Partial practical exponential stability of neutral stochastic functional differential equations with Markovian switching

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Abstract. In this paper we investigate the partial practical exponential stability of neutral stochastic functional differential equations with Markovian switching. The main tool used to prove the results is the Lyapunov method. We analyze an illustrative example to show the applicability and interest of the main results.

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1. Introduction

Neutral stochastic differential equations with Markovian switching have attracted much attention over the last decades within the scopes and main problems from the applied sciences and which are modeled by stochastic differential equations (see [1],[4],[8] and [9]). Neutral stochastic differential equations with Markovian switching are useful to model physical, biological and economical dynamical phenomena. In the literature, many authors have studied the existence and uniqueness of solution to neutral stochastic differential equations with a Markovian switching (see [10] and [11]). Stability is the most important concept in modern control theory, and switching systems can be used to model a wide type of physical and engineering systems in practice, hence, stability of neutral stochastic differential equations with a Markovian switching has received an increasing attention (see [6], [7] and [10]). However, in many physical systems, such stability is sometimes too strong to be satisfied. Therefore, the notion of stability with respect to part of the variables (i.e. partial stability) (see [5], [6], [12] and [13]) has been used, and the Lyapunov Method, as an indispensable tool, has been used to investigated the partial stability and stabilizability in various practically important problems. However, when the origin is not necessarily an equilibrium point, it is still possible to study the asymptotic stability of solutions with respect to a small neighborhood of the origin, which yields to the concept of practical stability (see [2] and [3]).

The structure of the paper is as follows. In Section 2 we introduce some basic notions and assumptions. Section 3 is devoted to prove some sufficient conditions ensuring practical partial p-th moment exponential stability of solutions to neutral stochastic functional differential equations with Markovian switching. We prove a sufficient condition ensuring the convergence of the solution (with respect to part of the variables) to a ball with radius $r > 0$ in p-th moment, even in the case that zero is not an equilibrium point. In Section 4 we prove a sufficient condition ensuring practical exponential instability in the q-th moment. Finally we analyze an example to illustrate our results in Section 5.

2. Preliminaries and definitions

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is right continuous and increasing and $\mathcal{F}_0$ contains all $\mathbb{P}$–null sets. $W(t)$ is an m-dimensional Brownian motion defined on the probability space. For a given $\tau > 0$, let $C([-\tau, 0]; \mathbb{R}^n)$ be the family of functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$ that are right-continuous and have limits on the left. $C([-\tau, 0]; \mathbb{R}^n)$ is equipped with the norm $\|\varphi\| = \sup_{-\tau\leq s\leq 0} |\varphi(s)|$ and $|x| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^n$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. Let $p > 0$, $t \geq 0$, $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\varphi(\theta)|^p < \infty$.

Let $\{r(t), t \in \mathbb{R}^+ = [0, +\infty[\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with a generator $\Gamma = (\gamma_{ij})_{\mathbb{N}\times\mathbb{N}}$ given by

$$\mathbb{P}\left(r(t+\Delta) = j | r(t) = i\right) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$, while

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

We assume that the Markov chain $r(t)$ is independent of the Brownian motion $W(t)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite sub-interval of $\mathbb{R}^+$.

Consider the following neutral stochastic functional differential equation with Markovian switching:

$$d\left[x(t) - G(x_t)\right] = f(t, x_t, r(t))dt + g(t, x_t, r(t))dW(t), \quad t > 0, \tag{2.1}$$

with the initial condition $x_0 = \xi = (\xi_1, \xi_2)^T \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ where $\xi_1 \in \mathbb{R}^k$ and $\xi_2 \in \mathbb{R}^p$, $k+p = n$, which is independent of $W(\cdot)$, $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^k \times \mathbb{R}^p$ and $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$. Let $\widetilde{x(t)} = x(t) - G(x_t)$. Here, we furthermore assume that

$$f : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times S \longrightarrow \mathbb{R}^n, \quad g : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times S \longrightarrow \mathbb{R}^{n\times m},$$
$$G : C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n.$$

Denote by $C^{1,2}\left([-\tau, +\infty[\times\mathbb{R}^n \times S; \mathbb{R}^+\right)$ the family of all non-negative functions $V(t, x, i)$ on $[-\tau, +\infty[\times\mathbb{R}^n \times S$, which are twice continuously differentiable with respect to x and once continuously differentiable with respect to t.

For any $(t, x, i) \in [-\tau, +\infty[\times\mathbb{R}^n \times S$, $x_t = \varphi \in C([-\tau, 0]; \mathbb{R}^n)$ and $\widetilde{\varphi}(\theta) = \varphi(\theta) - G(\varphi)$, define an operator $LV : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times S \to \mathbb{R}$ by (see [11])

$$\begin{aligned} LV(t, \widetilde{\varphi}(0), i) &= V_t(t, \widetilde{\varphi}(0), i) + V_x(t, \widetilde{\varphi}(0), i)f(t, \varphi, i) \\ &\quad + \frac{1}{2} trace\left(g^T(t, \varphi, i)V_{xx}(t, \widetilde{\varphi}(0), i)g(t, \varphi, i)\right) \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(t, \widetilde{\varphi}(0), j), \end{aligned}$$

where

$$V_t = \frac{\partial V(t, x, i)}{\partial t}, \quad V_x = \left(\frac{\partial V(t, x, i)}{\partial x_1}, \ldots, \frac{\partial V(t, x, i)}{\partial x_n}\right),$$
$$V_{xx} = \left(\frac{\partial^2 V(t, x, i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

For our purpose, we will state some assumptions which can ensure the existence and uniqueness of a solution, denoted by $x(t) = (x_1(t), x_2(t))$ on $t > 0$, for equation (2.1).

$\mathcal{A}_1$: (A local Lipschitz condition): For each $p = 1, 2, \ldots$ there is an $l_p > 0$ such that

$$|f(t, \varphi_1, i) - f(t, \varphi_2, i)| \vee |g(t, \varphi_1, i) - g(t, \varphi_2, i)| \leq l_p ||\varphi_1 - \varphi_2||,$$

for all $t \geq 0$, $i \in S$ and φ_1, $\varphi_2 \in C([-\tau, 0]; \mathbb{R}^n)$ with $||\varphi_1|| \vee ||\varphi_2|| \leq p$.

$\mathcal{A}_2$: There exist three functions $V \in C^{1,2}\left([-\tau, \infty) \times \mathbb{R}^n \times S; \mathbb{R}^+\right)$, U_1, $U_2 \in C\left([-\tau, \infty) \times \mathbb{R}^n; \mathbb{R}^+\right)$ and a probability measure m on $[-\tau, 0]$ satisfying $\int_{-\tau}^0 dm(\theta) = 1$, and nonnegative constants c'_1, c'_2, c'_3 with $c'_2 > c'_3$, such that

$$\lim_{|x|\to\infty} \left(\inf_{t\geq 0} U_1(t, x)\right) = \infty, \tag{2.2}$$

and for all $(t, x, i) \in \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \times S$, we have

$$U_1(t, x) \le V(t, x, i) \le U_2(t, x). \tag{2.3}$$

$$LV(t, \widetilde{\varphi}(0), i) \le c_1' - c_2' U_2(t, \widetilde{\varphi}(0)) + c_3' \int_{-\tau}^{0} U_2(t+\theta, \widetilde{\varphi}(\theta)) dm(\theta), \tag{2.4}$$

for all $i \in S$ and $x_t = \varphi \in C([-\tau, 0]; \mathbb{R}^n)$, $t \ge 0$ and where $\widetilde{\varphi}(\theta) = \varphi(\theta) - G(\varphi)$.

$\mathcal{A}_3$: There exists a constant $k \in [0, 1)$ such that

$$|G(\varphi_1) - G(\varphi_2)| \le k|\varphi_1(-\tau) - \varphi_2(-\tau)|, \tag{2.5}$$

for all φ_1, $\varphi_2 \in C([-\tau, 0]; \mathbb{R}^n)$.

Denote $x = (x_1, x_2)^T \in \mathbb{R}^n$, where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^p$, $k + p = n$, and the definitions of $\widetilde{x} = (\widetilde{x_1}, \widetilde{x_2})^T \in \mathbb{R}^n$ and $G(x_t) = (G_1(x_t), G_2(x_t))^T \in \mathbb{R}^n$ are similar to $x = (x_1, x_2)^T$. The domain $B_K = \{x \in \mathbb{R}^n : |x_1| < K\}$, and the stopping time $\tau_K = \inf\{t \ge t_0; x(t) \notin B_K\}$. Let $K \to \infty$, then $\tau_K \to \infty$. Denote the set of functions

$$\mathcal{K} := \left\{\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \text{continuous, monotonically increasing and } \phi(0) = 0\right\}.$$

We will study the partial stability of the neutral stochastic functional differential equation with Markovian switching when 0 is not an equilibrium point, but in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball $B_r := \{x \in \mathbb{R}^d : ||x_1|| \le r\}$, $r > 0$.

Definition 2.1. The solution $x(t) = (x_1(t), x_2(t))$ of equation (2.1) is said to be practically exponentially x_1^p-stable ($p > 0$), if there exist positive constants α, c, r such that, for any $(x_1, x_0) \in \mathbb{R}^k \times C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\mathbb{E}\left(|x_1|^p\right) \le c\mathbb{E}\left(||x_0||^p\right) \exp\left(-\alpha(t - t_0)\right) + r, \quad t \ge t_0. \tag{2.6}$$

When $p = 1$ (respectively $p = 2$), the solution $x(t)$ of stochastic system (2.1) is called exponential x_1-stable in the mean (respectively in the mean-square).

Remark 2.2. Note that, as the origin $x = 0$ may not be an equilibrium point of system (2.1), then we can no longer study the stability of the origin as an equilibrium point nor should we expect the p-th moment of the solution (with respect to part of the variables) of the system to approach the origin almost surely as $t \to +\infty$. Inequality (4.1) implies that $\mathbb{E}\left(|x_1|^p\right)$ will be ultimately bounded by a small bound $r > 0$, that is, $\mathbb{E}\left(|x_1|^p\right)$ will be small for sufficiently large t. This can be viewed as a robustness property of convergence almost surely to the origin provided that f and g satisfy $f(t, 0, r(t)) = 0$ and $g(t, 0, r(t)) = 0$, $\forall t \ge 0$. In this case the origin becomes an equilibrium point.

3. Main results

Now we can state and prove our main results.

Theorem 3.1. *Under assumptions $\mathcal{A}_1$-$\mathcal{A}_3$, for any given initial condition $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $i_0 \in S$, there exists a unique global solution $x(t)$, $t \ge -\tau$ of equation* (2.1).

Proof. See [10] and [11]. □

Lemma 3.2. *(i). Let $0 < p \leq 1$ and a, $b \in \mathbb{R}_+$. Then*

$$(a+b)^p \leq a^p + b^p.$$

(ii). Let $p > 1$, $\varepsilon > 0$ and a, $b \in \mathbb{R}_+$. Then

$$(a+b)^p \leq \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\left[a^p + \frac{b^p}{\varepsilon}\right].$$

Proof. See [9]. □

Let us rewrite now assumption $\mathcal{A}_3$ in a new way:
$\mathcal{A}_3'$: There exist constants $k \in [0,1)$, $p, \delta > 0$ such that

$$|G(\varphi_1) - G(\varphi_2)| \leq k e^{-\frac{\delta}{p}\tau} |\varphi_1(-\tau) - \varphi_2(-\tau)|, \tag{3.1}$$

for all φ_1, $\varphi_2 \in C([-\tau,0];\mathbb{R}^n)$.

Theorem 3.3. *Let c_1, c_2, c_3, δ be positive constants such that $\frac{c_3}{c_2} \leq \delta$ and $p > 0$. Assume that there exist $V(t,x,i) \in C^{1,2}\left(]-\tau,+\infty[\times\mathbb{R}^n \times S, \mathbb{R}^+\right)$ and $e^{\frac{c_3}{c_2}t}\rho(t) \in \mathbb{L}^1([0,+\infty[)$ such that,*

(i). $c_1|x_1|^p \leq V(t,x,i) \leq c_2|x_1|^p$ for $x_1 \in \mathbb{R}^k$.
(ii). $LV(t,\widetilde{\varphi}(0),i) \leq -c_3|\widetilde{\varphi}_1(0)|^p + \rho(t)$, $\forall (t,\varphi,i) \in \mathbb{R}^+ \times C([-\tau,0];\mathbb{R}^n) \times S$ where $\varphi = (\varphi_1,\varphi_2) \in C([-\tau,0];\mathbb{R}^k) \times C([-\tau,0];\mathbb{R}^s)$, $k+s=n$ and $\widetilde{\varphi}(0) = \varphi(0) - G(\varphi)$.

Let assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3'$ hold. Then, the solution $x(t) = (x_1(t), x_2(t))$ of equation (2.1) is practically exponentially x_1^p-stable if $0 < p \leq 1$. If $p > 1$, the same stability holds true if in addition there exists $\varepsilon > 0$ such that $k^p < \varepsilon\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{1-p}$.

Proof. According to the stopping time $\tau_K = \inf\{t \geq t_0; x(t) \notin B_K\}$, writing $x_t = \varphi = (\varphi_1,\varphi_2)$, $x_1(t) = \varphi_1(0)$, $\widetilde{\varphi}(0) = x(t) - G(x_t)$ and $\widetilde{\varphi}_1(0) = x_1(t) - G_1(x_t)$ for $t \geq t_0$, and applying Itô's formula (see [11]), we can derive for

$t \geq -\tau$

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}V(t\wedge\tau_K, x(t\wedge\tau_K) - G(x_{t\wedge\tau_K}), r(t\wedge\tau_K))\right)\\
&= \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right)\\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\left[\frac{c_3}{c_2}V(s, x(s) - G(x_s), r(s)) + LV(s, x(s) - G(x_s), r(s))\right]ds\right)\\
&\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right)\\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}V(s, x(s) - G(x_s), r(s)) - c_3|x_1(s) - G_1(x_s)|^p + \rho(s)\right)ds\right)\\
&\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right)\\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\left(\frac{c_3}{c_2}c_2|x_1(s) - G_1(x_s)|^p - c_3|x_1(s) - G_1(x_s)|^p + \rho(s)\right)ds\right)\\
&\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}V(t\wedge\tau_K, x(t\wedge\tau_K) - G(x_{t\wedge\tau_K}), r(t\wedge\tau_K))\right)\\
&\quad\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right).
\end{aligned}$$

Therefore, by inequality (i), we have

$$\begin{aligned}
&c_1\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}|x_1(t\wedge\tau_K) - G_1(x_{t\wedge\tau_K})|^p\right)\\
&\quad\leq \mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}V(t\wedge\tau_K, x(t\wedge\tau_K) - G(x_{t\wedge\tau_K}), r(t\wedge\tau_K))\right)\\
&\quad\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right).
\end{aligned}$$

Obviously $\tau_K \to \infty$ as $K \to \infty$. Then

$$\begin{aligned}
c_1\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t) - G_1(x_t)|^p\right) &\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right)\\
&+\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\rho(s)ds. \qquad (3.2)
\end{aligned}$$

First case: For $0 < p \leq 1$, by Lemma 3.2 and Assumption $\mathcal{A}_3'$, we have

$$
\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\leq \mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^p\right)+\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|G_1(x_t)|^p\right)\\
&\leq \mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^p\right)\\
&\quad +k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau))|^p\right)\\
&\leq \frac{1}{c_1}\mathbb{E}\left(V(t_0,x_0-G(x_0),r(t_0))\right)\\
&\quad +\frac{1}{c_1}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\rho(s)ds+k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\leq \frac{c_2}{c_1}\mathbb{E}\left(|x_0-G_1(x_0)|^p\right)\\
&\quad +\frac{1}{c_1}\int_{0}^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\leq \frac{c_2}{c_1}\mathbb{E}\left(\|x_0\|^p+|G_1(x_0)|^p\right)\\
&\quad +\frac{1}{c_1}\int_{0}^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\leq \frac{c_2}{c_1}\mathbb{E}\left(\|x_0\|^p+k^pe^{-\delta\tau}\|x_0\|^p\right)\\
&\quad +\frac{1}{c_1}\int_{0}^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\leq \frac{c_2}{c_1}\left(1+k^p\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad +\frac{1}{c_1}\int_{0}^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+k^pe^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right).
\end{aligned}
$$

For any $T>0$, we have

$$\begin{aligned}
&\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2}{c_1}(1+k^p)\,\mathbb{E}\left(\|x_0\|^p\right)+\frac{1}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +k^pe^{-\delta\tau}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\le \frac{c_2}{c_1}(1+k^p)\,\mathbb{E}\left(\|x_0\|^p\right)+\frac{1}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +k^pe^{\left(\frac{c_3}{c_2}-\delta\right)\tau}\sup_{t_0-\tau\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2}{c_1}(1+k^p)\,\mathbb{E}\left(\|x_0\|^p\right)+\frac{1}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +k^p\sup_{t_0-\tau\le t\le t_0}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\quad +k^p\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2}{c_1}(1+k^p)\,\mathbb{E}\left(\|x_0\|^p\right)+\frac{1}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +k^p\mathbb{E}\left(\|x_0\|^p\right)+k^p\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \left(\frac{c_2}{c_1}(1+k^p)+k^p\right)\mathbb{E}\left(\|x_0\|^p\right)+\frac{1}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +k^p\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right) &\le \frac{1}{1-k^p}\left(\frac{c_2}{c_1}(1+k^p)+k^p\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad +\frac{1}{c_1(1-k^p)}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\le C_1(k)\mathbb{E}\left(\|x_0\|^p\right)+C_2(k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds,
\end{aligned}$$

where $C_1(k)=\frac{1}{1-k^p}\left(\frac{c_2}{c_1}(1+k^p)+k^p\right)$ and $C_2(k)=\frac{1}{c_1(1-k^p)}$.
Letting $T\to+\infty$, we have

$$\sup_{t_0\le t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\le C_1(k)\mathbb{E}\left(\|x_0\|^p\right)+C_2(k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds.$$

Therefore, we obtain for all $t\ge t_0$

$$\mathbb{E}\left(|x_1(t)|^p\right)\le C_1(k)\mathbb{E}\left(\|x_0\|^p\right)e^{-\frac{c_3}{c_2}(t-t_0)}+C_2(k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds. \qquad (3.3)$$

Second case: For $p > 1$, by Lemma 3.2 and Assumption $\mathcal{A}_3'$, for $\varepsilon > 0$ such that $\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1} < 1$ and for $\lambda > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right) \\
&\leq \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^p\right) \\
&\quad + \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}\frac{|G_1(x_t)|^p}{\varepsilon}\right) \\
&\leq \left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^p\right) \\
&\quad + \frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right) \\
&\leq \frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\mathbb{E}\left(V(t_0, x_0-G(x_0), r(t_0))\right) \\
&\quad + \frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\rho(s)ds \\
&\quad + \frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right) \\
&\leq \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\mathbb{E}\left(|x_0-G_1(x_0)|^p\right) \\
&\quad + \frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_{0}^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds \\
&\quad + \frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right) \\
&\leq \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(\|x_0\|^p+\frac{|G_1(x_0)|^p}{\lambda}\right) \\
&\quad + \frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_{0}^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds \\
&\quad + \frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right) \\
&\leq \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)\mathbb{E}\left(\|x_0\|^p\right) \\
&\quad + \frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_{0}^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds \\
&\quad + \frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right).
\end{aligned}$$

Hence, for any $T > 0$, we have

$$
\begin{aligned}
&\sup_{t_0\le t\le T} \mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad+\frac{k^p e^{-\delta\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^p\right)\\
&\le \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad+\frac{k^p e^{\left(\frac{c_3}{c_2}-\delta\right)\tau}}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0-\tau\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad+\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0-\tau\le t\le t_0}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\quad+\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le \frac{c_2\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad+\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\
&\le C(\varepsilon,\lambda,k)\mathbb{E}\left(\|x_0\|^p\right)\\
&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad+\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right),
\end{aligned}
$$

where

$$C(\varepsilon,\lambda,k)=\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\left(\frac{c_2}{c_1}\left(1+\lambda^{\frac{1}{p-1}}\right)^{p-1}\left(1+\frac{k^p}{\lambda}\right)+\frac{k^p}{\varepsilon}\right).$$

Thus,

$$\begin{aligned}&\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\&\le\frac{C(\varepsilon,\lambda,k)}{1-\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}\mathbb{E}\left(\|x_0\|^p\right)\\&\quad+\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1\left(1-\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\right)}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds\\&\le C_3(\varepsilon,\lambda,k)\mathbb{E}\left(\|x_0\|^p\right)+C_4(\varepsilon,k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds,\end{aligned}$$

where

$$C_3(\varepsilon,\lambda,k)=\frac{C(\varepsilon,\lambda,k)}{1-\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}$$

and

$$C_4(\varepsilon,k)=\frac{\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}}{c_1\left(1-\frac{k^p}{\varepsilon}\left(1+\varepsilon^{\frac{1}{p-1}}\right)^{p-1}\right)}.$$

Letting $T\to+\infty$, we obtain

$$\begin{aligned}&\sup_{t_0\le t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^p\right)\\&\le C_3(\varepsilon,\lambda,k)\mathbb{E}\left(\|x_0\|^p\right)+C_4(\varepsilon,k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds,\end{aligned}$$

Then, we have for all $t\ge t_0$

$$\mathbb{E}\left(|x_1(t)|^p\right)\quad\le\quad C_3(\varepsilon,\lambda,k)\mathbb{E}\left(\|x_0\|^p\right)e^{-\frac{c_3}{c_2}(t-t_0)}+C_4(\varepsilon,k)\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds.$$

We see that for all $t\ge t_0$,

$$\begin{aligned}\mathbb{E}\left(|x_1(t)|^p\right)\quad&\le\quad\max\Big\{C_1(k),C_3(\varepsilon,\lambda,k)\Big\}\mathbb{E}\left(\|x_0\|^p\right)e^{-\frac{c_3}{c_2}(t-t_0)}\\&\quad+\max\Big\{C_2(k),C_4(\varepsilon,k)\Big\}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds.\end{aligned}$$

Setting

$$c=\max\Big\{C_1(k),C_3(\varepsilon,\lambda,k)\Big\},\qquad\alpha=\frac{c_3}{c_2},$$

$$r=\max\Big\{C_2(k),C_4(\varepsilon,k)\Big\}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds,$$

we conclude that the solution of system (2.1) is practically exponentially x_1^p-stable. The proof is therefore complete. □

In the following corollary we will study the partial stability in mean square of system (2.1) in the domain $D = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n; |x| < \infty\}$.
We will consider the new assumption:
$\mathcal{A}_3''$: There exists a constant $k \in [0,1)$ such that

$$|G(\varphi_1) - G(\varphi_2)| \leq ke^{-\frac{\delta}{2}\tau}|\varphi_1(-\tau) - \varphi_2(-\tau)|, \tag{3.4}$$

for all φ_1, $\varphi_2 \in C([-\tau,0];\mathbb{R}^n)$ and $\delta > 0$.

Corollary 3.4. *Let $x = (x_1,x_2)^T \in \mathbb{R}^n$, where $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^s$ and $n = k+s$, be the solution of system* (2.1). *Suppose that in the domain $D = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n; |x| < \infty\}$, along with a V-function, it is possible to specify a continuous vector $\mu(x)$-function, $\mu(0) = 0$ and $e^{\frac{c_3}{c_2}t}\rho(t) \in \mathbb{L}^1([0,+\infty[)$ such that, for all $(t,\varphi,i) \in \mathbb{R}^+ \times C([-\tau,0];\mathbb{R}^n) \times S$ and*

$$\widetilde{\varphi}(0) = \varphi(0) - G(\varphi) = x(t) - G(x_t),$$

we have

(i).
$$c_1|x_1|^2 \leq V(t,x,i) \leq c_2\left(|x_1|^2 + |\mu(x)|^2\right),$$

(ii).
$$LV(t,\widetilde{\varphi}(0),i) \leq -c_3\left(|\widetilde{\varphi_1}(0)|^2 + |\mu(\widetilde{\varphi}(0))|^2\right) + \rho(t). \tag{3.5}$$

Let assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3''$ hold. Assume that there exist $k \in [0,\frac{\sqrt{2}}{2})$ and $\delta > 0$ such that $\frac{c_3}{c_2} \leq \delta$. Then the solution of system (2.1) *is practically exponentially x_1-stable in mean-square.*

Proof. Proceeding as in the previous proof, writing $x_t = \varphi$, $x(t) = \varphi(0)$ and $x_1(t) = \varphi_1(0)$ for $t \geq t_0$, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}V(t\wedge\tau_K, x(t\wedge\tau_K) - G(x_{t\wedge\tau_K}), r(t\wedge\tau_K))\right)\\
&\leq \mathbb{E}\left(V(t_0,x_0 - G(x_0), r(t_0))\right)\\
&\quad + c_3\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\left(|x_1(s) - G_1(x(s))|^2 + |\mu\Big(x(s) - G(x(s))\Big)|^2\right)\right)\\
&\quad - c_3\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\left(|x_1(s) - G_1(x(s))|^2 + |\mu\Big(x(s) - G(x(s))\Big)|^2\right)\right)\\
&\quad + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right)\\
&\leq \mathbb{E}\left(V(t_0,x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right).
\end{aligned}$$

By inequality (i),

$$c_1\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)}|x_1(t\wedge\tau_K) - G_1(x_{t\wedge\tau_K})|^2\right)$$

$$\begin{aligned}
&\leq \mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_K - t_0)} V(t\wedge\tau_K, x(t\wedge\tau_K) - G(x_{t\wedge\tau_K}), r(t\wedge\tau_K))\right) \\
&\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_K} e^{\frac{c_3}{c_2}s}\rho(s)ds\right).
\end{aligned}$$

Obviously $\tau_K \to \infty$ as $K \to \infty$ then

$$\begin{aligned}
c_1\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t) - G_1(x_t)|^2\right) &\leq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) \\
&\quad + \int_{t_0}^{t} e^{\frac{c_3}{c_2}s}\rho(s)ds. \qquad (3.6)
\end{aligned}$$

Since μ is continuous and $\mu(0) = 0$ then there exist $\eta > 0$ such that $|\mu(x_0 - G(x_0))| \leq \eta|x_0 - G(x_0)|$ for a sufficiently small $\|x_0\|$.

$$\begin{aligned}
|\mu(x_0 - G(x_0))|^2 &\leq \eta^2|x_0 - G(x_0)|^2 \\
&\leq 2\eta^2\left(\|x_0\|^2 + |G(x_0)|^2\right) \\
&\leq 2\eta^2\left(\|x_0\|^2 + k^2 e^{-\frac{\delta}{\tau}}\|x_0\|^2\right) \\
&\leq 2\eta^2\left(\|x_0\|^2 + k^2\|x_0\|^2\right) \\
&\leq 2\eta^2(1 + k^2)\|x_0\|^2.
\end{aligned}$$

Then, for $x_0 = (x_{01}, x_{02}) \in C([-\tau, 0]; \mathbb{R}^k) \times C([-\tau, 0]; \mathbb{R}^s)$, we have

$$\begin{aligned}
|x_{01} - G_1(x_0))|^2 &\leq 2\left(\|x_{01}\|^2 + |G_1(x_0)|^2\right) \\
&\leq 2\left(\|x_0\|^2 + k^2\|x_0\|^2\right) \\
&\leq 2(1 + k^2)\|x_0\|^2.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left(|x_{01} - G_1(x_0)|^2 + |\mu(x_0 - G(x_0))|^2\right) \leq 2(1 + k^2)(1 + \eta^2)\mathbb{E}\left(\|x_0\|^2\right).$$

By Assumption $\mathcal{A}_3$ and (ii) in Lemma 3.2 (for $\varepsilon = 1$), we have for $k \in [0, \frac{\sqrt{2}}{2})$,

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right)\\
&\leq 2\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^2\right)+2\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|G_1(x_t)|^2\right)\\
&\leq 2\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)-G_1(x_t)|^2\right)+2k^2e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^2\right)\\
&\leq \frac{2}{c_1}\mathbb{E}\left(V(t_0,x_0-G(x_0),r(t_0))\right)+\frac{2}{c_1}\int_{t_0}^{t}e^{\frac{c_3}{c_2}s}\rho(s)ds\\
&\quad +2k^2e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^2\right)\\
&\leq \frac{2c_2}{c_1}\mathbb{E}\left(|x_{01}-G_1(x_0)|^2+|\mu(x_0-G(x_0))|^2\right)\\
&\quad +\frac{2}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+2k^2e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^2\right)\\
\leq\ &\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}\mathbb{E}\left(\|x_0\|^2\right)\\
&+\frac{2}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+2k^2e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^2\right).
\end{aligned}$$

Hence, for any $T > 0$, we have

$$\begin{aligned}
&\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right)\\
&\leq \frac{4c_2(1+k^2)(1+\eta^2)}{c_1}\mathbb{E}\left(\|x_0\|^2\right)\\
&\quad +\frac{2}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+2k^2e^{-\delta\tau}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t-\tau)|^2\right)\\
&\leq \frac{4c_2(1+k^2)(1+\eta^2)}{c_1}\mathbb{E}\left(\|x_0\|^2\right)\\
&\quad +\frac{2}{c_1}\int_0^{+\infty}e^{\frac{c_3}{c_2}s}\rho(s)ds+2k^2e^{\left(\frac{c_3}{c_2}-\delta\right)\tau}\sup_{t_0-\tau\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{4c_2(1+k^2)(1+\eta^2)}{c_1}\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+\frac{2}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds + 2k^2\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+2k^2\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right) \\
&\leq \left(\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}+2k^2\right)\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+\frac{2}{c_1}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds \\
&\quad+2k^2\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right) \\
&\leq \frac{1}{(1-2k^2)}\left(\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}+2k^2\right)\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+\frac{2}{c_1(1-2k^2)}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds.
\end{aligned}$$

Letting $T\to+\infty$, we obtain

$$\begin{aligned}
&\sup_{t_0\leq t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x_1(t)|^2\right) \\
&\leq \frac{1}{(1-2k^2)}\left(\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}+2k^2\right)\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+\frac{2}{c_1(1-2k^2)}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds.
\end{aligned}$$

Therefore, for all $t\geq t_0$, we have

$$\begin{aligned}
&\mathbb{E}\left(|x_1(t)|^2\right) \\
&\leq \frac{1}{(1-2k^2)}\left(\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}+2k^2\right)e^{-\frac{c_3}{c_2}(t-t_0)}\mathbb{E}\left(\|x_0\|^2\right) \\
&\quad+\frac{2}{c_1(1-2k^2)}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds.
\end{aligned}$$

Setting

$$c=\frac{1}{(1-2k^2)}\left(\frac{4c_2(1+k^2)(1+\eta^2)}{c_1}+2k^2\right),\quad \alpha=\frac{c_3}{c_2},$$

and

$$R=\frac{2}{c_1(1-2k^2)}\int_0^{+\infty} e^{\frac{c_3}{c_2}s}\rho(s)ds,$$

we prove that the solution of system (2.1) is practically exponentially x_1-stable in mean square. □

4. Practical exponential instability in the q-th moment

In this section we will prove a sufficient condition ensuring practical exponential instability of solutions in the q-th moment.

Definition 4.1. The solution $x(t) = (x_1(t), x_2(t))$ of equation (2.1) is said to be practically exponentially unstable in q-th moment ($q > 0$), if there exist positive constants α, c, r such that, for any $x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\mathbb{E}\left(|x|^q\right) \geq c\mathbb{E}\left(||x_0||^q\right)\exp\left(-\alpha(t-t_0)\right) + r, \quad t \geq t_0. \tag{4.1}$$

When $q = 2$, the solution $x(t)$ of stochastic system (2.1) is called practically exponentially unstable in mean square.

Remark 4.2. We can take in (4.1) a continuous nonnegative function $r(t)$ instead of r such that $\lim_{t\to+\infty} r(t) = 0$.

We will consider the new assumption:
$\mathcal{H}_3$: There exists a constant $k \in [0, 1)$ such that

$$|G(\varphi_1) - G(\varphi_2)| \leq k e^{-\frac{\delta}{q}\tau} |\varphi_1(-\tau) - \varphi_2(-\tau)|, \tag{4.2}$$

for all φ_1, $\varphi_2 \in C([-\tau, 0]; \mathbb{R}^n)$, $\delta > 0$ and $q > 0$.

Theorem 4.3. *Let c_1, c_2, c_3 and q be positive constants such that $\frac{c_3}{c_1} \geq \delta$. Assume that there exist $V(t, x, i) \in C^{1,2}\left(]-\tau, +\infty[\times\mathbb{R}^n \times S, \mathbb{R}^+\right)$ and $\gamma \in C(\mathbb{R}; \mathbb{R}_+)$ as well as a nonnegative constant ν, independent of t_0, such that*

- *(i). $c_1|x|^q \leq V(t, x, i) \leq c_2|x|^q$ for $x \in \mathbb{R}^n$.*
- *(ii). $LV(t, \widetilde{\varphi}(0), i) \geq -c_3|\widetilde{\varphi}(0)|^q + \gamma(t)$, $\forall (t, \varphi, i) \in \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times S$ where $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$ and $\widetilde{\varphi}(0) = \varphi(0) - G(\varphi)$.*
- *(iii). $\inf\limits_{t \geq t_0} \displaystyle\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds \geq \nu$.*

Let assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{H}_3$ hold. Assume that the constant k in assumption $\mathcal{H}_3$ verifies $k^q < \beta\left(1 + \beta^{\frac{1}{q-1}}\right)^{1-q}$ and $k^q < \alpha$; for α, $\beta > 0$ and $q > 1$. Then, the solution of equation (2.1) is practically q-th moment exponentially unstable.

Proof. Define the stopping time $\sigma_l = \inf\{t \geq t_0; |x(t)| \geq l\}$. Let $x_t = \varphi$, $x(t) = \varphi(0)$ and $\widetilde{\varphi}(0) = x(t) - G(x_t)$ for $t \geq t_0$. By Itô's formula, we obtain for

$t \geq -\tau$:

$$
\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_1}(t\wedge\sigma_l - t_0)}V(t\wedge\sigma_l, x(t\wedge\sigma_l) - G(x_{t\wedge\tau_K}), r(t\wedge\sigma_l))\right) \\
&= \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) \\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\sigma_l} e^{\frac{c_3}{c_1}s}\left[\frac{c_3}{c_1}V(s, x(s) - G(x_s), r(s)) + LV(s, x(s) - G(x_s), r(s))\right]ds\right) \\
&\geq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) \\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\sigma_l} e^{\frac{c_3}{c_1}s}\left(\frac{c_3}{c_1}c_1|x(s) - G(x_s)|^q - c_3|x(s) - G(x_s)|^q + \gamma(s)\right)ds\right) \\
&\geq \mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\sigma_l} e^{\frac{c_3}{c_1}s}\gamma(s)ds\right).
\end{aligned}
$$

Therefore, by inequality (i),

$$
\begin{aligned}
&\mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\sigma_l} e^{\frac{c_3}{c_1}s}\gamma(s)ds\right) \\
&\quad \leq \mathbb{E}\left(e^{\frac{c_3}{c_1}(t\wedge\sigma_l - t_0)}V(t\wedge\sigma_l, x(t\wedge\sigma_l) - G(x_{t\wedge\sigma_l}), r(t\wedge\sigma_l))\right) \\
&\quad \leq c_2\mathbb{E}\left(e^{\frac{c_3}{c_1}(t\wedge\sigma_l - t_0)}|x(t\wedge\sigma_l) - G(x_{t\wedge\sigma_l})|^q\right).
\end{aligned}
$$

Obviously $\sigma_l \to \infty$ as $l \to \infty$. Then

$$
\mathbb{E}\left(V(t_0, x_0 - G(x_0), r(t_0))\right) + \int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds \leq c_2\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t) - G(x_t)|^q\right). \tag{4.3}
$$

First case: For $0 < q \leq 1$, by Lemma 3.2 and Assumption $\mathcal{A}_3$, we have

$$
\begin{aligned}
\mathbb{E}\left(|x(t) - G(x_t)|^q\right) &\leq \mathbb{E}\left(|x(t)|^q\right) + \mathbb{E}\left(|G(x_t)|^q\right) \\
&\leq \mathbb{E}\left(|x(t)|^q\right) + k^q e^{-\delta\tau}\mathbb{E}\left(|x(t-\tau)|^q\right).
\end{aligned}
$$

On the other hand, by inequality (i), we obtain

$$
\begin{aligned}
\mathbb{E}\left(\|x_0\|^q\right) &\leq \mathbb{E}\left(|x_0 - G(x_0)|^q\right) + \mathbb{E}\left(|G(x_0)|^q\right) \\
&\leq \mathbb{E}\left(|x_0 - G(x_0)|^q\right) + k^q\mathbb{E}\left(\|x_0\|^q\right).
\end{aligned}
$$

Then,

$$
(1 - k^q)\mathbb{E}\left(\|x_0\|^q\right) \leq \mathbb{E}\left(|x_0 - G(x_0)|^q\right). \tag{4.4}
$$

This implies

$$
\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{1}{c_2}\mathbb{E}\left(V(t_0,x_0-G(x_0),r(t_0))\right)+\frac{1}{c_2}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad -k^q e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad -k^q e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right).
\end{aligned}
$$

Thus, for any $T>0$,

$$
\begin{aligned}
&\inf_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad +\inf_{t_0\leq t\leq T}\left(-k^q e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad +k^q e^{-\delta\tau}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad +k^q e^{(\frac{c_3}{c_1}-\delta)\tau}\sup_{t_0-\tau\leq t\leq T-\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad +k^q\sup_{t_0-\tau\leq t\leq T-\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right).
\end{aligned}
$$

Letting $T\to+\infty$, we obtain

$$
\begin{aligned}
&\inf_{t_0\leq t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{1}{c_2}\inf_{t_0\leq t<+\infty}\int_{t_0}^{t}e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad +k^q\sup_{t_0-\tau\leq t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0-G(x_0)|^q\right)+\frac{\nu}{c_2}\\
&\quad +k^q\sup_{t_0-\tau\leq t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)
\end{aligned}
$$

$$\geq \frac{c_1}{c_2}\mathbb{E}\left(|x_0 - G(x_0)|^q\right) + \frac{\nu}{c_2}$$
$$+k^q\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right).$$

Therefore, for all $t \geq t_0$, we have

$$\mathbb{E}\left(|x(t)|^q\right) \quad \geq \quad \frac{c_1}{c_2}\mathbb{E}\left(\|x_0\|^q\right)e^{-\frac{c_3}{c_1}(t-t_0)} + \frac{\nu}{c_2(1-k^q)}e^{-\frac{c_3}{c_1}t}.$$

Setting

$$C = \frac{c_1}{c_2}, \quad \alpha = \frac{c_3}{c_1} \quad \text{and} \quad r_1(t) = \frac{\nu}{c_2(1-k^q)}e^{-\frac{c_3}{c_1}t},$$

the result is proved.

Second case: For $q > 1$, by Lemma 3.2 and Assumption $\mathcal{A}_3$, for $\alpha > 0$ and $\beta > 0$ such that

$$\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1} < 1,$$

and $k^q < \alpha$, we have,

$$\begin{aligned}\mathbb{E}\left(|x(t)-G(x_t)|^q\right) &\leq \left(1+\alpha^{\frac{1}{q-1}}\right)^{q-1}\mathbb{E}\left(|x(t)|^q + \frac{|G(x_t)|^q}{\alpha}\right)\\ &\leq \left(1+\alpha^{\frac{1}{q-1}}\right)^{q-1}\mathbb{E}\left(|x(t)|^q\right)\\ &+ \frac{k^q}{\alpha}\left(1+\alpha^{\frac{1}{q-1}}\right)^{q-1}e^{-\delta\tau}\mathbb{E}\left(|x(t-\tau)|^q\right).\end{aligned}$$

By inequality (i),

$$\begin{aligned}\mathbb{E}\left(\|x_0\|^q\right) &\leq \left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\mathbb{E}\left(|x_0 - G(x_0)|^q + \frac{|G(x_0)|^q}{\beta}\right)\\ &\leq \left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\mathbb{E}\left(|x_0 - G(x_0)|^q + \frac{k^q}{\beta}\|x_0\|^q\right).\end{aligned}$$

Then,

$$\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right) \leq \mathbb{E}\left(|x_0 - G(x_0)|^q\right). \tag{4.5}$$

This implies

$$\begin{aligned}&\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\ &\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\mathbb{E}\left(|x_0 - G(x_0)|^q\right) + \frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds\\ &\quad -\frac{k^q}{\alpha}e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&\quad+\frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad-\frac{k^q}{\alpha}e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right).
\end{aligned}
$$

Hence, for any $T>0$, we have

$$
\begin{aligned}
&\inf_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&\quad+\frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad+\inf_{t_0\leq t\leq T}\left(-\frac{k^q}{\alpha}e^{-\delta\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\right)\\
&\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&\quad+\frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds\\
&\quad+\frac{k^q}{\alpha}e^{-\delta\tau}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t-\tau)|^q\right)\\
&\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&\quad+\frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\inf_{t_0\leq t\leq T}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds+\frac{k^q}{\alpha}\sup_{t_0-\tau\leq t\leq T-\tau}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right).
\end{aligned}
$$

Letting $T\to+\infty$, we obtain

$$
\begin{aligned}
&\inf_{t_0\leq t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\geq \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&\quad+\frac{\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\inf_{t_0\leq t<+\infty}\int_{t_0}^{t} e^{\frac{c_3}{c_1}s}\gamma(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k^q}{\alpha}\sup_{t_0-\tau\le t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\ge \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&+\frac{\nu\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}+\frac{k^q}{\alpha}\sup_{t_0-\tau\le t<+\infty}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right)\\
&\ge \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\mathbb{E}\left(\|x_0\|^q\right)\\
&+\frac{\nu\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2}+\frac{k^q}{\alpha}\mathbb{E}\left(e^{\frac{c_3}{c_1}(t-t_0)}|x(t)|^q\right).
\end{aligned}
$$

Therefore, for all $t\ge t_0$,

$$
\begin{aligned}
&\mathbb{E}\left(|x(t)|^q\right)\\
&\ge \frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2\left(1-\frac{k^q}{\alpha}\right)}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}\times\\
&\quad\times e^{-\frac{c_3}{c_1}(t-t_0)}\mathbb{E}\left(\|x_0\|^q\right)+\frac{\nu\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2\left(1-\frac{k^q}{\alpha}\right)}e^{-\frac{c_3}{c_1}t}.
\end{aligned}
$$

where

$$
C_1(\alpha,\beta,k)=\frac{c_1\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2\left(1-\frac{k^q}{\alpha}\right)}\left(1-\frac{k^q}{\beta}\left(1+\beta^{\frac{1}{q-1}}\right)^{q-1}\right)\left(1+\beta^{\frac{1}{q-1}}\right)^{1-q}
$$

and

$$
r_2(t)=\frac{\nu\left(1+\alpha^{\frac{1}{q-1}}\right)^{1-q}}{c_2\left(1-\frac{k^q}{\alpha}\right)}e^{-\frac{c_3}{c_1}t}.
$$

Finally

$$
\begin{aligned}
\mathbb{E}\left(|x(t)|^q\right) \quad\ge\quad & \min\Big\{C_1(\alpha,\beta,k),C\Big\}\mathbb{E}\left(\|x_0\|^q\right)e^{-\frac{c_3}{c_1}(t-t_0)}\\
&+\min\Big\{r_1(t),r_2(t)\Big\}.
\end{aligned}
$$

Setting $c=\min\Big\{C_1(\alpha,\beta,k),C\Big\}$, $\alpha=\dfrac{c_3}{c_1}$ and $r(t)=\min\Big\{r_1(t),r_2(t)\Big\}$, we conclude that the solution of system (2.1) is practically exponentially unstable in q-th moment ($q>0$).

□

5. Example

We consider the following neutral stochastic functional differential equation with Markovian switching:

$$\begin{cases} d[x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)] = f_1(t, x_t, r(t))dt + g_1(t, x_t, r(t))dW_1(t) \\ d[x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)] = f_2(t, x_t, r(t))dt + g_2(t, x_t, r(t))dW_2(t) \\ d[x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)] = f_3(t, x_t, r(t))dt + g_3(t, x_t, r(t))dW_3(t). \end{cases} \tag{5.1}$$

Let $t_0 = 0$, the initial data $x_0 = x(\lambda) = \xi = (\xi_1, \xi_2, \xi_3) \in C^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R}^3)$ and $\{W_i(t)\}_{i\in\{1,2,3\}}$ are one dimensional Brownian motions. Let

$$\begin{aligned} f_1(t, x_t, 1) &= -4\left(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right), \\ f_1(t, x_t, 2) &= -3\left(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right), \\ f_1(t, x_t, 3) &= -2\left(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right), \\ f_2(t, x_t, 1) &= -3\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right), \\ f_2(t, x_t, 2) &= -4\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right), \\ f_2(t, x_t, 3) &= -2\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right), \\ f_3(t, x_t, 1) &= -4\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right), \\ f_3(t, x_t, 2) &= -3\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right), \\ f_3(t, x_t, 3) &= -2\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right), \end{aligned}$$

$$\begin{aligned} g_1(t, x_t, 1) &= \int_{-1}^{0} \left| x_2(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1) \right| dm(\theta), \\ g_1(t, x_t, 2) &= \frac{1}{2}g_1(t, x_t, 1), \qquad g_1(t, x_t, 3) = \frac{1}{4}g_1(t, x_t, 1), \\ g_2(t, x_t, 1) &= \int_{-1}^{0} \left| x_1(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1) \right| dm(\theta), \\ g_2(t, x_t, 2) &= \frac{1}{2}g_2(t, x_t, 1), \qquad g_2(t, x_t, 3) = \frac{1}{4}g_2(t, x_t, 1), \\ g_3(t, x_t, 1) &= e^{-t}, \quad g_3(t, x_t, 2) = \frac{1}{2}e^{-t}, \quad g_3(t, x_t, 3) = \frac{1}{4}e^{-t}, \end{aligned}$$

where m is a probability measure defined on $[-1, 0]$ satisfying $\int_{-1}^{0} dm(\theta) = 1$. Let $\widetilde{\varphi}(0) = \widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), \widetilde{x}_3(t))$, where $\widetilde{x}_i(t) = x_i(t) - \frac{1}{2}e^{-\frac{1}{2}}x_i(t-1)$

for $i \in \{1,2,3\}$. Let $S = \{1,2,3\}$ and the matrix $\Gamma = (\gamma_{ij})_{1\leq i,j\leq 3}$ defined by

$$\begin{pmatrix} -8 & 4 & 4 \\ \gamma & -2\gamma & \gamma \\ 3 & 3 & -6 \end{pmatrix},$$

where $1 < \gamma < 2$.
We will prove that system (5.1) is practically exponentially unstable in mean square with respect to all variables.
Let $V(t,x,i) = \psi_i(x_1^2 + x_2^2 + x_3^2)$, for $i \in S$, where $\psi_1 = \psi_3 = 1$, $\psi_2 = \frac{1}{2}$. Then

$$\frac{1}{2}|x|^2 \leq V(t,x,i) \leq |x|^2 \tag{5.2}$$

Let $U_1(t,x) = \frac{1}{2}\left|x\right|^2$ and $U_2(t,x) = \left|x\right|^2$. By the definition of LV, we have for $i = 1$

$$\begin{aligned}
&LV(t,\widetilde{\varphi}(0),1)\\
&= -8\left(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2 - 6\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\\
&\quad -8\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2 - 2\left(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2\\
&\quad -2\left(\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2 + \left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right)\\
&\quad +\left(\int_{-1}^{0}\left|x_1(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right| dm(\theta)\right)^2 + e^{-2t}\\
&\quad +\left(\int_{-1}^{0}\left|x_2(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1)\right| dm(\theta)\right)^2\\
&= -10(x_1(t) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t-1))^2 - 8\left(x_2(t) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\\
&\quad -10\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\\
&\quad +\left(\int_{-1}^{0}\left|x_1(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right| dm(\theta)\right)^2 + e^{-2t}\\
&\quad +\left(\int_{-1}^{0}\left|x_2(t+\theta) - \frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1)\right| dm(\theta)\right)^2.
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}
&LV(t,\widetilde{\varphi}(0),1)\\
&\leq 1-8U_2(t,\widetilde{\varphi}(0))+\int_{-1}^{0}\left|x_2(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1)\right|^2 dm(\theta)\\
&\quad+\int_{-1}^{0}\left|x_1(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right|^2 dm(\theta)\\
&\leq 1-8U_2(t,\widetilde{\varphi}(0))+\int_{-1}^{0}U_2(t+\theta,\widetilde{\varphi}(0))dm(\theta).
\end{aligned}$$

For $i=2$, we have

$$\begin{aligned}
&LV(t,\widetilde{\varphi}(0),2)\\
&=-3\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2-4\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\\
&\quad+\frac{1}{2}\left(\int_{-1}^{0}\left|x_2(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1)\right|dm(\theta)\right)^2\\
&\quad+\frac{1}{2}\left(\int_{-1}^{0}\left|x_1(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right|dm(\theta)\right)^2+\frac{1}{8}e^{-2t}\\
&\quad-3\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2+\gamma\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2\\
&\quad+\gamma\left(\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2+\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right)\\
&=(\gamma-3)\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2+(\gamma-4)\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\\
&\quad+(\gamma-3)\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2+\frac{1}{8}e^{-2t}\\
&\quad+\frac{1}{2}\left(\int_{-1}^{0}\left|x_1(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right|dm(\theta)\right)^2.
\end{aligned}$$

By the Hölder inequality,

$$LV(t,\widetilde{\varphi}(0),2)\leq\frac{1}{8}+(\gamma-3)\,U_2(t,\widetilde{\varphi}(0))+\frac{1}{2}\int_{-1}^{0}U_2(t+\theta,\widetilde{\varphi}(0))dm(\theta).$$

For $i=3$, proceeding as $i=1$ and $i=2$, we have

$$
\begin{aligned}
&LV(t,\widetilde{\varphi}(0),3)\\
&= -4\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2-4\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\\
&\quad+\frac{1}{16}\left(\int_{-1}^{0}\left|x_2(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t+\theta-1)\right|dm(\theta)\right)^2\\
&\quad+\frac{1}{16}\left(\int_{-1}^{0}\left|x_1(t+\theta)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t+\theta-1)\right|dm(\theta)\right)^2\\
&\quad-4\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2+\frac{1}{16}e^{-2t}-\frac{3}{2}\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2\\
&\quad-\frac{3}{2}\left(\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2+\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right).
\end{aligned}
$$

By the Hölder inequality, we have

$$LV(t,\widetilde{\varphi}(0),3) \quad\leq\quad \frac{1}{16}-\frac{3}{2}U_2(t,\widetilde{\varphi}(0))+\frac{1}{16}\int_{-1}^{0}U_2(t+\theta,\widetilde{\varphi}(0))dm(\theta).$$

Then, for $i\in S$,

$$LV(t,\widetilde{\varphi}(0),i) \quad\leq\quad 1+(\gamma-3)\,U_2(t,\widetilde{\varphi}(0))+\int_{-1}^{0}U_2(t+\theta,\widetilde{\varphi}(0))dm(\theta).$$

Thus, assumption $\mathcal{A}_2$ is satisfied.
For all $\varphi_1=y_t$, $\varphi_2=z_t\in C([-\tau,0];\mathbb{R}^n)$.

$$|G(y_t)-G(z_t)|\leq\frac{1}{2}|y(t-1)-z(t-1)|, \tag{5.3}$$

then, assumption $\mathcal{A}_3$ is satisfied with $k=\frac{1}{2}$.
It is easy to verify assumptions $\mathcal{A}_1$. Therefore, by Theorem 3.1, system (5.1) has a unique solution $x(t)=(x_1(t),x_2(t),x_3(t))$.

$$
\begin{aligned}
LV(t,\widetilde{\varphi}(0),1) \quad&\geq\quad -10\left(\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2\right.\\
&\qquad\left.+\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2+\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right)\\
&\qquad+e^{-2t}\\
&\geq\quad -10|\widetilde{\varphi}(0)|^2+e^{-2t}.
\end{aligned}
$$

$$\begin{aligned}
&LV(t,\widetilde{\varphi}(0),2)\\
&\geq(\gamma-4)\left(\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2+\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2\right.\\
&\quad\left.+\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right)+\frac{1}{8}e^{-2t}\\
&\geq(\gamma-4)\,|\widetilde{\varphi}(0)|^2+\frac{1}{8}e^{-2t}.
\end{aligned}$$

$$\begin{aligned}
&LV(t,\widetilde{\varphi}(0),3)\\
&\geq-6\left(\left(x_1(t)-\frac{1}{2}e^{-\frac{1}{2}}x_1(t-1)\right)^2\right.\\
&\quad\left.+\left(x_2(t)-\frac{1}{2}e^{-\frac{1}{2}}x_2(t-1)\right)^2+\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2\right)\\
&\quad+\frac{1}{16}e^{-2t}\\
&\geq-6|\widetilde{\varphi}(0)|^2+\frac{1}{16}e^{-2t}.
\end{aligned}$$

Therefore, for $i\in S$

$$LV(t,\widetilde{\varphi}(0),i)\geq-\max\Big(10,4-\gamma,6\Big)|\widetilde{\varphi}(0)|^2+\frac{1}{16}e^{-2t}.$$

Finally,

$$LV(t,\widetilde{\varphi}(0),i)\geq-10|\widetilde{\varphi}(0)|^2+\frac{1}{16}e^{-2t}.$$

Hence, by Theorem 4.3, system (5.1) is practically exponentially unstable in mean square.

Now, we define V_1 for $i\in S$ for $\widetilde{\varphi}_1(0)=\widetilde{x}_3(t)$ by

$$V_1(t,x,i)=\frac{\psi_i}{2}x_3^2\quad\text{where }\psi_1=2\text{ and }\psi_2=\psi_3=1.$$

This implies that

$$\frac{1}{2}x_3^2\leq V_1(t,x,i)\leq x_3^2.$$

Then, for $i=1$,

$$\begin{aligned}
LV_1(t,\widetilde{\varphi}(0),1)\quad=\quad&-8\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2+e^{-2t}\\
&-4\left(x_3(t)-\frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2.
\end{aligned}$$

Finally, we obtain

$$LV_1(t,\widetilde{\varphi}(0),1)\leq-12|\widetilde{\varphi}_1(0)|^2+e^{-2t}.\tag{5.4}$$

In the same way, for $i = 2$,

$$LV_1(t, \widetilde{\varphi}(0), 2) = -3\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2 + \frac{1}{8}e^{-2t} + \frac{\gamma}{2}\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2.$$

Therefore,

$$LV_1(t, \widetilde{\varphi}(0), 2) \leq -\left(3 - \frac{\gamma}{2}\right)|\widetilde{\varphi}_1(0)|^2 + e^{-2t}. \tag{5.5}$$

On the other hand, for $i = 3$,

$$LV_1(t, \widetilde{\varphi}(0), 3) = -2\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2 + \frac{1}{32}e^{-2t} + \frac{3}{2}\left(x_3(t) - \frac{1}{2}e^{-\frac{1}{2}}x_3(t-1)\right)^2.$$

Therefore,

$$LV_1(t, \widetilde{\varphi}(0), 3) \leq -\frac{1}{2}|\widetilde{\varphi}_1(0)|^2 + e^{-2t}. \tag{5.6}$$

Then, from inequalities (5.4), (5.5) and (5.6), for any $i \in S$, we have

$$LV_1(t, \widetilde{\varphi}(0), i) \leq -\min\left(12, \left(3 - \frac{\gamma}{2}\right), \frac{1}{2}\right)|\widetilde{\varphi}_1(0)|^2 + e^{-2t}.$$

Hence,

$$LV_1(t, \widetilde{\varphi}(0), i) \leq -\frac{1}{2}|\widetilde{\varphi}_1(0)|^2 + e^{-2t}.$$

It is easy to verify that V_1 satisfies assumption $\mathcal{A}_2$ with $U_1(t,x) = \frac{1}{2}x_3^2$, $U_2(t,x) = x_3^2$, $c'_1 = 1$, $c'_2 = \frac{1}{2}$ and $c'_3 = 0$.

Thus, assumptions of Theorem 3.3 are satisfied with $c_1 = \frac{1}{2}$, $c_2 = 1$, $c_3 = \frac{1}{2}$, $p = 2$, $\delta = 1$, $\rho(t) = e^{-2t}$, $k \in [0, \frac{\sqrt{2}}{2})$ and $\frac{c_3}{c_2} \leq \delta$. Then system (5.1) is practically exponentially x_3-stable in mean square.

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