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Quasi-similarity of contractions having a 2×2 singular characteristic function ☆

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ABSTRACT

Let $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be a completely non-unitary contraction having a 2×2 singular characteristic function Θ_1; that is, $\Theta_1 = [\theta_{i,j}]_{i,j=1,2}$ with $\theta_{ij} \in H^\infty$ and $\det(\Theta_1) = 0$. As it is well known, Θ_1 is a singular matrix if and only if Θ_1 can be written as $\Theta_1 = w_1 m_1 \left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right][c_1 \quad d_1]$ where $w_1, m_1, a_1, b_1, c_1, d_1 \in H^\infty$ are such that (i) w_1 is an outer function with $|w_1| \leqslant 1$, (ii) m_1 is an inner function, (iii) $|a_1|^2 + |b_1|^2 = |c_1|^2 + |d_1|^2 = 1$, and (iv) $a_1 \wedge b_1 = c_1 \wedge d_1 = 1$ (here $\wedge$ stands for the greatest common inner divisor). Now consider a second completely non-unitary contraction $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$ having also a 2×2 singular characteristic function $\Theta_2 = w_2 m_2 \left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right][c_2 \quad d_2]$. We give necessary and sufficient conditions for T_1 and T_2 to be quasi-similar.

1. Statement of the Main Theorem

Introduction

Can one characterize the quasi-similarity of contractions in terms of their characteristic functions? Quasi-similarity is an equivalence relation between Hilbert space bounded operators which, being weaker than similarity, still preserves many interesting features as the eigenvalues, the spectral multiplicity or the non-triviality of the lattice of invariant subspaces (see [1,4,7] and references therein).

Two Hilbert space bounded operators $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ are said to be *quasi-similar* if there exist two bounded operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ and $W : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$XT_1 = T_2 X, \qquad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \qquad \ker(X) = \{0\},$$
$$T_1 W = W T_2, \qquad \operatorname{clos}\{W\mathcal{H}_2\} = \mathcal{H}_1, \qquad \ker(W) = \{0\}.$$

Such operators X and W are called *quasi-affinities* or *deformations.*

There has been several very deep and interesting approaches to find a characterization of quasi-similarity in terms of the characteristic functions of the operators involved. Namely, the Jordan model for C_0-contractions, completed by Bercovici, Sz.-Nagy and Foiaş and, independently, Müller, after pioneering work by Sz.-Nagy and Foiaş (see [1,7]); the Jordan model

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for weak contractions due to Wu [8,9]; and the classification, up to quasi-similarity, of C_{10}-contractions with finite defects and Fredholm index equal to -1 due to Makarov and Vasyunin [3]. More recently, we have given necessary and sufficient conditions for the quasi-similarity of contractions having a 2×1 characteristic function [2].

Framework

Let $T \in \mathcal{B}(\mathcal{H})$ be a completely non-unitary contraction having an $n \times n$ characteristic function Θ. This means, in particular, that T is a Fredholm operator with both defect indices equal to n and that its Fredholm index is 0. If $\det(\Theta) \neq 0$, then T is a weak contraction, and the characterization of the operators that are quasi-similar to T was given by Wu in [8,9]. Roughly speaking, if Θ is non-singular, then T is quasi-similar to a uniquely determined direct sum of a Jordan chain plus a finite number of operators of multiplication by the independent variable on spaces of type $\chi_\Omega L^2$, where Ω stands for a measurable subset of $\mathbb{T}$, the unit circle of the complex plane.

The purpose of this paper is to study, with the help of the coordinate-free function model developed by Nikolski and Vasyunin [6] (see also [4, Chapter 1]), the quasi-similarity of contractions having a 2×2 (non-zero) singular characteristic function. As we shall see, this case seems to be already somewhat difficult to manage, but we hope that it will provide hints to tackle the general case when the characteristic function is an $n \times n$ singular matrix. So let $T \in \mathcal{B}(\mathcal{H})$ be a completely non-unitary contraction having a characteristic function Θ which is a 2×2 singular matrix of functions in H^∞. As it is well known, such a function Θ can be written as $\Theta = wm\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right][c \quad d]$, where $w, m, a, b, c, d \in H^\infty$ are such that (i) w is an outer function with $|w| \leqslant 1$, (ii) m is an inner function, (iii) $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$, and (iv) $a \wedge b = c \wedge d = 1$ (here $\wedge$ stands for the greatest common inner divisor). Associated to these functions we also consider the set

$$\Omega := \{z \in \mathbb{T}\colon\ |w(z)| < 1\}$$

and the ideal $\mathcal{N}^+\{a, b\}$ generated by a pair of functions a and b from the Smirnov class $\mathcal{N}^+ := \{f/g\colon\ f, g \in H^\infty \text{ and } g \text{ is outer}\}$, that is

$$\mathcal{N}^+\{a, b\} := \{\nu a + \mu b\colon\ \nu, \mu \in \mathcal{N}^+\}.$$

Let us denote by $H^\infty_{2\times 2}$ and $\mathcal{N}^+_{2\times 2}$ the sets of all 2×2 matrices with entries in H^∞ and, respectively, the Smirnov class $\mathcal{N}^+$. For a function f from the Smirnov class, by f^{i} and f^{o} we denote the inner and outer parts of f. Let us also introduce the following notation: $\vartheta := \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $\varphi := [c \quad d]$, $\vartheta^{\mathrm{ad}} := [b \quad -a]$, $\varphi^{\mathrm{ad}} := \left[\begin{smallmatrix} d \\ -c \end{smallmatrix}\right]$ and

$$\det\left(\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} c \\ d \end{bmatrix}\right)^{\mathrm{i}} := \left\{(\det \Lambda)^{\mathrm{i}}\colon\ \Lambda \in \mathcal{N}^+_{2\times 2} \text{ and } \Lambda \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}\right\}.$$

For a matrix $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ the symbol M^{ad} denotes the adjugate matrix $\left[\begin{smallmatrix} \delta & -\beta \\ -\gamma & \alpha \end{smallmatrix}\right]$, so that the following equalities hold: $MM^{\mathrm{ad}} = M^{\mathrm{ad}}M = (\det M)I$. We fix this notation (with subindices when appropriate) throughout the paper.

Now consider two completely non-unitary contractions $T_i \in \mathcal{B}(\mathcal{H}_i)$ having 2×2 characteristic functions $\Theta_i = w_i m_i \left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right][c_i \quad d_i] = w_i m_i \vartheta_i \varphi_i$. Our main result in this paper is the following.

Main Theorem. *T_1 is quasi-similar to T_2 if and only if the following conditions hold*:

(i) $m_1 = m_2 = m$,
(ii) $\Omega_1 = \Omega_2$ *a.e.*,
(iii) *there exists* $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}} \cap \det(\varphi_1^{\mathrm{ad}} \to \varphi_2^{\mathrm{ad}})^{\mathrm{i}}$ *such that* $f \wedge m = 1$, *and*
(iv) *there exists* $g \in \det(\vartheta_1 \to \vartheta_2)^{\mathrm{i}} \cap \det(\varphi_2^{\mathrm{ad}} \to \varphi_1^{\mathrm{ad}})^{\mathrm{i}}$ *such that* $g \wedge m = 1$.

Remarks. We would like to underline at this point that one could think about the possibility of separating the outer and inner parts of $\Theta = wm\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right][c \quad d]$, that is, $\Theta^{\mathrm{o}} = w[c \quad d]$ and $\Theta^{\mathrm{i}} = m\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, in order to use the results from [2] to obtain quasi-similarity of operators having these characteristic functions separately. However, we will see (Proposition 4.1 below) that, in one of the most simplest cases, when $m = 1 = w$, the operators whose characteristic functions are

$$\begin{bmatrix} a \\ b \end{bmatrix} [c \quad d] \quad \text{and} \quad \begin{bmatrix} a \\ b \end{bmatrix} \oplus [c \quad d]$$

are quasi-similar if and only if there exist four functions $f_1, f_2, f_3, f_4 \in H^\infty$ such that $af_1 + bf_2 + cf_3 + df_4$ is an outer function; a condition that not always holds. This tells us that, unlike the 1×1 case, separating inner and outer parts is not the right way to tackle the proof.

Other terminology and notations

In what follows, clos{·} stands for the closure of the linear span of the set within the brackets. In particular, if T is a bounded operator defined in a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ is a linear subspace of $\mathcal{H}$, we shall frequently use that $\operatorname{clos}[T \operatorname{clos}\{\mathcal{M}\}] = \operatorname{clos}\{T\mathcal{M}\}$. Whenever we write L^2 or $L^2(\mathcal{H})$, our underlying measure space is assumed to be the unit circle $\mathbb{T}$ of the complex plane endowed with the Lebesgue measure; in particular, for two sets Ω_1 and Ω_2 we write $\Omega_1 = \Omega_2$ a.e., whenever these sets coincide up to a set of Lebesgue measure zero. We assume $0^{\mathrm{i}} = 0$, $0^{\mathrm{o}} = 1$, and $0 \wedge f = f^{\mathrm{i}}$.

Otherwise, our terminology and notations are standard. A label $(m.n)$ refers to the nth formula of section m.

2. Quasi-affinities in the coordinate-free function model

The coordinate-free function model

Since we shall make an intensive use of the properties and the notation of the coordinate-free function model for completely non-unitary contractions given in [6] (see also [4, Chapter 1]), we shall describe it briefly for the convenience of the reader.

Given a completely non-unitary contraction $T \in \mathcal{B}(\mathcal{H})$, let $D_T = (I - T^*T)^{1/2}$ be its defect operator and $\mathcal{D}_T = \operatorname{clos}\{D_T\mathcal{H}\}$ be its defect subspace, and take two auxiliary Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ such that

$$\dim(\mathcal{E}) = \dim(\mathcal{D}_T) \quad \text{and} \quad \dim(\mathcal{E}_*) = \dim(\mathcal{D}_{T^*}).$$

Now, let $U \in \mathcal{B}(\mathcal{K})$ be the minimal unitary dilation of T. Then U has a triangular matrix with respect to the canonical decomposition $\mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{G}$, where $\mathcal{G}$ and $\mathcal{G}_*$ are the so-called outgoing and incoming subspaces, respectively, and there exists a pair of functional embeddings

$$\Pi = (\pi_*, \pi) : L^2(\mathcal{E}_*) \oplus L^2(\mathcal{E}) \to \mathcal{K}$$

where, among other properties, the operator Π has dense range in $\mathcal{K}$ and π and π_* are isometries intertwining U and the operator M_z of multiplication by z in the corresponding L^2 space. Moreover,

$$\pi H^2(\mathcal{E}) = \mathcal{G} \perp \mathcal{G}_* = \pi_* H^2_-(\mathcal{E}_*)$$

and the operator $\Theta := \pi_*^*\pi \in \mathcal{B}(L^2(\mathcal{E}), L^2(\mathcal{E}_*))$ is the multiplication operator by a contractive-valued analytic function $z \mapsto \Theta(z) \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$; that is, $(\Theta f)(z) = \Theta(z)f(z)$, and this analytic function is equivalent to the characteristic function Θ_T of T defined by

$$\Theta_T(z) := \left(-T + zD_{T^*}\left(I - zT^*\right)^{-1}D_T\right)\big|\mathcal{D}_T.$$

We also have that T is unitarily equivalent to the model operator defined as the compression of U to the subspace $\mathcal{H}_\Theta$ of $\mathcal{K}$ defined as the orthogonal complement of the orthogonal sum $(\pi H^2(\mathcal{E}) \oplus \pi_* H^2_-(\mathcal{E}_*))$.

To describe the intertwining lifting theorem that we shall use, we need to introduce some more operators appearing in this model.

Define $\Delta := (I - \Theta^*\Theta)^{1/2}$. Then Δ is the positive part of the polar decomposition $\pi - \pi_*\Theta = \tau\Delta$ that also provides us with an isometry τ acting from the so-called residual subspace $L^2(\Delta\mathcal{E}) := \operatorname{clos}\{\Delta L^2(\mathcal{E})\}$ to $\mathcal{K}$. Similarly, for $\Delta_* := (I - \Theta\Theta^*)^{1/2}$ there is an isometry τ_* defined in $L^2(\Delta_*\mathcal{E}_*)$. These operators satisfy a number of relationships [6, p. 237], and some of them will be used time and again in the sequel, namely

$$\begin{aligned} &\tau\tau^* + \pi_*(\pi_*)^* = I, \qquad \tau^*\pi = \Delta, \qquad \tau^*\pi_* = 0, \qquad \tau^*\tau_* = -\Theta^*, \qquad \pi = \pi_*\Theta + \tau\Delta, \\ &\tau_*(\tau_*)^* + \pi\pi^* = I, \qquad (\tau_*)^*\pi_* = \Delta_*, \qquad (\tau_*)^*\pi = 0, \qquad (\tau_*)^*\tau = -\Theta, \qquad \pi_* = \pi\Theta^* + \tau_*\Delta_*. \end{aligned} \tag{2.1}$$

We also need the following equalities:

$$\begin{aligned} &\mathcal{G} = \pi H^2(\mathcal{E}), \qquad \mathcal{H} \oplus \mathcal{G} = \pi_* H^2(\mathcal{E}_*) \oplus \tau L^2(\Delta\mathcal{E}), \\ &\mathcal{G}_* = \pi_* H^2_-(\mathcal{E}_*), \qquad \mathcal{H} \oplus \mathcal{G}_* = \pi H^2_-(\mathcal{E}) \oplus \tau_* L^2(\Delta_*\mathcal{E}_*). \end{aligned} \tag{2.2}$$

Now let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be arbitrary completely non-unitary contractions. Let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator intertwining T_1 and T_2, that is, $T_2X = XT_1$. Then the liftings $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X intertwining the minimal unitary dilations of T_1 and T_2 and preserving the outgoing and incoming structure, in the sense that $Y\mathcal{G}_1 \subset \mathcal{G}_2$ and $Y^*\mathcal{G}_{*2} \subset \mathcal{G}_{*1}$, can be parametrized in either of the following forms [6, pp. 252–258]

$$Y = \pi_{*2}A_*(\pi_{*1})^* + \tau_2\Delta_2A\pi_1^* + \tau_2A_0(\tau_{*1})^* = \pi_2A\pi_1^* + \pi_{*2}A_*\Delta_{*1}(\tau_{*1})^* + \tau_2A_0(\tau_{*1})^*,$$

where $z \mapsto A(z) \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ and $z \mapsto A_*(z) \in \mathcal{B}(\mathcal{E}_{*1}, \mathcal{E}_{*2})$ are operator-valued, bounded analytic functions such that $A_*\Theta_1 = \Theta_2A$, and $z \mapsto A_0(z) \in \mathcal{B}(\Delta_{*1}\mathcal{E}_{*1}, \Delta_2\mathcal{E}_2)$ is an operator-valued, bounded measurable function, which can be regarded as a function in $\mathcal{B}(\mathcal{E}_{*1}, \Delta_2\mathcal{E}_2)$ equal to zero on $\operatorname{Ker}\Delta_{*1}$. This parametrization theorem will be essential in our computations.

Lifting quasi-affinities

The four lemmas that we give now tell us how to relate the conditions that define a quasi-affinity to the parameters of any of its liftings. Their complete proof can be found in [2].

Lemma 2.1. *Let $X:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded operator such that $XT_1=T_2X$ and let $Y=\pi_{*2}A_*(\pi_{*1})^*+\tau_2\Delta_2A\pi_1^*+\tau_2A_0(\tau_{*1})^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then $\operatorname{clos}\{X\mathcal{H}_1\}=\mathcal{H}_2$ if and only if*

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0\\ \Delta_2A\Theta_1^*+A_0\Delta_{*1} & \Delta_2 & \Delta_2A\Delta_1-A_0\Theta_1\end{bmatrix}\begin{bmatrix}H^2(\mathcal{E}_{*1})\\ H^2(\mathcal{E}_2)\\ L^2(\Delta_1\mathcal{E}_1)\end{bmatrix}\right\}=\begin{bmatrix}H^2(\mathcal{E}_{*2})\\ L^2(\Delta_2\mathcal{E}_2)\end{bmatrix}. \tag{2.3}$$

Moreover, in this case the operator $[\,A_*\quad \Theta_2\,]$ defined on $H^2(\mathcal{E}_{*1})\oplus H^2(\mathcal{E}_2)$ is outer, that is, its range is dense in $H^2(\mathcal{E}_{*2})$.

The next result gives a condition for the converse of the second part of Lemma 2.1 to hold.

Lemma 2.2. *Let $X:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded operator such that $XT_1=T_2X$ and let $Y=\pi_{*2}A_*(\pi_{*1})^*+\tau_2\Delta_2A\pi_1^*+\tau_2A_0(\tau_{*1})^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. If*

$$\operatorname{clos}\big\{(\Delta_2A\Delta_1-A_0\Theta_1)L^2(\Delta_1\mathcal{E}_1)\big\}=L^2(\Delta_2\mathcal{E}_2), \tag{2.4}$$

then the claim $\operatorname{clos}\{X\mathcal{H}_1\}=\mathcal{H}_2$ is equivalent to the assertion that the function $[\,A_*\quad \Theta_2\,]$ is outer.

Taking into account that $\ker(X)=\{0\}$ if and only if $\operatorname{clos}\{X^*\mathcal{H}_2\}=\mathcal{H}_1$ and that X^* is a compression of Y^*, the following lemmas follow directly from the previous ones.

Lemma 2.3. *Let $X:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded operator such that $XT_1=T_2X$ and let $Y=\pi_2A\pi_1^*+\pi_{*2}A_*\Delta_{*1}(\tau_{*1})^*+\tau_2A_0(\tau_{*1})^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then $\ker(X)=\{0\}$ if and only if*

$$\operatorname{clos}\left\{\begin{bmatrix} A^* & \Theta_1^* & 0\\ \Delta_{*1}A_*^*\Theta_2+A_0^*\Delta_2 & \Delta_{*1} & \Delta_{*1}A_*^*\Delta_{*2}-A_0^*\Theta_2^*\end{bmatrix}\begin{bmatrix}H^2_-(\mathcal{E}_2)\\ H^2_-(\mathcal{E}_{*1})\\ L^2(\Delta_{*2}\mathcal{E}_{*2})\end{bmatrix}\right\}=\begin{bmatrix}H^2_-(\mathcal{E}_1)\\ L^2(\Delta_{*1}\mathcal{E}_{*1})\end{bmatrix}.$$

Moreover, in this case the operator $\left[\begin{smallmatrix}A\\ \Theta_1\end{smallmatrix}\right]$ defined on $H^2(\mathcal{E}_1)$ is $*$-outer, that is, the range of its adjoint $[\,A^*\quad \Theta_1^*\,]$ defined on $H^2_-(\mathcal{E}_2)\oplus H^2_-(\mathcal{E}_{*1})$ is dense in $H^2_-(\mathcal{E}_1)$.

Lemma 2.4. *Let $X:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded operator such that $XT_1=T_2X$ and let $Y=\pi_2A\pi_1^*+\pi_{*2}A_*\Delta_{*1}(\tau_{*1})^*+\tau_2A_0(\tau_{*1})^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. If*

$$\operatorname{clos}\big\{\big(\Delta_{*1}A_*^*\Delta_{*2}-A_0^*\Theta_2^*\big)L^2(\Delta_{*2}\mathcal{E}_{*2})\big\}=L^2(\Delta_{*1}\mathcal{E}_{*1}),$$

then the claim $\ker(X)=\{0\}$ is equivalent to the assertion that the function $\left[\begin{smallmatrix}A\\ \Theta_1\end{smallmatrix}\right]$ is $*$-outer.

3. Proof of the Main Theorem

Main Theorem. *Let $T_i\in\mathcal{B}(\mathcal{H}_i)$ $(i=1,2)$ be completely non-unitary contractions having 2×2 characteristic functions $\Theta_i=w_im_i\left[\begin{smallmatrix}a_i\\ b_i\end{smallmatrix}\right][\,c_i\quad d_i\,]=w_im_i\vartheta_i\varphi_i$. T_1 is quasi-similar to T_2 if and only if the following conditions hold:*

(i) $m_1=m_2=m$,
(ii) $\Omega_1=\Omega_2$ *a.e.*,
(iii) *there exists* $f\in\det(\vartheta_2\to\vartheta_1)^{\mathrm{i}}\cap\det(\varphi_1^{\mathrm{ad}}\to\varphi_2^{\mathrm{ad}})^{\mathrm{i}}$ *such that* $f\wedge m=1$,
(iv) *there exists* $g\in\det(\vartheta_1\to\vartheta_2)^{\mathrm{i}}\cap\det(\varphi_2^{\mathrm{ad}}\to\varphi_1^{\mathrm{ad}})^{\mathrm{i}}$ *such that* $g\wedge m=1$.

The proof of the Main Theorem has been decomposed into a series of lemmas in order to make it more transparent the role of each condition in the network of implications.

Since our main tool will be the coordinate-free function model, we start by describing the functional representations of the residual subspaces for the minimal unitary dilation of an operator T with a characteristic function $\Theta=wm\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right][\,c\quad d\,]$.

If we consider the scalar outer function w as a 1×1 characteristic function, then we have $\Delta_w=\sqrt{1-|w|^2}$ and the corresponding residual subspace can be identified with $L^2(\Delta_w)=\operatorname{clos}\{\Delta_wL^2\}=\chi_\Omega L^2$, where $\Omega:=\{z\in\mathbb{T}\colon\ |w(z)|<1\}$ and χ_Ω is the indicator of the set Ω, i.e., $\chi_\Omega(\zeta)=1$ if $\zeta\in\Omega$ and $\chi_\Omega(\zeta)=0$ otherwise.

Since Θ is a 2×2 matrix with entries in H^∞, we can take as auxiliary spaces $\mathcal{E} = \mathcal{E}_* = \mathbb{C}^2$, therefore $L^2(\mathcal{E}) = L^2(\mathcal{E}_*) = L^2(\mathbb{C}^2) =: L^2_2$, $H^2(\mathcal{E}) = H^2(\mathcal{E}_*) =: H^2_2$ and $H^2_-(\mathcal{E}) = H^2_-(\mathcal{E}_*) = H^2_{2-} := L^2_2 \ominus H^2_2$.

With these, the proof of Lemma 3.1 following below—a straightforward routine computation—is omitted.

Lemma 3.1. *For $\Theta = wm\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right][\,c \quad d\,]$ the corresponding functions Δ and Δ_* in the function model are*

$$\Delta = \begin{bmatrix} d \\ -c \end{bmatrix}[\,\bar{d} \quad -\bar{c}\,] + \Delta_W \begin{bmatrix} \bar{c} \\ \bar{d} \end{bmatrix}[\,c \quad d\,] = \varphi^{\mathrm{ad}}\big(\varphi^{\mathrm{ad}}\big)^* + \Delta_W \varphi^* \varphi$$

and

$$\Delta_* = \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix}[\,b \quad -a\,] + \Delta_W \begin{bmatrix} a \\ b \end{bmatrix}[\,\bar{a} \quad \bar{b}\,] = \big(\vartheta^{\mathrm{ad}}\big)^* \vartheta^{\mathrm{ad}} + \Delta_W \vartheta \vartheta^*,$$

and the corresponding residual subspaces are

$$L^2\big(\Delta\mathbb{C}^2\big) = \begin{bmatrix} d \\ -c \end{bmatrix} L^2 \oplus \begin{bmatrix} \bar{c} \\ \bar{d} \end{bmatrix} L^2(\Delta_W) = \varphi^{\mathrm{ad}} L^2 \oplus \varphi^* \chi_\Omega L^2$$

and

$$L^2\big(\Delta_*\mathbb{C}^2\big) = \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix} L^2 \oplus \begin{bmatrix} a \\ b \end{bmatrix} L^2(\Delta_W) = \big(\vartheta^{\mathrm{ad}}\big)^* L^2 \oplus \vartheta \chi_\Omega L^2.$$

Moreover,

$$\operatorname{clos}\big\{\Delta L^2\big(\Delta\mathbb{C}^2\big)\big\} = L^2\big(\Delta\mathbb{C}^2\big) \quad \textit{and} \quad \operatorname{clos}\big\{\Delta_* L^2\big(\Delta_*\mathbb{C}^2\big)\big\} = L^2\big(\Delta_*\mathbb{C}^2\big).$$

Lemma 3.2. *There exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2 X$ and $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$ if and only if the following conditions hold*:

(i) *m_2 divides m_1,*
(ii) *$\Omega_2 \subseteq \Omega_1$ a.e., and*
(iii) *there exist two inner functions $f, u \in H^\infty$ such that $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}}$, $f \wedge m_2 = 1$ and $\frac{m_1}{m_2} u f \in \det(\varphi_1^{\mathrm{ad}} \to u\varphi_2^{\mathrm{ad}})^{\mathrm{i}}$.*

Proof. We suppose that there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2 X$ and $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$. Let

$$Y = \pi_{*2} A_* \pi_{*1}^* + \tau_2 \Delta_2 A \pi_1^* + \tau_2 A_0 \tau_{*1}^* = \pi_2 A \pi_1^* + \pi_{*2} A_* \Delta_{*1} \tau_{*1}^* + \tau_2 A_0 \tau_{*1}^*$$

be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then the parameters $A, A_* \in H^\infty_{2\times 2}$ satisfy (a) $\Theta_2 A = A_* \Theta_1$ and, according to Lemma 2.1, (b) $[\,A_* \quad \Theta_2\,]$ is outer. Multiply (a) by $\vartheta_2^{\mathrm{ad}}$ on the left and use that $\vartheta_2^{\mathrm{ad}} \vartheta_2 = 0$ and, consequently, that $\vartheta_2^{\mathrm{ad}} \Theta_2 = 0$, to obtain $m_1 w_1 \vartheta_2^{\mathrm{ad}} A_* \vartheta_1 \varphi_1 = 0$. As φ_1 is not a null vector and m_1 and w_1 are not zero a.e., the scalar function $\vartheta_2^{\mathrm{ad}} A_* \vartheta_1$ has to be zero. Analogously, multiplying by φ_1^{ad} on the right and using that $\varphi_1 \varphi_1^{\mathrm{ad}} = 0$ we obtain that $m_2 w_2 \vartheta_2 \varphi_2 A \varphi_1^{\mathrm{ad}} = A_* \Theta_1 \varphi_1^{\mathrm{ad}} = 0$, therefore, $\varphi_2 A \varphi_1^{\mathrm{ad}} = 0$. Since

$$\vartheta_2^{\mathrm{ad}} A_* \vartheta_1 = 0 \quad \text{and} \quad \varphi_2 A \varphi_1^{\mathrm{ad}} = 0,$$

we can use Lemma 5.2 from [2] with the components of the vectors $\vartheta_2^{\mathrm{ad}} A_*$, $A_* \vartheta_1$, $\varphi_2 A$ and $A \varphi_1^{\mathrm{ad}}$ to get four functions $f_1, f_2, f_3, f_4 \in H^\infty$ such that

$$\vartheta_2^{\mathrm{ad}} A_* = f_1 \vartheta_1^{\mathrm{ad}}, \qquad A_* \vartheta_1 = f_2 \vartheta_2, \qquad \varphi_2 A = f_3 \varphi_1, \quad \text{and} \quad A \varphi_1^{\mathrm{ad}} = f_4 \varphi_2^{\mathrm{ad}}. \tag{3.1}$$

Thus

$$(\det A_*) \vartheta_1 = A_*^{\mathrm{ad}} A_* \vartheta_1 = f_2 A_*^{\mathrm{ad}} \vartheta_2 = f_1 f_2 \vartheta_1$$

and

$$(\det A) \varphi_1^{\mathrm{ad}} = A^{\mathrm{ad}} A \varphi_1^{\mathrm{ad}} = f_4 A^{\mathrm{ad}} \varphi_2^{\mathrm{ad}} = f_4 f_3 \varphi_1^{\mathrm{ad}},$$

then we have

$$\det A_* = f_1 f_2 \quad \text{and} \quad \det A = f_3 f_4.$$

Making use of the equality

$$m_1 w_1 f_2 \vartheta_2 \varphi_1 = m_1 w_1 A_* \vartheta_1 \varphi_1 = A_* \Theta_1 = \Theta_2 A = m_2 w_2 \vartheta_2 \varphi_2 A = m_2 w_2 f_3 \vartheta_2 \varphi_1,$$

we have, multiplying by ϑ_2^* on the left, by φ_1^* on the right and using $\vartheta_2^*\vartheta_2 = 1 = \varphi_1\varphi_1^*$, that

$$m_1 w_1 f_2 = m_2 w_2 f_3. \tag{3.2}$$

On the other hand, as c_2 and d_2 are relatively prime (i.e. have no common inner factor), we have that $[\,c_2 \quad d_2\,]$ is outer (see the properties of inner and outer matrices of functions in [4,5] or [7]) and, consequently, $\operatorname{clos}\{w_2\varphi_2 H_2^2\} = H^2$. Now, using (b), we have

$$\begin{aligned} H_2^2 &= \operatorname{clos}\{[\,A_* \quad \Theta_2\,]H_4^2\} = \operatorname{clos}\{A_* H_2^2 + \Theta_2 H_2^2\} = \operatorname{clos}\{A_* H_2^2 + m_2 w_2 \vartheta_2 \varphi_2 H_2^2\} \\ &= \operatorname{clos}\{A_* H_2^2 + m_2\vartheta_2 \operatorname{clos}\{w_2\varphi_2 H_2^2\}\} = \operatorname{clos}\{[\,A_* \quad m_2\vartheta_2\,]H_3^2\}. \end{aligned}$$

Therefore, the matrix

$$[\,A_* \quad m_2\vartheta_2\,] = \begin{bmatrix} a_{*11} & a_{*12} & m_2 a_2 \\ a_{*21} & a_{*22} & m_2 b_2 \end{bmatrix}$$

is outer, in consequence, the three 2×2 minors are relatively prime or, equivalently, all the components of the vector

$$[\,\det A_* \quad m_2(b_2 a_{*11} - a_2 a_{*21}) \quad m_2(b_2 a_{*12} - a_2 a_{*22})\,] = [\,\det A_* \quad m_2 \vartheta_2^{\mathrm{ad}} A_*\,] = [\,f_1 f_2 \quad m_2 f_1 \vartheta_1^{\mathrm{ad}}\,]$$

are relatively prime. In particular, f_1 is an outer function and $f_2 \wedge m_2 = 1$. Using this in (3.2) we deduce that m_2 divides m_1. Let us point out here that the function f we are looking for is the inner part of f_2.

Let us see that $\Omega_2 \subseteq \Omega_1$ a.e. Since $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$, Lemma 2.1 tells us, using $\mathcal{E}_i = \mathcal{E}_{*i} = \mathbb{C}^2$ for $i = 1, 2$, that

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_2 A \Theta_1^* + A_0 \Delta_{*1} & \Delta_2 & \Delta_2 A \Delta_1 - A_0 \Theta_1 \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \\ L^2(\Delta_1 \mathbb{C}^2) \end{bmatrix}\right\} = \begin{bmatrix} H_2^2 \\ L^2(\Delta_2 \mathbb{C}^2) \end{bmatrix}. \tag{3.3}$$

Taking into account that, by Lemma 3.1,

$$\varphi_i L^2(\Delta_i \mathbb{C}^2) = \varphi_i(\varphi_i^{\mathrm{ad}} L^2 \oplus \varphi_i^* \chi_{\Omega_i} L^2) = \chi_{\Omega_i} L^2 \quad \text{for } i = 1, 2, \tag{3.4}$$

we have that $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \varphi_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} H^2 \\ L^2(\Delta_2\mathbb{C}^2) \end{smallmatrix}\right] = \left[\begin{smallmatrix} H^2 \\ \chi_{\Omega_2} L^2 \end{smallmatrix}\right]$ is a closed subspace. Therefore, if we apply the operator $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \varphi_2 \end{smallmatrix}\right]$ to the equality (3.3) above, we obtain

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0 \\ \varphi_2(\Delta_2 A \Theta_1^* + A_0 \Delta_{*1}) & \varphi_2 \Delta_2 & \varphi_2(\Delta_2 A \Delta_1 - A_0 \Theta_1) \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \\ L^2(\Delta_1 \mathbb{C}^2) \end{bmatrix}\right\} = \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2} L^2 \end{bmatrix}$$

which, using that

$$\varphi_i \Delta_i = \varphi_i(\varphi_i^{\mathrm{ad}}(\varphi_i^{\mathrm{ad}})^* + \Delta_{w_i}\varphi_i^*\varphi_i) = \Delta_{w_i}\varphi_i \quad \text{for } i = 1, 2,$$

is equivalent to

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_{w_2}\varphi_2 A \Theta_1^* + \varphi_2 A_0 \Delta_{*1} & \Delta_{w_2}\varphi_2 & \Delta_{w_2}\varphi_2 A \Delta_1 - \varphi_2 A_0 \Theta_1 \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \\ L^2(\Delta_1 \mathbb{C}^2) \end{bmatrix}\right\} = \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2} L^2 \end{bmatrix}.$$

Since $\varphi_2 A = f_3 \varphi_1$ and $\varphi_1 \Delta_1 = \Delta_{w_1}\varphi_1$, we have

$$\varphi_2 A \Theta_1^* = \varphi_2 A \overline{m}_1 \overline{w}_1 \varphi_1^* \vartheta_1^* = f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^*$$

and

$$\Delta_{w_2}\varphi_2 A \Delta_1 = \Delta_{w_2} f_3 \varphi_1 \Delta_1 = \Delta_{w_2} f_3 \Delta_{w_1}\varphi_1,$$

therefore, the space above can be written as

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_{w_2} f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^* + \varphi_2 A_0 \Delta_{*1} & \Delta_{w_2}\varphi_2 & (\Delta_{w_2} f_3 \Delta_{w_1} - m_1 w_1 \varphi_2 A_0 \vartheta_1)\varphi_1 \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \\ L^2(\Delta_1 \mathbb{C}^2) \end{bmatrix}\right\}.$$

Now, using that, by (3.4), $\varphi_1 L^2(\Delta_1\mathbb{C}^2) = \chi_{\Omega_1}L^2$, that the range of A_0 is included in $L^2(\Delta_2\mathbb{C}^2)$ and, hence, the range of $\varphi_2 A_0$ is included in $\varphi_2 L^2(\Delta_2\mathbb{C}^2) = \chi_{\Omega_2}L^2$, we conclude that

$$(\Delta_{w_2} f_3 \Delta_{w_1} - m_1 w_1 \varphi_2 A_0 \vartheta_1)\varphi_1 L^2(\Delta_1\mathbb{C}^2) = (\Delta_{w_2} f_3 \Delta_{w_1} - m_1 w_1 \varphi_2 A_0 \vartheta_1)\chi_{\Omega_1}L^2 \subseteq \chi_{\Omega_1\cap\Omega_2}L^2 = \chi_{\Omega_2}\chi_{\Omega_1}L^2.$$

Taking into account the orthogonal decomposition

$$\begin{bmatrix} H_2^2 \\ \chi_{\Omega_2}L^2 \end{bmatrix} = \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2\setminus(\Omega_1\cap\Omega_2)}L^2 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \chi_{\Omega_2}\chi_{\Omega_1}L^2 \end{bmatrix},$$

if we apply to the last equality the orthogonal projection from $\left[\begin{smallmatrix} H_2^2 \\ \chi_{\Omega_2}L^2 \end{smallmatrix}\right]$ onto the space

$$\begin{bmatrix} H_2^2 \\ \chi_{\Omega_2\setminus(\Omega_1\cap\Omega_2)}L^2 \end{bmatrix} = \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2}(1-\chi_{\Omega_1})L^2 \end{bmatrix},$$

whose matrix is $\left[\begin{smallmatrix} I & 0 \\ 0 & (1-\chi_{\Omega_1}) \end{smallmatrix}\right]$, we obtain

$$\begin{bmatrix} H_2^2 \\ \chi_{\Omega_2\setminus(\Omega_1\cap\Omega_2)}L^2 \end{bmatrix} = \operatorname{clos}\left\{ \begin{bmatrix} I & 0 \\ 0 & (1-\chi_{\Omega_1}) \end{bmatrix} \begin{bmatrix} A_* & \Theta_2 \\ \Delta_{w_2} f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^* + \varphi_2 A_0 \Delta_{*1} & \Delta_{w_2}\varphi_2 \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \end{bmatrix} \right\},$$

because $(\Delta_{w_2} f_3 \Delta_{w_1} - \varphi_2 A_0 \vartheta_1)\varphi_1 L^2(\Delta_1\mathbb{C}^2) \subseteq \chi_{\Omega_2}\chi_{\Omega_1}L^2$. Since $\operatorname{clos}\{\varphi_2 H_2^2\} = H^2$ we have

$$\begin{aligned} \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2\setminus(\Omega_1\cap\Omega_2)}L^2 \end{bmatrix} &= \operatorname{clos}\left\{ \begin{bmatrix} I & 0 \\ 0 & 1-\chi_{\Omega_1} \end{bmatrix} \begin{bmatrix} A_* & \Theta_2 \\ \Delta_{w_2} f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^* + \varphi_2 A_0 \Delta_{*1} & \Delta_{w_2}\varphi_2 \end{bmatrix} \begin{bmatrix} H_2^2 \\ H_2^2 \end{bmatrix} \right\} \\ &= \operatorname{clos}\left\{ \begin{bmatrix} I & 0 \\ 0 & 1-\chi_{\Omega_1} \end{bmatrix} \begin{bmatrix} A_* & m_2 w_2 \vartheta_2 \\ \Delta_{w_2} f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^* + \varphi_2 A_0 \Delta_{*1} & \Delta_{w_2} \end{bmatrix} \begin{bmatrix} H_2^2 \\ H^2 \end{bmatrix} \right\}. \end{aligned}$$

If we multiply the matrices and use that $\vartheta_1\vartheta_1^* + (\vartheta_1^{\mathrm{ad}})^*\vartheta_1^{\mathrm{ad}} = I$, that $w_1 H_2^2$ is dense in H_2^2 and that

$$(1-\chi_{\Omega_1})\Delta_{*1} = (1-\chi_{\Omega_1})\big(\big(\vartheta_1^{\mathrm{ad}}\big)^*\vartheta_1^{\mathrm{ad}} + \Delta_{w_1}\vartheta_1\vartheta_1^*\big) = (1-\chi_{\Omega_1})\big(\vartheta_1^{\mathrm{ad}}\big)^*\vartheta_1^{\mathrm{ad}},$$

the last equality can be written as

$$\begin{aligned} \begin{bmatrix} H_2^2 \\ \chi_{\Omega_2\setminus(\Omega_1\cap\Omega_2)}L^2 \end{bmatrix} &= \operatorname{clos}\left\{ \begin{bmatrix} A_*[\vartheta_1\vartheta_1^* + (\vartheta_1^{\mathrm{ad}})^*\vartheta_1^{\mathrm{ad}}] & m_2 w_2 \vartheta_2 \\ (1-\chi_{\Omega_1})(\Delta_{w_2} f_3 \overline{m}_1 \overline{w}_1 \vartheta_1^* + \varphi_2 A_0 (\vartheta_1^{\mathrm{ad}})^*\vartheta_1^{\mathrm{ad}}) & (1-\chi_{\Omega_1})\Delta_{w_2} \end{bmatrix} \begin{bmatrix} w_1 H_2^2 \\ H^2 \end{bmatrix} \right\} \\ &= \operatorname{clos}\left\{ \left[\begin{bmatrix} m_2 w_2 \vartheta_2 \\ (1-\chi_{\Omega_1})\Delta_{w_2} \end{bmatrix} [\, f_3 \overline{m}_1 \vartheta_1^* \quad 1\,] + \begin{bmatrix} A_*(\vartheta_1^{\mathrm{ad}})^* \\ (1-\chi_{\Omega_1})\varphi_2 A_0 (\vartheta_1^{\mathrm{ad}})^* \end{bmatrix} [\, w_1 \vartheta_1^{\mathrm{ad}} \quad 0\,] \right] H_3^2 \right\}, \end{aligned}$$

where we have also used that $(1-\chi_{\Omega_1})|w_1|^2 = (1-\chi_{\Omega_1})$ and, from (3.1) and (3.2), that

$$m_2 w_2 f_3 \vartheta_2 \overline{m}_1 \vartheta_1^* = w_1 f_2 \vartheta_2 \vartheta_1^* = w_1 A_* \vartheta_1 \vartheta_1^*.$$

Since the matrix above acting on H_3^2 is the sum of two rank one matrices, its rank must be at most two, thus $\chi_{\Omega_2\setminus\Omega_1\cap\Omega_2}L^2 = \{0\}$ or, equivalently, $\Omega_2 \subseteq \Omega_1$ almost everywhere.

Finally, taking $f = f_2^{\mathrm{i}}$, the inner part of f_2, we have $f \wedge m_2 = 1$. Moreover, since f_1 is an outer function, if f_4^{o} is the outer part of f_4, we have from (3.1) and (3.2) that

$$\frac{1}{f_1}A_*^{\mathrm{ad}},\ \frac{1}{f_4^{\mathrm{o}}}A \in \mathcal{N}_{2\times 2}^+, \qquad \frac{1}{f_1}A_*^{\mathrm{ad}}\vartheta_2 = \vartheta_1, \quad \text{and} \quad \frac{1}{f_4^{\mathrm{o}}}A\varphi_1^{\mathrm{ad}} = f_4^{\mathrm{i}}\varphi_2^{\mathrm{ad}}$$

with

$$\begin{aligned} &\left(\det\left(\frac{1}{f_1}A_*^{\mathrm{ad}}\right)\right)^{\mathrm{i}} = \big(\det A_*^{\mathrm{ad}}\big)^{\mathrm{i}} = (\det A_*)^{\mathrm{i}} = (f_1 f_2)^{\mathrm{i}} = f_2^{\mathrm{i}} = f \quad \text{and} \\ &\left(\det\left(\frac{1}{f_4^{\mathrm{o}}}A\right)\right)^{\mathrm{i}} = (\det A)^{\mathrm{i}} = (f_3 f_4)^{\mathrm{i}} = \frac{m_1}{m_2} f_2^{\mathrm{i}} f_4^{\mathrm{i}} = \frac{m_1}{m_2} f f_4^{\mathrm{i}}, \end{aligned}$$

thus $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}}$ and, taking $u = f_4^{\mathrm{i}}$, $\frac{m_1}{m_2}uf \in \det(\varphi_1^{\mathrm{ad}} \to u\varphi_2^{\mathrm{ad}})^{\mathrm{i}}$. This finishes the proof that the conditions are necessary.

Now, we suppose that m_2 divides m_1, that $\Omega_2 \subseteq \Omega_1$ a.e., and that there exist two inner functions $f, u \in H^\infty$ such that

$$f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}}, \qquad f \wedge m_2 = 1, \quad \text{and} \quad \frac{m_1}{m_2}uf \in \det\big(\varphi_1^{\mathrm{ad}} \to u\varphi_2^{\mathrm{ad}}\big)^{\mathrm{i}}.$$

We will prove that there exists a bounded operator $X:\mathcal{H}_1\to\mathcal{H}_2$ such that $XT_1=T_2X$ and $\operatorname{clos}\{X\mathcal{H}_1\}=\mathcal{H}_2$, by finding an adequate parametrization to produce a suitable lifting Y of X. According to Lemma 2.2 we need to build a lifting

$$Y=\pi_{*2}A_*(\pi_{*1})^*+\tau_2\Delta_2A\pi_1^*+\tau_2A_0(\tau_{*1})^*$$

whose parameters satisfy the hypothesis of that lemma. Those conditions are:

(1) $\Theta_2A=A_*\Theta_1$,
(2) $[\,A_*\quad\Theta_2\,]$ outer,
(3) $\operatorname{clos}\{(\Delta_2A\Delta_1-A_0\Theta_1)L^2(\Delta_1\mathbb{C}^2)\}=L^2(\Delta_2\mathbb{C}^2)$,

where $A, A_*\in H^\infty_{2\times2}$.

Since there exists a function $f\in\det(\vartheta_2\to\vartheta_1)^{\mathrm i}$ such that $f\wedge m_2=1$, it follows that there exists $\Lambda\in\mathcal{N}^+_{2\times2}$ satisfying $\Lambda\vartheta_2=\vartheta_1$ and $(\det\Lambda)^{\mathrm i}=f$. Let λ_* be an outer function such that $\lambda_*\Lambda\in H^\infty_{2\times2}$ and $\lambda_*\det\Lambda\in H^\infty$. If we denote $M^{\mathrm{ad}}=\lambda_*\Lambda$, we have $M^{\mathrm{ad}}\vartheta_2=\lambda_*\vartheta_1$. Let $f_1=(\lambda_*\det\Lambda)^{\mathrm o}$ be the outer part of $\lambda_*\det\Lambda$. Then

$$\lambda_*\det\Lambda=(\lambda_*\det\Lambda)^{\mathrm i}(\lambda_*\det\Lambda)^{\mathrm o}=ff_1,$$
$$\det M=\det M^{\mathrm{ad}}=\lambda_*^2\det\Lambda=\lambda_*ff_1,$$

and

$$M(\lambda_*\vartheta_1)=M\big(M^{\mathrm{ad}}\vartheta_2\big)=\big(\det M^{\mathrm{ad}}\big)\vartheta_2,$$

consequently

$$M\vartheta_1=\frac{\det M^{\mathrm{ad}}}{\lambda_*}\vartheta_2=ff_1\vartheta_2.$$

Let h be an inner function such that $m_1=hm_2$. Since $huf\in\det(\varphi_1^{\mathrm{ad}}\to u\varphi_2^{\mathrm{ad}})^{\mathrm i}$ we have, analogously, a matrix $\Gamma\in\mathcal{N}^+_{2\times2}$ such that $\Gamma\varphi_1^{\mathrm{ad}}=u\varphi_2^{\mathrm{ad}}$ and $(\det\Gamma)^{\mathrm i}=huf$. Thus there exists an outer function λ such that $N=\lambda\Gamma\in H^\infty_{2\times2}$, $\lambda\det\Gamma\in H^\infty$, $N\varphi_1^{\mathrm{ad}}=\lambda u\varphi_2^{\mathrm{ad}}$ and $\det N=\lambda^2\det\Gamma=\lambda(\lambda\det\Gamma)=\lambda huff_2$, where $f_2=(\lambda\det\Gamma)^{\mathrm o}$. Moreover, it follows that $N^{\mathrm{ad}}\varphi_2^{\mathrm{ad}}=hff_2\varphi_1^{\mathrm{ad}}$ and, therefore, $\varphi_2N=hff_2\varphi_1$.

We choose $A_*=w_2f_2M$ and $A=w_1f_1N$. Let us check that our three conditions hold.

(1) The equality $\Theta_2A=A_*\Theta_1$ holds because

$$A_*\Theta_1=w_2f_2Mm_1w_1\vartheta_1\varphi_1=m_1w_1w_2f_2(M\vartheta_1)\varphi_1=m_1w_1w_2f_2(ff_1\vartheta_2)\varphi_1$$

and

$$\Theta_2A=m_2w_2\vartheta_2\varphi_2w_1f_1N=m_2w_1w_2f_1\vartheta_2(\varphi_2N)=m_2w_1w_2f_1\vartheta_2(hff_2\varphi_1).$$

(2) To prove that $[\,A_*\quad\Theta_2\,]$ is outer, we will check that

$$\operatorname{clos}\big\{[\,A_*\quad\Theta_2\,]H^2_4\big\}=H^2_2.$$

Now, since w_2, f_2 and φ_2 are outer functions and $A_*=w_2f_2M$, we have

$$\operatorname{clos}\big\{[\,A_*\quad\Theta_2\,]H^2_4\big\}=\operatorname{clos}\big\{[\,M\quad m_2\vartheta_2\,]H^2_3\big\},$$

consequently, it is enough to prove that $[\,M\quad m_2\vartheta_2\,]$ is outer or, equivalently, that the three 2×2 minors have no common inner divisors or, in other words, that the components of the vector

$$[\,\det M\quad m_2\vartheta_2^{\mathrm{ad}}M\,]=[\,\lambda_*ff_1\quad m_2\lambda_*\vartheta_1^{\mathrm{ad}}\,]=[\,\lambda_*ff_1\quad m_2\lambda_*b_1\quad -m_2\lambda_*a_1\,]$$

have no common inner divisors. But this is true because $f\wedge m_2=1$, a_1 and b_1 have no common inner divisor, and λ_* and f_1 are outer functions.

(3) To check the third condition we need to specify the parameter A_0. We take $A_0=a_0\chi_{\Omega_2}\varphi_2^*\vartheta_1^*$, where a_0 is chosen depending on f, namely, we put $a_0=0$ if $f\neq0$ and $a_0=1$ if $f=0$. Since $L^2(\Delta_i\mathbb{C}^2)=\varphi_i^{\mathrm{ad}}L^2\oplus\varphi_i^*\chi_{\Omega_i}L^2$, we can rewrite the required equality as follows

$$\operatorname{clos}\big\{(\Delta_2A\Delta_1-A_0\Theta_1)\varphi_1^{\mathrm{ad}}L^2,\ (\Delta_2A\Delta_1-A_0\Theta_1)\varphi_1^*\chi_{\Omega_1}L^2\big\}=\varphi_2^{\mathrm{ad}}L^2\oplus\varphi_2^*\chi_{\Omega_2}L^2. \tag{3.5}$$

Let us consider the first term. Using formulas from Lemma 3.1 and the definition of the functions A and A_0 we get

$$\operatorname{clos}\big\{(\Delta_2A\Delta_1-A_0\Theta_1)\varphi_1^{\mathrm{ad}}L^2\big\}=\operatorname{clos}\big\{w_1f_1\Delta_2N\varphi_1^{\mathrm{ad}}L^2\big\}=\operatorname{clos}\big\{w_1f_1\lambda u\Delta_2\varphi_2^{\mathrm{ad}}L^2\big\}=\operatorname{clos}\big\{w_1f_1\lambda u\varphi_2^{\mathrm{ad}}L^2\big\}=\varphi_2^{\mathrm{ad}}L^2.$$

Thus, to prove (3.5) it is sufficient to check that the orthogonal projection of the second term in the left-hand side of (3.5) onto $\varphi_2^*\chi_{\Omega_2}L^2$ gives the whole subspace, i.e., that

$$\varphi_2^*\chi_{\Omega_2}\varphi_2 \operatorname{clos}\{(\Delta_2 A\Delta_1 - A_0\Theta_1)\varphi_1^*\chi_{\Omega_1}L^2\} = \varphi_2^*\chi_{\Omega_2}L^2.$$

Let us check this identity:

$$\begin{aligned}\varphi_2^*\chi_{\Omega_2}\varphi_2 \operatorname{clos}\{(\Delta_2 A\Delta_1 - A_0\Theta_1)\varphi_1^*\chi_{\Omega_1}L^2\} &= \varphi_2^* \operatorname{clos}\{\chi_{\Omega_2}[(\varphi_2\Delta_2)(w_1 f_1 N)(\Delta_1\varphi_1^*) - \varphi_2(a_0\varphi_2^*\vartheta_1^*)(m_1 w_1\vartheta_1\varphi_1)\varphi_1^*]L^2\}\\ &= \varphi_2^* \operatorname{clos}\{\chi_{\Omega_2}[w_1 f_1 \Delta_{w_2}\Delta_{w_1}\varphi_2 N\varphi_1^* - a_0 m_1 w_1]L^2\}\\ &= \varphi_2^* \operatorname{clos}\{\chi_{\Omega_2}[w_1 f_1 \Delta_{w_2}\Delta_{w_1} h f f_2 - a_0 m_1 w_1]L^2\}.\end{aligned}$$

Note that the function within the brackets is different from zero almost everywhere on Ω_2. Indeed, since $\Omega_2 \subset \Omega_1$ and $\Delta_i \neq 0$ on Ω_i, all the functions in the first summand are different from zero on Ω_2, except possibly the function f. If $f \neq 0$, then being an analytic function in the unit disc, f is different from zero a.e. on the circle, and we have non-zero first summand with the second equal to zero, because, in this case, we took $a_0 = 0$. If, on the other hand, $f = 0$, then the second summand is nonzero. In either case we have that $\varphi_2^*\chi_{\Omega_2}L^2$, which is what we need. □

Lemma 3.3. *There exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$ and $\ker(X) = \{0\}$ if and only if the following conditions hold*:

(i) *m_1 divides m_2,*
(ii) *$\Omega_1 \subseteq \Omega_2$ a.e., and*
(iii) *there exist two inner functions $g, v \in H^\infty$ such that $g \in \det(\varphi_1^{\mathrm{ad}} \to \varphi_2^{\mathrm{ad}})^{\mathrm{i}}$, $g \wedge m_1 = 1$, and $\frac{m_2}{m_1} v g \in \det(\vartheta_2 \to v\vartheta_1)^{\mathrm{i}}$.*

Proof. To consider an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$ and $\ker(X) = \{0\}$ we apply Lemma 3.2 to the operator $X^* : \mathcal{H}_2 \to \mathcal{H}_1$, for which we have $T_1^* X^* = X^* T_2^*$ and $\operatorname{clos}\{X^*\mathcal{H}_2\} = \mathcal{H}_1$.

If we denote $\overline{\Omega} = \{\bar{z}:\ z \in \Omega\}$ for a domain Ω and $\widetilde{A}(z) = A^*(\bar{z})$ for any operator-valued analytic function A then the characteristic functions of T_i^* are $\widetilde{\Theta}_i = \widetilde{m}_i \widetilde{w}_i \widetilde{\varphi}_i \widetilde{\vartheta}_i$ and, for the corresponding sets Ω_i, $\overline{\Omega}_i$ are the supports of the functions Δ_{w_i}. According to Lemma 3.2 the existence of such operator X^* is equivalent to the conditions:

(1) $\widetilde{m}_1$ divides $\widetilde{m}_2$,
(2) $\overline{\Omega}_1 \subseteq \overline{\Omega}_2$ a.e., and
(3) there exist two inner functions $f, u \in H^\infty$ such that $f \in \det(\widetilde{\varphi}_1 \to \widetilde{\varphi}_2)^{\mathrm{i}}$, $f \wedge \widetilde{m}_1 = 1$ and $\frac{\widetilde{m}_2}{\widetilde{m}_1} u f \in \det((\widetilde{\vartheta}_2)^{\mathrm{ad}} \to u(\widetilde{\vartheta}_1)^{\mathrm{ad}})^{\mathrm{i}}$.

It is clear that $\widetilde{m}_1$ divides $\widetilde{m}_2$ if and only if m_1 divides m_2 and that $\overline{\Omega}_1 \subseteq \overline{\Omega}_2$ if and only if $\Omega_1 \subseteq \Omega_2$. Finally, it is easy to see that (iii) and (3) are equivalent by taking $g = \widetilde{f}$ and $v = \widetilde{u}$. □

Lemma 3.4. *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ be completely non-unitary contractions having 2×2 characteristic functions $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} [\,c_i \quad d_i\,] = w_i m_i \vartheta_i \varphi_i$. There exists a bounded operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying*

$$XT_1 = T_2X, \qquad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \quad \textit{and} \quad \ker(X) = \{0\}$$

if and only if the following conditions hold:

(i) *$m_1 = m_2 = m$,*
(ii) *$\Omega_1 = \Omega_2 = \Omega$ a.e., and*
(iii) *there exists $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}} \cap \det(\varphi_1^{\mathrm{ad}} \to \varphi_2^{\mathrm{ad}})^{\mathrm{i}}$ such that $f \wedge m = 1$.*

Proof. We suppose that there exists a bounded operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ satisfying

$$XT_1 = T_2X, \qquad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \quad \text{and} \quad \ker(X) = \{0\}.$$

Using Lemmas 3.2 and 3.3, we know that

(i) $m_2 = m_1 = m$,
(ii) $\Omega_2 = \Omega_1$ a.e., and
(iii) there exist two inner functions $f, u \in H^\infty$ such that $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}}$, $f \wedge m = 1$ and $uf \in \det(\varphi_1^{\mathrm{ad}} \to u\varphi_2^{\mathrm{ad}})^{\mathrm{i}}$.

Starting as in the proof of Lemma 3.2, there exist four functions $f_1, f_2, f_3, f_4 \in H^\infty$ such that the parameters $A, A_* \in H^\infty_{2\times 2}$ of the lifting of X satisfy (3.1), i.e.,

$$\vartheta_2^{\mathrm{ad}} A_* = f_1 \vartheta_1^{\mathrm{ad}}, \qquad A_* \vartheta_1 = f_2 \vartheta_2, \qquad \varphi_2 A = f_3 \varphi_1, \quad \text{and} \quad A\varphi_1^{\mathrm{ad}} = f_4 \varphi_2^{\mathrm{ad}},$$

where, moreover, $u = f_4^{\mathrm{i}}$.

Now, according to Lemma 2.3, the matrix $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer, hence

$$H_2^2 = \operatorname{clos}\{[\,A^T \quad \Theta_1^T\,]H_4^2\} = \operatorname{clos}\{A^T H_2^2 + m_1 w_1 \varphi_1^T \vartheta_1^T H_2^2\}.$$

As w_1 is outer and $a_1 \wedge b_1 = 1$, we have $\operatorname{clos}\{w_1 \vartheta_1^T H_2^2\} = H^2$ and, therefore,

$$H_2^2 = \operatorname{clos}\{[\,A^T \quad \Theta_1^T\,]H_4^2\} = \operatorname{clos}\{A^T H_2^2 + m_1 \varphi_1^T \operatorname{clos}\{w_1 \vartheta_1^T H_2^2\}\} = \operatorname{clos}\{[\,A^T \quad m_1 \varphi_1^T\,]H_3^2\},$$

thus the matrix

$$[\,A^T \quad m_1 \varphi_1^T\,] = \begin{bmatrix} a_{11} & a_{21} & m_1 c_1 \\ a_{12} & a_{22} & m_1 d_1 \end{bmatrix}$$

is outer and, consequently, the three components of the vector

$$\begin{aligned}[\,\det A^T \quad m_1(d_1 a_{11} - c_1 a_{12}) \quad m_1(d_1 a_{21} - c_1 a_{22})\,] &= [\,\det A \quad m_1(\varphi_1^T)^{\mathrm{ad}} A^T\,] = [\,\det A \quad m_1 (A\varphi_1^{\mathrm{ad}})^T\,] \\ &= [\,f_3 f_4 \quad m_1 f_4 (\varphi_2^{\mathrm{ad}})^T\,]\end{aligned}$$

have no common inner divisor. In particular, f_4 is an outer function. We conclude that $u = f_4^{\mathrm{i}} = 1$ and, therefore, $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}} \cap \det(\varphi_1^{\mathrm{ad}} \to \varphi_2^{\mathrm{ad}})^{\mathrm{i}}$. This finishes the proof that the conditions are necessary.

To prove that the conditions are sufficient we will use again the proof of Lemma 3.2. Bearing in mind that $u = 1$ and $\frac{m_1}{m_2} = 1$, take the parameters for the lifting of X as chosen in that lemma, that is, $A_* = w_2 f_2 M$ and $A = w_1 f_1 N$, where $M, N \in H^\infty_{2\times 2}$ satisfy

$$\begin{aligned}&M^{\mathrm{ad}} \vartheta_2 = \lambda_* \vartheta_1, \qquad \det M = \lambda_* f f_1, \qquad M\vartheta_1 = f f_1 \vartheta_2, \\ &N\varphi_1^{\mathrm{ad}} = \lambda \varphi_2^{\mathrm{ad}}, \qquad \det N = \lambda f f_2, \qquad \varphi_2 N = f f_2 \varphi_1,\end{aligned}$$

with $f_1, f_2, \lambda_*, \lambda \in H^\infty$ being outer functions.

According to Lemmas 3.2 and 3.4, we have to prove that $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer and that the following equality holds $\operatorname{clos}\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*) L^2(\Delta_{*2}\mathbb{C}^2)\} = L^2(\Delta_{*1}\mathbb{C}^2)$, where $A_0 = a_0 \chi_\Omega \varphi_2^* \vartheta_1^*$, and we choose $a_0 = 0$ if f is not a null function and $a_0 = 1$ otherwise.

To show that $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer, it is enough to prove that $[\,A^T \quad \Theta_1^T\,]$ is outer. Now, since $f \wedge m = 1$, $c_2 \wedge d_2 = 1$, and the functions λ and f_2 are outer, it follows that the elements of the vector

$$[\,\det N^T \quad m(\varphi_1^T)^{\mathrm{ad}} N^T\,] = [\,\lambda f f_2 \quad m\lambda(\varphi_2^{\mathrm{ad}})^T\,] = [\,\lambda f f_2 \quad m\lambda d_2 \quad -m\lambda c_2\,]$$

have no common inner divisor. This implies that

$$\operatorname{clos}\{[\,N^T \quad m\varphi_1^T\,]H_3^2\} = H_2^2.$$

Therefore, since $A = w_1 f_1 N$, w_1 and f_1 are outer functions and $\operatorname{clos}\{\vartheta_1^T H_2^2\} = H^2$, we conclude that $[\,A^T \quad \Theta_1^T\,]$ is outer.

Using the functional representations given in Lemma 3.1, we reformulate the required identity $\operatorname{clos}\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*) L^2(\Delta_{*2}\mathbb{C}^2)\} = L^2(\Delta_{*1}\mathbb{C}^2)$ as follows

$$\operatorname{clos}\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)(\vartheta_2^{\mathrm{ad}})^* L^2,\ (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)\vartheta_2 \chi_\Omega L^2\} = (\vartheta_1^{\mathrm{ad}})^* L^2 \oplus \vartheta_1 \chi_\Omega L^2. \tag{3.6}$$

For the first term we have

$$\begin{aligned}\operatorname{clos}\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)(\vartheta_2^{\mathrm{ad}})^* L^2\} &= \operatorname{clos}\{\Delta_{*1} \overline{w}_2 \overline{f}_2 M^* (\vartheta_2^{\mathrm{ad}})^* L^2\} = \operatorname{clos}\{\Delta_{*1} \overline{w}_2 \overline{f}_2 (\vartheta_2^{\mathrm{ad}} M)^* L^2\} \\ &= \operatorname{clos}\{\Delta_{*1} \overline{w}_2 \overline{f}_2 (\lambda_* \vartheta_1^{\mathrm{ad}})^* L^2\} = (\vartheta_1^{\mathrm{ad}})^* L^2.\end{aligned}$$

And the projection onto the second component of the second term in (3.6) gives us

$$\begin{aligned}\operatorname{clos}\{\vartheta_1 \chi_\Omega \vartheta_1^* (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)\vartheta_2 \chi_\Omega L^2\} &= \vartheta_1 \chi_\Omega \operatorname{clos}\{(\Delta_{w_1} \Delta_{w_2} \overline{w}_2 \overline{f}_2 \vartheta_1^* M^* \vartheta_2 - \vartheta_1^* a_0 \vartheta_1 \varphi_2 \overline{m}\overline{w}_2 \varphi_2^* \vartheta_2^* \vartheta_2) L^2\} \\ &= \vartheta_1 \chi_\Omega \operatorname{clos}\{(\Delta_{w_1} \Delta_{w_2} \overline{w}_2 \overline{f}_2 \overline{f}_1 \overline{f} - a_0 \overline{m}\overline{w}_2) L^2\} \\ &= \vartheta_1 \chi_\Omega L^2.\end{aligned}$$

This finishes the proof of the lemma. □

Finally, let us note that Lemma 3.4 directly implies the Main Theorem.

4. Concluding remarks

The conditions (iii) and (iv) in the Main Theorem, namely,

(iii) there exists $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}} \cap \det(\varphi_1^{\mathrm{ad}} \to \varphi_2^{\mathrm{ad}})^{\mathrm{i}}$ such that $f \wedge m = 1$, and
(iv) there exists $g \in \det(\vartheta_1 \to \vartheta_2)^{\mathrm{i}} \cap \det(\varphi_2^{\mathrm{ad}} \to \varphi_1^{\mathrm{ad}})^{\mathrm{i}}$ such that $g \wedge m = 1$

are unpleasant because they mix the roles of factors $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]$ and $[\, c_i \quad d_i \,]$.

It is obvious that if there exists $f \in \det(\vartheta_2 \to \vartheta_1)^{\mathrm{i}}$, then $\mathcal{N}^+\{a_1, b_1\} \subset \mathcal{N}^+\{a_2, b_2\}$. This lead us to conjecture that it would be possible to substitute conditions (iii) and (iv) by the following pair of conditions:

(iii′) $\mathcal{N}^+\{a_1, b_1\} = \mathcal{N}^+\{a_2, b_2\}$, and
(iv′) $\mathcal{N}^+\{c_1, d_1\} = \mathcal{N}^+\{c_2, d_2\}$.

These conditions are the most natural ones for the problem at hand because, according to [2], condition (iii′) is equivalent to the assertion that the parts of operators corresponding to the inner $*$-outer factors $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]$ are quasi-similar and condition (iv′) is equivalent to the assertion that the parts of operators corresponding to the outer $*$-inner factors $[\, c_i \quad d_i \,]$ are quasi-similar as well.

More precisely, we have the following conjecture.

Conjecture. *Conditions* (iii′) *and* (iv′) *imply conditions* (iii) *and* (iv) *for every inner function m.*

If the conjecture is true, the Main Theorem states that the quasi-similarity of the operators is equivalent to the separate quasi-similarity of each of its parts m_i, w_i, ϑ_i and φ_i. However, as we mentioned in the introductory part, our next result tells us that this would not imply that each operator is quasi-similar to the direct sum of its parts.

Consider the characteristic functions

$$\Theta_1 = \begin{bmatrix} ma \\ mb \end{bmatrix} [\, wc \quad wd \,] = (m\vartheta)(w\varphi), \qquad \Theta_2 = \begin{bmatrix} ma & 0 & 0 \\ mb & 0 & 0 \\ 0 & wc & wd \end{bmatrix} = \begin{bmatrix} m\vartheta & 0 \\ 0 & w\varphi \end{bmatrix}, \tag{4.1}$$

where $a, b, c, d \in H^\infty$ are such that $a \wedge b = c \wedge d = 1$ and $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$, m is inner and w is outer. We can take the auxiliary spaces as $\mathcal{E}_1 = \mathcal{E}_{*1} = \mathbb{C}^2$ and $\mathcal{E}_2 = \mathcal{E}_{*2} = \mathbb{C}^3$. Then

$$\Delta_1 = \varphi^{\mathrm{ad}} \left(\varphi^{\mathrm{ad}}\right)^* + \Delta_w \varphi^* \varphi, \qquad \Delta_{*1} = \left(\vartheta^{\mathrm{ad}}\right)^* \vartheta^{\mathrm{ad}} + \Delta_w \vartheta \vartheta^*,$$
$$\Delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_1 \end{bmatrix}, \qquad \Delta_{*2} = \begin{bmatrix} (\vartheta^{\mathrm{ad}})^* \vartheta^{\mathrm{ad}} & 0 \\ 0 & \Delta_w \end{bmatrix},$$
$$L^2(\Delta_1 \mathcal{E}_1) = \varphi^{\mathrm{ad}} L^2 \oplus \varphi^* \chi_\Omega L^2, \qquad L^2(\Delta_{*1} \mathcal{E}_{*1}) = \left(\vartheta^{\mathrm{ad}}\right)^* L^2 \oplus \vartheta \chi_\Omega L^2,$$
$$L^2(\Delta_2 \mathcal{E}_2) = \begin{bmatrix} 0 \\ \varphi^{\mathrm{ad}} L^2 \oplus \varphi^* \chi_\Omega L^2 \end{bmatrix}, \qquad L^2(\Delta_{*2} \mathcal{E}_{*2}) = \begin{bmatrix} (\vartheta^{\mathrm{ad}})^* L^2 \\ \chi_\Omega L^2 \end{bmatrix}.$$

Proposition 4.1. *The operators T_1 and T_2 with respective characteristic functions given in* (4.1) *are quasi-similar if and only if $\mathcal{N}^+\{ma, mb, c, d\} = \mathcal{N}^+$, i.e., if there exist four functions $f_1, f_2, f_3, f_4 \in H^\infty$ such that $maf_1 + mbf_2 + cf_3 + df_4$ is an outer function.*

Proof. We suppose that T_1 and T_2 are quasi-similar, then there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$, $\ker(X) = \{0\}$ and $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$. The parameters A, A_*, and A_0 of its lifting

$$Y = \pi_{*2} A_* \pi_{*1}^* + \tau_2 \Delta_2 A \pi_1^* + \tau_2 A_0 \tau_{*1}^* = \pi_2 A \pi_1^* + \pi_{*2} A_* \Delta_{*1} \tau_{*1}^* + \tau_2 A_0 \tau_{*1}^*$$

satisfy $\Theta_2 A = A_* \Theta_1$ and $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer.

If we denote $A = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right]$ and $A_* = \left[\begin{smallmatrix} A_{*1} \\ A_{*2} \end{smallmatrix}\right]$ with $A_1 \in H^\infty_{1\times 2}$, $A_2 \in H^\infty_{2\times 2}$, $A_{*1} \in H^\infty_{2\times 2}$, $A_{*2} \in H^\infty_{1\times 2}$, then we have

$$\Theta_2 A = \begin{bmatrix} m\vartheta & 0 \\ 0 & w\varphi \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = A_* \Theta_1 = \begin{bmatrix} A_{*1} \\ A_{*2} \end{bmatrix} m w \vartheta \varphi,$$

thus

$$\vartheta A_1 = w A_{*1} \vartheta \varphi, \tag{4.2}$$
$$\varphi A_2 = m A_{*2} \vartheta \varphi. \tag{4.3}$$

Multiplying (4.2) by ϑ^{ad} from the left we get $\vartheta^{\mathrm{ad}} A_{*1}\vartheta = 0$ and, by Lemma 5.2 from [2], there exists a function $\alpha \in H^\infty$ such that $A_{*1}\vartheta = \alpha\vartheta$, and therefore (4.2) yields

$$A_1 = \alpha w \varphi. \tag{4.4}$$

Rewriting (4.3) as $\varphi[A_2 - m(A_{*2}\vartheta)I] = 0$ we conclude, again according to Lemma 5.2 from [2], that there exists a function $\psi \in H^\infty_{1\times 2}$ such that $A_2 - m(A_{*2}\vartheta)I = \varphi^{\mathrm{ad}}\psi$, i.e.,

$$A_2 = m(A_{*2}\vartheta)I + \varphi^{\mathrm{ad}}\psi. \tag{4.5}$$

Note that $[\varphi^{\mathrm{ad}}\psi - (\psi\varphi^{\mathrm{ad}})I]\varphi^{\mathrm{ad}} = 0$, whence $\varphi^{\mathrm{ad}}\psi - (\psi\varphi^{\mathrm{ad}})I = \delta\varphi$ for some $\delta \in H^\infty_{2\times 1}$. Thus, denoting $\beta = mA_{*2}\vartheta + \psi\varphi^{\mathrm{ad}}$ we can rewrite (4.5) in the form $A_2 = \beta I + \delta\varphi$. Together with (4.4) this yields

$$A = \begin{bmatrix} \alpha w \\ \delta \end{bmatrix}\varphi + \beta\begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{4.6}$$

and therefore

$$\begin{bmatrix} A \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} \alpha w \\ \delta \\ mw\vartheta \end{bmatrix}\varphi + \beta\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}.$$

Since the first summand is of rank one, all minors of the $*$-outer matrix $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ have a common factor β, and hence this function has to be outer. Recalling that $\beta = mA_{*2}\vartheta + \psi\varphi^{\mathrm{ad}}$ we conclude that $\mathcal{N}^+\{ma, mb, c, d\} = \mathcal{N}^+$, i.e., this condition is necessary for quasi-similarity.

To prove that the condition is sufficient we suppose that for some f_i, $f_i \in H^\infty$, the function $\beta = f_1 ma + f_2 mb + f_3 c + f_4 d$ is outer. We need to find two bounded operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ and $X' : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$XT_1 = T_2 X, \qquad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \qquad \ker(X) = \{0\},$$
$$T_1 X' = X' T_2, \qquad \operatorname{clos}\{X'\mathcal{H}_2\} = \mathcal{H}_1, \qquad \ker(X') = \{0\}.$$

It will be enough to find two suitable liftings $Y = \pi_{*2}A_*\pi^*_{*1} + \tau_2\Delta_2 A\pi_1^* + \tau_2 A_0\tau^*_{*1}$ and $Y' = \pi_{*1}A'_*\pi^*_{*2} + \tau_1\Delta_1 A'\pi_2^* + \tau_1 A'_0\tau^*_{*2}$ of X and X', respectively. According to Lemmas 2.2 and 2.4 it is sufficient to find six matrix-valued functions $A \in H^\infty_{3\times 2}$, $A_* \in H^\infty_{3\times 2}$, $A_0 \in L^\infty_{3\times 2}$, $A' \in H^\infty_{2\times 3}$, $A'_* \in H^\infty_{2\times 3}$, and $A'_0 \in L^\infty_{2\times 3}$ satisfying the following ten conditions:

(1) $\Theta_2 A = A_*\Theta_1$,
(2) $[\,A_* \quad \Theta_2\,]$ is outer,
(3) $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer,
(4) $\operatorname{clos}\{(\Delta_2 A\Delta_1 - A_0\Theta_1)L^2(\Delta_1\mathcal{E}_1)\} = L^2(\Delta_2\mathcal{E}_2)$,
(5) $\operatorname{clos}\{(\Delta_{*1}A^*_*\Delta_{*2} - A^*_0\Theta^*_2)L^2(\Delta_{*2}\mathcal{E}_{*2})\} = L^2(\Delta_{*1}\mathcal{E}_{*1})$,
(6) $\Theta_1 A' = A'_*\Theta_2$,
(7) $[\,A'_* \quad \Theta_1\,]$ is outer,
(8) $\left[\begin{smallmatrix} A' \\ \Theta_2 \end{smallmatrix}\right]$ is $*$-outer,
(9) $\operatorname{clos}\{(\Delta_1 A'\Delta_2 - A'_0\Theta_2)L^2(\Delta_2\mathcal{E}_2)\} = L^2(\Delta_1\mathcal{E}_1)$,
(10) $\operatorname{clos}\{(\Delta_{*2}(A'_*)^*\Delta_{*1} - (A'_0)^*\Theta^*_1)L^2(\Delta_{*1}\mathcal{E}_{*1})\} = L^2(\Delta_{*2}\mathcal{E}_{*2})$.

Checking the first five conditions is easy by taking the following matrices:

$$A = \begin{bmatrix} wc & wd \\ f_1 ma + f_2 mb + f_4 d & -f_3 d \\ -f_4 c & f_1 ma + f_2 mb + f_3 c \end{bmatrix} = \begin{bmatrix} w \\ -f_3 \\ -f_4 \end{bmatrix}\varphi + \beta\begin{bmatrix} 0 \\ I \end{bmatrix},$$
$$A_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_1 & f_2 \end{bmatrix} = \begin{bmatrix} I \\ [\,f_1 \quad f_2\,] \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0 & 0 \\ a_0\bar{a}\bar{c} & a_0\bar{b}\bar{c} \\ a_0\bar{a}\bar{d} & a_0\bar{b}\bar{d} \end{bmatrix} = \begin{bmatrix} 0 \\ a_0\varphi^*\vartheta^* \end{bmatrix},$$

where the function a_0 is chosen as follows: if $f_1 a + f_2 b = 0$, then $a_0 = 1$; otherwise, $a_0 = 0$.

(1) Let us compute

$$\Theta_2 A = \begin{bmatrix} m\vartheta & 0 \\ 0 & w\varphi \end{bmatrix}\left(\begin{bmatrix} w \\ -\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \end{bmatrix}\varphi + \beta\begin{bmatrix} 0 \\ I \end{bmatrix}\right) = \begin{bmatrix} mw\vartheta \\ w(\beta - f_3 c - f_4 d) \end{bmatrix}\varphi = \begin{bmatrix} \vartheta \\ f_1 a + f_2 b \end{bmatrix} mw\varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_1 & f_2 \end{bmatrix}\vartheta\varphi$$
$$= A_*\Theta_1.$$

(2) The matrix

$$[\,A_* \quad \Theta_2\,] = \begin{bmatrix} 1 & 0 & ma & 0 & 0 \\ 0 & 1 & mb & 0 & 0 \\ f_1 & f_2 & 0 & wc & wd \end{bmatrix}$$

is outer because two of its 3×3 minors are the functions wc and wd, which have no common inner divisors.

(3) The matrix $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer because the matrix A is. Indeed, two of its 2×2 minors are the functions $wc\beta$ and $-wd\beta$, which have no common inner divisors.

(4) We have

$$\operatorname{clos}\{(\Delta_2 A \Delta_1 - A_0 \Theta_1) L^2(\Delta_1 \mathcal{E}_1)\} = \operatorname{clos}\{(\Delta_2 A \Delta_1 - A_0 \Theta_1)\varphi^{\mathrm{ad}} L^2, (\Delta_2 A \Delta_1 - A_0 \Theta_1)\varphi^* \chi_\Omega L^2\}$$

and the first term yields

$$\operatorname{clos}\left\{\begin{bmatrix} 0 & 0 \\ 0 & \Delta_1 \end{bmatrix}\left(\begin{bmatrix} w \\ -\begin{bmatrix} f_3 \\ f_4 \end{bmatrix}\end{bmatrix}\varphi + \beta \begin{bmatrix} 0 \\ I \end{bmatrix}\right)\varphi^{\mathrm{ad}} L^2\right\} = \begin{bmatrix} 0 \\ \operatorname{clos}\{\beta \Delta_1 \varphi^{\mathrm{ad}} L^2\} \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi^{\mathrm{ad}} L^2 \end{bmatrix}.$$

And the projection onto the orthogonal complement of the second term gives us

$$\begin{aligned}
&\operatorname{clos}\left\{\begin{bmatrix} 0 & 0 \\ 0 & \chi_\Omega \varphi^* \varphi \end{bmatrix}(\Delta_2 A \Delta_1 - A_0 \Theta_1)\varphi^* \chi_\Omega L^2\right\} \\
&\quad = \operatorname{clos}\left\{\begin{bmatrix} 0 & 0 \\ 0 & \chi_\Omega \varphi^* \varphi \end{bmatrix}\left(\begin{bmatrix} 0 & 0 \\ 0 & \Delta_w^2 \end{bmatrix}\left(\begin{bmatrix} w \\ -\begin{bmatrix} f_3 \\ f_4 \end{bmatrix}\end{bmatrix} + \beta \begin{bmatrix} 0 \\ \varphi^* \end{bmatrix}\right) - A_0 m w \vartheta\right)\chi_\Omega L^2\right\} \\
&\quad = \begin{bmatrix} 0 \\ \operatorname{clos}\{\chi_\Omega \varphi^* (\Delta_w^2 (\beta - f_3 c - f_4 d) - m w a_0) L^2\} \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 \\ \operatorname{clos}\{\chi_\Omega \varphi^* (\Delta_w^2 m (f_1 a + f_2 b) - m w a_0) L^2\} \end{bmatrix} = \begin{bmatrix} 0 \\ \chi_\Omega \varphi^* L^2 \end{bmatrix}.
\end{aligned}$$

Indeed, in order to see that the function $\Delta_w^2 m(f_1 a + f_2 b) - m w a_0$ is different from zero almost everywhere on Ω, simply consider the cases $af_1 + bf_2 = 0$, so that $a_0 = 0$, and $af_1 + bf_2 \neq 0$, so that $a_0 = 1$.

(5) In a similar way, we have

$$\operatorname{clos}\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*) L^2(\Delta_{*2} \mathcal{E}_{*2})\} = \operatorname{clos}\left\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)\begin{bmatrix} (\vartheta^{\mathrm{ad}})^* L^2 \\ \chi_\Omega L^2 \end{bmatrix}\right\}.$$

The first component is

$$\operatorname{clos}\left\{(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)\begin{bmatrix} (\vartheta^{\mathrm{ad}})^* L^2 \\ 0 \end{bmatrix}\right\} = \operatorname{clos}\left\{\Delta_{*1} A_*^* \begin{bmatrix} (\vartheta^{\mathrm{ad}})^* L^2 \\ 0 \end{bmatrix}\right\} = \operatorname{clos}\{\Delta_{*1} (\vartheta^{\mathrm{ad}})^* L^2\} = (\vartheta^{\mathrm{ad}})^* L^2.$$

And the projection of the second component onto the orthogonal complement is

$$\begin{aligned}
\operatorname{clos}\left\{\chi_\Omega \vartheta \vartheta^* (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)\begin{bmatrix} 0 \\ \chi_\Omega L^2 \end{bmatrix}\right\} &= \operatorname{clos}\left\{\left(\vartheta \vartheta^* \Delta_w^2 \begin{bmatrix} \overline{f}_1 \\ \overline{f}_2 \end{bmatrix} - [\,0 \quad \bar{a}_0 \vartheta \varphi\,]\begin{bmatrix} 0 \\ \overline{w} \varphi^* \end{bmatrix}\right)\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\{\vartheta (\Delta_w^2 (\bar{a} \overline{f}_1 + \bar{b} \overline{f}_2) - \bar{a}_0 \overline{w}) \chi_\Omega L^2\} = \vartheta \chi_\Omega L^2.
\end{aligned}$$

This completes the verification of the first five conditions.

It is a bit more difficult to chose parameters to satisfy conditions (6)–(10). Since the functions a and b are mutually prime, according to Lemma 5.3 of [2], we can find a pair of numbers γ_1 and γ_2 such that $(\gamma_1 a + \gamma_2 b) \wedge m = 1$ and, analogously, another pair δ_1 and δ_2 such that $(\delta_1 c + \delta_2 d) \wedge m = 1$. Again by Lemma 5.3 of [2], we can find a number t such that $(cf_3 + df_4) + t(\gamma_1 a + \gamma_2 b)(\delta_1 c + \delta_2 d) \wedge m = 1$. Then we take

$$A_0' = 0, \qquad A' = \begin{bmatrix} f_3 + t\delta_1(\gamma_1 a + \gamma_2 b) & 1 & 0 \\ f_4 + t\delta_2(\gamma_1 a + \gamma_2 b) & 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$A_*' = \vartheta[\,wt\gamma_1(\delta_1 c + \delta_2 d) - wmf_1 \quad wt\gamma_2(\delta_1 c + \delta_2 d) - wmf_2 \quad m\,] + w\beta[\,I \quad 0\,].$$

(6) First we check the intertwining relation

$$\begin{aligned}
A_*' \Theta_2 &= mw\vartheta[\,t(\gamma_1 a + \gamma_2 b)(\delta_1 c + \delta_2 d) - m(af_1 + bf_2) \quad \varphi\,] + w\beta[\,m\vartheta \quad 0\,] \\
&= mw\vartheta[\,t(\gamma_1 a + \gamma_2 b)(\delta_1 c + \delta_2 d) + (cf_3 + df_4) \quad \varphi\,] \\
&= mw\vartheta\varphi \begin{bmatrix} f_3 + \delta_1 t(\gamma_1 a + \gamma_2 b) & 1 & 0 \\ f_4 + \delta_2 t(\gamma_1 a + \gamma_2 b) & 0 & 1 \end{bmatrix} = \Theta_1 A'.
\end{aligned}$$

(7) The matrix $[\,A'_* \quad \Theta_1\,]$ is outer because the matrix A'_* is. Indeed, its three 2×2 minors are $w^2\beta(t(\gamma_1 a+\gamma_2 b)(\delta_1 c+\delta_2 c)+(cf_3+df_4))$, $-mwa\beta$, $mwb\beta$ and due to our choice of the parameter t these functions have no common inner factor.

(8) Since the three 3×3 minors of the matrix

$$\begin{bmatrix} A' \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} f_3+t\delta_1(\gamma_1 a+\gamma_2 b) & 1 & 0 \\ f_4+t\delta_2(\gamma_1 a+\gamma_2 b) & 0 & 1 \\ ma & 0 & 0 \\ mb & 0 & 0 \\ 0 & wc & wd \end{bmatrix}$$

including the first two lines are ma, mb, and $t(\gamma_1 a+\gamma_2 b)(\delta_1 c+\delta_2 d)+(cf_3+df_4)$, the matrix is $*$-outer.

(9) We have

$$\operatorname{clos}\{\Delta_1 A'\Delta_2 L^2(\Delta_2\mathcal{E}_2)\} = \operatorname{clos}\operatorname{span}\left\{\Delta_1 A'\begin{bmatrix} 0 \\ \varphi^{\mathrm{ad}}L^2 \end{bmatrix}, \Delta_1 A'\Delta_w\begin{bmatrix} 0 \\ \varphi^*\chi_\Omega L^2 \end{bmatrix}\right\}.$$

The first term is

$$\operatorname{clos}\left\{\Delta_1 A'\begin{bmatrix} 0 \\ \varphi^{\mathrm{ad}}L^2 \end{bmatrix}\right\} = \operatorname{clos}\{\Delta_1\varphi^{\mathrm{ad}}L^2\} = \varphi^{\mathrm{ad}}L^2,$$

and the projection of the second one onto the orthogonal complement is

$$\operatorname{clos}\left\{\varphi^*\varphi\chi_\Omega\Delta_1 A'\begin{bmatrix} 0 \\ \varphi^*\chi_\Omega L^2 \end{bmatrix}\right\} = \operatorname{clos}\{\varphi^*\varphi\chi_\Omega\Delta_w\varphi^* L^2\} = \varphi^*\chi_\Omega L^2.$$

Finally, (10) is proven in a similar way. □

Example 1. Consider the characteristic functions

$$\Theta_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}[\,c_1 \quad d_1\,] = \vartheta_1\varphi_1 \quad \text{and} \quad \Theta_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}[\,c_2 \quad d_2\,] = \vartheta_2\varphi_2$$

and assume that $\mathcal{N}^+\{a_i,b_i,c_i,d_i\} = \mathcal{N}^+$ for $i=1,2$. This implies, by direct application of Proposition 4.1, that each operator T_{Θ_i} is quasi-similar to the direct sum $T_{\vartheta_i}\oplus T_{\varphi_i}$.

In this case, the conjecture is true: The conditions $\mathcal{N}^+\{a_1,b_1\} = \mathcal{N}^+\{a_2,b_2\}$ and $\mathcal{N}^+\{c_1,d_1\} = \mathcal{N}^+\{c_2,d_2\}$ are both necessary and sufficient for T_{Θ_i} to be quasi-similar because, if these conditions hold, then, by our results in [2], T_{ϑ_1} is quasi-similar to T_{ϑ_2} and T_{φ_1} is quasi-similar to T_{φ_2}, so it follows that $T_{\vartheta_1}\oplus T_{\varphi_1}$ is quasi-similar to $T_{\vartheta_2}\oplus T_{\varphi_2}$ and, by the assumption above, T_{Θ_1} and T_{Θ_2} are quasi-similar.

On the other hand, as announced in the Introduction, the necessary and sufficient assumption of Proposition 4.1 may fail, as the following example shows.

Example 2. There exist functions $a,b,c,d\in H^\infty$ such that $a\wedge b = 1 = c\wedge d$ and $|a|^2+|b|^2 = 1 = |c|^2+|d|^2$ that do not satisfy the conditions of Proposition 4.1: for instance, if we take the functions

$$a(z) = c(z) = \frac{1}{\sqrt{2}}\exp\left(-\frac{1+z}{1-z}\right) \quad \text{and} \quad b(z) = d(z) = \frac{1}{\sqrt{2}}\prod_n\left(\frac{\lambda_n - z}{1-\bar{\lambda}_n z}\right),$$

where $\lambda_n = 1-\frac{1}{2^n}$, then, as easily seen, there exist no functions $f_1,f_2,f_3,f_4\in H^\infty$ such that $af_1+bf_2+cf_3+df_4$ is an outer function.

Finally, let S be the shift operator of multiplicity one and consider $S\oplus S^*$. Then we obtain the following nice corollary of our Main Theorem and Proposition 4.1.

Corollary 4.2. *Let $\Theta = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right][\,c \quad d\,]$, where $a,b,c,d\in H^\infty$ are such that $|a|^2+|b|^2 = |c|^2+|d|^2 = 1$, and $a\wedge b = c\wedge d = 1$. Let further M_z denote the multiplication operator by the independent variable in L^2 and $\Omega = \{z\in\mathbb{T}\colon |w(z)|<1\}$. Then the operator T_Θ is quasi-similar to $S\oplus S^*$ if and only if $\mathcal{N}^+\{a,b\} = \mathcal{N}^+\{c,d\} = \mathcal{N}^+$, i.e., there exist $f,g,h,k\in H^\infty$ such that $af+bg$ and $ch+dk$ are outer.*

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