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## Extensions of topological algebras

ANTONIO FERNANDEZ, MIGUEL FLORENCIO AND PEDRO J. PAUL\*

*Depto. Matemática Aplicada, E. S. Ingenieros Industriales, Av. Reina Mercedes s/n, 41012 Sevilla, Spain*

VLADIMÍR MÜLLER

*MÚ ČSAV, Žitná 25, 11567 Praha 1, Czechoslovakia*

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### ABSTRACT

We prove that, in the class of commutative topological algebras with separately continuous multiplication, an element is permanently singular if and only if it is a topological divisor of zero. This extends the result given by R. Arens [1] for the Banach algebra case. We also give sufficient conditions for non-removability of ideals in commutative topological algebras with jointly continuous multiplication.

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### Introduction

By a topological algebra we mean a topological vector space with a jointly continuous multiplication making of it a complex algebra. The topology of a topological algebra  $A$  can be given by a system  $\mathcal{U}$  of zero-neighbourhoods satisfying the following properties:

- (i) For every  $V \in \mathcal{U}$ , there exists  $W \in \mathcal{U}$  such that  $W + W \subset V$ .

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- (ii) For every  $V \in \mathcal{U}$  and  $\alpha \in \mathbb{C}$  with  $|\alpha| \leq 1$ ,  $\alpha V \subset V$ .
- (iii) Every  $V \in \mathcal{U}$  is absorbent.
- (iv) For every  $V \in \mathcal{U}$ , there exists  $W \in \mathcal{U}$  such that  $W \cdot W \subset V$ .

Every algebra in this paper will be a commutative complex algebra with unit element denoted usually by  $e$ .

A locally convex algebra is a topological algebra with a system of convex zero-neighbourhoods. The topology of a locally convex algebra  $A$  can be given by a directed system of seminorms  $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$  (in this case, (iv) above can be written as follows: for every  $\alpha \in \mathcal{D}$  there exists  $\beta \in \mathcal{D}$  such that  $|xy|_\alpha \leq |x|_\beta |y|_\beta$  for all  $x, y \in A$ ).

Let  $A$  and  $B$  be topological algebras with units  $e_A$  and  $e_B$ , respectively. We say that  $B$  is an extension of  $A$  if there exists a unit preserving, injective algebra homomorphism  $f : A \rightarrow B$  such that  $A$  is topologically isomorphic to its image  $f(A)$ . In this case, we identify  $A$  with  $f(A)$  and simply write  $A \subset B$ .

Let  $A$  be a topological algebra and  $I \subset A$  an ideal. We say that  $I$  is removable if there exists an extension  $B \supset A$  such that  $I$  is not contained in any proper ideal of  $B$ . It is easy to see that this condition is equivalent to the existence of a finite number of elements  $x_1, \dots, x_k \in I$  and  $y_1, \dots, y_k \in B$  such that  $x_1 y_1 + \dots + x_k y_k = e$ . An ideal which is not removable will be called non-removable. The notion of non removable ideal was introduced by R. Arens [2]. Non-removable ideals in commutative Banach algebras have been studied, e.g., in [2, 6, 4, 5] and in topological algebras in [8].

## 1. Non-removable ideals in topological algebras

The aim of this section is to give a sufficient condition for an ideal in a topological algebra to be non-removable. This condition will be shown to be more general than the one given in [8]. However, it seems that there is no simple necessary and sufficient condition characterizing non-removability. Our result will be reformulated also for permanently singular elements.

### Theorem 1

*Let  $A$  be a commutative topological algebra with unit  $e$ , and  $\mathcal{U}(A)$  a system of zero-neighbourhoods defining the topology of  $A$  and satisfying (i)–(iv). Let  $I \subset A$  be an ideal such that*

- (1) *For every finite subset  $\{x_1, \dots, x_k\} \subset I$  there exists  $V \in \mathcal{U}(A)$  such that for all  $W \in \mathcal{U}(A)$  there is an  $n \geq 1$  such that, for every  $r > 0$ , there exists  $u \in A \setminus V$  satisfying  $ux_i^n \in rW$  ( $i = 1, \dots, k$ )*

*then  $I$  is non-removable.*

*Proof.* Suppose, on the contrary, that there exists an extension  $B \supset A$ , and elements  $x_1, \dots, x_k \in I$ ,  $y_1, \dots, y_k \in B$  such that  $x_1 y_1 + \dots + x_k y_k = e$ . Let  $\mathcal{U}(B)$  be a system of zero-neighbourhoods for the topology of  $B$ . Let  $V \in \mathcal{U}(A)$  be the neighbourhood given by condition (1). Take  $V', W' \in \mathcal{U}(B)$  such that  $V' \cap A \subset V$  and

$$\underbrace{W'W' + \dots + W'W'}_{k \text{ times}} \subset V',$$

and  $W \in \mathcal{U}(A)$  satisfying  $W \subset W' \cap A$ . Let  $n$  be the integer from condition (1) (for  $V$  and  $W$ ) and  $m = k(n-1) + 1$ . Then we have

$$e = e^m = \left( \sum_{i=1}^k x_i y_i \right)^m = \sum_{i_1 + \dots + i_k = m} \frac{m!}{i_1! \dots i_k!} (x_1 y_1)^{i_1} \dots (x_k y_k)^{i_k}.$$

In every term of this sum at least one exponent  $i_j \geq n$ , so that, for some  $v_i \in B$ , we may write

$$e = \sum_{i=1}^k x_i^n v_i.$$

Take  $s > 0$  such that  $v_i \subset sW'$  for  $i = 1, \dots, k$ , let  $r = s^{-1}$  and take  $u \in A \setminus V$  given by condition (1). Then

$$u \in A \setminus V \subset B \setminus V'$$

but, on the other hand

$$u x_i^n v_i - (u x_i^n) v_i \in rW' \cdot sW' \subset W' \cdot W' \quad (i = 1, \dots, k)$$

and therefore

$$u - uc = \sum_{i=1}^k u x_i^n v_i \in \underbrace{W'W' + \dots + W'W'}_{k \text{ times}} \subset V',$$

a contradiction.  $\square$

*Remark 1.* For a locally convex algebra  $A$ , with the topology given by a system of seminorms  $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$  condition (1) can be reformulated as follows:

(1') For every finite subset  $\{x_1, \dots, x_k\} \subset I$ , there exists  $\alpha \in \mathcal{D}$ , such that, for every  $\beta \in \mathcal{D}$ , there is some  $n \geq 1$  satisfying

$$\inf \left\{ \sum_{i=1}^k |u x_i^n|_\beta : u \in A, |u|_\alpha = 1 \right\} = 0.$$

Therefore, if  $I$  is an ideal in  $A$  satisfying (1') then it is non removable.

*Remark 2.* In [8, Prop. 2.18] was given the following sufficient condition for the non-removability of an ideal  $I$  in a topological algebra  $A$  with a system of zero-neighbourhoods  $\mathcal{U}$ :

- (2)  $I$  is contained in an ideal  $J = I_1 + I_s(A)$  where:  
 $I_1$  consists locally of joint topological divisors of zero, i.e., for every finite  $\{y_1, \dots, y_r\} \subset I_1$  there exists a net  $\{u_\gamma\}_\gamma \subset A$  such that  $u_\gamma \not\equiv 0$  but  $u_\gamma y_i \equiv 0$ , for  $i = 1, \dots, r$ .  
 $I_s(A)$  is the set of all elements of  $A$  with small powers:  $z \in A$  is said to have small powers if for every zero-neighbourhood  $V$  there exists an integer  $n \geq 1$  such that  $\lambda z^n \in V$  for all  $\lambda \in \mathbb{C}$ .

### Proposition

Let  $A$  be a topological algebra and  $I \subset A$  an ideal satisfying (2), then  $I$  satisfies (1).

*Proof.* Let  $\mathcal{U}$  be a system of zero neighbourhoods in  $A$  satisfying (i)–(iv). To see that  $I$  satisfies condition (1), take  $x_1, \dots, x_k \in I$ . Then, since  $I$  satisfies condition (2), we can find  $y_1, \dots, y_k \in I_1$  and  $z_1, \dots, z_k \in I_s(A)$  such that  $x_i = y_i + z_i$  for  $i = 1, \dots, k$ . It is easy to see that the  $y_i$ 's and the  $z_i$ 's satisfy the following conditions:

- (a) There exists  $V \in \mathcal{U}$  such that, for every  $W \in \mathcal{U}$ , there is some  $u \in A \setminus V$  satisfying  $uy_i \in W$  for  $i = 1, \dots, k$ .  
(b) For every  $U \in \mathcal{U}$ , there is some  $n \geq 1$  satisfying  $z_i^n \in \bigcap_{r>0} rU$  for  $i = 1, \dots, k$ .

Let  $V \in \mathcal{U}$  be given by (a), and for  $W \in \mathcal{U}$  arbitrary take  $U \in \mathcal{U}$  such that  $UU \div UU \subset W$ . Let  $n \geq 1$  be the integer from (b), then we can write:

$$x_i^n = (y_i + z_i)^n = z_i^n + y_i \left[ \sum_{j=1}^n \binom{n}{j} y_i^{j-1} z_i^{n-j} \right] = z_i^n + y_i v_i \quad (i = 1, \dots, k)$$

for some  $v_1, \dots, v_k \in A$ . Fix  $r > 0$  and let  $s > 0$  be such that  $v_i \in sU$  for  $i = 1, \dots, k$ , then by using (a) we can find  $u \in A \setminus V$  such that

$$uy_i \in rs^{-1}U \quad (i = 1, \dots, k).$$

Therefore, we can write  $ux_i^n = uz_i^n + (uy_i)v_i$  where, for some  $t > 0$ ,

$$uz_i^n \in (tU) \left( \bigcap_{r'>0} r't \right) \subset \bigcap_{r'>0} r'tU \subset r'tU$$

and, on the other hand,

$$(uy_i)v_i \subset rs^{-1}U \cdot sU \subset r'tU.$$

Hence

$$ux_i^n \subset r'tU + r'tU \subset r'W \quad (i = 1, \dots, k),$$

which proves that  $I$  satisfies (1).  $\square$

An element  $x$  of a topological algebra  $A$  is called permanently singular if  $x$  is singular in every extension  $B \supset A$ . Clearly,  $x \in A$  is permanently singular if and only if the ideal  $xA$  generated by  $x$  is non-removable. Therefore Theorem 1 yields the following

**Corollary**

Let  $A$  be a commutative topological algebra with unit  $e$ , and a system of zero-neighbourhoods  $\mathcal{U}$  satisfying (i)-(iv). Suppose that  $x \in A$  satisfies the following condition

- (3) there exists  $V \in \mathcal{U}$  such that, for each  $W \in \mathcal{U}$ , there is some  $n \geq 1$  satisfying  $(A \setminus V)x^n \cap rW \neq \emptyset$  for every  $r > 0$ ,

then  $x$  is permanently singular.

*Remark.* If  $A$  is a locally convex algebra, and  $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$  is the corresponding system of seminorms, condition (3) may be written as follows:

- (3') there exists  $\alpha \in \mathcal{D}$  such that, for each  $\beta \in \mathcal{D}$ , there is some  $n \geq 1$  satisfying

$$\inf\{|zx^n|_\beta : z \in A, |z|_\alpha = 1\} > 0.$$

We construct now an example showing that condition (1) is more general than (2) even in the case of simply generated ideals in locally convex algebras.

EXAMPLE. Let  $A$  be the algebra of all polynomials with complex coefficients in the variable  $x$ , endowed with the topology given by the system of seminorms  $|\cdot|_k$ ,  $k = 1, 2, \dots$  defined by:

$$\left| \sum_{i=0}^{\infty} \alpha_i x^i \right|_k = \sum_{i=0}^{\infty} c_{ki} |\alpha_i| \quad (k = 1, 2, \dots)$$

(actually, all sums are finite) where  $c_{ki}$  ( $k = 1, 2, \dots; i = 0, 1, 2, \dots$ ) are positive numbers satisfying:

- (a)  $c_{k,0} = 1$
- (j)  $c_{k,i+j} \leq c_{k+1,i} c_{k+1,j}$
- (γ)  $c_{k+1,i} \geq c_{k,i}$
- (δ)  $c_{k+1,i+1} \geq c_{k,i}$
- (c)  $\inf \left\{ \frac{c_{k,k+j}}{c_{1,j}} : j = 0, 1, \dots \right\} > 0.$

Conditions  $(\beta)$  and  $(\gamma)$  imply that  $A$  is a locally convex algebra. It is clear, since all  $c_{ik} > 0$ , that  $|\sum \alpha_i x^i|_k > 0$  for every non-zero polynomial  $\sum \alpha_i x^i$  and every index  $k = 1, 2, \dots$ , this means that there are no elements with small powers in  $A$ . Condition  $(\delta)$  implies  $|ax|_{k+1} \geq |a|_k$  for all  $a \in A$  and  $k = 1, 2, \dots$ , i.e.  $x$  is not a topological divisor of zero in  $A$ . Therefore, the ideal  $xA$  does not satisfy (2).

On the other hand,  $x$  satisfies (3'): take  $a = 1$ , and for arbitrary seminorm  $|\cdot|_k$  put  $n = k$ , then, by condition (c):

$$\inf \{ |ux^k|_k : u \in A, |u|_1 = 1 \} < \inf_{j \geq 0} \left\{ \frac{|x^{k+j}|_k}{|x^j|_1} \right\} = \inf_{j \geq 0} \left\{ \frac{c_{k,k+j}}{c_{1,j}} \right\} = 0.$$

It remains to show that it is possible to find numbers  $c_{ki}$  satisfying  $(\alpha) - (c)$ . To see this, assume we can construct sets  $M_k \subset \{0, 1, 2, \dots\}$ ,  $k = 1, 2, \dots$  satisfying  $0 \in M_k$  and

- a)  $M_{k+1} + M_{k+1} \subset M_k$ .
- b)  $M_{k+1} - 1 \subset M_k$ .
- c)  $M_{k+1} \subset M_k$  and
- d) For every pair  $n, k$  such that  $k \geq 1$ ,  $n \geq k$ , there exists  $m$  such that  $m, m+1, \dots, m+n \notin M_1$ , and  $m+n+k \in M_k$ .

Now take

$$c_{k,i} = 2^{i - \max\{j \leq i, j \in M_k\}} \quad (i = 0, 1, \dots, \quad k = 1, 2, \dots).$$

It is a matter of routine to check that the above properties of the sets  $M_k$  imply  $(\alpha) - (b)$  for  $c_{ki}$ . To prove (c) consider the infimum over those  $j = n + m$ , where  $m$  is the index existing for given  $k$  and  $n \geq k$ , i.e.,

$$\inf \left\{ \frac{c_{k,n+j}}{c_{1,j}} : j > 0 \right\} \leq \inf \left\{ \frac{c_{k,n+m+k}}{c_{1,n+m}}, n \geq k \right\} \leq \inf \left\{ \frac{1}{2^n}, n \geq k \right\} = 0.$$

The sets  $M_k$  can be constructed as follows: put

$$N_k^{(0)} = \{2^{2^i}, i \geq k\}$$

$$N_k^{(r)} = N_{k+1}^{(r-1)} \cup \left( N_{k+1}^{(r-1)} - 1 \right) \cup \bigcup_{s=0}^{r-1} \left( N_{k+1}^{(r-1)} + N_{k+1}^{(s)} \right)$$

for  $k, r = 1, 2, \dots$ , and now take

$$M_k = \bigcup_{r=0}^{\infty} N_k^{(r)} \cup \{0\} \quad (k = 1, 2, \dots).$$

Clearly  $M_{k+1} \subset M_k$  since  $N_{k+1}^{(r-1)} \subset N_k^{(r)}$  for  $r = 1, 2, \dots$ . The properties  $M_{k+1} - \{0\} \subset M_k$  and  $M_{k+1} + M_{k+1} \subset M_k$  can be checked analogously.

Finally, fix  $k$  and  $n > k$  and put  $m = 2^{2^n} - 2n$ , then  $2^{2^n} \in M_n$ ,  $2^{2^n} - 1 \in M_{n-1}$  and by induction  $m + n + k = 2^{2^n} - (n - k) \in M_k$ . It remains to prove that  $m, m + 1, \dots, m + n \notin M_1$ . First note that for  $j = 1, 2, \dots$   $r = 0, 1, 2, \dots$  we have:

$$\min N_j^{(r)} = \min N_{j+1}^{(r-1)} - 1 = \dots = \min N_{j+r}^{(0)} \quad r = 2^{j+r} - r.$$

Further, the open interval  $(2^{2^n - 1}, 2^{2^n})$  and  $N_j^{(0)}$  are disjoint:

$$N_j^{(0)} \cap (2^{2^n - 1}, 2^{2^n}) = \emptyset,$$

and it is easy to prove, by induction on  $r$ , that, as a matter of fact, we have:

$$N_j^{(r)} \cap (2^r 2^{2^n - 1}, 2^{2^n} - r) = \emptyset$$

for every  $r = 0, 1, \dots, n - 1$  and  $j = 1, 2, \dots$ . Therefore,

$$\begin{aligned} M_1 \cap [2^{2^n} - 2n, 2^{2^n} - n] &= \bigcup_{r=0}^{n-1} \{N_1^{(r)} \cap [2^{2^n} - 2n, 2^{2^n} - n]\} \subset \\ &\subset \bigcup_{r=0}^{n-1} \{N_1^{(r)} \cap (2^r 2^{2^n - 1}, 2^{2^n} - r)\} \\ &= \emptyset. \end{aligned}$$

Hence  $m = 2^{2^n} - 2n, m + 1, \dots, m + n = 2^{2^n} - n \notin M_1$ .

## 2. Topological divisors of zero in s-algebras

In this section we deal with algebras having multiplication only separately continuous. These algebras have been also called topological algebras by some authors [7]. To avoid misunderstanding, these algebras will be called s-algebras in this paper.

In terms of zero-neighbourhoods, the difference is that for an s-algebra  $A$  we assume (i) (iii) plus the following (iv') which is weaker than (iv):

(iv') For every  $V \in \mathcal{U}$  and  $x \in A$ , there exists  $W \in \mathcal{U}$  such that  $xW \subset V$ .

An element  $x$  of an s-algebra  $A$  is said to be a topological divisor of zero if there exists a net  $\{u_\alpha\}_\alpha \subset A$  such that  $u_\alpha \neq 0$  but  $u_\alpha x \rightarrow 0$ . Clearly,  $x$  is not a topological



divisor of zero if and only if the mapping  $f_x(a) := xa$  is a homeomorphism from  $A$  onto  $xA$ . The notion of  $s$ -extension is defined analogously to the notion of extension for topological algebras. Let  $A$  be an  $s$ -algebra and  $x \in A$  be a topological divisor of zero, then  $x$  is singular in any  $s$ -extension  $B \supset A$ . If this were not the case, we could find an  $s$ -extension  $B \supset A$  and  $y \in B$  such that  $xy = e$ . But for  $(u_\alpha)_\alpha$ , the net in  $A$  such that  $u_\alpha \neq 0$  and  $u_\alpha x = 0$  we would have  $u_\alpha e = (u_\alpha x)y = 0$  (by the separate continuity of multiplication in  $B$ ), a contradiction.

The purpose of this section is to prove the converse of the statement above. This will mean that in the class of  $s$ -algebras there exists a simple characterization of permanently singular elements, similar to the one that holds for Banach algebras (recall that if  $A$  is a Banach  $s$ -algebra, then  $A$  is a Banach algebra by the Banach-Steinhaus theorem).

Let  $A$  be an  $s$ -algebra with unit  $e$  and  $\mathcal{U}$  a system of zero-neighbourhoods in  $A$  satisfying (i) (iii) and (iv'). Let  $A[x]$  be the algebra of all polynomials with coefficients from  $A$  in one variable  $x$ . We define a topology in  $A[x]$  in the following way: Let  $V = (V_i)_{i=0}^\infty$  be a sequence from  $\mathcal{U}$  and define

$$N_V = \left\{ \sum_{i=0}^n a_i x^i \in A[x] : a_i \in V_i, i = 0, 1, \dots \right\}.$$

Let  $\mathcal{V}$  be the set of all  $N_V$  obtained from all sequences  $V$ . It is easy to see that  $\mathcal{V}$  satisfies (i), (ii), (iii) and (iv'), therefore  $A[x]$  is an  $s$ -algebra (if we identify  $A[x]$  with the countable direct sum of copies of  $A$  by means of

$$\sum_{i=0}^n a_i x^i \in A[x] \longrightarrow (a_0, a_1, \dots, a_n, 0, \dots) \in \bigoplus_{m=0}^\infty A$$

the topology defined above is precisely the direct sum topology). By identifying elements of  $A$  with constant polynomials we see that  $A[x]$  is an  $s$ -extension of  $A$ . Moreover, if  $A$  is a locally convex  $s$ -algebra, then  $A[x]$  is also locally convex.

Let  $A$  be an  $s$ -algebra and  $I \subset A$  a closed ideal. Then  $A/I$  is again an  $s$ -algebra. To see this we only need to prove (iv'): let  $a + I \in A/I$  and let  $V + I$  be a zero-neighbourhood in  $A/I$ . Take  $W$  such that  $aW \subset V$ , then

$$(a + I)(W + I) \subset aW + aI + IW + I^2 \subset V + I.$$

## Theorem 2

*Let  $A$  be an  $s$ -algebra with unit  $e$  and  $u \in A$ . Then  $u$  is invertible in some  $s$ -extension  $B \supset A$  if and only if  $u$  is not a topological divisor of zero in  $A$ .*

*Proof.* One implication was proved above. Conversely, assume that  $u$  is not a topological divisor of zero in  $A$ , i.e., that  $a \mapsto au$  is a homeomorphism from  $A$  onto  $uA$ . This means that for every  $V \in \mathcal{U}$  there exists  $V' \in \mathcal{U}$  such that  $V' \cap uA = uV$ . Consider the  $s$ -algebra  $A[x]$  and let  $I$  be the ideal generated by  $e - ux$ ,  $I = (e - ux)A[x]$ . We prove firstly that  $I$  is closed in  $A[x]$ : Let  $(p_\alpha)_\alpha$  be a net of elements from  $I$ ,

$$p_\alpha = (e - ux) \sum_{i=0}^{\infty} b_i^{(\alpha)} x^i = b_0^{(\alpha)} + \sum_{i=0}^{\infty} (b_i^{(\alpha)} - ub_{i-1}^{(\alpha)}) x^i$$

(where only a finite number of coefficients  $b_i^{(\alpha)}$  are non-zero for every  $\alpha$ ) and suppose that  $p_\alpha \rightarrow p = \sum_{i=0}^n a_i x^i$  in the topology of  $A[x]$ . Then, coordinatewise, we have:

$$\begin{aligned} b_0^{(\alpha)} &\rightarrow a_0, \\ b_i^{(\alpha)} - ub_{i-1}^{(\alpha)} &\rightarrow a_i \quad \text{for } i = 1, \dots, n, \\ b_i^{(\alpha)} - ub_{i-1}^{(\alpha)} &\rightarrow 0 \quad \text{for } i > n. \end{aligned}$$

Since  $b_0^{(\alpha)}u \rightarrow a_0u$ , we have  $b_1^{(\alpha)} \rightarrow a_1 + a_0u$ , and inductively:

$$\begin{aligned} b_i^{(\alpha)} &\rightarrow c_i := a_i + a_{i-1}u + \dots + a_0u^i \quad \text{for } i = 0, \dots, n, \\ b_i^{(\alpha)} &\rightarrow a_nu^{i-n} + a_{n-1}u^{i-n+1} + \dots + a_0u^i = c_nu^{i-n} \quad \text{for } i > n \end{aligned}$$

where  $c_n = a_n + a_{n-1}u + \dots + a_0u^n$ . Suppose  $c_n \neq 0$  and let  $V_0 \in \mathcal{U}$  such that  $c_n \notin V_0$ . Let  $W_0 \in \mathcal{U}$  such that  $W_0 + W_0 \subset V_0$ . Construct  $V_i, W_i \in \mathcal{U}$  such that:

$$V_{i+1} \cap uA \subset uW_i \quad \text{and} \quad W_{i+1} + W_{i+1} \subset V_{i+1} \quad \text{for } i = 0, 1, 2, \dots$$

and consider the zero-neighbourhood  $N_W$  in  $A[x]$  given by the sequence

$$\underbrace{(W_0, \dots, W_0)}_{n \text{ times}}, W_0, W_1, W_2, \dots.$$

Since  $p_\alpha \rightarrow p$  and  $b_n^{(\alpha)} \rightarrow c_n$ , there exists an index  $\alpha$  such that

$$b_{n+i}^{(\alpha)} - ub_{n+i-1}^{(\alpha)} \in W_i \quad (i = 1, 2, \dots)$$

and also

$$b_n^{(\alpha)} \dots c_n \in W_0.$$

This implies  $b_n^{(\alpha)} \notin W_0$  since  $c \notin V_0$ . We prove now, by induction, that  $b_{n+i}^{(\alpha)} \notin W_i$  for  $i = 0, 1, \dots$ : suppose  $b_{n+i}^{(\alpha)} \notin W_i$ , then  $ub_{n+i}^{(\alpha)} \notin uW_i$ , and so  $ub_{n+i}^{(\alpha)} \notin V_{i+1}$ . Write

$$ub_{n+i}^{(\alpha)} = \left(-b_{n+i+1}^{(\alpha)} + ub_{n+i}^{(\alpha)}\right) + b_{n+i+1}^{(\alpha)}$$

to deduce that  $b_{n+i+1}^{(\alpha)} \notin W_{i+1}$ . Therefore, we have that  $b_{n+i}^{(\alpha)} \notin W_i$  for  $i = 0, 1, \dots$  which implies  $b_{n+i}^{(\alpha)} \neq 0$  for all  $i \geq 0$  and this contradicts the fact that  $\sum b_i^{(\alpha)} x^i$  is a polynomial and, consequently, has only a finite number of non-zero coefficients. We have proved that  $c_n = 0$  and therefore  $p$ , the limit of  $p_\alpha$ , can be written as:

$$p = \sum_{i=0}^n a_i x^i = (c - ux) \sum_{i=0}^{n-1} c_i x^i \in I = (c - ux)A[x].$$

Now, let  $q : A[x] \rightarrow A[x]/I$  be the canonical homomorphism and let  $g : A \rightarrow A[x]$  be the natural embedding. Denote by  $f := q \circ g$ . Since  $c - ux \in I$ , we have  $(u+I)(x+I) = c+I$ , hence  $f(u)$  is invertible in  $A[x]/I$ . Finally, we must check that  $A[x]/I$  is an s-extension of  $A$ . Clearly  $f$  is a continuous algebra homomorphism. To prove that  $f$  is 1-1 and  $f(A)$  is topologically isomorphic to  $A$ , it suffices to prove that for all  $V \in \mathcal{U}$  we can find  $N_W \in \mathcal{V}$  such that  $(N_W + I) \cap f(A) \subset V$ : For  $V \in \mathcal{U}$ , let  $W_0 \in \mathcal{U}$  be such that  $W_0 + W_0 \subset V$  (hence  $W_0 \subset V$ ). Choose  $V_1 \in \mathcal{U}$  such that  $ua \in V_1$  implies  $a \in W_0$ , and take  $W_1 \in \mathcal{U}$  such that  $W_1 + W_1 \subset V_1$ . Define inductively neighbourhoods  $V_i, W_i \in \mathcal{U}$  such that

$$ua \in V_{i+1} \text{ implies } a \in W_i,$$

$$W_{i+1} + W_{i+1} \subset V_{i+1}, \text{ (hence } W_{i+1} \subset V_{i+1}).$$

Let  $N_W \in \mathcal{V}$  be the neighbourhood corresponding to the sequence  $(W_i)_{i=0}^\infty$ . To show that  $(N_W + I) \cap f(A) \subset V$ , take  $a \in A$ ,

$$p = (c - xu) \sum_{i=0}^n b_i x^i \in I$$

and  $a + p \in N_W$ . We have:

$$a + p = (a + b_0) + x(b_1 - ub_0) + x^2(b_2 - ub_1) + \dots + x^{n+1}(-ub_n).$$

Since  $ub_n \in W_{n+1} \subset V_{n+1}$  we have  $b_n \in W_n$ . Furthermore,

$$b_{n-1}u = (b_{n-1}u - b_n) + b_n \in W_n + W_n \subset V_n,$$

so that  $b_{n-1} \in W_{n-1}$ . We continue in the same way and obtain  $b_i \in W_i$  for  $i = n-1, \dots, 1, 0$ . Finally, since  $b_0 \in W_0$ ,  $a = a + b_0 - b_0 \in W_0 + W_0 \subset V$ .  $\square$

References

1. R. Arens, Extensions of Banach algebras, *Pacific J. Math.* **10** (1960), 1–16.
2. R. Arens, Ideals in Banach algebras extensions, *Studia Math.* **31** (1968), 29–34.
3. B. Bollobás, Adjointly inverses to commutative Banach algebras, *Trans. Amer. Math. Soc.* **181** (1973), 165–174.
4. V. Müller, Non-removable ideals in commutative Banach algebras, *Studia Math.* **74** (1982), 97–104.
5. V. Müller, Removability of ideals in commutative Banach algebras *Studia Math.* **78** (1984), 297–307.
6. W. D. Żelazko, Concerning a problem of Arens on removable ideals in Banach algebras, *Collect. Math.* **30** (1974), 127–131.
7. W. D. Żelazko, Metric Generalizations of Banach Algebras, *Rozprawy Math.* **47** (1965).
8. W. D. Żelazko, On non-removable ideals in commutative locally convex algebras, *Studia Math.* **77** (1983), 133–154.

