**15**] also shows that $\kappa(G) \leq \delta(G)$ for every connected graph G.

Although Hager worked on a similar concept in [**5**], the generalized k-connectivity was introduced by Chartrand *et al.* in [**4**]. The interest of the generalized k-connectivity lies in the fact that it is a natural generalization of the connectivity $\kappa(G)$ and represents a measure of the capability of a network to connect sets of vertices. Let G be a connected graph with n vertices and $S \subseteq V(G)$. A tree T is called an *S-tree* if $S \subseteq V(T)$. A family of trees $T_1, \ldots, T_\ell$ are internally disjoint S-trees if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$, for $1 \leq i < j \leq \ell$. For an integer k, with $2 \leq k \leq n$, the *generalized k-connectivity* $\kappa_k(G)$ of G is defined as $\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G), |S| = k\}$, where $\kappa(S)$ is the maximum number of internally disjoint S-trees in G. Clearly, $\kappa_2(G) = \kappa(G)$, the classical connectivity of G. If $n < k$, $\kappa_k(G) = 1$ is adopted. In [**9**] the sharp bound $\kappa_3(G) \leq \kappa(G)$ is proved for any connected graph G. The generalized connectivity of complete graphs and complete bipartite graphs was studied in [**4**] and [**7, 12**], respectively.

Since our purpose is to study the 3-generalized connectivity of the strong product of graphs, let us remember that the *strong product* $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which two vertices (x_1, x_2) and (y_1, y_2) are adjacent if $x_1 = y_1$ and $x_2 y_2 \in E(G_2)$, or $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_1 y_1 \in E(G_1)$ and $x_2 y_2 \in E(G_2)$. Obviously, the strong product of two graphs is commutative. Observe that for every $v \in V(G_2)$, the subgraph of $G_1 \boxtimes G_2$ induced by the set $\{(u, v) : u \in V(G_1)\}$ is isomorphic to G_1. For this reason, this subgraph is called G_1-copy and denoted by G_1^v. Analogously, for each $u \in V(G_1)$, the set $\{(u, v) : v \in V(G_2)\}$ induces a subgraph isomorphic to G_2 and it is called G_2-copy and denoted by G_2^u.

Product graphs provide important methods to construct bigger graphs and play a key role in design and analysis of network. For a better knowledge of this topic, we refer the reader to the book by Hammack, Imrich and Klavžar [**6**].

Some research on the connectivity of the Cartesian and of the strong product of graphs can be found in [**1, 2, 8, 13, 14, 16**]. Regarding the generalized 3-connectivity of the Cartesian product graph $\kappa_3(G_1 \Box G_2)$, Li, Li and Sun [**8**] for connected graphs G_1 and G_2 such that $\kappa_3(G_1) \geq \kappa_3(G_2)$ obtain the following sharp bounds
(i) If $\kappa(G_1) = \kappa_3(G_1)$, *then* $\kappa_3(G_1 \Box G_2) \geq \kappa_3(G_1) + \kappa_3(G_2) - 1$.
(ii) If $\kappa(G_1) > \kappa_3(G_1)$, *then* $\kappa_3(G_1 \Box G_2) \geq \kappa_3(G_1) + \kappa_3(G_2)$.

In this paper, we study the 3-generalized connectivity of the strong product of two connected graphs.

1.1 A summary of our main results

For connected graphs G_1 and G_2 with at least three vertices and girth and least five, we construct internally disjoint trees that connect any three vertices $x, y, z \in V(G_1 \boxtimes G_2)$. The number of such trees is expressed in terms of the connectivity, generalized 3-connectivity or minimum degree of the factor graphs. As a direct consequence, we derive the following result.

Theorem 3.1 *Let* G_1 *and* G_2 *be connected graphs with at least three vertices and girth at least five. Then,*

$$\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1.$$

The bound is sharp.
The sharpness of this bound is confirmed when the factor graphs also satisfy that $\kappa_3(G_1) = \delta(G_1)$ and $\kappa_3(G_2) = \delta(G_2)$

Corollary 3.1 *Let* G_1 *and* G_2 *be connected graphs with at least three vertices, girth at least five and such that* $\kappa_3(G_1) = \delta(G_1)$, $\kappa_3(G_2) = \delta(G_2)$. *Then,*

$$\delta(G_1 \boxtimes G_2) - 1 \leq \kappa_3(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2).$$

Both bounds are sharp.

2. SPECIFIC NOTATION AND REMARK

Before proceeding with the main results of this work, we need to introduce some basic definitions, specific notation as well as a useful observation.
Given an $\{x, y, z\}$-tree T, where $x, y, z \in V(T)$, simply deleting extra vertices, we can construct an $\{x, y, z\}$-tree $\widetilde{T} \subseteq T$ with the minimum number of vertices (see [**8**]). This tree $\widetilde{T}$ is called a *minimal* $\{x, y, z\}$-tree. In this paper, a minimal

tree T is called an *xyz-path* when T is an xz-path with y as an internal vertex, and T is said an *r-rooted* $\{x, y, z\}$-tree, for $r \in V(T)$, when r is the root of T and x, y, z are the leaves.
Also, we need to introduce a kind of trees which need an special treatment in this paper.

Definition 2.1. *Let G be a connected graph and x, y, z three distinct vertices of G. An $\{x, y, z\}$-tree T in G is said to be* special *if either T is an r-rooted tree with edge set $E(T) = \{rx, ry, rz\}$ or T is a path such that $d_T(x, y) \leq 2$ or $d_T(y, z) \leq 2$ or $d_T(x, z) \leq 2$.*

Remark 2.1. *For distinct vertices x, y, z of a graph G with $g(G) \geq 5$ and $\kappa_3(G) \geq 2$, let us notice that:*

(i) If G contains the r-rooted tree with edge-set $\{rx, ry, rz\}$, then any other $\{x, y, z\}$-tree of G is not special.

(ii) If there exist special *xyz-paths T_1 and T_2 such that $d_{T_1}(x, y) \leq 2$ and $d_{T_2}(y, z) \leq 2$, combining T_1 and T_2, we can construct T_1' and T_2' another pair of internally disjoint xyz-paths such that only T_1' is* special *(see the first case in Figure 1).*

(iii) If there exist three special *paths T_1, T_2 and T_3 in G such that $d_{T_1}(x, y) \leq 2$, $d_{T_2}(y, z) \leq 2$ and $d_{T_3}(x, z) \leq 2$, combining these paths, we can construct another set of internally disjoint paths T_1', T_2' and T_3' such that at most two of these paths are* special *and satisfying that $V(T_1' \cup T_2' \cup T_3') = V(T_1 \cup T_2 \cup T_3)$ (see the second and third cases in Figure 1).*

Notice that the third case in Figure 1 is a combination of the two previous ones. To illustrate some constructions in Section 3 we use the structure of Figure 2 as the most general way to represent two *special* paths in a graph with girth at least five. It must be taken into account that lines between two vertices in Figure 2 may be edges or paths.

Our goal is to study the maximum number of internally disjoint trees that connect vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in $G_1 \boxtimes G_2$, for vertices $x_1, y_1, z_1 \in V(G_1)$ and $x_2, y_2, z_2 \in V(G_2)$. To do that, throughout the proofs below, $P_1, \ldots, P_{\ell_1}$ denote internally disjoint minimal $\{x_1, y_1, z_1\}$-trees in G_1 and $Q_1, \ldots, Q_{\ell_2}$ internally disjoint minimal $\{x_2, y_2, z_2\}$-trees in G_2. We always assume that $P_1, \ldots, P_{\ell_1}$ and $Q_1, \ldots, Q_{\ell_2}$ contain the minimum number of *special* trees, that is, at most two. Without loss of generality, in cases (i) and (ii) of Remark 2.1 we denote by P_1 (or Q_1, respectively) the unique *special* tree, whereas in case

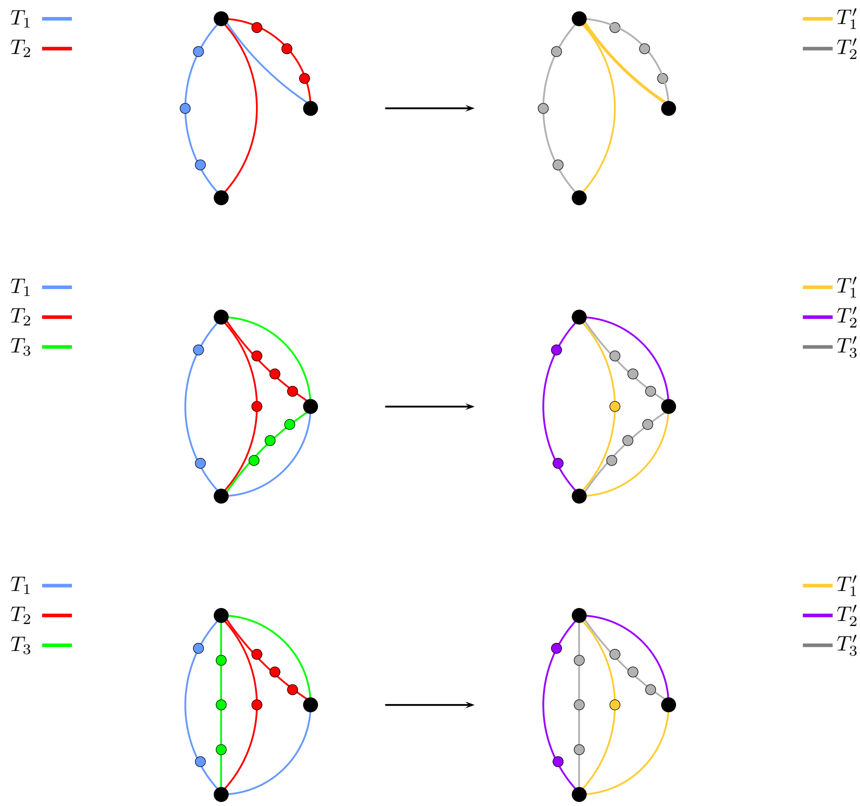

Figure 1: It can be considered that there exist at most two *special* $\{x, y, z\}$-trees in a connected graph with girth at least five and 3-connectivity at least two.

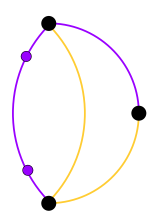

Figure 2: General way to represent two *special* paths in a graph with girth at least five.

(iii), we consider that P_1 and P_2 are the *special* trees of G_1, and the same consideration holds for Q_1 and Q_2 in G_2. When no tree is *special*, we assume that $|V(P_1)| = \min\{|V(P_i)| : 1 \leq i \leq \ell_1\}$ and $|V(Q_1)| = \min\{|V(Q_j)| : 1 \leq j \leq \ell_2\}$.

We say that a *tree* T_{ij} *in* $G_1 \boxtimes G_2$ *is associated to trees* P_i *in* G_1 *and* Q_j *in* G_2 when every vertex $(u, v) \in V(T_{ij})$ is such that $u \in P_i$ and $v \in Q_j$.

Let us also give a general idea of the notation used to describe trees in this paper. If P_i is an $x_1y_1z_1$-path, we denote it as $P_i : x_1\overline{x}_1^i \ldots \underline{y}_1^i y_1 \overline{y}_1^i \ldots \underline{z}_1^i z_1$ and if P_i is an r^i-rooted $\{x_1, y_1, z_1\}$-tree, we write $P_i : r^i \ldots \underline{x}_1^i x_1 \cup r^i \ldots \underline{y}_1^i y_1 \cup r^i \ldots \underline{z}_1^i z_1$ (see Figure 3). Similarly, we write $Q_j : x_2\overline{x}_2^j \ldots \underline{y}_2^j y_2 \overline{y}_2^j \ldots \underline{z}_2^j z_2$ when Q_j is an $x_2y_2z_2$-path, and $Q_j : s^j \ldots \underline{x}_2^j x_2 \cup s^j \ldots \underline{y}_2^j y_2 \cup s^j \ldots \underline{z}_2^j z_2$ when Q_j is an s^j-rooted $\{x_2, y_2, z_2\}$-tree. From the definition of *special* trees, it follows that $\overline{x}_1^i \neq \underline{y}_1^i$ and $\overline{y}_1^i \neq \underline{z}_1^i$, when P_i is not an *special* $x_1y_1z_1$-path, and that at least one element of the set $\{\underline{x}_1^i, \underline{y}_1^i, \underline{z}_1^i\}$ is distinct to r^i, when P_i is not an *special* r^i-rooted $\{x_1, y_1, z_1\}$-tree. Similar considerations follow for a not *special* tree Q_j.

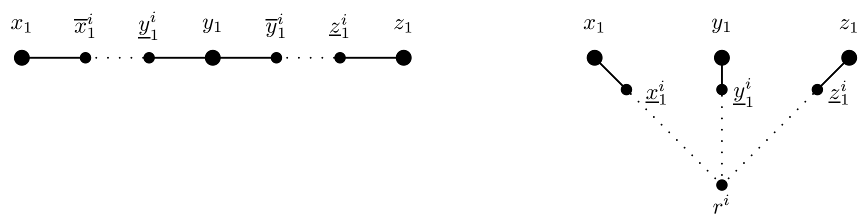

Figure 3: Description of an $x_1y_1z_1$-path and an r^i-rooted $\{x_1, y_1, z_1\}$-tree P_i.

To complete the notation used in the paper, notice that given $u \in V(G_1)$, every $\{x_2, y_2, z_2\}$-tree Q_j of G_2 induces an $\{(u, x_2), (u, y_2), (u, z_2)\}$-tree Q_j^u in the copy G_2^u such that $V(Q_j^u) = \{(u, v) : v \in V(Q_j)\}$ and $E(Q_j^u) = \{(u, v_1)(u, v_2) : v_1v_2 \in E(Q_j)\}$.

3. LOWER BOUNDS ON $\kappa_3(G_1 \boxtimes G_2)$

To estimate $\kappa_3(G_1 \boxtimes G_2)$, we construct internally disjoint trees connecting any three distinct vertices $x, y, z \in V(G_1 \boxtimes G_2)$.
First, we assume that x, y, z come from a single vertex in G_1 and three distinct vertices in G_2 or vice versa.

Lemma 3.1. *Let G_1 and G_2 be connected graphs with at least three vertices and girth at least five. Let $x_i, y_i, z_i \in V(G_i)$ be three distinct vertices, for $i = 1, 2$. The following assertions hold:*

(i) There exist at least $\delta(G_1)\kappa_3(G_2) + \delta(G_1) + \kappa_3(G_2)$ internally disjoint trees connecting vertices (x_1, x_2), (x_1, y_2), (x_1, z_2) in $G_1 \boxtimes G_2$.

(ii) There exist at least $\kappa_3(G_1)\delta(G_2) + \kappa_3(G_1) + \delta(G_2)$ *internally disjoint trees connecting vertices* (x_1, x_2), (y_1, x_2), (z_1, x_2) *in* $G_1 \boxtimes G_2$.

Proof. Due to the commutativity of the strong product of graphs, it suffices to prove (i). Denote $x = (x_1, x_2)$, $y = (x_1, y_2)$ and $z = (x_1, z_2)$. Since x, y, z belong to a unique copy $G_2^{x_1}$ in $G_1 \boxtimes G_2$ and x_2, y_2, z_2 are connected at least by $\ell_2 = \kappa_3(G_2)$ internally disjoint trees $Q_1, \ldots, Q_{\ell_2}$ in G_2, trees $Q_1^{x_1}, \ldots, Q_{\ell_2}^{x_1}$ are ℓ_2 internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

To construct other $\delta(G_1)\kappa_3(G_2)$ trees, we next define an $\{x, y, z\}$-tree T_j^u for each $u \in N_{G_1}(x_1)$ and $j \in \{1, \ldots, \ell_2\}$. To do that, we distinguish if none, one or two trees of the family $Q_1, \ldots, Q_{\ell_2}$ contain direct edges between the vertices of the set $\{x_2, y_2, z_2\}$.

For each tree Q_j such that $x_2y_2 \notin E(Q_j)$, $y_2z_2 \notin E(Q_j)$ and $x_2z_2 \notin E(Q_j)$, let us denote

$$\ddot{Q}_j^u : \; Q_j^u - \{(u, x_2), (u, y_2), (u, z_2)\}.$$

If Q_1 is an $x_2y_2z_2$-path such that x_2y_2 or y_2z_2 belong to $E(Q_1)$, then

$$\ddot{Q}_1^u : \; Q_1^u - \{(u, x_2), (u, z_2)\}.$$

Suppose that Q_2 also contains a direct edge between two vertices of the set $\{x_2, y_2, z_2\}$. For instance, we assume that $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$. Then

$$\ddot{Q}_2^u : \; Q_2^u - \{(u, x_2), (u, y_2)\}.$$

By the definition of the strong product of graphs, for $j \in \{1, \ldots, \ell_2\}$, each end vertex of $\ddot{Q}_j^u$ is adjacent to at least one vertex of the set $\{x, y, z\}$. We define T_j^u as a tree contained in $G_1 \boxtimes G_2$ such that $V(T_j^u) = V(\ddot{Q}_j^u) \cup \{x, y, z\}$. (See the blue illustration in Figure 4).

Therefore $Q_1^{x_1}, \ldots, Q_{\ell_2}^{x_1}$, $T_1^u, \ldots, T_{\ell_2}^u$ are at least $\kappa_3(G_2) + \delta(G_1)\kappa_3(G_2)$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$. If there exits a vertex $u \in N_{G_1}(x_1)$ such that $d_{G_1}(u) = 1$, then $d_{G_1}(x_1) \geq 2$ and the previous bound leads to

$$(1 + d_{G_1}(x_1))\kappa_3(G_2) \geq 3\kappa_3(G_2) \geq 2\kappa_3(G_2) + 1 = (1 + \delta(G_1))\kappa_3(G_2) + \delta(G_1),$$

which proves (i). Otherwise, for each $u \in N_{G_1}(x_1)$, we consider a vertex $w_u \in N_{G_1}(u)\backslash\{x_1\}$. Since $g(G_1) \geq 5$, vertices $w_u \neq w_v$ for $u, v \in N_{G_1}(x_1)$ with $u \neq v$. This fact makes feasible the construction of another $\{x, y, z\}$-tree $\widetilde{T}^u$, for every $u \in N_{G_1}(x_1)$.

If $x_2y_2 \notin E(Q_1 \cup Q_2)$, $y_2z_2 \notin E(Q_1 \cup Q_2)$ and $x_2z_2 \notin E(Q_1 \cup Q_2)$, then (see the

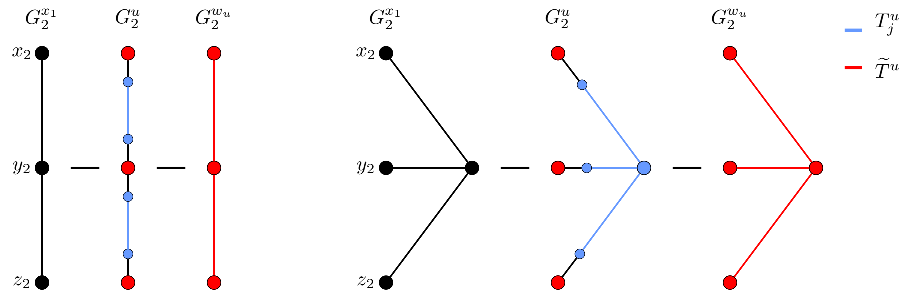

Figure 4: Trees T_j^u and $\widetilde{T}^u$ associated to an $\{x_2, y_2, z_2\}$-tree Q_j such that $x_2y_2 \notin E(Q_j)$, $y_2z_2 \notin E(Q_j)$ and $x_2z_2 \notin E(Q_j)$.

red illustration in Figure 4)

$$\widetilde{T}^u : Q_1^{w_u} \cup (x_1, x_2)(u, x_2)(w_u, x_2) \cup (x_1, y_2)(u, y_2)(w_u, y_2) \cup (x_1, z_2)(u, z_2)(w_u, z_2).$$

In case Q_1 is an $x_2y_2z_2$-path such that $x_2y_2 \in E(Q_1)$, then

$$\widetilde{T}^u : Q_1^{w_u} \cup (x_1, x_2)(u, x_2)(w_u, x_2) \cup (x_1, y_2)(u, x_2) \cup (x_1, z_2)(u, z_2)(w_u, z_2).$$

A similar tree can be constructed in the symmetrical case $y_2z_2 \in E(Q_1)$.

If $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$, then the vertex (u, x_2) is adjacent to the three vertices x, y, z, (see Figure 5), and therefore, we define

$$\widetilde{T}^u : (x_1, x_2)(u, x_2) \cup (x_1, y_2)(u, x_2) \cup (x_1, z_2)(u, x_2).$$

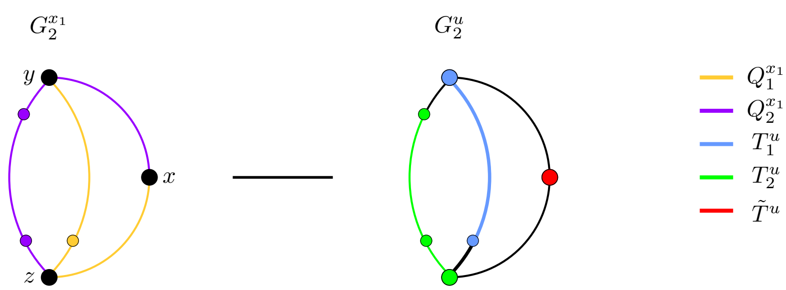

Figure 5: Five $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ when $x_2y_2 \in E(Q_1)$ and $x_2z_2 \in E(Q_2)$.

Trees $Q_1^{x_1}, \ldots, Q_{\ell_2}^{x_1}$, $T_1^u, \ldots, T_{\ell_2}^u$, $\tilde{T}^u$ for $u \in N_{G_1}(x_1)$ provide the desired result. □

Now, we consider that x, y, z come from two different vertices in G_1 and from other two different ones in G_2.

Lemma 3.2. *Let G_1 and G_2 be connected graphs with at least three vertices and girth at least five. For distinct vertices $x = (x_1, x_2)$, $y = (x_1, z_2)$ and $z = (z_1, z_2)$*

in $G_1 \boxtimes G_2$, *there exist at least* $\kappa(G_1)\kappa(G_2) + \kappa(G_1) + \kappa(G_2) - 1$ *internally disjoint* $\{x, y, z\}$*-trees in* $G_1 \boxtimes G_2$.

Proof. Notice that $x, y \in G_2^{x_1}$ while $z \in G_2^{z_1}$. Consider $k_1 = \kappa(G_1)$ internally disjoint x_1z_1-paths $P_1, ..., P_{k_1}$ in G_1 and $k_2 = \kappa(G_2)$ internally disjoint x_2z_2-paths $Q_1, \ldots, Q_{k_2}$ in G_2. Assume that $|V(P_1)| = \min\{|V(P_i)| : 1 \leq i \leq k_1\}$ and $|V(Q_1)| = \min\{|V(Q_j)| : 1 \leq j \leq k_2\}$. It may occur that $\overline{x}_1^1 = z_1$, $\underline{z}_2^1 = x_2$.

(I) First, we construct $2k_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$.

Associated to paths P_1 and Q_1, (see Figure 6), we have

$T_{11} : Q_1^{x_1} \cup (x_1, z_2) \ldots (z_1, z_2)$.

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (x_1, z_2)(\overline{x}_1^1, \underline{z}_2^1) \ldots (\underline{z}_1^1, \underline{z}_2^1)(z_1, z_2)$.

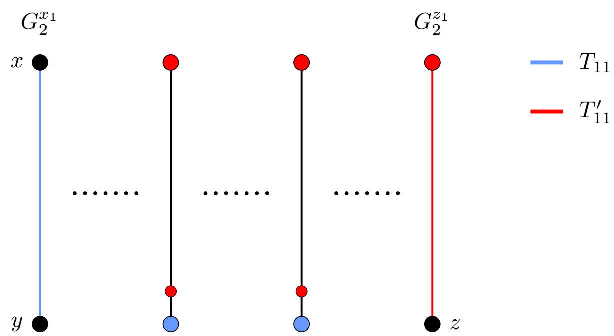

Figure 6: Trees T_{11}, T'_{11} associated to paths P_1 and Q_1.

Associated to paths P_1 and Q_j, we construct trees T_{1j}, T'_{1j} for $j \in \{2, \ldots, k_2\}$, when $k_2 \geq 2$. The minimality of Q_1 and the hypothesis $g(G_2) \geq 5$ guarantee that $\overline{x}_2^j \neq \underline{z}_2^j$.

If $x_1z_1 \in E(P_1)$, then

$T_{1j} : Q_j^{x_1} \cup (x_1, \underline{z}_2^j)(z_1, z_2)$.

$T'_{1j} : (x_1, x_2)(z_1, \overline{x}_2^j) \ldots (z_1, z_2) \cup (z_1, \underline{z}_2^j)(x_1, z_2)$.

If $x_1z_1 \notin E(P_1)$, (see Figure 7), then

$T_{1j} : (x_1, x_2) \ldots (x_1, \underline{\underline{z}}_2^j)(\overline{x}_1^1, \underline{z}_2^j) \ldots (\underline{z}_1^1, \underline{z}_2^j)(z_1, z_2) \cup (\overline{x}_1^1, \underline{z}_2^j)(x_1, z_2)$.

$T'_{1j} : (x_1, x_2)(\overline{x}_1^1, \overline{x}_2^j) \ldots (z_1, \overline{x}_2^j) \ldots (z_1, z_2) \cup (x_1, z_2)(x_1, \underline{z}_2^j)(\overline{x}_1^1, \underline{\underline{z}}_2^j) \ldots (\overline{x}_1^1, \overline{x}_2^j)$.

Trees $T_{11}, \ldots T_{1k_2}$, $T'_{11}, \ldots T'_{1k_2}$ are $2k_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ and the result is proved when $k_1 = 1$.

(II) If $k_1 \geq 2$, we construct the remaining trees associating them to paths P_i and

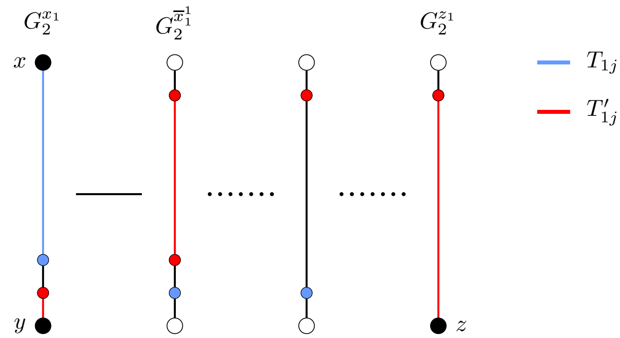

Figure 7: Trees T_{1j}, T'_{1j} associated to paths P_1 and Q_j when $x_1z_1 \notin E(P_1)$ and $j \geq 2$.

Q_j, for $i \in \{2, \dots, k_1\}$ and $j \in \{1, \dots, k_2\}$. Let us notice that $\overline{x}_1^i \neq \underline{z}_1^i$ because $g(G_1) \geq 5$.

Trees T_{i1}, T'_{i1} have a symmetrical construction to T_{1j}, T'_{1j} due to the symmetrical position of vertices x, y, z in this lemma and to the commutativity of the strong product of graphs. Hence, the proof is complete when $k_2 = 1$. For $i \in \{2, \dots, k_1\}$ and $j \in \{2, \dots, k_2\}$, we consider

$$T_{ij} : (x_1, x_2)(\overline{x}_1^i, \overline{x}_2^j) \dots (\overline{x}_1^i, \underline{z}_2^j)(x_1, z_2) \cup (\overline{x}_1^i, \underline{z}_2^j) \dots (\underline{z}_1^i, \underline{z}_2^j)(z_1, z_2).$$

Notice that trees T_{ij}, for $i \in \{1, \dots, k_1\}$, $j \in \{1, \dots, k_2\}$, $T'_{11}, \dots, T'_{1k_2}$ and $T'_{21}, \dots, T'_{k_1 1}$ prove the result. □

Now, we study the number of internally disjoint trees in $G_1 \boxtimes G_2$ that join vertices x, y, z which come from three vertices in G_1 and from two in G_2 or vice versa.

Lemma 3.3. *Let G_1 and G_2 be connected graphs with at least three vertices and girth at least five. Let $x_i, y_i, z_i \in V(G_i)$ be distinct vertices, for $i = 1, 2$. The following assertions hold:*

- *(i) There exist at least $\kappa(G_1)\kappa_3(G_2) + \kappa(G_1) + \kappa_3(G_2) - 1$ internally disjoint trees connecting vertices (x_1, x_2), (x_1, y_2), (z_1, z_2) in $G_1 \boxtimes G_2$.*
- *(ii) There exist at least $\kappa_3(G_1)\kappa(G_2) + \kappa_3(G_1) + \kappa(G_2) - 1$ internally disjoint trees connecting vertices (x_1, x_2), (y_1, x_2), (z_1, z_2) in $G_1 \boxtimes G_2$.*

Proof. Due to the commutativity of the strong product of graphs, it suffices to prove (i). Denote $x = (x_1, x_2)$, $y = (x_1, y_2)$, $z = (z_1, z_2)$. Notice that $x, y \in G_2^{x_1}$

while $z \in G_2^{z_1}$. Consider $k_1 = \kappa(G_1)$ internally disjoint x_1z_1-paths $P_1, ..., P_{k_1}$ in G_1 and $\ell_2 = \kappa_3(G_2)$ internally disjoint $\{x_2, y_2, z_2\}$-trees $Q_1, \ldots, Q_{\ell_2}$ in G_2.

Assume that $|V(P_1)| = \min\{|V(P_i)| : 1 \leq i \leq k_1\}$ and that at most Q_1 and Q_2 are *special* trees.

(I) First, we construct $2\ell_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to P_1, $Q_1, \ldots, Q_{\ell_2}$.

If only Q_1 is an *special* tree, we consider

$T_{11} : Q_1^{x_1} \cup (x_1, z_2) \ldots (z_1, z_2)$.

$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (x_1, y_2) \ldots (z_1, y_2)$.

If both Q_1 and Q_2 are *special* trees, we construct four trees associated to P_1, Q_1 and Q_2. Depending on whether x, y play a symmetrical role or not, Figure 8 depicts with different color the vertices which belong to each of these four trees.

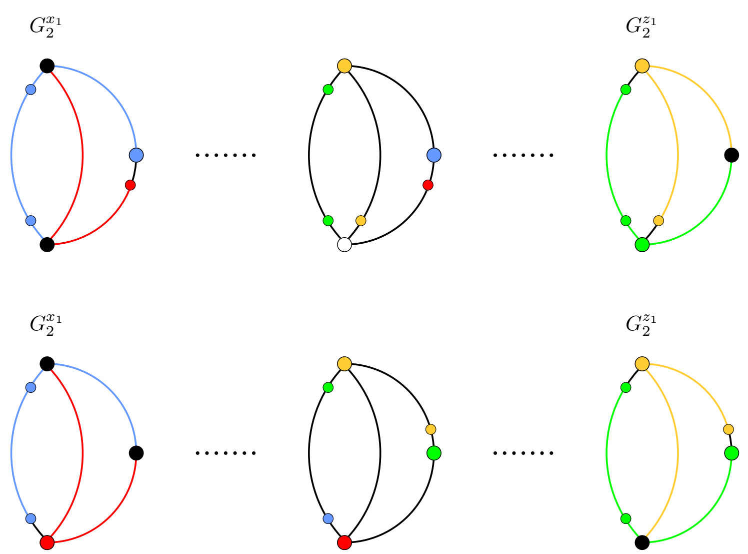

Figure 8: Four $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to a path P_1 in G_1 and to *special* trees Q_1, Q_2 in G_2

For every not *special* tree Q_j, for $j \in \{1, \ldots, \ell_2\}$, we construct two $\{x, y, z\}$-trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$ associated to P_1 and Q_j. First, we focus on a particular case. Assume that $x_1z_1 \notin E(P_1)$ and that Q_j is an s^j-rooted tree such that $d_{Q_j}(s^j, x_2) \geq 2$, $s^jy_2 \in E(Q_j)$, $s^jz_2 \in E(Q_j)$. It means that $Q_j : s^j \ldots \underline{\underline{x}}_2^j \underline{x}_2^j x_2 \cup s^jy_2 \cup s^jz_2$ where $\underline{\underline{x}}_2^j$ may be equal to s^j. Then

$$T_{1j} : (x_1, x_2)(\overline{x}_1^1, \underline{x}_2^j)(x_1, \underline{\underline{x}}_2^j) \dots (x_1, y_2) \cup (\overline{x}_1^1, \underline{\underline{x}}_2^j) \dots (z_1, \underline{x}_2^j) \dots (z_1, z_2).$$

$$T'_{1j} : (x_1, x_2)(x_1, \underline{x}_2^j)(\overline{x}_1^1, \underline{\underline{x}}_2^j) \dots (\overline{x}_1^1, s^j)(x_1, y_2) \cup (\overline{x}_1^1, s^j) \dots (\underline{z}_1^1, s^j)(z_1, z_2).$$

Notice that a symmetrical construction holds when $s^j x_2 \in E(Q_j)$, $d_{Q_j}(s^j, y_2) \geq 2$ and $s^j z_2 \in E(Q_j)$.

In any other case, we consider trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$ such that

$$\begin{aligned} V(T_{1j}) = \quad & \{x, y, z\} \cup V(Q_j^{x_1} - (x_1, z_2))\,\}) \cup \\ & \{(u, v) : u \in V(P_1 - \{x_1, z_1\}), v \in N_{Q_j}(z_2)\}. \\ V(T'_{1j}) = \quad & \{x, y, z\} \cup V(\,Q_j^{z_1} - \{(z_1, x_2), (z_1, y_2)\,\}) \cup \\ & \{(u, v) : u \in V(P_1 - \{x_1, z_1\}), v \in N_{Q_j}(x_2) \cup N_{Q_j}(y_2)\}. \end{aligned}$$

Hence, if $k_1 = 1$, we have constructed $2\ell_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ and the proof is complete.

(II) If $k_1 \geq 2$, we construct the remaining trees associating them to paths $P_2, \dots, P_{k_1}$ in G_1 and to trees $Q_1, \dots, Q_{\ell_2}$ in G_2. We assume $i \in \{2, \dots, k_1\}$ and denote $P_i : x_1 \overline{x}_1^i \dots \underline{z}_1^i z_1$. Notice that $x_1 \neq \overline{x}_1^i \neq \underline{z}_1^i \neq z_1$ due to the minimality of P_1 and that $g(G_1) \geq 5$.

Associated to P_i, Q_1, we construct trees T_{i1}, T'_{i1}. If Q_1 is an $x_2 y_2 z_2$-path such that $x_2 y_2 \in E(Q_1)$, then

$$V(T_{i1}) = \{x, y, z\} \cup V(\,Q_1^{\overline{x}_1^i} - (\overline{x}_1^i, x_2)\,) \cup \{(u, z_2) : u \in V(P_i - \{x_1, z_1\})\}.$$

$$V(T'_{i1}) = \{x, y, z\} \cup V(\,Q_1^{\underline{z}_1^i} - (\underline{z}_1^i, z_2)\,) \cup \{(u, x_2) : u \in V(P_i - \{x_1, z_1\})\}.$$

In any other case, trees T_{i1}, T'_{i1} have sets of vertices

$$V(T_{i1}) = \{x, y, z\} \cup V(\,Q_1^{\overline{x}_1^i} - \{(\overline{x}_1^i, x_2), (\overline{x}_1^i, y_2)\}\,) \cup \{(u, z_2) : u \in V(P_i - \{x_1, z_1\})\}.$$

$$V(T'_{i1}) = \{x, y, z\} \cup V(\,Q_1^{\underline{z}_1^i} - (\underline{z}_1^i, z_2)\,) \cup \{(u, v) : u \in V(P_i - \{x_1, z_1\}), v \in \{x_2, y_2\}\}.$$

If $\ell_2 = 1$, trees $T_{11}, \dots, T_{k_1 1}, T'_{11}, \dots, T'_{k_1 1}$ prove the lemma. Finally, we consider that $k_1 \geq 2$ and $\ell_2 \geq 2$. If Q_1 and Q_2 are *special* trees, we construct three $\{x, y, z\}$-trees T_{i1}, T'_{i1}, T_{i2} associated to P_i, Q_1 and Q_2 as it is shown in Figure 9.

If Q_j is not an *special* tree, for $j \in \{2, \dots, \ell_2\}$, we consider a tree T_{ij} such that

$$\begin{aligned} V(T_{ij}) = \quad & \{x, y, z\} \cup V(\,Q_j^{\overline{x}_1^i} - \{(\overline{x}_1^i, x_2), (\overline{x}_1^i, y_2), (\overline{x}_1^i, z_2)\}\,) \cup \\ & \{(u, v) : u \in V(P_i - \{x_1, z_1\}), v \in N_{G_2}(z_2)\}. \end{aligned}$$

Notice that trees T_{ij}, for $i \in \{1, \dots, k_1\}$, $j \in \{1, \dots, \ell_2\}$, $T'_{11}, \dots, T'_{1\ell_2}$ and $T'_{21}, \dots, T'_{k_1 1}$ prove the result. □

Finally, we assume that x, y, z come from three different vertices in G_1 and G_2.

Lemma 3.4. *Let G_1 and G_2 be connected graphs with at least three vertices and girth at least five. For distinct vertices $x_i, y_i, z_i \in V(G_i)$, with $i = 1, 2$, there exist*

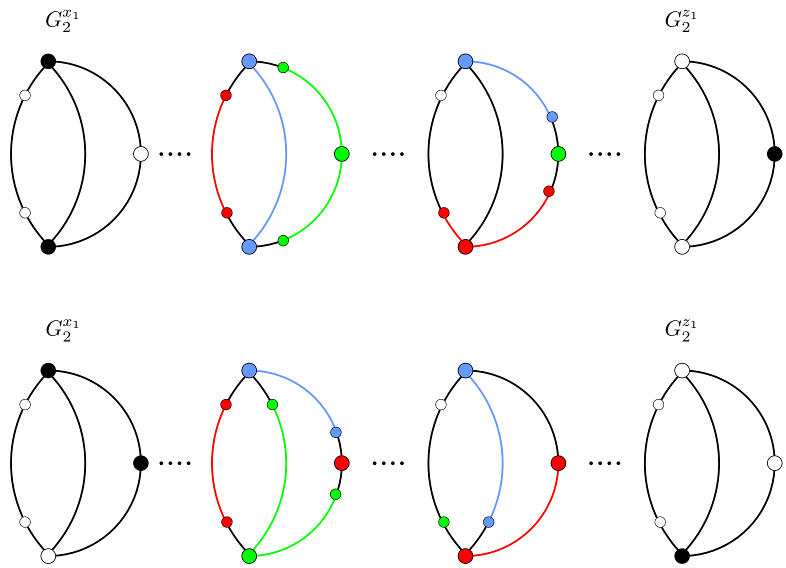

Figure 9: Three $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to a path P_i in G_1 ($i \geq 2$) and to *special* trees Q_1, Q_2 in G_2.

at least $\kappa_3(G_1)\kappa_3(G_2)+\kappa_3(G_1)+\kappa_3(G_2)-1$ *internally disjoint trees joining vertices* $x = (x_1, x_2)$*,* $y = (y_1, y_2)$*,* $z = (z_1, z_2)$ *in* $G_1 \boxtimes G_2$*.*

Proof. Notice that vertices x, y, z belong to distinct copies $G_2^{x_1}$, $G_2^{y_1}$, $G_2^{z_1}$, respectively. Consider $\ell_1 = \kappa_3(G_1)$ internally disjoint $\{x_1, y_1, z_1\}$-trees $P_1, \ldots, P_{\ell_1}$ in G_1 and $\ell_2 = \kappa_3(G_2)$ internally disjoint $\{x_2, y_2, z_2\}$-trees $Q_1, \ldots, Q_{\ell_2}$ in G_2. From Remark 2.1, we know that at most P_1, P_2, Q_1, Q_2 are *special* trees. For simplicity, denote $\ddot{Q}_j^{x_1} = Q_j^{x_1} - \{(x_1, y_2), (x_1, z_2)\}$, $\ddot{Q}_j^{y_1} = Q_j^{y_1} - \{(y_1, x_2), (y_1, z_2)\}$ and $\ddot{Q}_j^{z_1} = Q_j^{z_1} - \{(z_1, x_2), (z_1, y_2)\}$. Without loss of generality, when P_i is a path, we assume that it is the $x_1y_1z_1$-path.

(I) First, we construct $2\ell_2$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to P_1 and $Q_1, \ldots, Q_{\ell_2}$.

a) If at most Q_1 is an *special* tree, we construct trees T_{11} and T'_{11} as follows.

If P_1 is a path, we have

$$T_{11} : Q_1^{x_1} \cup (x_1, y_2) \ldots (y_1, y_2) \cup (x_1, z_2) \ldots (z_1, z_2).$$

$$T'_{11} : Q_1^{z_1} \cup (x_1, x_2) \ldots (z_1, x_2) \cup (y_1, y_2) \ldots (z_1, y_2).$$

If P_1 is an r^1-rooted tree,

$$T_{11}: Q_1^{x_1} \cup (x_1, y_2) \dots (r^1, y_2) \dots (y_1, y_2) \cup (x_1, z_2) \dots (r^1, z_2) \dots (z_1, z_2).$$
$$T'_{11}: (x_1, x_2) \dots (r^1, x_2) \dots (y_1, x_2) \dots (y_1, y_2) \cup (r^1, x_2) \dots (z_1, x_2) \dots (z_1, z_2).$$

b) If both Q_1 and Q_2 are *special* trees, we construct four trees associated to P_1, Q_1 and Q_2. Depending on whether P_1 is a path or a tree, Figure 10 or Figure 11 shows with different color the vertices that belong to each tree, respectively.

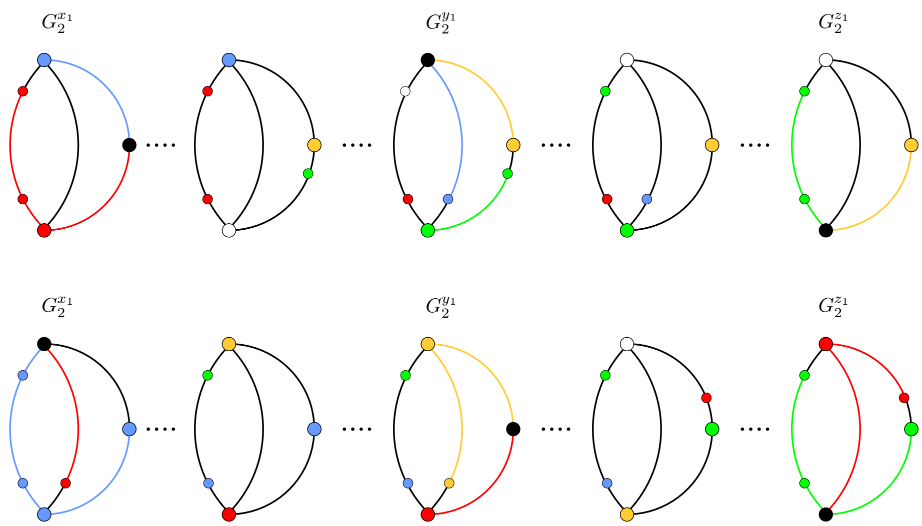

Figure 10: Four $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to a path P_1 in G_1 and to *special* trees Q_1, Q_2 in G_2.

c) For each $j \in \{2, \dots, \ell_2\}$ such that Q_j is not an *special* tree, we construct two trees T_{1j}, T'_{1j} in $G_1 \boxtimes G_2$ associated to trees P_1 and Q_j as follows.

Consider that Q_j is either an $x_2z_2y_2$-path or an s^j-rooted tree such that $d_{P_1}(x_1, y_1) \geq 2$ and $\underline{x}_2^j = \underline{y}_2^j = s^j$, denoting in this last case $\overline{z}_2^j = \underline{z}_2^j$.

$$\begin{aligned} V(T_{1j}) = \quad & \{x, y, z\} \cup (x_1, x_2) \dots (x_1, \underline{z}_2^j) \cup (y_1, y_2) \dots (y_1, \overline{z}_2^j) \cup \\ & \{(u, v) : u \in V(P_1 - \{x_1, z_1\}), v \in N_{Q_j}(z_2)\}. \\ V(T'_{1j}) = \quad & \{x, y, z\} \cup V(\ddot{Q}_j^{z_1}) \cup \{(u, v) : u \in V(P_1 - \{x_1, z_1\}), \\ & v \in N_{Q_j}(x_2) \cup N_{Q_j}(y_2)\}. \end{aligned}$$

Similar constructions hold when $d_{P_1}(y_1, z_1) \geq 2$ and $\underline{z}_2^j = \underline{y}_2^j = s^j$.

In any other case, to unify the description of the trees T_{1j} and T'_{1j}, without loss of generality, we denote $\overline{y}_2^j = \underline{y}_2^j$ when Q_j is an s^j-rooted tree and provide an specific role to the vertex $y = (y_1, y_2)$. Concretely, if P_1 is a path, we consider that P_1 is an

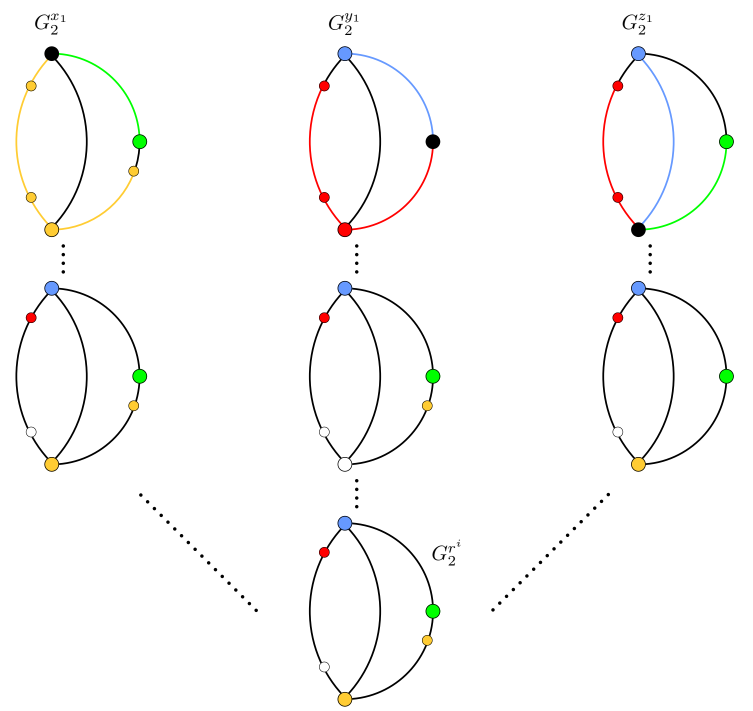

Figure 11: Four $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to a path P_1 in G_1 and to *special* trees Q_1, Q_2 in G_2.

$x_1y_1z_1$-path and if P_1 is an r^1-rooted tree, we assume either that $d_{Q_j}(s^j, y_2) \geq 2$ or that Q_j is an $x_2y_2z_2$-path.

Under these assumptions, to construct the trees T_{1j}, T'_{1j} it is enough to consider that

$$\begin{aligned} V(T_{1j}) = \quad & \{x, y, z\} \cup V(\ddot{Q}_j^{y_1}) \cup \{(u, v) : u \in V(P_1 - \{x_1, y_1, z_1\}), \\ & v \in N_{Q_j}(x_2) \cup N_{Q_j}(z_2)\}. \\ V(T'_{1j}) = \quad & \{x, y, z\} \cup \{(x_1, x_2), \ldots, (x_1, \underline{y}_2)\} \cup \{(z_1, z_2), \ldots, (z_1, \overline{y}_2)\} \cup \\ & \{(u, v) : u \in V(P_1 - \{x_1, y_1, z_1\}), v \in N_{Q_j}(y_2)\}. \end{aligned}$$

When $\ell_1 = 1$, trees $T_{11}, \ldots, T_{1\ell_2}, T'_{11}, \ldots, T'_{1\ell_2}$ are $2\ell_2$ internally disjoint $\{x, y, z\}$-trees, as desired.

(II) Now, we assume that P_1 and P_2 are *special* trees. Associated to P_1, P_2, $Q_1, \ldots, Q_{\ell_2}$ we construct $3\ell_2 + 1$ internally $\{x, y, z\}$-trees.

The construction of four trees associated to P_1, P_2, Q_1 follows from the one described for P_1, Q_1, Q_2 in (Ib) due to the symmetrical position of the vertices x, y, z in this lemma and to the symmetry of the strong product of graphs.

If Q_1 and Q_2 are also *special* trees, Figures 12 and 13 shows seven internally disjoint trees associated to *special* trees P_1, P_2, Q_1, Q_2.

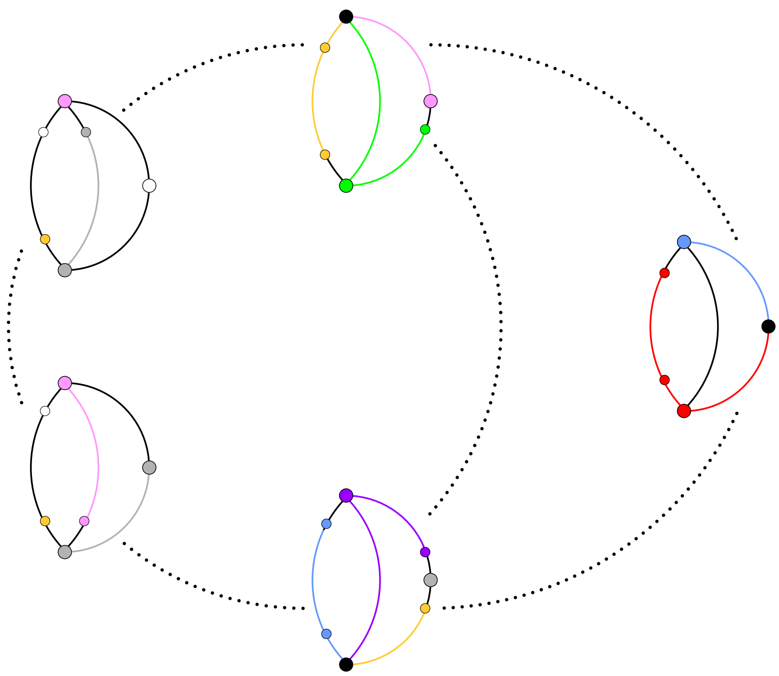

Figure 12: Seven $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to *special* trees P_1,P_2 in G_1 and Q_1, Q_2 in G_2.

Associated to P_1, P_2, Q_j, when Q_j is not an *special* tree for $j \geq 2$, we construct internally disjoint trees T_{1j}, T'_{1j}, T_{2j} such that

$$\ddot{Q}_j^{x_1} \subset V(T_{1j}), \ddot{Q}_j^{y_1} \subset V(T'_{1j}), \ddot{Q}_j^{z_1} \subset V(T_{2j})$$

and take into account the equality $V(T_{1j} \cup T'_{1j} \cup T_{2j}) = V(\ddot{Q}_j^{x_1} \cup \ddot{Q}_j^{y_1} \cup \ddot{Q}_j^{z_1}) \cup N$, where $N = \{(u, v) : u \in V(P_1 \cup P_2) - \{x_1, y_1, z_1\},\ v \ \in \ N_{Q_j}(x_2) \cup N_{Q_j}(y_2) \cup N_{Q_j}(z_2)\}$.

The proof is finished when $\ell_1 = 2$, or symmetrically, when $\ell_2 = 2$.

(III) Assume that P_i is not an *special* tree, for $i \in \{2, \ldots, \ell_1\}$. Associated to it, we construct $\ell_2 + 1$ internally disjoint $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$. To do that, it has to

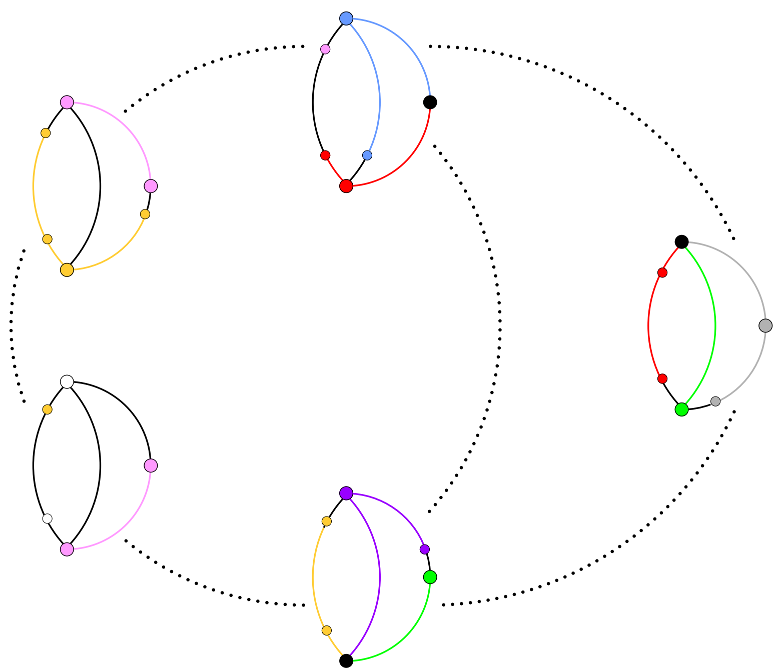

Figure 13: Seven $\{x, y, z\}$-trees in $G_1 \boxtimes G_2$ associated to *special* trees P_1,P_2 in G_1 and Q_1, Q_2 in G_2.

be distinguished whether P_i is an r^i-rooted tree or a path.

Consider P_i is an r^i-rooted tree with leaves x_1, y_1, z_1.

If Q_2 is not an *special* tree, notice that trees T_{i1}, T'_{i1}, associated to P_i and Q_1, are symmetrical to the trees T_{1j}, T'_{1j} constructed in (Ic). When both Q_1 and Q_2 are *special* trees, T_{i1}, T'_{i1}, T_{i2} are symmetrical to T_{1j}, T'_{1j} T_{2j}, constructed in (II). When T_{ij} is associated to P_i and to a *non-special* tree Q_j, then

$$\begin{aligned} V(T_{ij}) = \quad & \{x, y, z\} \cup V(Q_j^{r^i} - \{(r^i, x_2), (r^i, y_2), (r^i, z_2)\}) \cup \\ & \{(u, v) : u \in V(P_i - \{x_1, y_1, z_1\}), v \in N_{Q_j}(x_2) \cup N_{Q_j}(y_2) \cup N_{Q_j}(z_2)\}. \end{aligned}$$

Finally, consider that P_i is an $x_1y_1z_1$-path. Notice that, for every $u \in P_i$, all the trees $Q_1^u, \ldots, Q_{\ell_2}^u$ contain the three vertices $(u, x_2), (u, y_2), (u, z_2)$, and hence, we cannot proceed as usual to construct internally disjoint trees associated to $P_i, Q_1, \ldots, Q_{\ell_2}$.

Instead, as we have mentioned in the Introduction, according to [**10, 15**] we can consider $R_1, \ldots, R_{k_2}$ internally disjoint x_2y_2-paths and $S_1, \ldots, S_{k_2}$ internally disjoint y_2z_2-paths in G_2.

Since R_j and S_j may not be internally disjoint, they will be used in different copies of G_2. We denote $R_j : x_2\overline{x}_2^j \ldots \underline{y}_2^j y_2$ and $S_j : y_2\overline{y}_2^j \ldots \underline{z}_2^j z_2$ for $j \in \{1, \ldots, k_2\}$. In particular, if R_1 and S_1 are the shortest ones, it may occur that $R_1 : x_2y_2$ or $S_1 : y_2z_2$.

Associated to P_i, R_1, S_1, we consider
$T_{i1} : (x_1, x_2)(\overline{x}_1^i, \overline{x}_2^1) \ldots (\overline{x}_1^i, y_2) \ldots (\underline{y}_1^i, y_2)(y_1, y_2)(\overline{y}_1^i, \overline{y}_2^1) \ldots (\overline{y}_1^i, z_2) \ldots (\underline{z}_1^i, z_2)(z_1, z_2)$.
$T'_{i1} : (x_1, x_2)(\overline{x}_1^i, x_2) \ldots (\underline{y}_1^i, x_2) \ldots (\underline{y}_1^i, \underline{y}_2^1)(y_1, y_2)(\overline{y}_1^i, y_2) \ldots (\underline{z}_1^i, y_2) \ldots (\underline{z}_1^i, \underline{z}_2^1)(z_1, z_2)$.

For paths P_i, R_j, S_j, $i \in \{2, \ldots, \ell_1\}$, $j \in \{2, \ldots, \ell_2\}$, we consider
$T_{ij} : (x_1, x_2)(\overline{x}_1^i, \overline{x}_2^j) \ldots (\overline{x}_1^i, \underline{y}_2^j) \ldots (\underline{y}_1^i, \underline{y}_2^j)(y_1, y_2)(\overline{y}_1^i, \overline{y}_2^j) \ldots (\overline{y}_1^i, \underline{z}_2^j) \ldots (\underline{z}_1^i, \underline{z}_2^j)(z_1, z_2)$.
□

Now, we are ready to prove the main result of the paper.

Theorem 3.1. *Let G_1 and G_2 be connected graphs with at least three vertices and girth at least five. Then, $\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1$. The bound is sharp.*

Proof. The bound $\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1$ is consequence of the inequality $\delta(G) \geq \kappa(G) \geq \kappa_3(G)$ and Lemmas 3.1, 3.2, 3.3 and 3.4. To see that the bound is sharp, it is enough to check out that $\kappa_3(\mathcal{P} \boxtimes \mathcal{P}) = 2$, where $\mathcal{P}$ denotes a path with three vertices. □

The equality $\delta(G_1 \boxtimes G_2) = \delta(G_1)\delta(G_2) + \delta(G_1) + \delta(G_2)$ together with Theorem 3.1 provide an accurate result on $\kappa_3(G_1 \boxtimes G_2)$ when the generator graphs also satisfy that $\kappa_3(G_1) = \delta(G_1)$ and $\kappa_3(G_2) = \delta(G_2)$.

Corollary 3.1. *Let G_1 and G_2 be connected graphs with at least three vertices, girth at least five and such that $\kappa_3(G_1) = \delta(G_1)$ and $\kappa_3(G_2) = \delta(G_2)$. Then,*

$$\delta(G_1 \boxtimes G_2) - 1 \leq \kappa_3(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2).$$

Both bounds are sharp.

4. CONCLUSIONS

This paper is the first study of the generalized 3-connectivity on the strong product graph. For two connected graphs G_1 and G_2 with the requirements of having at least three vertices and girth at least five, we have constructed internally disjoint trees that connect any three vertices $x, y, z \in V(G_1 \boxtimes G_2)$. They provide constructive sharp lower bounds on $K_3(G_1 \boxtimes G_2)$ in terms of well known parameters of the factor graphs.

From these results we have obtained our main Theorem 3.1 where we have showed that $\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2) - 1$, for two connected graphs G_1 and G_2 with at least three vertices and girth at least five. Moreover, we have deduced that $\delta(G_1 \boxtimes G_2) - 1 \leq \kappa_3(G_1 \boxtimes G_2) \leq \delta(G_1 \boxtimes G_2)$, when also the factor graphs verify that $\kappa_3(G_1) = \delta(G_1)$ and $\kappa_3(G_2) = \delta(G_2)$.

Although we do not show it in this paper due to the high number of cases involved, we have managed to prove that $\kappa_3(G_1 \boxtimes G_2) \geq \kappa_3(G_1)\kappa_3(G_2) + \kappa_3(G_1) + \kappa_3(G_2)$ for connected graphs G_1 and G_2 with at least *four* vertices and girth at least five. As consequence of this inequality, it follows that $\kappa_3(G_1 \boxtimes G_2) = \delta(G_1 \boxtimes G_2)$ for graphs G_i with at least *four* vertices, girth at least five and such that $\kappa_3(G_i) = \delta(G_i)$ for $i \in \{1, 2\}$.

As future work, we would like to establish some general results about the generalized k-connectivity of the strong product of graphs for $k \geq 4$. Also, it would also be interesting to keep exploring the generalized k-connectivity on other kind of product graphs.

5. ACKNOWLEDGMENT

This research was supported by the Ministry of Economy and Competitiveness, Spain, under project MTM2014-60127-P.

REFERENCES

1. E. Abajo, R. M. Casablanca, A. Diánez, P. García-Vázquez: *On average connectivity of the strong product of graphs.* Discrete Appl. Math. **161** (2013), 2795–2801.
2. C. Balbuena, P. García-Vázquez, X. Marcote: *Reliability of interconnection networks modeled by a product of graphs.* Networks **48(3)** (2006), 114–120.
3. G. Chartrand, L. Lesniak, P. Zhang: *Graphs and Digraphs.* Chapman and Hall/CRC, U.S.A., 2010.

4. G. Chartrand, F. Okamoto, P. Zhang: *Rainbow trees in graphs and generalized connectivity.* Networks **55(4)** (2010), 360–367.

5. M. Hager: *Pendant tree-connectivity.* J. Comb. Theory **38** (1985), 179–189.

6. R. Hammack, W. Imrich, S. Klavžar: *Handbook of Product Graphs, Second Edition.* CRC Press, U.S.A., 2011.

7. S. Li, W. Li, X. Li: *The generalized connectivity of complete bipartite graphs.* Ars Combin. **104** (2012), 65–79.

8. H. Li, X. Li and Y. Sun: *The generalized* 3*-connectivity of Cartesian product graphs.* Discrete Mathematics and Theoretical Computer Science **14(1)** (2012), 43–54.

9. S. Li, X. Li, W. Zhou: *Sharp bounds for the generalized connectivity* $\kappa_3(G)$. Discrete Math. **310** (2010), 2147–2163.

10. K. Menger: *Zur allgemeinen Kurventheorie.* Fund. Math. **10** (1927), 96–115.

11. K. Menger: *Kurventheorie.* Teubner, (1932).

12. F. Okamoto, P. Zhang: *The tree connectivity of regular complete bipartite graphs.* J. Combin. Math. Combin. Comput. **74** (2010), 279–293.

13. S. Špacapan: *Connectivity of Cartesian products of graphs.* Appl. Math. Lett. **21** (2008), 682–685.

14. S. Špacapan: *Connectivity of strong products of graphs.* Graphs and Combinatorics **26(3)** (2010), 457–467.

15. H. Whitney: *Congruent graphs and the connectivity of graphs.* Amer. J. Math. **54** (1932), 150–168.

16. J.M. Xu, C. Yang: *Connectivity of Cartesian product graphs.* Discrete Math. **306(1)** (2006), 159–165.

Encarnación Abajo
Escuela Técnica Superior de Arquitectura,
Departamento de Matemática Aplicada I,
Universidad de Sevilla,
Avenida de Reina Mercedes, 2,
41012, Sevilla,
Spain.
E-mail: *eabajo@us.es*

(Received 26.07.2016)
(Revised 16.11.2017)

Rocío Casablanca
Escuela Técnica Superior de Arquitectura,
Departamento de Matemática Aplicada I,
Universidad de Sevilla,
Avenida de Reina Mercedes, 2,
41012, Sevilla,
Spain.
E-mail: *rociomc@us.es*

Ana Diánez
Escuela Técnica Superior de Arquitectura,
Departamento de Matemática Aplicada I,
Universidad de Sevilla,
Avenida de Reina Mercedes, 2,
41012, Sevilla,
Spain.
E-mail: *anadianez@us.es*

Pedro García-Vázquez
Escuela Técnica Superior de Arquitectura,
Departamento de Matemática Aplicada I,
Universidad de Sevilla,
Avenida de Reina Mercedes, 2,
41012, Sevilla,
Spain.
E-mail: *pgvazquez@us.es*