# The space of Denjoy-Dunford integrable functions is ultrabornological

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*Dedicado a mi amigo Pepe Bonet con cariño y admiración.*

**Abstract**

Let $X$ be a Banach space. We prove that the normed and non-complete space of all functions $f\colon [a,b] \to X$ which are integrable in the sense of Denjoy-Dunford (introduced by Gordon in 1989) is ultrabornological.

## 1 Introduction

For functions $f\colon [a,b] \to \mathbb{R}$ there exist more integrals than that of Lebesgue's, although maybe not so widespread; namely, the integrals attached to the names of Denjoy (two types), Henstock, Khinchin, Kurzweil, Luzin, McShane, and Perron. The relationships between these integrals are by now well understood and they can be classified, roughly speaking, with respect to the kind of derivatives that can be integrated by means of the corresponding Fundamental Theorem of Calculus, so

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that

$$\boxed{\text{Lebesgue} \Leftrightarrow \text{McShane}}$$
$$\Downarrow \quad \not\Uparrow$$
$$\boxed{\text{Denjoy (strict)} \Leftrightarrow \text{Henstock} \Leftrightarrow \text{Kurzweil} \Leftrightarrow \text{Luzin} \Leftrightarrow \text{Perron}}$$
$$\Downarrow \quad \not\Uparrow$$
$$\boxed{\text{Denjoy (general)} \Leftrightarrow \text{Khinchin}}$$

meaning that the lower in this arrangement, the fewer conditions are imposed on the notion of derivability for a derivative function to be integrable. We refer the reader to the excellent monographs on the subject written by Gordon [Go3] and Henstock [H], and also to Saks's classic [Sa].

All of these integrals have been extended to functions $f\colon [a,b] \to X$ taking their values in a Banach space $X$, but integrals that were equivalent in $\mathbb{R}$ may be different in a general Banach space $X$. For instance, we have the integrals of Bochner, Dunford, and Pettis [DU, II.2–3], the integral of McShane [F1], [F2], [F3], [FM], [Go2] and [Sw], and the integrals of Denjoy-Dunford, Denjoy-Pettis, Denjoy-Henstock, and Kurzweil [A], [F1], [F2], [FM], [Ga], [GaM], [Gi] and [Go1].

From the point of view of Functional Analysis, each of these notions of vector-valued integration produces its own linear space of vector-valued integrable functions together with a norm defined on it in a natural way. However, these normed spaces are usually non-complete; the space of Bochner integrable functions being the exception. Fortunately, during the last years most of these non-complete normed spaces of integrable functions have been shown to be ultrabornological.

Let us recall at this point that an absolutely convex and bounded subset $B$ of a locally convex space $E$ is said to be a Banach disc if the linear span of $B$, usually denoted by $E_B$, endowed with the normed topology defined by the gauge of $B$ is a Banach space [PB, 3.2]. A locally convex space $E$ is said to be ultrabornological if every absolutely convex set that absorbs the Banach discs of $E$ is a zero-neighborhood. A locally convex space is ultrabornological if and only if it is the inductive limit of a family of Banach spaces [PB, 6.1]. Their importance lies in the fact that ultrabornological spaces form a good class of domain spaces for which the following form of the closed graph theorem holds [K, §35.2.(2)]: "*If $E$ and $F$ are locally convex spaces such that $E$ is ultrabornological and $F$ is webbed, then every linear mapping from $E$ into $F$ is continuous provided that it has closed graph.*" (The definition of webbed space, due to De Wilde, is a bit too involved to be given here. Note, however, that almost all relevant spaces are webbed; this is the case of the space of test functions and the space of distributions [K, §35.1].)

Going back to the spaces of vector-valued integrable functions we were discussing, Gilioli [Gi] proved that the space of Kurzweil integrable functions and the space of real-valued (i.e., $X = \mathbb{R}$) Denjoy-Khinchin integrable functions are ultrabornological. It was proved in [DFFP] that the space of Dunford integrable functions and the space of Pettis integrable functions (both in a general $\sigma$-finite measure space) are ultrabornological. The theorem for the McShane integral was given in [DFFP] and extended to the integral over $\mathbb{R}$ by Swartz [Sw]. Finally, Gámez [Ga, Cap. 4] gave a unified proof for integrals satisfying some general principles, but the vector-valued

Denjoy-Dunford integral was explicitly excluded.

The proofs of these theorems consist in different versions of du Bois-Reymond, Lebesgue, and Toeplitz's sliding hump techniques which, roughly speaking, can be described as follows: to prove that an absolutely convex set $A$ that absorbs Banach discs is a zero-neighborhood, proceed by contradiction and start by finding, with the help of a suitable family of projections, a sequence of elements in the unit ball $(x_n)$, either disjoint or with decreasing support, such that $x_n \notin nA$. Then use the elements $(x_n)$ to cook up a Banach disc containing them and, consequently, not absorbed by $A$. (This technique works also for some curious subspaces of $C[a, b]$, the interested reader is referred to [BS] and [Gi]).

As we have pointed out, there is a gap in the list, namely, the space of Denjoy-Khinchin integrable functions is known to be ultrabornological in the scalar-valued case, but it is not known whether the Denjoy-Dunford extension to the vector-valued case given by Gordon [Go1] produces an ultrabornological space of integrable functions. The purpose of this note is to prove that the answer is also affirmative.

## 2 Vector-Valued Extensions of the Denjoy-Khinchin Integral

We start by recalling the Denjoy-Khinchin integral for real valued functions (alias "Denjoy integral" [Go1], "Denjoy integral in the wide sense" and "Khinchin integral" [Go3, Ch. 15]; note, in particular, that the terminologies in [Go1] and [Go3] do not match exactly) and its relationship with the usual Lebesgue integration (see [Go1], [Go3] or [Sa] for the proofs). Afterwards, we shall describe the extensions to the vector-valued case proposed by Gordon [Go1]. In what follows, $X$ stands for a Banach space with dual $X^*$ and bidual $X^{**}$.

**Definitions** Let $F\colon [a, b] \to X$ be a function and let $S$ be a subset of $[a, b]$.

(1) The function $F$ is said to be *absolutely continuous on* $S$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i) - F(c_i)\| < \varepsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals having endpoints in $S$ and such that $\sum(d_i - c_i) < \delta$. The set of all absolutely continuous functions on $S$ will be denoted by $AC(S, X)$.

(2) The function $F$ is said to be *absolutely continuous in the generalized sense on* $S$ if $F$ is continuous on $S$ and $S$ can be expressed as a countable union $S = \bigcup_n S_n$ such that $F \in AC(S_n, X)$ for all $n \in \mathbb{N}$. The set of all functions which are absolutely continuous in the generalized sense on $S$ will be denoted by $ACG(S, X)$.

(3) A point $t \in (a, b)$ is said to be a *point of density of a set* $S \subset [a, b]$ if

$$\lim_{h\to 0^+} \frac{\mu^*(S \cap [t-h, t+h])}{2h} = 1$$

where $\mu^*$ stands for the outer Lebesgue measure on $\mathbb{R}$.

(4) The function $F$ is said to be *approximately derivable at a point* $t \in (a, b)$ with *approximate derivative* $F'_{\mathrm{ap}}(t) \in X$ if there exists a measurable set $S \subset [a, b]$

that has $t$ as a point of density and such that

$$\lim_{\substack{s\to t\\ s\in S}} \frac{F(s)-F(t)}{s-t} = F'_{\mathrm{ap}}(t).$$

As it is well known, the Fundamental Theorem of Calculus for the Lebesgue integral tells us that a function $f\colon [a,b]\to\mathbb{R}$ is Lebesgue integrable in $[a,b]$ if and only if there exists $F\in AC([a,b],\mathbb{R})$ such that $F'=f$ almost everywhere on $[a,b]$. The Denjoy-Khinchin integral is obtained by replacing $AC$ by $ACG$ and "derivative" by "approximate derivative" in this theorem.

**Definitions** A function $f\colon [a,b]\to\mathbb{R}$ is said to be *Denjoy-Khinchin integrable on* $[a,b]$ if there exists $F\in ACG([a,b],\mathbb{R})$ such that $F'_{\mathrm{ap}}=f$ almost everywhere on $[a,b]$ in which case the *Denjoy-Khinchin integral of* $f$ *on* $[a,b]$ is defined by

$$\int_a^b f := F(b)-F(a)$$

(there is no ambiguity here because if $F\in ACG([a,b],\mathbb{R})$, then $F$ is approximately derivable almost everywhere and, moreover, if $F'_{\mathrm{ap}}=0$ almost everywhere, then $F$ is constant on $[a,b]$). It is worth noting that every Denjoy-Khinchin integrable function is measurable.

The function $f$ is said to be *Denjoy-Khinchin integrable on a set* $S\subset[a,b]$ if $f\chi_S$ is Denjoy-Khinchin integrable on $[a,b]$ in which case $\int_S f := \int_a^b f\chi_S$.

We denote by $DK([a,b],\mathbb{R})$ the linear space of all Denjoy-Khinchin integrable (classes of almost everywhere equal) functions on $[a,b]$. This space is topologized in a natural way using the *scalar Alexiewicz norm* $\|\cdot\|_A$ defined by

$$\|f\|_A := \sup\left\{\left|\textstyle\int_c^d f\right| : c,d\in[a,b], c<d\right\}.$$

This is equivalent to the norm defined by $\|f\|_A^* := \|F\|_\infty$ where $F\in ACG([a,b],\mathbb{R})$ is such that $F'_{\mathrm{ap}}=f$ almost everywhere on $[a,b]$ and $F(a)=0$; indeed $\|f\|_A^*\le\|f\|_A\le 2\|f\|_A^*$. The space $DK([a,b],\mathbb{R})$ endowed with the scalar Alexiewicz norm is a non-complete, ultrabornological normed space [Ga], [Gi].

Bochner, Dunford and Pettis integrals are the best known extensions of Lebesgue integral to functions $f\colon[a,b]\to X$ (see [DU, II.2–3], for instance). Based on these, the Denjoy-Bochner, Denjoy-Dunford and Denjoy-Pettis integrals were defined by Gordon [Go1] as follows.

**Definitions** A function $f\colon[a,b]\to X$ is said to be *Denjoy-Bochner integrable on* $[a,b]$ if there exists $F\in ACG([a,b],X)$ such that $F'_{\mathrm{ap}}=f$ almost everywhere on $[a,b]$ in which case the *Denjoy-Bochner integral of* $f$ *on* $[a,b]$ is defined by $\int_a^b f := F(b)-F(a)$ (there is no ambiguity here because, again, if $F\in ACG([a,b],X)$ is such that $F'_{\mathrm{ap}}=0$ almost everywhere on $[a,b]$, then $F$ is constant on $[a,b]$).

A function $f\colon[a,b]\to X$ is said to be *Denjoy-Dunford integrable on* $[a,b]$ if the composition $t\to\langle x^*,f(t)\rangle$ is Denjoy-Khinchin integrable for every $x^*\in X^*$. Gámez and Mendoza [GaM, Thm. 3] proved that for every interval $[c,d]\subset[a,b]$ there is an element $\int_c^d f\in X^{**}$ called *the Denjoy-Dunford integral of* $f$ *on* $[c,d]$ such that

$$\left\langle x^*, \textstyle\int_c^d f\right\rangle = \int_c^d \langle x^*,f\rangle = \int_a^b \langle x^*,f\rangle\,\chi_{[c,d]}.$$

(N. B. The existence of these elements $\int_c^d f \in X^{**}$ was assumed by Gordon as a part of his original definition of Denjoy-Dunford integral [Go1, Def. 25].)

A function $f\colon [a,b] \to X$ is said to be *Denjoy-Pettis integrable on* $[a,b]$ if it is Denjoy-Dunford integrable and the Denjoy-Dunford integral $\int_c^d f$ is in $X$ for every $[c,d] \subset [a,b]$.

As in the scalar case, these spaces can be topologized in a natural way using the *vectorial Alexiewicz norm* $\|\cdot\|_A$ defined by

$$\begin{aligned}\|f\|_A :&= \sup\left\{\left\|\int_c^d f\right\| : c,d \in [a,b], c<d\right\}\\ &= \sup\left\{\left|\int_c^d \langle x^*, f\rangle\right| : c,d \in [a,b], c<d; x^* \in X^*, \|x^*\| \leq 1\right\}\end{aligned}$$

followed by the appropriate quotient, namely, two functions $f, g\colon [a,b] \to X$ are in the same equivalence class if $\langle x^*, f-g\rangle = 0$ almost everywhere for every $x^* \in X^*$.

None of these spaces is complete, but Gámez [Ga, 4.2.13] proved, by means of a clever adaptation of Gilioli's technique [Gi], that the spaces of Denjoy-Bochner and Denjoy-Pettis integrable functions are ultrabornological; the Denjoy-Dunford case remaining open.

**Theorem.** *The space $DD([a,b],X)$ of Denjoy-Dunford integrable functions endowed with the vectorial Alexiewicz norm is ultrabornological.*

*Proof.* The sliding hump technique that we shall employ is, essentially, the one we used in [DFFP, Thm. 1]. However, the measure theoretic framework of that paper is too general for our purposes here, so we need to make some major adjustments, mainly contained in the Lemma below, to adapt it to the case at hand.

Let $A$ be an absolutely convex subset of $DD([a,b],X)$ that absorbs all Banach discs. To prove that $A$ is a zero-neighborhood, we have to check that $A$ absorbs the unit ball $B$ of $DD([a,b],X)$. Assume, on the contrary, that $A$ does not absorb $B$. Let $c$ be the midpoint of $[a,b]$ and write $B = \chi_{[a,c]}B + \chi_{[c,d]}B$, then $A$ does not absorb $\chi_{[a,c]}B$ or $A$ does not absorb $\chi_{[c,d]}B$. Let $I_1$ be a half of $I_0 = [a,b]$ such that $A$ does not absorb $\chi_{I_1}B$. The obvious inductive procedure tells us that we can find a sequence of nested intervals $(I_n)$ such that for every $n \in \mathbb{N}$ the set $A$ does not absorb $\chi_{I_n}B$ and $\mathrm{length}(I_n) = 2^{-n}(b-a)$. Denote by $t_0$ the unique point belonging to all of the intervals $(I_n)$. For every $n \in \mathbb{N}$, take a function $f_n$ in (the set of equivalence classes) $\chi_{I_n}B$, hence $\|f_n\|_A \leq 1$, such that $f_n(t_0) = 0$ and $f_n \notin nA$. Now take a sequence $(\alpha_n) \in \ell^1$. If $t$ is a point in $[a,b]$ such that $t \neq t_0$, then the series $\sum_n \alpha_n f_n(t)$ contains only a finite number of non-zero terms because the supports of the functions $f_n$ decrease to $\{t_0\}$, so we may define a function $f\colon [a,b] \to X$ by $f(t) := \sum_n \alpha_n f_n(t)$. We claim —it will be proved in the Lemma below— that for every $x^* \in X^*$ the scalar function given by

$$\langle x^*, f(t)\rangle = \sum_n \alpha_n \langle x^*, f_n(t)\rangle \qquad (t \in [a,b])$$

is Denjoy-Khinchin integrable on $[a,b]$ and that the series $\sum_n \alpha_n \langle x^*, f_n\rangle$ converges to $\langle x^*, f\rangle$ in the scalar Alexiewicz norm. This implies that $f$ is Denjoy-Dunford integrable. Let us also see that $\sum_n \alpha_n f_n$ converges to $f$ in the vectorial Alexiewicz norm. This follows from the convergence in the scalar Alexiewicz norm by noting that if $c < d$ are points in $[a, b]$ and $x^*$ is in the unit ball of $X^*$, then

$$\begin{aligned} \left| \int_c^d \left\langle x^*, f - \sum_{n=1}^m \alpha_n f_n \right\rangle \right| &= \left| \sum_{n=m+1}^\infty \alpha_n \int_c^d \langle x^*, f_n \rangle \right| \\ &\le \sum_{n=m+1}^\infty |\alpha_n| \left| \int_c^d \langle x^*, f_n \rangle \right| \\ &\le \sum_{n=m+1}^\infty |\alpha_n| \, \|f_n\|_A \le \sum_{n=m+1}^\infty |\alpha_n| \underset{m\to\infty}{\longrightarrow} 0. \end{aligned}$$

Since the series $\sum_n \alpha_n f_n$ converges for every sequence $(\alpha_n) \in \ell^1$, it follows, as it is well known, that the set $C := \{\sum_n \alpha_n f_n : \|(\alpha_n)\|_1 \le 1\}$ is a Banach disc in $DD([a, b], X)$ which is not absorbed by $A$ because $f_n \in C$ but $f_n \notin nA$ for all $n \in \mathbb{N}$. A contradiction. ■

**Lemma.** *Let $(I_n)$ be a sequence of nested intervals in $[a, b]$ with a unique common point $t_0$. Let $(f_n)$ be a sequence of Denjoy-Khinchin integrable functions such that $f_n$ is supported in $I_n$ and $f_n(t_0) = 0$ for each $n \in \mathbb{N}$. If $\sum_n \|f_n\|_A < \infty$, then the function defined by $f(t) = \sum_n f_n(t)$ is Denjoy-Khinchin integrable and the series $\sum f_n$ converges to $f$ in the scalar Alexiewicz norm.*

*Proof.* We may assume that $t_0 = b$ (if not, reason as follows in $[a, t_0]$ and symmetrically in $[t_0, b]$) so that we may write $I_n = [c_n, b]$ where $(c_n)$ is increasing and $\lim_n c_n = b$. First note, as above, that $f$ is well-defined because for every $t \in [a, b)$ there is only a finite number of non-zero terms in the series $\sum_n f_n(t)$. Now, for each $n \in \mathbb{N}$ take the primitive $F_n \in ACG([a, b], \mathbb{R})$ such that $(F_n)'_{\text{ap}} = f_n$ and $F_n(a) = 0$. Since $\sum_n \|F_n\|_\infty < \infty$ because $\|F_n\|_\infty = \|f_n\|_A^* \le \|f_n\|_A$, it follows that the series $\sum_n F_n$ converges in $C([a, b])$ to a continuous function $F$. We shall prove that $F \in ACG([a, b], \mathbb{R})$ and that $F'_{\text{ap}} = f$ almost everywhere on $[a, b]$. This will show that $f$ is Denjoy-Khinchin integrable on $[a, b]$.

Note that if $k \ge n$ then $I_k$ and $[a, c_n)$ are disjoint. Therefore, $f_k$ is zero in $[a, c_n)$ so that $F_k$ must be constant in this interval. Using that $F_k$ is continuous and that $F_k(a) = 0$, it follows that $F_k = 0$ in $[a, c_n]$. This tells us, on the one hand, that $F$ is zero in $[a, c_1]$, and on the other hand, that $F = \sum_{k=1}^n F_k$ in $[c_n, c_{n+1}]$ for each $n \in \mathbb{N}$. Therefore, we have that $F\chi_{[c_n, c_{n+1}]} \in ACG([c_n, c_{n+1}], \mathbb{R})$ for each $n \in \mathbb{N}$ and this proves that $F \in ACG([a, b], \mathbb{R})$. The same equality $F = \sum_{k=1}^n F_k$ in $[c_n, c_{n+1}]$ also shows that

$$F'_{\text{ap}} = \sum_{k=1}^n (F_k)'_{\text{ap}} = \sum_{k=1}^n f_k = f \qquad \text{almost everywhere on } [c_n, c_{n+1}],$$

so that $F'_{\text{ap}} = f$ almost everywhere on $[a, b]$. Finally,

$$\left\| f - \sum_{n=1}^m f_m \right\|_A \le 2 \left\| f - \sum_{n=1}^m f_m \right\|_A^* = 2 \left\| F - \sum_{n=1}^m F_n \right\|_\infty \le 2 \sum_{n=m+1}^\infty \|F_n\|_\infty \underset{m\to\infty}{\longrightarrow} 0,$$

which shows that the series $\sum_n f_n$ converges to $f$ in the scalar Alexiewicz norm. ■

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