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Complemented copies of c_0 in vector-valued Köthe-Dieudonné function spaces

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Abstract

Let Λ be a barrelled perfect (in the sense of J. Dieudonné) Köthe space of measurable functions defined on an atomless finite Radon measure space. Let X be a Banach space containing a copy of c_0, then the space $\Lambda(X)$ of Λ-Bochner integrable functions contains a complemented copy of c_0.

G. Emmanuele [5] proved that if a Banach space X contains a copy of c_0, then the spaces $L^p(\mu, X)$ $(1 \leq p < \infty)$ of p-Bochner integrable functions contain a complemented copy of c_0 when μ is any atomless finite measure space. Here, we want to show that the same phenomenon occurs in the space $\Lambda(X)$ of Λ-Bochner integrable functions, where Λ is a barrelled perfect (in the sense of J. Dieudonné [3]) Köthe space of measurable functions defined on an atomless finite Radon measure space. We shall give the proof in the case of the Lebesgue measure on the unit interval; it can be modified easily to cover the general case.

Although we refer the reader to [2], [3] and [8] for the terminology used in this paper, we will start by recalling the main definitions.

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In what follows, μ stands for the Lebesgue measure defined on the unit interval $[0,1]$. For a subset $\Lambda \subset L^1(\mu)$ its Köthe-dual is the set $\Lambda^\times$ of all $f \in L^1(\mu)$ such that $f \cdot g$ is integrable for each $g \in \Lambda$. Note that $L^\infty(\mu) \subset \Lambda^\times$. We have $\Lambda \subset \Lambda^{\times\times} := (\Lambda^\times)^\times$ and Λ is said to be a perfect Köthe space if $\Lambda = \Lambda^{\times\times}$ [3]. If Λ is perfect, then Λ and $\Lambda^\times$ are put into duality by the canonical bilinear form:

$$(f,g) \in \Lambda \times \Lambda^\times \longrightarrow \langle f,g \rangle := \int_{[0,1]} f(t) \cdot g(t)\, d\mu(t).$$

The strong topology $\beta(\Lambda,\Lambda^\times)$ equals the polar topology generated by the family $\mathcal{B}$ of all solid (or normal) and $\sigma(\Lambda^\times,\Lambda)$-bounded subsets of $\Lambda^\times$, and it is given by the seminorms

$$p_M : f \in \Lambda \longrightarrow p_M(f) := \sup\left\{ \int_{[0,1]} \left|f(t) \cdot g(t)\right| d\mu(t) \ : \ g \in M \right\} \qquad (M \in \mathcal{B}).$$

In what follows, Λ remains a perfect Köthe space endowed with its strong topology $\beta(\Lambda,\Lambda^\times)$. Let X be a Banach space with dual X'. We define $\Lambda(X)$ as the space of (classes of) strongly measurable functions $f : [0,1] \to X$ such that the composition $\|f(\cdot)\|$ is a function in Λ; such functions are called Λ-Bochner integrable. From the strong topology $\beta(\Lambda,\Lambda^\times)$ on Λ and the norm on X, we can generate in $\Lambda(X)$ a natural topology defined by the seminorms

$$q_M(f) := p_M(\|f(\cdot)\|) = \sup\left\{ \int_{[0,1]} \|f(t)\| \cdot |g(t)|\, d\mu(t) \ : \ g \in M \right\} \qquad (M \in \mathcal{B}).$$

We will refer to this topology as the natural topology of $\Lambda(X)$. When $\Lambda = L^p(\mu)$ with $1 \leq p \leq +\infty$, we obtain the space $L^p(\mu, X)$ of p-Bochner integrable functions [2, (p. 97)]. In particular, $L^\infty(\mu, X)$ is the space of strongly measurable, essentially bounded functions from $[0,1]$ into X with the ess-sup norm.

The locally convex structure of these spaces has been studied by A. L. Macdonald [9], N. Phuong-Các [10] and two of the present authors in collaboration with C. Sáez [6], [7], among others. In particular, we know that $\Lambda(X)$ is complete [10, Prop. 2] and that it is barrelled if so is Λ [7, Thm. 3]. We shall make use of these two facts in the proof below.

Theorem

Let X be a Banach space containing a copy of c_0. Let Λ be a perfect Köthe function space such that $\Lambda(\beta(\Lambda,\Lambda^\times))$ is barrelled. Then $\Lambda(X)$ contains a complemented copy of c_0.

Proof. Let (x_n) be a sequence in X equivalent to the standard basis of c_0, i.e. there are positive constants α and β such that for all finite sequences $a_1, a_2, \ldots, a_n$ of real numbers we have

$$\alpha \max\{|a_i| : 1 \le i \le n\} \le \left\| \sum_{i=1}^{n} a_i x_i \right\| \le \beta \max\{|a_i| : 1 \le i \le n\}.$$

Consider the sequence (r_n) of Rademacher functions and define a sequence of simple functions by $f_n := r_n x_n$ $(n \in \mathbb{N})$. Since $|r_n(t)| = 1$ for all $t \in [0,1]$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\alpha \max\{|a_i| : 1 \le i \le n\} \le & \left\| \sum_{i=1}^{n} a_i r_i(t) x_i \right\| \qquad \text{for all } t \in [0,1], \\ & \le \beta \max\{|a_i| : 1 \le i \le n\}\end{aligned}$$

whenever $a_1, a_2, \ldots, a_n$ are real numbers. Now let $M \in \mathcal{B}$. Multiplication by functions in M, integration over $[0,1]$, and taking supremum over M yield

$$\begin{aligned}\alpha \max\{|a_i| : 1 \le i \le n\}\, p_M(1) \le\; & q_M\Big(\sum_{i=1}^{n} a_i f_i\Big) \\ & \le \beta \max\{|a_i| : 1 \le i \le n\}\, p_M(1),\end{aligned}$$

where $p_M(1)$ is the value of the seminorm p_M on the constant function 1. Since $\Lambda(X)$ is complete, this shows that the closed linear span E of the sequence (f_n) is isomorphic to c_0.

Let us show that E is complemented in $\Lambda(X)$. Since (x_n) is equivalent to the unit basis on c_0, there is a bounded sequence (x_n^*) in X' such that $\langle x_m^*, x_n\rangle = \delta_{mn}$ (Kronecker's delta, of course) for all $m, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider the simple functions given by $f_n^* := r_n x_n^*$. For $f \in \Lambda(X)$, define

$$\langle f_n^*, f\rangle := \int_{[0,1]} r_n(t)\, \langle x_n^*, f(t)\rangle \, d\mu(t).$$

If M is any solid bounded subset of $\Lambda^\times$ containing the constant function 1 (recall that $L^\infty(\mu) \subset \Lambda^\times$), then we have that $\left|\langle f_n^*, f\rangle\right| \le \|x_n^*\|\, q_M(f)$, hence the formula above defines an element in $\Lambda(X)'$, the topological dual of $\Lambda(X)$. Let us see that the sequence (f_n^*) converges to zero in the weak topology $\sigma\big(\Lambda(X)', \Lambda(X)\big)$. Take $f \in \Lambda(X)$. The function f is Bochner integrable, and so is each $r_n f$, because $\Lambda(X) \subset L^1(\mu, X)$. Bearing this in mind, we have

$$\left|\langle f_n^*, f\rangle\right| := \left|\int_{[0,1]} r_n(t)\,\langle x_n^*, f(t)\rangle \,d\mu(t)\right| \leq \|x_n^*\| \left\|\int_{[0,1]} r_n(t) f(t)\,d\mu(t)\right\|.$$

This latter expression tends to zero as $n \to \infty$ because (x_n^*) is a bounded sequence and, as it is well-known, $\lim_n \left\|\int_{[0,1]} r_n(t)f(t)\,d\mu(t)\right\| = 0$ for every Bochner integrable function f. Since (f_n^*) converges to zero in the weak topology $\sigma\big(\Lambda(X)', \Lambda(X)\big)$ and $|\langle f_m^*, f_n\rangle| = \delta_{mn}$ for all $m, n \in \mathbb{N}$, it follows that the mapping

$$P : f \in \Lambda(X) \longrightarrow P(f) := \sum_{n=1}^{\infty} \langle f_n^*, f\rangle\, f_n \in E$$

is a well-defined linear projection from $\Lambda(X)$ onto $E \cong c_0$. It is easy to see that P has closed graph. As $\Lambda(X)$ is barrelled and E is a Banach space, we can use the Closed Graph Theorem for barrelled spaces [8, §34.6.(9)] to deduce that P is continuous. And this tells us that $E = P(\Lambda(X))$ is complemented. □

Corollary

Let X be a Banach space containing a copy of c_0. Let Λ be a perfect Köthe function space such that $\Lambda\left(\beta(\Lambda, \Lambda^\times)\right)$ is barrelled. Then $\Lambda(X)$ is neither injective nor a Grothendieck space.

Remarks. We know [1] that if $L^\infty(\mu, X)$ contains a complemented copy of c_0 then X must contain a copy of c_0 so that the statement "If Λ contains a copy of c_0 then $\Lambda(X)$ contains a complemented copy of c_0," does not hold in general.

In the case when Λ is also a Banach space, P. Dowling [4] has recently proved that $\Lambda(X)$ contains a subspace isomorphic to c_0 if and only if either Λ or X contains a copy of c_0. This result and our theorem above suggest the following conjecture: "Let Λ be a barrelled perfect Köthe function space and X be a Banach space, if $\Lambda(X)$ contains a complemented copy of c_0 then either Λ or X contains a copy of c_0."

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