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Barrelledness of Spaces with Toeplitz Decompositions[1]

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DEDICAMOS ESTE ARTÍCULO CON TODO CARIÑO A NUESTRO MAESTRO Y AMIGO MIGUEL FLORENCIO LORA

A Toeplitz decomposition of a locally convex space E into subspaces (E_k) with projections (P_k) is a decomposition of every $x \in E$ as $x = \sum_k P_k x$, where ordinary summability has been replaced by summability with respect to an infinite and lower triangular regular matrix. We extend to the setting of Toeplitz decompositions a couple of results about barrelledness of Schauder decompositions. The first result, given for Schauder decompositions by Noll and Stadler, links the barrelledness of a normed space E to the barrelledness of the pieces E_k via the fact that E' is big enough so as to coincide with its summability dual. Our second theorem, given for Schauder decompositions by Díaz and Miñarro, links the quasibarrelledness of an $\aleph_0$-quasibarrelled (in particular, (DF)) space E to the quasibarrelledness of the pieces E_k via the fact that the decomposition is simple. © 1999 Academic Press

Key Words: decompositions of locally convex spaces; barrelledness; summability and bases; (DF)-spaces; sequence spaces.

INTRODUCTION

Up to what point can one substitute ordinary summability by a matrix summability method in the definition of a Schauder decomposition, and, still, obtain nice results about the locally convex structure of the space in terms of the locally convex structure of its pieces? Our purpose here is to

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extend two theorems about the barrelledness of certain locally convex spaces having a Schauder decomposition—obtained, respectively, by Noll and Stadler [14] (main hypothesis: a normed space with a shrinking decomposition) and Díaz and Miñarro [5] (main hypothesis: the space is (DF) and the decomposition is equicontinuous)—to the setting of decompositions defined in terms of more general matrix summability methods. In a similar framework (scalar sequence spaces having a generalized sectional convergence scheme), several interesting barrelledness results have been given by Ruckle and Saxon [17]. Although we refer the reader to this very interesting paper, we shall make some comments about it at the end of our work.

Terminology and Notation. Although our notation and terminology will be mostly standard, e.g., φ is the space of finitely non-zero sequences, c is the space of convergent sequences, $e^{[k]}$ stands for the kth unit sequence (we refer the reader to [8, 15, 16, 18, 22]), let us recall a few facts from summability theory. Let $T = [t_{nk}]$ be an infinite matrix of scalars from the field $\mathbb{K}$ of real or complex numbers. The matrix T is said to be *row-finite* if each row of T is in φ, an *Sp_1-matrix* if each column of T is convergent to 1, and *reversible* if for every sequence $y \in c$ the infinite system of linear equations $T \cdot x = y$ has unique solution. It is well known [22, 5.4.5–5.4.9] that each row-finite and reversible T has a unique two-sided inverse matrix T^{-1} such that each row of T^{-1} is in l^1 and for each $y \in c$ the unique solution of $T \cdot x = y$ is $T^{-1} \cdot y$. An important particular case is that of a triangle. Following Wilansky [22], a lower triangular infinite matrix with non-zero diagonal entries is called a triangle. A triangle is always row-finite and reversible, and its inverse is also a triangle.

Let E be a locally convex space. The *convergence field of T in E* is the space $c_T(E)$ of all sequences (x_k) from E such that the product $T \cdot (x_k)$ is a convergent sequence in E. The sequences in $c_T(E)$ are said to be *T-convergent*. For $(x_k) \in c_T(E)$ the limit of the sequence $T \cdot (x_k)$ is called the *T-limit of* (x_k) and will be denoted by T-lim x_k, in other words

$$T\text{-lim } x_k := \lim_n \sum_k t_{nk} x_k.$$

We simply denote by c_T the convergence field of T in $\mathbb{K}$. If T is a row-finite and reversible matrix then the norm $\|x\|_T := \|T \cdot x\|_\infty$ makes c_T a Banach space (isomorphic to c).

DEFINITIONS. Let $T = [t_{nk}]$ be a row-finite infinite matrix of scalars. A sequence (P_k) of non-trivial, mutually orthogonal, and continuous linear projections defined on a locally convex space E is said to be a *Toeplitz decomposition* of E with respect to the matrix T or, for short, a *T-decom-*

position of E, if

$$x = T\text{-lim } P_k x \qquad \text{for every } x \in E.$$

Alternatively, if we define the sequence of operators

$$T_n: x \in E \to T_n(x) := \sum_k t_{nk} P_k x \in E,$$

then (P_k) is a T-decomposition of E whenever $\lim_n T_n x = x$ for every $x \in E$. If Σ stands for the triangle with all of its lower triangular entries equal to 1, then Σ-decompositions are the familiar Schauder decompositions.

It is important to note that the operators T_n are not projections in general (they are increasing projections in the case of a Schauder decomposition), however we do have $T_n P_k = P_k T_n = t_{nk} P_k$ for all $n, k \in \mathbb{N}$. Note also that the sequence of operators (T_n) is precisely the product $T \cdot (P_k)$. Hence saying that $\lim_n T_n x = x$ is the same as saying that the sequence $T \cdot (P_k x)$ converges to x. Call $E_k := P_k(E)$. Since E_k does not reduce to the zero subspace and for every $x_k \in E_k$ we have $x_k = \lim_n T_n x_k = \lim_n t_{nk} x_k$, it follows that $\lim_n t_{nk} = 1$, i.e., T is an Sp_1-matrix. Let us also point out, for later use, that, as is easy to see, we can write $T_n(E) = E_{n1} \oplus E_{n2} \oplus \cdots$, where $E_{nk} = E_k$ if $t_{nk} \neq 0$ and $E_{nk} = \{0\}$ if $t_{nk} = 0$. If T is a triangle then $T_n(E)$ is a complemented subspace of $E_1 \oplus E_2 \oplus \cdots \oplus E_n$.

Still another way of looking at a Toeplitz decomposition is the following: Every E_k is a complemented subspace of E and we can identify every $x \in E$ with the vector-valued sequence $(P_k x) \in \prod E_k$, so that E becomes a linear subspace of $\prod E_k$ that, with the topology translated from E, has the set of all finite sequences as a dense subspace because $\lim_n T_n x = x$ for every $x \in E$ and T is row-finite.

We extend now some of the terminology commonly used for Schauder decompositions (see [9, 10, 18, 20, 21]): A T-decomposition (P_k) of a locally convex space E is said to be *finite-dimensional* if every E_k is finite-dimensional, *equicontinuous* if the sequence of operators (T_n) is equicontinuous, and *complete* if for each sequence $(x_k) \in \prod E_k$ such that the product $T \cdot (x_k)$ is a Cauchy sequence in E there exists $x \in E$ such that $x_k = P_k x$ for every $k \in \mathbb{N}$, and a fortiori, $T \cdot (x_k)$ converges to x.

Using primes to denote adjoint operators, for every $x \in E$ and every $u \in E'$ we can write

$$\begin{aligned} \langle x, u \rangle &= \lim_n \langle T_n x, u \rangle = \lim_n \sum_k t_{nk} \langle P_k x, u \rangle = \lim_n \sum_k t_{nk} \langle x, P_k' u \rangle \\ &= \lim_n \langle x, T_n' u \rangle. \end{aligned}$$

This shows that (P_k') is also a T-decomposition of E' endowed with the weak topology $\sigma(E', E)$. If we call $E_k := P_k(E)$ and $E_k' := P_k'(E')$ then the dual of E_k can be identified with E_k'. The computation above also shows that the sequence $(T_n'u)$ is $\sigma(E', E)$-bounded.

A T-decomposition (P_k) of a locally convex space E is said to be *shrinking* if (P_k') is also a T-decomposition of E' endowed with the strong topology $\beta(E', E)$ and is said to be *simple* if $(T_k'u)$ is a $\beta(E', E)$-bounded sequence for every $u \in E'$.

EXAMPLE. A K-space is a locally convex sequence space $\lambda \supset \varphi$ such that the kth projection defined by $\pi_k((x_n)_n) := x_k e^{[k]}$ is continuous for every $k \in \mathbb{N}$. A K-space λ is said to have property T-AK if $x = T\text{-lim}\, x_k e^{[k]}$ for every sequence $x = (x_k) \in \lambda$. Thus, a sequence space λ has property T-AK if and only if the sequence (π_k) is a (one-dimensional) T-decomposition of λ or, in other words, the sequence of operators defined by $\tau_n := \sum_k t_{nk}\pi_k$, i.e., $(\tau_n) := T \cdot (\pi_k)$, satisfies $x = \lim_n \tau_n(x_k)$ for every sequence $x = (x_k) \in \lambda$ (see [3, 4, 12]). (When dealing with scalar sequence spaces, we shall keep the notations (π_k) and (τ_k) throughout the paper). In particular, λ has the property Σ-AK means precisely that $(e^{[k]})$ is a Schauder basis of λ.

The matrices T such that c_T has property T-AK where characterized by Buntinas [4, Theorems 8–10].

BUNTINAS'S THEOREM. *Let T be a row-finite and reversible Sp_1-matrix. Then the following conditions are equivalent:*

(1) *The sequence of coordinate projections (π_k) is a T-decomposition of c_T.*

(2) *The sequence of operators (τ_n) is equicontinuous on c_T.*

(3) *If we denote T^{-1} by $[s_{nk}]$ then*

$$\sup\left\{ \sum_j \left| \sum_k t_{mk} t_{nk} s_{kj} \right| : m, n \in \mathbb{N} \right\} < \infty.$$

(4) *The dual $(c_T)'$ can be identified with the multiplier space $(c_T \to c_T)$ formed by the sequences y such that the coordinatewise product xy is in c_T for every $x \in c_T$ and, in this case, the bilinear form of the dual pair is given by*

$$\langle x, y \rangle_{(c_T, (c_T)')} = T\text{-lim}\, xy.$$

The first non-trivial examples of matrices T such that c_T has property T-AK are the series-to-sequence Cesàro matrices of order $\alpha \geq 0$; this was proved by Zeller [23]. Therefore, to avoid clumsy repetitions, a row-finite and reversible Sp_1-matrix T such that c_T has property T-AK will be called a *Zeller–Buntinas matrix*. Note that if T is a Zeller–Buntinas matrix then c_T is a *sum space* in the sense of Ruckle [16].

BARRELLEDNESS OF NORMED SPACES WITH SHRINKING TOEPLITZ DECOMPOSITIONS

Our first purpose is to extend to the setting of Toeplitz decompositions with respect to triangles a result due to Noll and Stadler [14] that links the barrelledness of a normed space E to the barrelledness of the pieces E_k via the fact that E' is big enough.

DEFINITION [14]. Let E be a locally convex space having a T-decomposition (P_k). The β_T-dual of E is the space E^{β_T} defined by any of the equivalent formulations

$$E^{\beta_T} := \left\{ (u_k) \in \prod_{k=1}^{\infty} E'_k : (\langle x, u_k \rangle)_k \text{ is } T\text{-convergent for all } x \in E \right\}$$

$$= \left\{ (u_k) \in \prod_{k=1}^{\infty} E'_k : T \cdot (u_k) \text{ is } \sigma(E', E)\text{-Cauchy} \right\}.$$

Every $(u_k) \in E^{\beta_T}$ defines a linear form in E, namely, $x \in E \to \lim_n \sum_k t_{nk} \langle x, u_k \rangle$ and, in this sense, $E' \subset E^{\beta_T}$. It is clear that the equality $E' = E^{\beta_T}$ holds if and only if (P'_k) is a complete T-decomposition of E' endowed with the weak topology $\sigma(E', E)$.

THEOREM 1. *Let $E[\|\cdot\|]$ be normed space such that $E[\sigma(E, E')]$ has a shrinking Toeplitz decomposition (P_k) with respect to a triangle T. Then (P_k) is an equicontinuous Toeplitz decomposition of $E[\|\cdot\|]$ and the following conditions are equivalent*:

(i) *E is barrelled;*

(ii) *$E' = E^{\beta_T}$ and E_k is barrelled for every $k \in \mathbb{N}$.*

Proof. As is well known, every projection P_k is norm-continuous because it is $\sigma(E, E')$-continuous. Now, since (P_k) is shrinking and E' is a Banach space, it follows that the sequence of operators (T'_n) is norm-equicontinuous in E' and a standard duality argument shows that (T_n) is also norm-equicontinuous in E. Finally, since the linear span of the subspaces (E_k) is dense in $E[\|\cdot\|]$ and for every $x_k \in E_k$ the convergence to x_k of the sequence $(T_n x_k)_n$ holds in the norm topology (because it reduces to the convergence to 1 of the columns of T), a standard equicontinuity argument originally due to Mazur (see [11, Sect. 39.4.(1)] or [7]) shows that $(T_n x)$ converges to x in the norm-topology for every $x \in E$.

That (i) implies (ii) is an easy consequence of the Banach–Steinhaus theorem. To prove that (ii) implies (i), let us carefully refine Noll and Stadler's original proof. Let us begin by noting that for each $n \in \mathbb{N}$ we have that $T_n(E)$ is a barrelled space because, as we pointed out above, $T_n(E)$ is a complemented subspace of $E_1 \oplus E_2 \oplus \cdots \oplus E_n$. Proceed now by contradiction and assume that there is a $\sigma(E', E)$-bounded sequence (v_m) such that $\|v_m\| \geq m2^m$ for each $m \in \mathbb{N}$. Take $y_m := m^{-1}v_m$ for $m = 1, 2, \ldots$ to define a sequence (y_m) that $\sigma(E', E)$-converges to zero but such that $\|y_m\| \geq 2^m$ for each $m \in \mathbb{N}$. We shall prove by induction that there are a couple of increasing sequences (m_j) and (n_j) from $\mathbb{N}$ such that for every $j \in \mathbb{N}$,

$$\|y_m - T_n' y_m\| \leq 2^{-j} \qquad \text{for all } m = 1, 2, \ldots, m_{j-1} \text{ and all } n \geq n_j, \quad (1)$$

$$\|T_n' y_m\| \leq 2^{-j} \qquad \text{for all } n = 1, 2, \ldots, n_j \text{ and all } m \geq m_j. \quad (2)$$

Assume that m_j and n_j have been found such that (1) and (2) hold. Since (P_k) is shrinking, we have that $\lim_n \|y_m - T_n' y_m\| = 0$ for all $m = 1, 2, \ldots, m_j$ so there is an index $n_{j+1} > n_j$ such that (1) holds with j replaced by $j + 1$. On the other hand, for each $n = 1, 2, \ldots, n_{j+1}$ the sequence $(T_n' v_m)_m$ is $\sigma(T_n'(E'), T_n(E))$-bounded in the dual $T_n'(E')$ of the barrelled space $T_n(E)$. Therefore, each of the sequences $(T_n' v_m)_m$ $(n = 1, 2, \ldots, n_{j+1})$ is norm-bounded and, since $\|T_n' y_m\| = m^{-1}\|T_n' v_m\|$, it follows that there is an index m_{j+1} such that (2) also holds with j replaced by $j + 1$.

For each $j \in \mathbb{N}$ call $\alpha_j := \|y_{m_j}\|^{-1} \leq 2^{-m_j}$. Now fix $n \in \mathbb{N}$ and take j_0 such that $n \leq n_{j_0}$. Inequality (2) yields

$$\sum_{j \geq j_0} \alpha_j \|T_n' y_{m_j}\| \leq 2^{-j_0} \sum_{j \geq j_0} 2^{-m_j} < \infty$$

so that the series $\sum_{j=1}^{\infty} \alpha_j T_n' y_{m_j}$ converges in the Banach space $T_n'(E')$ to some element z_n. We shall prove now that (z_n) is a $\sigma(E', E)$-Cauchy sequence. Fix $x \in E$ with $\|x\| \leq 1$ and for each n take $j = j(n)$ such that $n_{j(n)} \leq n < n_{j(n)+1}$. Then we may write

$$\langle x, z_n \rangle = \sum_{j=1}^{j(n)-1} \alpha_j \langle x, T_n' y_{m_j} \rangle + \alpha_{j(n)} \langle x, T_n' y_{m_{j(n)}} \rangle + \sum_{j=j(n)+1}^{\infty} \alpha_j \langle x, T_n' y_{m_j} \rangle. \tag{3}$$

Let us see that the three summands in the right hand side converge as $n \to \infty$. The first one can be written as

$$\sum_{j=1}^{j(n)-1} \alpha_j \langle x, T_n' y_{m_j} \rangle = \sum_{j=1}^{j(n)-1} \alpha_j \langle x, y_{m_j} \rangle + \sum_{j=1}^{j(n)-1} \alpha_j \langle x, T_n' y_{m_j} - y_{m_j} \rangle. \quad (4)$$

The first term in the right side of equality (4) converges because (y_m) is a $\sigma(E', E)$-bounded sequence and $\alpha_j \le 2^{-m_j}$ for each $j \in \mathbb{N}$. The second term converges because inequality (1) tells us that

$$\sum_{j=1}^{j(n)-1} \left| \alpha_j \langle x, T_n' y_{m_j} - y_{m_j} \rangle \right| \le 2^{-j(n)} \sum_{j=1}^{j(n)-1} 2^{-m_j}.$$

This shows that the first term in equality (3) converges. The central term in equality (3) can be written as

$$\begin{aligned} \alpha_{j(n)} \langle x, T_n' y_{m_{j(n)}} \rangle &= \alpha_{j(n)} \langle T_n x, y_{m_{j(n)}} \rangle \\ &= \langle T_n x - x, \alpha_{j(n)} y_{m_{j(n)}} \rangle + \langle x, \alpha_{j(n)} y_{m_{j(n)}} \rangle, \end{aligned}$$

where the first summand converges to zero because $\|\alpha_{j(n)} y_{m_{j(n)}}\| = 1$ and $T_n x$ converges to x in the norm, and the second summand also converges to zero because (y_m) is $\sigma(E', E)$-convergent to zero and $\alpha_j \le 2^{-m_j}$. This shows that the central term in equality (3) also converges. Finally inequality (2) tells us that

$$\left| \sum_{j=j(n)+1}^{\infty} \alpha_j \langle x, T_n' y_{m_j} \rangle \right| \le \sum_{j=j(n)+1}^{\infty} 2^{-m_j} 2^{-j(n)}$$

tends to zero as $n \to \infty$. This shows that the third summand in the right side of (3) also converges.

Define now the sequence $(u_k) := T^{-1} \cdot (z_n)$ (recall that T^{-1} is also a triangle). We shall prove that (u_k) is in E^{β_T}. Denote the entries of the matrix T^{-1} by s_{kn} so that $u_k = \sum_n s_{kn} z_n$. To see that u_k is in E_k' it is enough to check that if $i \ne k$ then $P_i' u_k = 0$. Put $P_i' u_k = \sum_n s_{kn} P_i' z_n$. Now, since the series $\sum_j \alpha_j T_n' y_{m_j}$ is norm-convergent to z_n, since the sum in n is finite, and since $P_i' T_n' = t_{ni} P_i'$ for every $i, n \in \mathbb{N}$ because the projections (P_n') are mutually orthogonal, we have

$$\begin{aligned} P_i' u_k &= \sum_n s_{kn} P_i' z_n = \sum_n s_{kn} \sum_j \alpha_j P_i' T_n' y_{m_j} \\ &= \sum_j \alpha_j \Big(\sum_n s_{kn} t_{ni} \Big) P_i' y_{m_j} = 0. \end{aligned}$$

On the other hand, $T \cdot (u_k) = T \cdot T^{-1} \cdot (z_n) = (z_n)$ that, as we have seen, is a $\sigma(E', E)$-Cauchy sequence. This shows that $(u_k) \in E^{\beta_T}$. By our hypothesis, there exists some element $u \in E'$ such that $u_k = P_k' u$ for every

$k \in \mathbb{N}$. Since the decomposition is shrinking, we have that $(z_n) = T \cdot (u_k) = T \cdot (P_k' u)$ is norm-convergent to u, and

$$z_{n_j} = \sum_{i=1}^{j-1} \alpha_i \left(T_{n_j}' y_{m_i} - y_{m_i} \right) + \sum_{i=1}^{j-1} \alpha_i y_{m_i} + \sum_{i=j}^{\infty} \alpha_i T_{n_j}' y_{m_i}.$$

In norm the first and last terms in the right side do not exceed $2^{-j} \sum_i \alpha_i \leq 2^{-j}$ by (1) and (2), and thus tend to zero. Therefore, $u = \lim_j z_{n_j} = \lim_j \sum_{i=1}^{j-1} \alpha_i y_{m_i}$, but this is a contradiction because $\| \alpha_i y_{m_i} \| = 1$. ∎

COROLLARY 1.1. *Let E be normed space such that $E[\sigma(E, E')]$ has a finite-dimensional shrinking Toeplitz decomposition (P_k) with respect to a triangle T. Then (P_k) is an equicontinuous Toeplitz decomposition of E and the following conditions are equivalent*:

(i) *E is barrelled;*

(ii) $E' = E^{\beta_T}$.

Remarks. (1) The natural hypothesis $E' = E^{\beta_T}$ is itself a sort of weak barrelledness condition. It can be proved that if T is a Zeller–Buntinas triangle and (P_k) is a T-decomposition of a locally convex space E with barrelled subspaces E_k, then $E' = E^{\beta_T}$ if and only if a barrel $U \subset E$ is a zero-neighborhood for the Mackey topology in E provided that T_n converges pointwise to the identity for the normed topology generated by the gauge of U.

(2) The hypothesis that the space is normed cannot be suppressed in general. For instance, $c_0[\sigma(c_0, l^1)]$ satisfies all the hypotheses of Theorem 1 except that of being normed and, obviously, it is not barrelled. Metrizability is not enough, either, because the sequence space $E := \{x \in \omega : \sup_n n|x_{2n} - x_{2n+1}| < \infty\}$ is not a barrelled subspace of ω but $E^{\beta_T} = E' = \varphi$ [1].

We do not know if Theorem 1 holds for (DF)-spaces. The strong dual of a (DF)-space is a Fréchet space and one could try to adapt Stadler and Noll's proof to this case, but the crucial step of passing from (v_m) to (y_m) cannot be done in a non-normable Fréchet space. More concretely, a Fréchet space E turns out to be a Banach space if and only if whenever (v_n) is an unbounded sequence from E, there is an unbounded subsequence (v_{n_k}) and a sequence of scalars (α_k) convergent to zero such that $(\alpha_j v_{n_j})$ is bounded but does not converge to zero.

(3) Within the framework of scalar sequence spaces and ordinary convergence, the class of spaces to which Theorem 1 applies resembles the spaces with the so-called Wilansky property [1, 14]. This concept has been

extended to T-convergence by Noll [13]: Let T be an Sp_1 triangle, an FK-space λ containing φ is said to have the *Wilansky* $(\beta_T\text{-}W)$ *property* if whenever μ is a subspace of λ such that $\mu^{\beta_T} = \lambda^{\beta_T}$ then μ is barrelled. Theorem 1 implies that if λ is a BK-space such that both λ and λ' have property T-AK for a triangle T, then λ has property $(\beta_T\text{-}W)$, a result that can also be deduced from Noll's results in [13]. More generally we have the following result.

COROLLARY 1.2. *Let (P_k) be a shrinking Toeplitz decomposition of a Banach space E with respect to a triangle T. Let F be a subspace of E containing all subspaces (E_k). Then F is barrelled if and only if $F^{\beta_T} = E^{\beta_T}$.*

BARRELLEDNESS OF (DF)-SPACES WITH TOEPLITZ DECOMPOSITIONS

Our second purpose is to extend to the setting of Toeplitz decompositions a result due to Díaz and Miñarro [5] that links the quasibarrelledness of a (DF)-space E to the quasibarrelledness of the pieces E_k via the fact that the decomposition is equicontinuous. This result was motivated by the problem of the stability in tensor products of Fréchet spaces of some topological vector space properties (see also [2]).

THEOREM 2. *Let E be an $\aleph_0$-quasibarrelled space with a simple Toeplitz decomposition (P_k) with respect to a row-finite matrix T. Then the decomposition is equicontinuous. Moreover, if every E_k is quasibarrelled then E is quasibarrelled.*

Proof. We start by proving that the decomposition is equicontinuous. According to [8, 12.2.1], it will be enough to prove that if B is a bounded subset of E then $\bigcup_n T_n(B)$ is also bounded. But this follows from

$$\sup\{|\langle T_n x, u\rangle| : x \in B,\, n \in \mathbb{N}\} = \sup\{|\langle x, T_n' u\rangle| : x \in B,\, n \in \mathbb{N}\}$$

and the latter supremum is finite because, since the decomposition is simple, we know that $(T_k' u)$ is a $\beta(E', E)$-bounded sequence for every $u \in E'$.

Let now W be a bornivorous barrel in E. We must prove that W is a zero-neighborhood. Each $T_n(E)$ is quasibarrelled, being complemented in a quasibarrelled space of the form $E_1 \oplus \cdots \oplus E_k$, so $W \cap T_n(E)$ is a zero-neighborhood in $T_n(E)$, and $T_n^{-1}(W) = T_n^{-1}(W \cap T_n(E))$ is a barrel and a zero-neighborhood in E. Hence, by hypothesis, $U := \bigcap_n T_n^{-1}(W)$ is a zero-neighborhood in E, provided U is bornivorous. Moreover, $U \subset W$

because $x \in U$ means that each $T_n x \in W$, and W closed implies that $\lim_n T_n x = x \in W$. Therefore, it remains only to prove that U absorbs an arbitrary bounded set B. As we proved above, $C := \bigcup_n T_n(B)$ is also bounded. Thus for some $\varepsilon > 0$ we have $\varepsilon C \subset W$. This proves that $\varepsilon C \subset U$ and, since U is closed, it follows $\varepsilon B \subset U$ so that U is, indeed, bornivorous. ∎

COROLLARY 2.1. *Let E be a (DF)-space with a simple Toeplitz decomposition with respect to a row-finite matrix T. If every E_k is quasibarrelled then E is quasibarrelled.*

COROLLARY 2.2. *Let E be a sequentially complete (DF)-space with a Toeplitz decomposition (P_k) with respect to a row-finite matrix T. Then E is barrelled if and only if every E_k is barrelled.*

Proof. Since E is sequentially complete, it follows from the Banach–Mackey theorem [15, 3.4.1] that the decomposition (P_k) is simple and we may use Theorem 2 above, together with the Banach–Mackey theorem again, to prove the equivalence. ∎

Recall at this point that a locally convex space E is said to be distinguished if its strong dual $E'[\beta(E', E)]$ is barrelled.

COROLLARY 2.3. *Let E be a Fréchet space with a shrinking Toeplitz decomposition (P_k) with respect to a row-finite matrix T. Then E is distinguished if and only if every E_k is distinguished.*

Proof. By our hypothesis, (P_k') is a Toeplitz decomposition of $E'[\beta(E', E)]$ so that $(T_n' u)$ is a $E'[\beta(E', E)]$-bounded sequence for every $u \in E'$. On the other hand, we know that the operators (T_n') are $\sigma(E', E)$-continuous in E'. A simple duality argument that uses these facts shows now that (P_k') is a simple, even equicontinuous, decomposition of $E'[\beta(E', E)]$. But it is well known that $E'[\beta(E', E)]$ is a sequentially complete (DF)-space, so the conclusion follows from Corollary 2.2. ∎

COROLLARY 2.4. *Let E be a Fréchet space with a finite-dimensional shrinking Toeplitz decomposition (P_k) with respect to a row-finite matrix T. Then E is distinguished.*

EXAMPLES. The $\aleph_0$-quasibarrelledness hypothesis cannot be suppressed in the results of this section. For instance, the unit vectors $(e^{[k]})$ form an equicontinuous Schauder basis of l^∞ endowed with its Mackey topology $\mu(l^\infty, l^1)$ and this space is not barrelled [19, Chap. 2, Sect. 1.4(26, 28), Sect. 1.5(7, 8), Sect. 1.6(7)]. Another example is $\varphi[\sigma(\varphi, \omega)]$.

APPLICATION TO CESÀRO BASES

Next in importance to Schauder bases are Cesàro bases. If Schauder bases can be seen as one-dimensional decompositions with respect to the matrix Σ of ordinary summability, then Cesàro bases are one-dimensional decompositions with respect to the matrix C_1 of Cesàro summability,

$$C_1 = \begin{bmatrix} 1 & & & & \\ 1 & 1/2 & & & \\ 1 & 2/3 & 1/3 & & \\ 1 & 3/4 & 2/4 & 1/4 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

As we pointed out above, C_1 is the first post-Σ example of a Zeller–Buntinas triangle. If E is a locally convex space having a Cesàro basis then its corresponding β_{C_1}-dual is called its c-dual [6]. Our results above read as follows for this particular case.

THEOREM 3. *Let E be a locally convex space having a Cesàro basis.*

(i) *If E is normed and the basis is shrinking, then E is barrelled if and only if $E' = E^c$.*

(ii) *If E is a Banach space and the basis is shrinking, then a subspace F of E is barrelled provided that $F^c = E^c$.*

(iii) *If E is a (DF)-space, then E is quasibarrelled.*

(iv) *If E is a sequentially complete (DF)-space, then E is barrelled.*

(v) *If E is a Fréchet space and the basis is shrinking, then E is distinguished.*

RELATIONSHIP WITH GENERALIZED SECTIONAL CONVERGENCE

As we mentioned in the Introduction, Ruckle and Saxon's paper [17] contains a number of interesting barrelledness results obtained in a framework that is similar to the Toeplitz decomposition environment used here. Let us describe it briefly.

A *generalized sectional convergence scheme* is a sequence (S_k) of finitely nonzero matrices that converges coordinatewise to the identity matrix. A K-space λ is said to have property (S_k)-AK if the sequence $(S_k x)$ converges to x for every x in λ. So a triangle T generates a generalized sectional convergence scheme by taking S_k as the diagonal matrix whose diagonal elements are the elements of the kth row of T; i.e., what Ruckle and Saxon call a *diagonal sectional convergence scheme*. In this case, property (S_k)-AK equals property T-AK. In this sense, Ruckle and Saxon's framework is bigger than ours; on the other hand, our framework is bigger as we deal with Toeplitz decompositions and not with Toeplitz bases.

Their barrelledness results are linked to the notion of the $\beta\varphi$-topology: Every vector sequence space λ containing φ can be topologized by means of the strong topology $\beta(\lambda, \varphi)$, called the $\beta\varphi$-topology of λ [16]. They show, for instance, that "if λ is a barrelled K-space with property (S_k)-AK, then λ carries its $\beta\varphi$ topology." And also that "if λ is a sequentially complete K-space with property (S_k)-AK, then λ is barrelled if and only if it carries its $\beta\varphi$ topology."

As a corollary of these theorems and our theorems above we may give the following result.

PROPOSITION 1. *Let λ be a K-space with property T-AK with respect to a triangle T.*

(i) *If λ is normed, λ' has also property T-AK, and $\lambda' = \lambda^{\beta_T}$, then λ carries its $\beta\varphi$-topology.*

(ii) *If λ is a sequentially complete (DF)-space, then λ carries its $\beta\varphi$-topology.*

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