BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 40, No. 3, 1992

*FUNCTIONAL ANALYSIS*

# On the Mikusiński-Antosik Diagonal Theorem

by

Miguel FLORENCIO, Pedro J. PAÚL (*) and J. M. VIRUÉS

*Presented by C. BESSAGA on November 11, 1992*

**Summary.** We give an improvement of the Mikusiński-Antosik Diagonal Theorem. From this result we obtain as corollaries the other matrix lemmas used by P. Antosik and C. Swartz, showing that they have a common character.

One of the main purposes of the work done by P. Antosik and C. Swartz along many years has been to give non-categorical proofs and generalizations of such well-known classical results in functional analysis as the Banach-Steinhaus Theorem, the Nikodym Boundedness Theorem, the Orlicz-Pettis Theorem, the Vitali-Hahn-Saks Theorem or the Schur and Philips Lemmas, to mention only a few. Their main tools have been several variations of the sliding-hump technique given as theorems about the behaviour of the diagonal of certain infinite matrices with elements in a locally convex space or, more generally, in a topological group. Indeed, their monography [4] comprising most of these results is called "Matrix Methods in Analysis".

The use of these matrix theorems was started by J. Mikusiński [7]. His result was generalized by Antosik who obtained the Mikusiński-Antosik Diagonal Theorem [1, Diagonal Theorem (II) on p. 306]. Antosik and Swartz shifted to the so-called Basic Matrix Theorem [4, Thm 2 on p. 7] as the main tool in a series of papers that culminated in [4]. Later, Antosik has been using (see [2, Lemma in §2 on p. 25] and [3, Lemma in §2 on p. 89]) a different version that we shall call here the Antosik Diagonal Lemma.

---

AMS Classification: primary 46A03, secondary 22A10.

Key words and phrases: Mikusiński-Antosik Diagonal Theorem, sliding-hump techniques, topological groups, normed groups.

This research has been supported by La Consejería de Educación y Ciencia de la Junta de Andalucía.

(*) Author to whom correspondence should be addressed.

Antosik and Swartz claimed that "The Basic Matrix Theorem has a very different character than the Mikusiński-Antosik Diagonal Theorem in that the hypothesis and the conclusions have very different forms" [4, p. 9]. We think, however, that this is not quite so, and our purpose in this work is to give an improvement, Theorem 1 below, of the Mikusiński-Antosik Diagonal Theorem that yields both the Basic Matrix Theorem and the Antosik Diagonal Lemma. Theorem 1 below has been obtained independently by P. Antosik and S. Saeki [5]. They use deep results about measuroids — an interesting generalization of the notion of measure introducted by Saeki [9] — and their proof is quite different from the one we give here. We think that both of them are interesting. Let us also remark that J. Pochciał has made a similar attempt [8]. His result [8, Theorem 3] yields the Antosik Diagonal Lemma and the proof can be modified to give the Mikusiński-Antosik Diagonal Theorem. His methods — Ramsey's Theorem — are entirely different from ours.

The results below are stated for normed groups (that we always assume to be Abelian) for the sake of simplicity. Nevertheless, since the topology of any topological group is always generated by a family of quasi-norms [6], the reader can check that these results hold true for arbitrary topological groups.

THEOREM (The Mikusiński-Antosik Diagonal Theorem). *Let $(X, |\cdot|)$ be a normed group, and $(x_{ij})$ an infinite matrix in $X$ such that its rows converge to zero, i.e. $\lim_j x_{ij} = 0$ for each $i \in \mathbb{N}$. Then, there exist an infinite set $I \subset \mathbb{N}$ and a set $J \subset I$ such that the following hold for all $i \in I$:*

$$\sum_{j\in J} |x_{ij}| < +\infty \quad \text{and} \quad \Big|\sum_{j\in J} x_{ij}\Big| \geqslant \frac{1}{2}|x_{ii}|.$$

*Remark.* If $X$ is complete or $J$ is finite, the meaning of $|\sum_{j\in J} x_{ij}|$ in the statement above is clear. If $X$ is not complete and $J$ is infinite, we understand that

$$\Big|\sum_{j\in J} x_{ij}\Big| := \lim_{n\to\infty}\Big|\sum_{j\in J_n} x_{ij}\Big|,$$

where $J_n := J \cap \{1, 2, \dots, n\}$. This definition makes sense because $\sum_{j\in J} |x_{ij}| < +\infty$. We shall maintain this convention throughout the paper.

Observe that the result above gives the existence of two sets $J \subset I$ and shows two things about series indexed in $I \times J$. In particular, the proof of Antosik shows that $J$ is either finite or $J = I$. Now, if $J'$ is a subset of $J$, does the same conclusion of the theorem hold for $J'$? The absolute convergence of the series holds, so what about the second inequality? Our main result, Theorem 1 below, is a generalization of the Diagonal Theorem that gives a positive answer to this question. This key fact enables us to deduce the other matrix results. The proof is based on a variation of Antosik's own proof (cf. [1]).

THEOREM 1. *Let $(X, |\cdot|)$ be a normed group, and $(x_{ij})$ an infinite matrix in $X$ such that its rows converge to zero, i.e.* $\lim_j x_{ij} = 0$ *for each* $i \in \mathrm{N}$. *Then, at least one of the following assertions is true:*

(a) *There exists an infinite set $I \subset \mathrm{N}$ and a finite set $\Delta \subset I$ such that*

$$\Big|\sum_{j\in\Delta} x_{ij}\Big| \geqslant \frac{1}{2}|x_{ii}|, \quad \textit{for all } i \in I.$$

(b) *There exists an infinite set $I \subset \mathrm{N}$ such that for each infinite set $J \subset I$ the series $\sum_{j\in J} |x_{ij}|$ converges for all $i \in I$ and*

$$\Big|\sum_{j\in J} x_{ij}\Big| \geqslant \frac{1}{2}|x_{ii}| \quad \textit{for all } i \in J. \tag{1}$$

Proof. Assume that $(x_{ij})$ does not satisfy (a). We have to see that (b) holds. Since (a) does not hold, we have an additional hypothesis on the matrix $(x_{ij})$, namely that for each finite subset $\Delta$ of N such that

$$\Big|\sum_{j\in\Delta} x_{ij}\Big| \geqslant \frac{1}{2}|x_{ii}| \quad \text{for all } i \in \Delta,$$

there exists $r > \max(\Delta)$, such that

$$\Big|\sum_{j\in\Delta} x_{ij}\Big| < \frac{1}{2}|x_{ii}| \quad \text{for all } i \geqslant r.$$

We start by finding simultaneously an increasing sequence of indices $(i_n)$ and a sequence $(\varepsilon_n)$ of positive scalars such that

$$\max\Big\{\Big|\sum_{j\in\Delta} x_{i_n j}\Big| : \Delta \subset \{i_1, i_2, \dots, i_{n-1}\}\Big\} = \Big(\frac{1}{2} - \varepsilon_n\Big)|x_{i_n i_n}| \tag{2}$$

$$\text{for all } n \geqslant 2$$

and

$$|x_{i_n i_{n+q}}| < 2^{-q}\varepsilon_n |x_{i_n i_n}| \quad \text{for all } n, q \geqslant 1. \tag{3}$$

Proceed by induction. From the additional assumption, it follows that there exists some index $r_0$ such that $|x_{ii}| > 0$ for all $i \geqslant r_0$. Take $i_1 = r_0$ and $\varepsilon_1 = 1/2$. Apply now the additional assumption to $\Delta = \{i_1\}$ to obtain an index $r_1 > i_1$ such that $|x_{ii_1}| < \frac{1}{2}|x_{ii}|$ for $i > r_1$. Use that the row $i_1$ of the matrix converges to zero to select an index $i_2 > r_1$ such that

$|x_{i_1i_2}| < 2^{-1}\varepsilon_1|x_{i_1i_1}|$. Since $i_2 > r_1 > i_1 = r_0$, we can find a positive $\varepsilon_2$ such that $|x_{i_2i_1}| = (\frac{1}{2} - \varepsilon_2)|x_{i_2i_2}|$. Plainly, inequality (3) holds for $n = q = 1$.

Assume that for $p \geqslant 2$ we have already found $i_1, i_2, \ldots, i_p$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ satisfying (2) for $2 \leqslant n \leqslant p-1$ and (3) for $1 \leqslant n \leqslant p-1$ and $n+q \leqslant p$. Apply the additional assumption to each of the subsets of $\{i_1, i_2, \ldots, i_p\}$. Then, there is an index $r_p > i_p$ (the largest of the $r$'s found for each of these subsets) such that

$$(4)\qquad \max\Big\{\Big|\sum_{j\in\Delta} x_{ij}\Big| : \Delta \subset \{i_1, i_2, \ldots, i_p\}\Big\} < \frac{1}{2}|x_{ii}|, \quad \text{for all } i \geqslant r_p.$$

Use that the rows $i_1, i_2, \ldots, i_p$ of the matrix converge to zero and that $|x_{ii}| > 0$ for all $i \geqslant r_0$ to find $i_{p+1} > r_p$ such that

$$|x_{i_ni_{p+1}}| < 2^{n-p-1}\varepsilon_n|x_{i_ni_n}|, \quad \text{for } 1 \leqslant n \leqslant p.$$

Take $\varepsilon_{p+1}$ such that

$$\max\Big\{\Big|\sum_{j\in\Delta} x_{i_{p+1}j}\Big| : \Delta \subset \{i_1, i_2, \ldots, i_p\}\Big\} = \Big(\frac{1}{2} - \varepsilon_{p+1}\Big)|x_{i_{p+1}i_{p+1}}|.$$

Since $i_{p+1} > r_p$, we have that $\varepsilon_{p+1} > 0$ by (4). Now, it is easy to check that (2) holds for $2 \leqslant n \leqslant p$ and (3) holds for $1 \leqslant n \leqslant p$ and $n+q \leqslant p+1$. This finishes the induction.

To see that (b) holds, define $I := \{i_1, i_2, \ldots\}$ and let $J = \{j_1, j_2, \ldots\}$ be any infinite subset of $I$. Fix $i_n \in I$. By (3), from one term on, the terms of the series $\sum_{k=1}^{\infty} |x_{i_nj_k}|$ are bounded by the terms of a convergent geometric series. This proves that the series converges. To prove (1) assume that $i_n \in J$, say $i_n = j_p$ then, by (2) and (3), we have

$$\Big|\sum_{k=1}^{\infty} x_{i_nj_k}\Big| = \Big|\sum_{k=1}^{\infty} x_{j_pj_k}\Big| \geqslant |x_{j_pj_p}| - \Big|\sum_{k<p} x_{j_pj_k}\Big| - \Big|\sum_{k>p} x_{j_pj_k}\Big|$$
$$\geqslant \Big(1 - \Big(\frac{1}{2} - \varepsilon_n\Big) - \varepsilon_n\Big)|x_{j_pj_p}| = \frac{1}{2}|x_{j_pj_p}| = \frac{1}{2}|x_{i_ni_n}|. \qquad \square$$

*Remark.* The crucial difference between our proof and Antosik's one is the fact that in equality (2) he only takes one of the sums, namely $\sum_{k=1}^{n-1} x_{i_ni_k}$.

COROLLARY 1 (The Antosik Diagonal Lemma). *Let $(X, |\cdot|)$ be a normed group, and $(x_{ij})$ an infinite matrix in $X$ such that for every increasing sequence $(m_i)$ in* N *there exists a subsequence $(n_i)$ of $(m_i)$ such that*

$$(5)\qquad \lim_{i\to\infty} x_{n_in_j} = 0 \quad \textit{for all } j \in \mathbb{N}, \quad \textit{and}$$

$$(6)\qquad \lim_{i\to\infty} \sum_{j=1}^{\infty} x_{n_in_j} = 0.$$

*Then* $\lim_{i\to\infty} x_{ii} = 0$.

Proof. Proceed by contradiction and assume that there exists $\varepsilon > 0$ and an increasing sequence of indices $(m_i)$ such that $|x_{m_i m_i}| > \varepsilon$. Find the corresponding subsequence $(n_i)$ given by the hypothesis. Then the submatrix $(x_{n_i n_j})$ still satisfies the hypothesis, $|x_{n_i n_i}| > \varepsilon$ for all $i \in \mathrm{N}$ and, by the selection of $(n_i)$ such that (5) and (6) hold, all the rows and the columns of $(x_{n_i n_j})$ converge to zero. Therefore, we may reduce it to the case when the original matrix already satisfies these additional assumptions.

Let us see that condition (a) of Theorem 1 does not hold. Indeed, if $I$ is an infinite set and $\Delta$ is a finite subset of $I$, then

$$\lim_{i\to\infty, i\in I} \Big|\sum_{j\in\Delta} x_{ij}\Big| = 0,$$

because the columns of the matrix converge to zero. Since $|x_{ii}| > \varepsilon$ for all $i \in \mathrm{N}$, condition (a) cannot hold.

Theorem 1 tells us that condition (b) must hold. Thus, there exists an infinite set $I \subset \mathrm{N}$ such that for every infinite set $J \subset I$ we have

$$\Big|\sum_{j\in J} x_{ij}\Big| \geqslant \frac{1}{2}|x_{ii}| > \frac{1}{2}\varepsilon, \quad \text{for all } i \in J.$$

This is a contradiction with the fact that if we write $I = \{m_1, m_2, \ldots\}$ then, by the hypothesis, there exists a subsequence $J = \{n_1, n_2, \ldots\}$ such that (6) holds. □

COROLLARY 2 (The Basic Matrix Theorem). *Let* $(X, |\cdot|)$ *be a normed group, and* $(x_{ij})$ *an infinite matrix in* $X$ *such that*

(a) *it has convergent columns, i.e.* $x_j = \lim_{i\to\infty} x_{ij}$ *exists for each* $j \in \mathrm{N}$, *and*

(b) *for every increasing sequence* $(m_i)$ *in* $\mathrm{N}$ *there exists a subsequence* $(n_i)$ *of* $(m_i)$ *such that the sequence*

$$\Big(\sum_{j=1}^{\infty} x_{in_j}\Big)_{i=1}^{\infty}$$

*is a Cauchy sequence.*

*Then the convergence* $x_j = \lim_{i\to\infty} x_{ij}$ *is uniform with respect to* $j \in \mathrm{N}$. *In particular,* $\lim_{i\to\infty} x_{ii} = 0$.

Proof. Assume, by contradiction, that the convergence is not uniform in $j \in \mathrm{N}$. Then there is $\varepsilon > 0$ such that for each $k \in \mathrm{N}$ there exist $i_k, i'_k \in \mathrm{N}$, with $k < i_k < i'_k$, such that

$$\sup\{|x_{i_k j} - x_{i'_k j}| : j \in \mathrm{N}\} > \varepsilon.$$

Then, we can find $j_1 < j_2 < \ldots$ in N such that

$$|x_{i_k j_k} - x_{i'_k j_k}| > \varepsilon, \quad \text{for all } k \in \mathbf{N}.$$

Now, consider the infinite matrix $(y_{mn})$ in $X$ defined by

$$y_{mn} := x_{i_m j_n} - x_{i'_m j_n} \qquad (m, n \in \mathbf{N}).$$

Note, in particular, that $|y_{mm}| > \varepsilon$ for all $m \in \mathbf{N}$, and that this will give us the desired contradiction if we prove that $(y_{mn})$ satisfies the hypothesis of Corollary 1.

The first part is obvious since for all $n \in \mathbf{N}$, we have

$$\lim_{m\to\infty} y_{mn} = \lim_{m\to\infty} (x_{i_m j_n} - x_{i'_m j_n}) = x_{j_n} - x_{j_n} = 0.$$

On the other hand, let $(m_q)$ be any increasing sequence in N. Consider the sequence $(j_{m_q})$. By the hypothesis (b), there is a subsequence $(j_{n_q})$ (i.e. $(n_q)$ is a subsequence of $(m_q)$) such that

$$\left(\sum_{q=1}^{\infty} x_{i j_{n_q}}\right)_{i=1}^{\infty}$$

is a Cauchy sequence. This means, in particular, that

$$\lim_{p\to\infty} \sum_{q=1}^{\infty} y_{n_p n_q} = \lim_{p\to\infty} \sum_{q=1}^{\infty} (x_{i_{n_p} j_{n_q}} - x_{i'_{n_p} j_{n_q}}) = 0$$

which finishes the proof.

*Acknowledgements.* This research has its origin in a comment made by L. Drewnowski to the authors during a scientific stay in Poznań. We want to thank him for his hospitality and helpful discussions. We are also indebted to the referee for pointing out the work done by Antosik and Saeki [5] and for supplying us with reference [8]. Finally, we thank S. Saeki (Manhattan, Kansas) for sending us the unpublished manuscript [5].

E. S. INGENIEROS INDUSTRIALES, AVDA, REINA MERCEDES S/N, 41012 SEVILLA, SPAIN

REFERENCES

[1] P. Antosik, *On the Mikusiński diagonal theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **19** (1971) 305–310.

[2] P. Antosik, *On K, M and KM-sequences and uniform convergence*, Convergence Structures 1984, Proc. Conf. on Convergence, Bechyně, Czechoslovakia, Akademie-Verlag, Berlin, (1985) 25–31.

[3] P. Antosik, *A lemma on matrices and its applications*, Contemporary Math., **52** (1986) 89–95.

[4] P. Antosik, C. Swartz, *Matrix methods in analysis*, Lecture Notes in Mathematics, Vol. 1113, Springer-Verlag, Berlin 1985.

[5] P. Antosik, S. Saeki, *A diagonal theorem for measuroids and its applications*, unpublished manuscript (personal communication 1992).

[6] J. Burzyk, P. Mikusiński, *On normability of semigroups*, Bull. Acad. Polon. Sci. Sér. Sci. Math., **28** (1980), 33–35.

[7] J. Mikusiński, *A theorem on vector matrices and its applications in measure theory and functional analysis*, Bull. Acad. Polon. Sci, Sér. Sci. Math. Astronom. Phys., **18** (1970) 193–196.

[8] J. Pochciał, *On Mikusiński-Antosik diagonal theorems*, Generalized Function and Convergence, World Scientific, (1990) 356–359.

[9] S. Saeki, *The Vitali-Hahn-Saks theorem and measuroids*, Proc. Amer. Math. Soc., **114** (1992) 775–782.