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BARRELLEDNESS IN λ-SUMS OF NORMED SPACES

Miguel Florencio, Pedro J. Paúl and Carmen Sáez
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Let E_n $n=1,2,...$ be a sequence of normed spaces and λ an AK-Köthe sequence space. The λ-sum of the spaces E_n is defined by $\lambda\{E_n\}:=\{(x_n)_n: x_n$ is in E_n and $(|x_n|)_n\in\lambda\}$. Starting from the topology in λ, $\lambda\{E_n\}$ can be given a natural topology. In this paper we characterize the dual of $\lambda\{E_n\}$ as the $\lambda^\times$-sum of E'_n and give conditions for $\lambda\{E_n\}$ to be quasi-barrelled, barrelled, reflexive, bornological or distinguished.

1. λ-SUMS OF NORMED SPACES

Let λ be a sequence space which is normal in the sense of Köthe [7,§30], [13,Ch.2]. A family M which covers $\lambda^\times$ and consists of normal, absolutely convex, closed and $\sigma(\lambda^\times,\lambda)$-bounded subsets of $\lambda^\times$ induces on λ the topology τ of uniform convergence on M which, because of the normality, is given by the seminorms:

$$q_M:\beta=(\beta_n)_n\in\lambda\to q_M(\beta):=\sup\{\sum_{n=1}^{\infty}|\beta_n\alpha_n|:\alpha\in M\}$$

as M runs through M [3], [11].

If $E_n(|\cdot|)$ $n=1,2,...$ is a sequence of normed spaces over the field $\mathbb{K}$ of real or complex numbers, we define the λ-sum of E_n as $\lambda\{E_n\}:=\{(x_n)_n: x_n\in E_n$ and $(|x_n|)_n\in\lambda\}$, note that if $|x_n|\leq 1$ $n=1,2,...$ and $\alpha\in\lambda$ then $\alpha x:=(\alpha_n x_n)_n$ is in $\lambda\{E_n\}$. Also $\lambda\{E_n\}$ contains the algebraic direct sum, $\phi\{E_n\}$ where ϕ is the space of finitely non-zero sequences, of the spaces E_n. Starting from τ we define the topology τ also on $\lambda\{E_n\}$ (there is little danger of confusion) by means of the seminorms:

$$p_M: x=(x_n)_n \in \lambda\{E_n\} \to p_M(x) := q_M((\|x_n\|)_n)$$

as M runs through $\mathcal{M}$. ℓ^p-sums and c_0-sums of a sequence of normed spaces have been widely studied [2], [5,§17], [7,§26.8], [9,§4.9] or [12,III §§12,15]. On the other hand, λ-sums of spaces E_n all equal to a fixed Hausdorff lcs E have been introduced by Pietsch [10] and studied in [3], [5,§19.4], [9,§4] or [11]. Our purpose here is to study barrelledness and related properties in the space $\lambda\{E_n\}(\tau)$, our immediate precursors being the paper by Lurje [8], where the barrelledness of $\ell^p\{E_n\}$ is studied, and our [4], where we study barrelledness in $\lambda\{E\}$.

In what follows, λ stands for a perfect sequence space [7,§30], [13,Ch.2] and we shall consider λ and $\lambda\{E_n\}$ endowed with the topology τ. By using the continuity of the maps

$$I_k: x=(x_n)_n \in \lambda\{E_n\} \to I_k(x) := x_k \in E_k$$

$$J_k: x_k \in E_k \to J_k(x_k) := x_k e_k \in \lambda\{E_n\}$$

$$I: x=(x_n)_n \in \lambda\{E_n\} \to I(x) := (\|x_n\|)_n \in \lambda$$

where $k \in \mathbb{N}$ and e_k stands for the k-th unit sequence, standard arguments [3,3.8] or [11,2.(3)] show that the completion of $\lambda\{E_n\}$ is $\lambda\{\tilde{E}_n\}$. An element $x=(x_n)_n$ in $\lambda\{E_n\}(\tau)$ is said to have the property AK if x is the τ-limit of the sequence of its finite sections $P_k(x) := (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$. We shall be mainly concerned with the subspace of $\lambda\{E_n\}(\tau)$ formed by those elements having AK. Clearly, this space is equal to the λ_r-sum of E_n, where λ_r is the subspace of sequences having AK in $\lambda(\tau)$, and coincides also with the τ-closure of $\phi\{E_n\}$ in $\lambda\{E_n\}$. In other terms:

$$\lambda_r\{E_n\} = \{x=(x_n)_n \in \lambda\{E_n\}:\ x=\tau\text{-}\lim_k P_k(x)\} = \overline{\phi\{E_n\}}^{\tau}$$

If $\mathcal{M}$ is the family of all normal, absolutely convex, closed and $\sigma(\lambda^\times,\lambda)$-bounded subsets of $\lambda^\times$, τ coincides with the strong topology $\beta(\lambda,\lambda^\times)$. In this case, λ_r is called the regular subspace of λ and has some interesting properties, e.g. $c_0 \cdot \lambda \subset \lambda_r$, λ_r is normal, $(\lambda_r)^\times = \lambda^\times$, $\beta(\lambda,\lambda^\times)$ induces on λ_r its own strong topology, λ_r endowed with the Mackey topology $\mu(\lambda_r,\lambda^\times)$ is barrelled and, in fact, $\lambda=\lambda_r$ if and only if $\lambda(\beta(\lambda,\lambda^\times))$ is barrelled [6], [7,§30], [13,Ch.2].

THEOREM 1. Define the generalized Köthe α-dual of $\lambda\{E_n\}$ by

$$(\lambda\{E_n\})^\times := \{u=(u_n)_n : u_n \in E_n' \text{ and } \sum_n |\langle u_n, x_n\rangle| < \infty\}$$

Then we have the following chain of equalities:

$$(\lambda_r\{E_n\}(\tau))' = (\lambda\{E_n\})^\times = (\lambda_r\{E_n\})^\times = \lambda^\times\{E_n'\} = \lambda^\times \cdot \ell^\infty\{E_n'\}$$

Moreover, a subset $A \subset (\lambda_r\{E_n\})'$ is equicontinuous if and only if it is contained in $rM\cdot B_1$ for some $r>0$ and $M \in \mathcal{M}$, B_1 standing for the unit ball of the normed space $\ell^\infty\{E_n'\}$.

PROOF. We shall prove the "$\subset$" inclusions from the left to the right and then close the circle:

(1) If $f \in (\lambda_r\{E_n\}(\tau))'$ take $u_n := fJ_n \in E_n'$ for each n. Take arbitrary $y=(y_n)_n$ in $\lambda\{E_n\}$ and $\alpha=(\alpha_n)_n$ in c_0 and take $|\beta_n|=1$ such that $|\alpha_n\langle u_n,y_n\rangle| = \alpha_n\beta_n\langle u_n,y_n\rangle$ then we have that $\alpha\beta y = (\alpha_n\beta_n y_n)_n$ is in $\lambda_r\{E_n\}$ hence

$$f(\alpha\beta y) = \lim_k f(P_k(\alpha\beta y)) = \sum_n |\alpha_n\langle u_n,y_n\rangle| < \infty$$

since α was arbitrary in c_0, we obtain $(|\langle u_n,y_n\rangle|)_n \in \ell^1$.

(2) This is immediate.

(3) Let $u=(u_n)_n$ in $(\lambda_r\{E_n\})^\times$ and $\alpha=(\alpha_n)_n$ arbitrary in λ. For each n we find $|x_n|=1$ such that

$$|\alpha_n||u_n| \le |\alpha_n\langle u_n,x_n\rangle| + 1/2^n$$

Since $(\alpha_n x_n)_n$ is in $\lambda\{E_n\}$, we have

$$\sum_n |\alpha_n||u_n| \le \sum_n |\langle u_n,\alpha_n x_n\rangle| + \sum_n 1/2^n < \infty$$

Since α was arbitrary, $(|u_n|)_n$ is in $\lambda^\times$, i.e. $u \in \lambda^\times\{E_n'\}$.

(4) Given $u=(u_n)_n$ in $\lambda^\times\{E_n'\}$ define $v_n := u_n/|u_n|$ if $u_n \ne 0$ and $v_n := 0$ otherwise (this notation will be often used in the sequel). Then $u = (|u_n|)_n \cdot (v_n)_n$ where $(|u_n|)_n \in \lambda^\times$ and $v := (v_n)_n$ is in $B_1 \subset \ell^\infty\{E_n'\}$.

(5) Take $u=\alpha w$ where $\alpha \in \lambda^\times$ and for some $r>0$, $|w_n| \le r$ for

all n. Since $\mathcal{M}$ covers $\lambda^{\times}$, take M in $\mathcal{M}$ containing α, then for all x in $\lambda_r\{E_n\}$ we have

$$\sum_n |\langle u_n, x_n\rangle| \leq r\sum_n |\alpha_n| |x_n| \leq r p_M(x)$$

thus $\langle u,x\rangle := \sum_n \langle u_n, x_n\rangle$ defines a τ-continuous linear form on $\lambda_r\{\hat{E}_n\}$.

Now if A is equicontinuous, take r>0 and M in $\mathcal{M}$ such that $|\langle u,x\rangle| \leq r p_M(x)$ for all u in A and x in $\lambda_r\{E_n\}$. Take the corresponding v in B_1, let us check that $(|u_n|)_n$ is in rM for all u in A. Indeed, take α in M° and $\varepsilon > 0$. Choose $|x_n| = 1$ such that $|\alpha_n u_n| \leq \langle \alpha_n u_n, x_n\rangle + \varepsilon/2^n$. Then for all k we have

$$\sum_{n=1}^{k} \alpha_n |u_n| \leq \sum_{n=1}^{k} \langle u_n, \alpha_n x_n\rangle + \varepsilon \sum_{n=1}^{k} 1/2^n \leq$$
$$\leq \langle u, (\alpha_1 x_1, \ldots, \alpha_k x_k, 0, 0, \ldots)\rangle + \varepsilon \leq$$
$$\leq r p_M(\alpha_1 x_1, \ldots, \alpha_k x_k, 0, 0, \ldots) + \varepsilon \leq$$
$$\leq r q_M(\alpha) + \varepsilon$$

since k and ε were arbitrary, $\sum_n \alpha_n |u_n| \leq r$ for all α in M° this implies that $(|u_n|)_n$ is in $rM^{\circ\circ} = rM$. Finally, if $A = rM \cdot B_1$ it is clear that $|\langle u, \cdot\rangle| \leq r p_M(\cdot)$ for all $u \in A$.
QED

<u>COROLLARY</u>. <u>The strong dual of</u> $\lambda_r\{E_n\}(\tau)$ <u>is the space</u> $\lambda^{\times}\{E_n'\}(\beta)$ <u>where</u> β <u>is the topology of uniform convergence on the family</u> $\mathcal{B}$ <u>of all normal</u>, <u>absolutely convex</u>, <u>closed and</u> $\sigma(\lambda,\lambda^{\times})$-<u>bounded subsets of</u> λ.

<u>PROOF</u>. Let A be a $\sigma(\lambda_r\{E_n\}, \lambda^{\times}\{E_n'\})$-bounded set. It is easy to see that $I(A) = \{(|x_n|)_n : x \in A\}$ is $\sigma(\lambda,\lambda^{\times})$-bounded: recall that $\lambda^{\times}\{E_n'\} = \lambda^{\times} \cdot \ell^{\infty}\{E_n'\}$. Thus if u is in $\lambda^{\times}\{E_n'\}$ then for x in A

$$|\langle u,x\rangle| \leq \sum_n |u_n| |x_n| \leq q_H((|u_n|)_n) = p_H(u)$$

where H, the absolutely convex, closed and normal hull of A, is in $\mathcal{B}$. Hence the strong topology in $\lambda^{\times}\{E_n'\}$ is coarser than β. On the other hand, take H in $\mathcal{B}$. Since H is normal, the set $H^* := \{P_k(\alpha) : \alpha \in H, k \in \mathbb{N}\}$ is also normal and included in both H and λ_r, and $p_H(\cdot) = p_{H^*}(\cdot)$

in $\lambda^{\times}\{E_n'\}$. Take $A:=\{\beta y : \beta \in H^*, y_n$ in the unit ball of $E_n\}$, then A is contained in $\lambda_r\{E_n\}$ and for u in $\lambda^{\times}\{E_n'\}$ we have

$$\sup\{|\langle u,\beta y\rangle| : \beta y \in A\} = \sup\{|\textstyle\sum_n \beta_n \langle u_n, y_n\rangle| : \beta y \in A\} =$$

$$= \sup\{\textstyle\sum_n |\beta_n| \, |u_n| : \beta \in H^*\} = p_{H^*}(u) = p_H(u)$$

second equality: H^* is normal and y_n can be arbitrarily selected from the unit ball of E_n. Thus A is weakly bounded in $\lambda_r\{E_n\}$ and $\sup\{|\langle u,x\rangle| : x \in A\} = p_H(u)$, this means that every β-continuous seminorm is also strongly continuous. QED

2. BARRELLED AND BORNOLOGICAL $\lambda_r\{E_n\}$

In this section, M (resp. B) stands for the family of all normal, absolutely convex, closed and $\sigma(\lambda^{\times},\lambda)$--bounded (resp. $\sigma(\lambda,\lambda^{\times})$-bounded) sets in $\lambda^{\times}$ (resp. λ), and we keep the notation τ (resp. β) for the corresponding topologies. In this case $\lambda_r(\tau)$ is always barrelled and, under certain conditions, also bornological [6].

THEOREM 2. (1) $\lambda_r\{E_n\}(\tau)$ is quasi-barrelled.
(2) If every E_n is barrelled then $\lambda_r\{E_n\}(\tau)$ is barrelled.
(3) If λ_r and every E_n are reflexive, then $\lambda_r\{E_n\}(\tau)$ is also reflexive.

PROOF. (1) Let A be a strongly bounded subset of the dual of $\lambda_r\{E_n\}(\tau)$. According to the preceding result, A is β-bounded, thus H, the absolutely convex, closed and normal hull of I(A), is in M, and by writing every u in A as $(|u_n|)_n \cdot (v_n)_n$ (notation from (4) in Theorem 1) we obtain $A \subset H \cdot B_1$. Now, according to Theorem 1, A is equicontinuous.

(2) By Banach-Mackey theorem [7, §21, 11.(8)], it suffices to prove that $(\lambda_r\{E_n\}(\tau))'$ is weakly sequentially complete. By Theorem 1, this dual is the generalized Köthe α-dual. Now, let $u^{(p)} = (u_n^{(p)})_n$ $p=1,2,\dots$ be a Cauchy se-

quence for the weak topology $\sigma((\lambda_r\{E_n\})^\times,\lambda_r\{E_n\})$. It is clear that coordinatewise each sequence $u_n^{(p)}$ $p=1,2,\dots$ is $\sigma(E_n',E_n)$-Cauchy. Since E_n is barrelled, all these sequences are weakly convergent. Call $u_n:=\sigma\text{-}\lim_p u_n^{(p)}$ for every n. Define $u:=(u_n)_n$, we shall prove that u is in $(\lambda_r\{E_n\})^\times$ and $u=\sigma\text{-}\lim_p u^{(p)}$. Indeed: if $x=(x_n)_n\in\lambda_r\{E_n\}$ then $\alpha^{(p)}:=(\langle u_n^{(p)},x_n\rangle)_n$ is in ℓ^1 for all $p=1,2,\dots$ and these terms form a $\sigma(\ell^1,\ell^\infty)$-Cauchy sequence (note that if $\beta\in\ell^\infty$ then βx is in $\lambda_r\{E_n\}$ and $|\langle\alpha^{(p)}-\alpha^{(q)},\beta\rangle| = |\langle u^{(p)}-u^{(q)},\beta x\rangle|$). By Schur theorem [7,§22,4.(2)] $\alpha^{(p)}$ $p=1,2,\dots$ converges in the ℓ^1-norm to some $\alpha=(\alpha_n)_n\in\ell^1$. Necessarily, $\alpha_n=\lim_p\langle u_n^{(p)},x_n\rangle=\langle u_n,x_n\rangle$. Thus $(\langle u_n,x_n\rangle)_n$ is in ℓ^1, i.e. $u\in(\lambda_r\{E_n\})^\times$, and we have $\lim_p\langle u_n^{(p)},x\rangle=\langle u,x\rangle$

(3) If $\lambda_r(\tau)$ is reflexive then $\lambda_r=(\lambda^\times(\beta))'=\lambda$ and also $\lambda^\times=(\lambda^\times)_r$ [7,§30,7.(4)]. Then, by the result above, the strong dual of $\lambda_r\{E_n\}(\tau)$ is $\lambda^\times\{E_n'\}(\beta)=(\lambda^\times)_r\{E_n'\}(\beta)$, whose strong dual is, in turn, $\lambda\{E_n''\}(\tau)=\lambda_r\{E_n\}(\tau)$. QED

<u>COROLLARY</u>. <u>If</u> $\lambda_r(\tau)$ <u>is a DF-space</u>, <u>then</u> $\lambda_r\{E_n\}(\tau)$ <u>is also a DF-space</u>.

<u>PROOF</u>. If A_m $m=1,2,\dots$ is a countable fundamental system of bounded sets in $\lambda_r(\tau)$ and V_1 stands for the unit ball in $\ell^\infty\{E_n\}$, then $A_m\cdot V_1$ $m=1,2,\dots$ is the corresponding fundamental system in $\lambda_r\{E_n\}(\tau)$. Indeed, if A is τ-bounded in $\lambda_r\{E_n\}$ then $I(A)$ is τ-bounded in λ_r. Take A_m such that $I(A)\subset A_m$ and for each $x=(x_n)_n$ in A write $x=(\|x_n\|)_n\cdot(y_n)_n$ with $(y_n)_n$ in V_1. Then x is in $I(A)\cdot V_1$ which is contained in $A_m\cdot V_1$. On the other hand, $\lambda_r\{E_n\}$ is, as we have seen, quasi-barrelled, thus it is a DF--space. QED

<u>THEOREM 3</u>. <u>If</u> $\lambda_r(\tau)$ <u>is bornological</u>, <u>then</u> $\lambda_r\{E_n\}(\tau)$ <u>is also bornological</u>.

<u>PROOF</u>. By Theorem 2, τ is the Mackey topology on $\lambda_r\{E_n\}$. Let $f:\lambda_r\{E_n\}\to\mathbb{K}$ be a locally bounded linear form (i.e. f is bounded on τ-bounded sets). According to [7,§28,1.(3)] we must prove that f is τ-continuous.

For every y in V_1 (the unit ball in $\ell^\infty\{E_n\}$) we define

$$I_y:\alpha \in \lambda_r(\tau) \to I_y(\alpha):=(\alpha_n y_n)_n \in \lambda_r\{E_n\}(\tau)$$

every I_y is linear and continuous. Now take

$$N:=\{\alpha \text{ in } \lambda_r: |f(I_y(\alpha))| \leq 1 \text{ for all } y \text{ in } V_1\}$$

N is absolutely convex and bornivorous, for if A is τ--bounded in λ_r, then $A\cdot V_1$ is τ-bounded in $\lambda_r\{E_n\}$ and we can find $r>0$ such that $|f(I_y(\alpha))| \leq r$ for all y in V_1 and all α in A, therefore $A\subset rN$. Since $\lambda_r(\tau)$ is bornological, N is a zero-neighborhood in $\lambda_r(\tau)$ or, equivalently, we can find M in $\mathcal{M}$ such that $|f(I_y(\alpha))|\leq q_M(\alpha)$ for all y in V_1 and α in λ_r. Finally, if x is in $\lambda_r\{E_n\}$ and we write $x=(|x_n|y_n)_n$ with $y=(y_n)_n$ in V_1, we obtain

$$|f(x)| = |f(I_y((|x_n|)_n))| \leq q_M((|x_n|)_n) = p_M(x)$$

thus f is τ-continuous. QED

Our next result extends, using a different approach, Proposition 2.3 in Bierstedt and Bonet's celebrated paper [1], where it is used as a tool for their result about distinguished Köthe echelon spaces.

<u>THEOREM 4</u>. <u>If</u> $\lambda_r(\tau)$ <u>is distinguished, then</u> $\lambda_r\{E_n\}(\tau)$ <u>is also distinguished</u>.

<u>PROOF</u>. Since λ_r is weakly dense in λ, the strong topologies $\beta(\lambda^\times,\lambda)$ and $\beta(\lambda^\times,\lambda_r)$ coincide on $\lambda^\times$. Then the strong dual of $\lambda_r(\tau)$ is the complete and barrelled space $\lambda^\times(\beta(\lambda^\times,\lambda))$. Now, by the Corollary to Theorem 1, the strong dual of $\lambda_r\{E_n\}(\tau)$ is $\lambda^\times\{E_n'\}(\beta)$ which is complete since so are all E_n'. Thus it suffices to prove that this space is quasi-barrelled (we cannot apply Theorem 2 since it is not true in general that $(\lambda^\times)_r=\lambda^\times$, e.g. ℓ^∞). Let N be a bounded subset of the strong dual of $\lambda^\times\{E_n'\}(\beta)$. For each v in B_1, the unit ball in $\ell^\infty\{E_n'\}$, define the map

$$I_v:\alpha \in \lambda^\times \to I_v(\alpha):=(\alpha_n v_n)_n \in \lambda^\times\{E_n'\}$$

for f in N and v in B_1, $f \circ I_v$ is $\beta(\lambda^\times,\lambda)$-continuous on $\lambda^\times$. Moreover, the set

$$N^* := \{f \circ I_v : f \text{ in } N \text{ and } v \text{ in } B_1\}$$

is bounded in the strong dual of $\lambda^\times(\beta(\lambda^\times,\lambda))$: indeed, if A is bounded in $\lambda^\times$ then $A \cdot B_1$ is bounded in $\lambda^\times\{E_n'\}$ and we can find $r>0$ such that $|f(I_v(\alpha))| \leq r$ for all α in A, v in B_1 and f in N. Therefore N^* is equicontinuous and we can find M in $\mathcal{B}$ such that $|f(I_v(\cdot))| \leq q_M(\cdot)$ for all v in B_1 and f in N. Finally, given u in $\lambda^\times\{E_n'\}$, we write $u=(|u_n|v_n)_n$ with v in B_1 to obtain

$$|f(u)| = |f(I_v((|u_n|)_n))| \leq q_M((|u_n|)_n) = p_M(u)$$

for all f in N. Thus N is equicontinuous. QED

REMARK. The proof above shows indeed that "If $\lambda(\tau)$ is quasi-barrelled, then $\lambda\{E_n\}(\tau)$ is also quasi-barrelled" without AK assumptions.

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E.S. Ingenieros Industriales
Av. Reina Mercedes s/n
41012-SEVILLA (Spain)

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