Statistical solutions and piecewise Liouville theorem for the impulsive reaction-diffusion equations on infinite lattices *

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Abstract

We first verify the global well-posedness of the impulsive reaction-diffusion equations on infinite lattices. Then we establish that the generated process by the solution operators has a pullback attractor and a family of Borel invariant probability measures. Furthermore, we formulate the definition of statistical solution for the addressed impulsive system and prove the existence. Our results show that the statistical solution of the impulsive system satisfies merely the Liouville type theorem piecewise, and the Liouville type equation for impulsive system will not always hold true on the interval containing any impulsive point.

Keywords: Statistical solution; Impulsive lattice system; Reaction-diffusion equation; Piecewise Liouville theorem; Pullback attractor.

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1 Introduction

Lattice system has the discretization characteristic in spatial or time variables [18]. It has wide applications in the real world (see e.g. [17, 29]). The asymptotic theory of lattice system has been extensively studied (see e.g. [13,25,28,36,47,48] and the references therein). Recently, reference [37,40] investigated the invariant measures for discrete long-wave-short-wave resonance equations and discrete Klein-Gordon-Schrödinger equations.

The statistical solutions and invariant measures are important objectives in the investigation of the turbulence (see e.g. [12,16,22,23,33,35]). We first recall some relevant results about the statistical solutions for evolution equations. Firstly, Foias *et al.* studied systematically the statistical solutions and its properties for the three-dimensional (3D) incompressible Navier-Stokes (NS) equations in [23, 24] (and the references therein). The limiting behavior of statistical solutions for the 3D NS α-model when $\alpha \to 0^+$ was investigated in [11]; Reference [31] constructed the statistical solutions for the 2D NS equations; the abstract framework concerning the theory of statistical solutions for general

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evolution equations was given in [10, 12]; the statistical solutions and invariant measures for the 3D globally modified NS equations was studied in [14, 30, 38, 41]; the sufficient conditions leading to the existence of trajectory statistical solutions for autonomous evolution equations was established in [42], and the results of [42] was applied to some concrete evolution equations in [43–46]. In addition, the invariant measures for lattice system was studied in [37,40,49]. However, to the best of our knowledge, there are no references studying the statistical solutions for impulsive differential equations.

This article studies the statistical solutions for the following impulsive reaction-diffusion equations on infinite lattices

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} + \nu(2u_k - u_{k+1} - u_{k-1}) + \lambda u_k + f(u_k) = g_k(t), \;\; t > s, \, t \neq t_j, \;\; k, j \in \mathbb{Z}, \tag{1.1}$$

$$u_k(t_j^+) - u_k(t_j) = \phi_{kj}(u_k(t_j)), \;\; k, j \in \mathbb{Z}, \;\; t_j \in \mathbb{R}, \tag{1.2}$$

with the initial conditions

$$u_k(s^+) = \lim_{\theta \to s^+} u_k(\theta) = u_{k,s^+}, \;\; s \in \mathbb{R}, \;\; k \in \mathbb{Z}, \tag{1.3}$$

where $u_k(\cdot) \in \mathbb{R}$ is the unknown function, the functions $f(\cdot)$, $g_k(\cdot)$ and $\phi_{kj}(\cdot)$ are given and assumed to satisfy some conditions, $\{t_j\}_{j\in\mathbb{Z}}$ is a given sequence of impulsive points satisfying

$$t_{j+1} - t_j \geqslant \eta, \;\; j \in \mathbb{Z}, \;\; \text{and} \;\; \lim_{j\to+\infty} t_j = +\infty, \;\; \lim_{j\to-\infty} t_j = -\infty, \tag{1.4}$$

for a given positive constant η. In addition, $\nu > 0$, $\lambda > 0$ are constants, and $\mathbb{R}$ and $\mathbb{Z}$ stand for the sets of real and integer numbers, respectively.

The feature of the impulsive equations (1.1)-(1.3) is that its solutions have the first type of discontinuities at the given impulsive points $\{t_j\}_{j\in\mathbb{Z}}$, are left continuous on $\mathbb{R}$ and are continuous on $\{t \in \mathbb{R} : t \neq t_j, j \in \mathbb{Z}\}$. Note that, instead of the initial condition $u_k(t)|_{t=s} = u_{k,s}$, we impose the limiting condition $u_k(s^+) = u_{k,s^+}$ in (1.3), which is natural for equations (1.1)-(1.2) since may be $s = t_j$ for some $j \in \mathbb{Z}$. Equation (1.2) describes the impulsive effect of the system, which leads to the piecewise continuity of the solutions and produces difficult in the investigation of the impulsive differential equations .

Equations (1.1)-(1.2) are discretization with respect to the spatial variable of the following reaction-diffusion equations with fixed impulsive moments on the real line $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \lambda u + f(u) = g(t), \;\; t > s, \;\; t \neq t_j, \, j \in \mathbb{Z} \\ u(t_j^+) - u(t_j) = \phi_j(u(t_j)), \;\; j \in \mathbb{Z}, \;\; t_j \in \mathbb{R}. \end{cases} \tag{1.5}$$

Let us mention some relevant results about the long-term behavior of the impulsive equations. For instance, in [34], Schmalfuss studied the attractor for random dynamical system perturbed by impulses. In [26], Iovane and Kapustyan used the theory of multi-valued process to investigate the impulsive equations, establishing that autonomous equations with values of the impulsive perturbation vanishes (called damped impulsive effects) has the L^2 global attractor, and that each element within the global attractor lies to some trajectory of the non-perturbed evolution equations. In [27], Iovane, Kapustyan and Valero proved that non-autonomous equations without damped impulsive effects possesses the L^2 global attractor. Recently, Yan, Wu and Zhong in [39] studied the impulsive reaction-diffusion equations (1.5) and proved the existence of the uniform attractors in $L^p(\Omega)$, $L^{2p-2}(\Omega)$ and $H_0^1(\Omega)$.

The theory of impulsive differential equations, including the effects of impulses on dynamical systems, has been intensively studied, and one can refer to [1–3, 5–9, 19–21]. For example, Ciesielski studied the isomorphisms, semicontinuity and stability of the impulsive dynamical systems in [19, 20]. Bonotto and his cooperators studied systematically the impulsive semi-dynamical systems and impulsive systems, including the notations of various attractors, their existence and properties such as upper-continuity, lower-continuity and stability, etc. We can refer to the series of articles [1, 5–8] and the references therein. Nevertheless, up to our knowledge, it seems that there is no references investigating the statistical solutions for impulsive differential equations or impulsive lattice system.

The main contribution of this article is the existence and piecewise Liouville type theorem of the statistical solutions for the impulsive reaction-diffusion equations on infinite lattices. We are concerned about the Borel probability measures of its solutions in the phase space. Firstly, we prove that the impulsive equations (1.1)-(1.3) is global well-posed. Then we verify that the process generated by the solutions operator has a pullback attractor and a family of Borel invariant probability measures. Furthermore, we put forward the concept of statistical solution for the addressed impulsive system and prove its existence. It seems that this is the first article investigating the statistical solutions and invariant measures for the impulsive evolution system. Our results reveal that the statistical solution of the impulsive system satisfies merely the Liouville type theorem piecewise, which indicates that the Liouville type equation for impulsive system will not always hold true on an interval containing any impulsive point. There are two main difficulties caused by the impulses in our investigations.

Firstly, the impulses naturally lead to the discontinuity of the solutions. This discontinuity will result in some difficulties when we estimate the solution to obtain its global existence via the extension. Also this discontinuity will cause some difficulties when we prove the pullback bounded absorbing property and the pullback asymptotically nullness of the generated process $\{U(t,s)\}_{t\geqslant s}$ in the phase space ℓ^2. These are because the Gronwall's inequality is no longer valid on the interval containing any impulsive point. To estimate the piecewise continuous solutions, we will extend the impulsive inequality (see [3, Lemma 2.2]) to the function sets $PC^1(\mathbb{R},\mathbb{R})$(see notation in §2).

Secondly, due to the impulsive effect, the evolution equation containing impulsive is essentially non-autonomous despite that the equation is autonomous (that is g is independent of time t). The non-autonomous system has some intrinsic differences with the autonomous one. For example, the non-autonomous system depends on both the initial time and the present time. As for the impulsive problem, the ℓ^2-valued mapping $t \mapsto S(t,s)u$ is left-continuous for $t \in [s,+\infty)$ with each $s \in \mathbb{R}$ and $u \in \ell^2$, but $\|S(t,s)u-u\|$ is still dependent of s. Factually, the solution $S(t,s)u$ depends simultaneously on initial time s when $s \to t^-$. This is due to the non-autonomy caused by the impulses. We will use the structure of the discussed impulsive system to establish the following piecewise continuity of the ℓ^2-valued mapping $s \mapsto S(t,s)u$: it is continuous on $\{s<t : s \neq t_j, j \in \mathbb{Z}\}$, is left-continuous on $(-\infty,t]$ and has the first kind of discontinuities at the impulsive points $\{t_j\}_{j\in\mathbb{Z}}$. This type of continuity of the ℓ^2-valued mapping $s \mapsto S(t,s)u$ plays a key role when we construct the invariant measures $\{m_s\}_{s\in\mathbb{R}}$ for $\{S(t,s)\}_{t\geqslant s}$ on ℓ^2.

It is worth mentioning that the impulses will result in the discontinuity of the invariant measures $\{m_s\}_{s\in\mathbb{R}}$, that is, the function $s \to \int_{\ell^2}\psi(u)\mathrm{d}m_s(u)$ is not necessarily continuous for $\psi \in C(\ell^2)$. We are thus prompted to amend the definition of statistical solution (see [12, Definition 3.2]) such that it is suitable for the impulsive problem discussed here. It seems that the definition of statistical solution formulated by us is also feasible for other impulsive differential equations. In our definition of the statistical solution, the function $s \to \int_{\ell^2}\psi(u)\mathrm{d}m_s(u)$ is piecewise continuous and the statistical solution $\{m_s\}_{s\in\mathbb{R}}$ satisfies merely the Liouville type theorem piecewise, which implies that the Liouville type equation for impulsive systems will not always hold on the interval containing any impulsive point. These phenomena are essentially due to the impulsive effects.

The rest of the article is arranged as follows. In §2, we first verify the global well-posedness of the solutions to problem (1.1)-(1.3). In §3, we first establish that the solutions operator of problem (1.1)-(1.3) forms a process $\{S(t,s)\}_{t\geqslant s}$, which is continuous and has a bounded pullback absorbing set on ℓ^2. Then we prove that $\{S(t,s)\}_{t\geqslant s}$ has pullback asymptotically nullness and a pullback attractor. In §4, we first prove some type of piecewise continuity of $\{S(t,s)\}_{t\geqslant s}$ with respect to s. Then we refine [32, Theorem 3.1] to construct a family of Borel invariant probability measures for the process $\{S(t,s)\}_{t\geqslant s}$ on ℓ^2. Furthermore, we verify that the family of invariant measures meets piecewise the Liouville theorem and is exactly a statistical solution of the impulsive problem (1.1)-(1.3).

2 Estimates and well-posedness of the solutions

This section verifies some estimates and global well-posedness of problem (1.1)-(1.3).

Let's introduce the mathematical settings of our problem. Besides the sets of real numbers $\mathbb{R}$ and integer numbers $\mathbb{Z}$ introduced in previous section, we denote by $\mathbb{Z}_+$ the set of positive integers. We will take

$$\ell^2 = \Big\{ v = (v_k)_{k\in\mathbb{Z}} : v_k \in \mathbb{R},\ \sum_{k\in\mathbb{Z}} v_k^2 < +\infty \Big\}$$

as the phase space in our investigation, and endow ℓ^2 with the inner product and norm as

$$(v,u) = \sum_{k\in\mathbb{Z}} v_k u_k,\ \ \|v\|^2 = \sum_{k\in\mathbb{Z}} v_k^2,\ \ v = (v_k)_{k\in\mathbb{Z}},\ \ u = (u_k)_{k\in\mathbb{Z}} \in \ell^2.$$

Obviously, $(\ell^2, (\cdot,\cdot))$ is a Hilbert space. We introduce three operators A, B and B^* as

$$\begin{cases} (Av)_k = 2v_k - v_{k+1} - v_{k-1},\ \ k\in\mathbb{Z},\ \ v = (v_k)_{k\in\mathbb{Z}}, \\ (Bv)_k = v_{k+1} - v_k,\ \ k\in\mathbb{Z},\ \ v = (v_k)_{k\in\mathbb{Z}}, \\ (B^*v)_k = v_{k-1} - v_k,\ \ k\in\mathbb{Z},\ \ v = (v_k)_{k\in\mathbb{Z}}. \end{cases}$$

These operators have the following properties

$$(B^*Bv, u) = (Bv, Bu) = (Av, u),\ \ \forall\, v, u \in \ell^2, \tag{2.1}$$

and

$$\begin{cases} \|Bv\| = \|B^*v\| \leqslant 2\|v\|,\ \ \forall\, v\in\ell^2, \\ \|Av\| \leqslant 4\|v\|,\ \ \forall\, v\in\ell^2. \end{cases} \tag{2.2}$$

At the same time, to describe the type of continuity of the solutions for the impulsive differential equations, we introduce two sets $PC(I;\mathbb{R})$ and $PC(I;\ell^2)$ of piecewise continuous functions from interval $I\subset\mathbb{R}$ to $\mathbb{R}$ and to ℓ^2 respectively as follows.

$$\begin{aligned} PC(I;\mathbb{R}) =& \{y(\cdot)\in\mathbb{R} : y(\cdot) \text{ is continuous for } t\in I, t\neq t_j, j\in\mathbb{Z}, \text{is left continuous for } t\in I \text{ and} \\ & \text{has the first kind of discontinuities at the impulsive points } t_j\in I, j\in\mathbb{Z}\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} PC(I;\ell^2) =& \{u(\cdot)\in\ell^2 : u(\cdot) \text{ is continuous for } t\in I, \tau\neq t_j, j\in\mathbb{Z}, \text{is left continuous for } t\in I \text{ and} \\ & \text{has the first kind of discontinuities at the impulsive points } t_j\in I, j\in\mathbb{Z}\}. \end{aligned} \tag{2.4}$$

In addition, $PC^1(I;\mathbb{R})$ and $PC^1(I;\ell^2)$ denotes the set of functions whose first derivative belongs to $PC(I;\mathbb{R})$ and $PC(I;\ell^2)$, respectively.

In order to write problem (1.1)-(1.3) in an abstract form, we set

$$\lambda u = (\lambda u_k)_{k\in\mathbb{Z}},\ \ g(t) = (g_k(t))_{k\in\mathbb{Z}},\ \text{ and }\ \phi_j(u(t_j)) = (\phi_{kj}(u_k(t_j)))_{k\in\mathbb{Z}},\ j\in\mathbb{Z}.$$

In addition, we define

$$\tilde{f}(u) = (f(u_k))_{k\in\mathbb{Z}},\ \ \forall\, u = (u_k)_{k\in\mathbb{Z}} \in \ell^2. \tag{2.5}$$

Then $\tilde{f}$ is called the Nemytskii operator associated with f. Using these notations and operators, we write equations (1.1)-(1.3) as

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + \lambda u + \tilde{f}(u) = g(t), \ \ t > s, \, t \neq t_j, \, j \in \mathbb{Z}, \tag{2.6}$$

$$u(t_j^+) = u(t_j) + \phi_j(u(t_j)), \quad j \in \mathbb{Z}, \tag{2.7}$$

$$u(s^+) = u_{s^+}, \ \ s \in \mathbb{R}. \tag{2.8}$$

To ensure the well-posedness of problem (2.6)-(2.8), we assume that the functions f, g and $\phi_j = (\phi_{kj})_{k\in\mathbb{Z}}$ satisfy the following conditions.

(H1) $f(\cdot) \in C^1(\mathbb{R})$, $f(\theta)\theta \geqslant 0$, $f(0) = 0$ and $f'(\theta) \geqslant -\lambda_0 > -\lambda$ for a constant $\lambda_0 > 0$, $\forall\, \theta \in \mathbb{R}$.

(H2) For any $k, j \in \mathbb{Z}$, $\phi_{kj}(0) = 0$, and there is a constant $L > 0$ such that

$$|\phi_{kj}(\theta') - \phi_{kj}(\theta'')| \leqslant L|\theta' - \theta''|, \ \ \forall\, \theta', \theta'' \in \mathbb{R}, \tag{2.9}$$

$$\frac{1}{\eta} \ln(2 + 2L^2) < \lambda, \tag{2.10}$$

where the constant λ is from equation (1.1) and η is from (1.4).

(H3) $g(\cdot) \in C(\ell^2)$ and $e^{\sigma s} \int_{-\infty}^{s} e^{\sigma\theta} \|g(\theta)\|^2 \mathrm{d}\theta < +\infty$ for every $s \in \mathbb{R}$, where

$$\sigma = \lambda - \frac{1}{\eta} \ln(2 + 2L^2) > 0. \tag{2.11}$$

Some examples illustrating the existence of functions f and g satisfying (H1) and (H3), can be found in [4] and [40, Example 3.1], respectively. It is also not difficult to see the existence of the function $\phi_j = (\phi_{kj})_{k\in\mathbb{Z}}$ satisfying (H2).

With assumptions (H1)-(H3), we next verify that problem (2.6)-(2.8) has a unique local solution.

Lemma 2.1. *Suppose that assumptions* (H1)-(H3) *hold true. Then for every given initial time s and initial value $u_{s^+} \in \ell^2$, there uniquely exists a solution to problem* (2.6)-(2.8) *which satisfies*

$$u(\cdot) \in PC([s, T); \ell^2) \cap PC^1((s, T); \ell^2),$$

where $T > s$. Furthermore, $\lim_{\theta \to T^-} \|u(\theta)\| = +\infty$ provided $T < +\infty$.

Proof. Since $u \longmapsto Au$, $u \longmapsto \lambda u$ are bounded linear operators from ℓ^2 to ℓ^2 and $g(\cdot) \in C(\ell^2)$, we just need to show that the function $\tilde{f}(\cdot)$ defined by (2.5) is locally Lipschitz with respect to u. Let $\mathfrak{B} \subset \ell^2$ be a bounded subset, then by using (H1) and the differential mean-value theorem, we have that

$$\|\tilde{f}(u) - \tilde{f}(v)\|^2 = \sum_{k\in\mathbb{Z}} |f'(\theta_k)|^2 |u_k - v_k|^2 \leqslant L_f(\mathfrak{B}) \|u - v\|^2, \ \ \forall\, u, \, v \in \mathfrak{B}, \tag{2.12}$$

where θ_k locates between u_k and v_k, and $L_f(\mathfrak{B}) = \sup_{\theta \in [0, 3\sup_{u\in\mathfrak{B}} \|u\|]} |f'(\theta)|$ is a constant which depends only on f and $\mathfrak{B}$. By the classical theory (see e.g. [3, Theorems 2.3 and 2.6]) of the impulsive differential equations, we get the results of Lemma 2.1. $\square$

To prove that the above local solution exists globally on $[s, +\infty)$, we will use the following impulsive inequality to estimate the solution and the extension theorem (see [3, Theorem 2.6]).

Lemma 2.2. *Let $\xi(\cdot) \in PC^1(\mathbb{R};\mathbb{R})$ satisfy*

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} \leqslant q(t)\xi(t) + p(t), \ \ t \neq t_j, j \in \mathbb{Z}, \\ \xi(t_j^+) \leqslant \kappa_j \xi(t_j), \ \ j \in \mathbb{Z}, \\ \xi(s^+) \leqslant \xi_0, \ \ s \in \mathbb{R}, \end{cases} \tag{2.13}$$

where $q(\cdot)$, $p(\cdot) \in PC(\mathbb{R};\mathbb{R})$, $\kappa_j > 0$ and ξ_0 are constant. Then

$$\xi(t) \leqslant \xi_0 \prod_{s<t_j<t} \kappa_j \exp\big(\int_s^t q(\vartheta)\mathrm{d}\vartheta\big) + \int_s^t \prod_{\vartheta \leqslant t_j < t} \kappa_j \exp\big(\int_\vartheta^t q(\theta)\mathrm{d}\theta\big) p(\vartheta)\mathrm{d}\vartheta, \ \ \forall\, t > s, \tag{2.14}$$

where $\prod_{s\leqslant t_j<t} \kappa_j$ denotes the product of the numbers t_j for the integer j such that $t_j \in [s,t)$, and in the case the number of the members of the sequence $\{t_j\}_{j\in\mathbb{Z}}$ lying in $[s,t)$ is zero, $\prod_{s\leqslant t_j<t} \kappa_j = 1$.

Proof. This lemma is an extension of [3, Lemma 2.2] which considers the case $\xi(\cdot) \in PC^1(\mathbb{R}_+;\mathbb{R})$. Here we consider the function $\xi(\cdot) \in PC^1(\mathbb{R};\mathbb{R})$ since we investigate the pullback asymptotic behavior and consider $s \to -\infty$. We can prove this Lemma via an induction argument. In fact, for a given $s \in \mathbb{R}$, there exists some $j_0 \in \mathbb{Z}$ such that $s \in (t_{j_0}, t_{j_0+1}]$, then by assumption $\xi(\cdot)$ satisfies the differential inequality in (2.13) on (s, t_{j_0+1}). Applying the Gronwall inequality on (s, t_{j_0+1}) and using the left-continuity of $\xi(t)$, we see that $\xi(\cdot)$ satisfies (2.14) on $(s, t_{j_0+1}]$. Then we consider (2.13) on the interval $(t_{j_0+1}, t_{j_0+2}]$, with the initial value $\xi(t_{j_0+1}^+) \leqslant \kappa_{t_{j_0+1}} \xi(t_{j_0+1})$. Also applying the Gronwall inequality on (t_{j_0+1}, t_{j_0+2}) and using the left-continuity of $\xi(\cdot)$, we see that $\xi(\cdot)$ also satisfies (2.14) on $(t_{j_0+1}, t_{j_0+2}]$. Analogously, we can prove that $\xi(t)$ satisfies (2.14) on $(t_{j_0+k}, t_{j_0+k+1}]$ for any $k \in \mathbb{Z}_+$. The detailed proof is omitted here. □

Directly from Lemma 2.2, we get

Lemma 2.3. *Let $\xi(\cdot) \in PC^1(\mathbb{R};\mathbb{R})$ satisfy*

$$\begin{cases} \dfrac{\mathrm{d}\xi(t)}{\mathrm{d}t} + \alpha\xi(t) \leqslant q(t), \ \ t \neq t_j, j \in \mathbb{Z}, \\ \xi(t_j^+) - \xi(t_j) \leqslant \beta \xi(t_j), \ \ j \in \mathbb{Z}, \\ \xi(s^+) \leqslant \xi_0, \ \ s \in \mathbb{R}, \end{cases} \tag{2.15}$$

where $q(\cdot) \in PC(\mathbb{R};\mathbb{R})$, $\alpha > 0$, $\beta > 0$ and y_0 are constants. Then

$$\xi(t) \leqslant \xi_0 (1+\beta)^{i<s,t>} e^{-\alpha(t-s)} + \int_s^t (1+\beta)^{i\langle\vartheta,t)} e^{-\alpha(t-\vartheta)} q(\vartheta)\mathrm{d}\vartheta, \ \ \forall\, t > s, \tag{2.16}$$

hereinafter $i < s,t >$ and $i\langle\vartheta,t)$ denote the number of members of the impulsive points $\{t_j\}_{j\in\mathbb{Z}}$ lying in the intervals (s,t) and $[\vartheta,t)$, respectively.

Lemma 2.4. *Suppose that assumptions* (H1)-(H3) *hold true. Then for each given $s \in \mathbb{R}$ and $u_{s^+} \in \ell^2$ the corresponding solution (guaranteed by Lemma 2.1) of problem* (2.6)-(2.8) *meets*

$$\|u(t)\|^2 \leqslant \|u_{s^+}\|^2 e^{-\sigma(t-s)} + \frac{1}{\lambda}\int_s^t e^{-\sigma(t-\theta)} \|g(\theta)\|^2 \mathrm{d}\theta, \ \ s < t \leqslant T, \tag{2.17}$$

hereafter σ is the constant given by (2.11).

Proof. We denote by $u(\cdot) = u(\cdot; s, u_{s^+})$ the solution of problem (2.6)-(2.8) with the initial value u_{s^+} at initial time s. Using $u(\cdot)$ to take inner product of (2.6) with in ℓ^2 gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Bu\|^2 + \lambda\|u\|^2 + (\tilde{f}(u), u) \leqslant \frac{\lambda\|u\|^2}{2} + \frac{\|g(t)\|^2}{2\lambda}, \ \ t \neq t_j, j \in \mathbb{Z}. \tag{2.18}$$

By (H1), $(\tilde{f}(u), u) = \sum_{k\in\mathbb{Z}} f(u_k)u_k \geqslant 0$. Thus, (2.18) implies that

$$\frac{\mathrm{d}\|u(t)\|^2}{\mathrm{d}t} + \lambda\|u(t)\|^2 \leqslant \frac{1}{\lambda}\|g(t)\|^2, \;\; t \neq t_j, j \in \mathbb{Z}. \tag{2.19}$$

Now, for the impulsive condition, we have by (H2) that

$$\begin{aligned} \|u(t_j^+)\|^2 &= \sum_{k\in\mathbb{Z}} |u_k(t_j^+)|^2 = \sum_{k\in\mathbb{Z}} \big(u_k(t_j) + \phi_{kj}(u_k(t_j))\big)^2 \\ &\leqslant 2\sum_{k\in\mathbb{Z}} |u_k(t_j)|^2 + 2\sum_{k\in\mathbb{Z}} |\phi_{kj}(u_k(t_j))|^2 \\ &\leqslant (2+2L^2)\sum_{k\in\mathbb{Z}} |u_k(t_j)|^2 = (2+2L^2)\|u(t_j)\|^2. \end{aligned} \tag{2.20}$$

Applying Lemma 2.3 to (2.19)-(2.20) for $\xi(t) = \|u(t)\|^2$, we obtain

$$\|u(t)\|^2 \leqslant \|u_{s^+}\|^2(2+2L^2)^{i<s,t>}e^{-\lambda(t-s)} + \frac{1}{\lambda}\int_s^t (2+2L^2)^{i\langle\theta,t)}e^{-\lambda(t-\theta)}\|g(\theta)\|^2\mathrm{d}\theta, \;\; \forall\, t > s. \tag{2.21}$$

Now (1.4) implies that

$$i < s,t > \leqslant \frac{t-s}{\eta} \;\text{ and }\; i\langle\theta,t) \leqslant \frac{t-\theta}{\eta}.$$

Thus, we have by (2.10)-(2.11) that

$$(2+2L^2)^{i<s,t>}e^{-\lambda(t-s)} \leqslant e^{-\sigma(t-s)} \;\text{ and }\; (2+2L^2)^{i\langle\theta,t)}e^{-\lambda(t-\theta)} \leqslant e^{-\sigma(t-\theta)}. \tag{2.22}$$

Inserting (2.22) into (2.21) gives (2.17). This ends the proof. □

Now, combining Lemmas 2.1 and 2.4, we obtain the global existence and uniqueness result of the solutions to problem (2.6)-(2.8).

Theorem 2.1. *Suppose that assumptions* (H1)-(H3) *hold true. Then, for each given $s \in \mathbb{R}$ and $u_{s^+} \in \ell^2$, there uniquely exists a solution $u \in PC([s,+\infty);\ell^2) \cap PC^1((s,+\infty);\ell^2)$ to problem* (2.6)-(2.8) *satisfying*

$$\|u(t)\|^2 \leqslant \|u_{s^+}\|^2 e^{-\sigma(t-s)} + \frac{1}{\lambda}\int_s^t e^{-\sigma(t-\theta)}\|g(\theta)\|^2\mathrm{d}\theta, \;\; \forall\, t > s. \tag{2.23}$$

We next establish that the solution of problem (2.6)-(2.8) depends continuously on its initial value.

Theorem 2.2. *Suppose that assumptions* (H1)-(H3) *hold true. Denote by $u^{(j)}(\cdot) = u^{(j)}(\cdot; s, u_{s^+}^{(j)})$, $j = 1,2$, the solutions of problem* (2.6)-(2.8) *corresponding to the initial values $u_{s^+}^{(j)}$ $(j = 1,2)$. Then*

$$\|u^{(1)}(t) - u^{(2)}(t)\|^2 \leqslant \|u_{s^+}^{(1)} - u_{s^+}^{(2)}\|^2 e^{-(\sigma+\lambda-\lambda_0)(t-s)}, \;\; \forall\, t > s, \tag{2.24}$$

hereinafter the constant λ_0 is from (H1).

Proof. Let $u^{(j)}(\cdot) = u^{(j)}(\cdot; s, u_{s^+}^{(j)})$ $(j = 1,2)$ be the solutions of problem (2.6)-(2.8) with the initial values $u_{s^+}^{(j)}$, $j = 1,2$, and set $v(\cdot) = u^{(1)}(\cdot) - u^{(2)}(\cdot)$. Then $v(\cdot)$ satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu Av + \lambda v + \tilde{f}(u^{(1)}) - \tilde{f}(u^{(2)}) = 0, \;\; t \neq t_j,\, j \in \mathbb{Z}, \;\; t > s, \tag{2.25}$$

$$v(t_j^+) - v(t_j) = \phi_j(u^{(1)}(t_j)) - \phi_j(u^{(2)}(t_j)), \;\; j \in \mathbb{Z}, \tag{2.26}$$

$$v(s^+) = u_{s^+}^{(1)} - u_{s^+}^{(2)}, \;\; s \in \mathbb{R}. \tag{2.27}$$

Using v to take inner product with (2.25) in ℓ^2 yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\nu\|Bv\|^2+\lambda\|v\|^2+(\tilde{f}(u^{(1)})-\tilde{f}(u^{(2)}),v)=0,\ \ t\neq t_j, j\in\mathbb{Z}. \tag{2.28}$$

By (H1) we have

$$(\tilde{f}(u^{(1)})-\tilde{f}(u^{(2)}),v)=\sum_{k\in\mathbb{Z}}f'(\theta_k)(u_k^{(1)}-u_k^{(2)})v_k\geqslant-\sum_{k\in\mathbb{Z}}\lambda_0v_k^2=-\lambda_0\|v\|^2, \tag{2.29}$$

where θ_k is located between $u_k^{(1)}$ and $u_k^{(2)}$. Inserting (2.29) into (2.28) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+2(\lambda-\lambda_0)\|v(t)\|^2\leqslant 0,\ \ t\neq t_j, j\in\mathbb{Z}. \tag{2.30}$$

For the impulsive condition of $\|v(\cdot)\|^2$, we have by using (H2) that

$$\begin{aligned}\|v(t_j^+)\|^2=&\sum_{k\in\mathbb{Z}}v_k^2(t_j^+)=\sum_{k\in\mathbb{Z}}\big(v_k(t_j)+\phi_{kj}(u_k^{(1)}(t_j))-\phi_{kj}(u_k^{(2)}(t_j))\big)^2\\ \leqslant&2\sum_{k\in\mathbb{Z}}v_k^2(t_j)+2\sum_{k\in\mathbb{Z}}\big(\phi_{kj}(u_k^{(1)}(t_j))-\phi_{kj}(u_k^{(2)}(t_j))\big)^2\\ =&2\sum_{k\in\mathbb{Z}}v_k^2(t_j)+2\sum_{k\in\mathbb{Z}}L^2\big(u_k^{(1)}(t_j)-u_k^{(2)}(t_j)\big)^2\\ \leqslant&(2+2L^2)\|v(t_j)\|^2.\end{aligned} \tag{2.31}$$

Applying Lemma 2.3 to (2.30)-(2.31) for $\xi(t)=\|v(t)\|^2$ implies

$$\|v(t)\|^2\leqslant\|v_{s^+}\|^2(2+2L^2)^{i<s,t>}e^{-2(\lambda-\lambda_0)(t-s)},\ \ \forall\, t>s. \tag{2.32}$$

Now, from (2.22) we see that

$$(2+2L^2)^{i<s,t>}e^{-2(\lambda-\lambda_0)(t-s)}\leqslant e^{-(\sigma+\lambda-\lambda_0)(t-s)}. \tag{2.33}$$

Inserting (2.33) into (2.32) implies (2.24). The proof of Theorem 2.2 is complete. $\square$

3 Existence of the pullback attractor

In this section, we first show that the solution operator of (2.6)-(2.8) forms a process $\{S(t,s)\}_{t\geqslant s}$ which is continuous and possesses a bounded pullback absorbing set on ℓ^2. Then we verify that $\{S(t,s)\}_{t\geqslant s}$ has pullback asymptotically nullness and a pullback attractor.

From Theorem 2.1 we find that the solutions operator

$$S(t,s):u_{s^+}\in\ell^2\longmapsto S(t,s)u_{s^+}=u(t;s,u_{s^+}) \tag{3.1}$$

of problem (2.6)-(2.8) forms a process on ℓ^2, hereafter $u(\cdot;s,u_{s^+})$ stands for the solution of problem (2.6)-(2.8) with the initial condition u_{s^+} at initial time s. Theorem 2.2 tells us that the process $\{S(t,s)\}_{t\geqslant s}$ is continuous on ℓ^2, that is, the map $S(t,s):\ell^2\longmapsto\ell^2$ is continuous for every given t and s with $s\leqslant t$.

In this article, $Z(\ell^2)$ denotes the family which contains all subsets of ℓ^2. We will use $\mathcal{D}_\sigma$ to denote the class of families $\widehat{D}=\{D(\theta):\theta\in\mathbb{R}\}\subseteq Z(\ell^2)$ which satisfies

$$\lim_{\theta\to-\infty}(e^{\sigma\theta}\sup_{u\in D(\theta)}\|u\|^2)=0. \tag{3.2}$$

For the basic theory of process and pullback attractors, one can refer to [15]. Since the definitions of pullback $\mathcal{D}_\sigma$-absorbing set, pullback $\mathcal{D}_\sigma$-asymptotically nullness and pullback $\mathcal{D}_\sigma$-attractor for a continuous process are well known, we omit them here.

Lemma 3.1. *Suppose that assumptions* (H1)-(H3) *hold true. Then the process* $\{S(t,s)\}_{t\geqslant s}$ *defined by* (3.1) *possesses in* ℓ^2 *a bounded pullback* $\mathcal{D}_\sigma$*-absorbing set.*

Proof. Set

$$R_\sigma(s) := \big(1+\frac{1}{\lambda}\int_{-\infty}^{s} e^{-\sigma(s-\theta)}\|g(\theta)\|^2\mathrm{d}\theta\big)^{1/2},\ \ s\in\mathbb{R}. \tag{3.3}$$

Then the family of time-dependent closed balls $\hat{\mathcal{B}}(s)=\{\mathcal{B}(s)=\mathcal{B}(0,R_\sigma(s)):s\in\mathbb{R}\}$ is the bounded pullback $\mathcal{D}_\sigma$-absorbing set, where $\mathcal{B}(0,R_\sigma(s))$ is the closed ball in ℓ^2, with center zero and radius $R_\sigma(s)$. Factually, for any $\widehat{D}=\{D(\theta):\theta\in\mathbb{R}\}\in\mathcal{D}_\sigma$ and any $u_{s^+}\in D(s)$ for $s\in\mathbb{R}$, we find from (2.23) and (3.2) that there is a time $s_1=s_1(t,\widehat{D})<t$ yields

$$\begin{aligned}\|U(t,s)u_{s^+}\|^2\leqslant&\|u_{s^+}\|^2e^{-\sigma(t-s)}+\frac{1}{\lambda}\int_{s}^{t}e^{-\sigma(t-\theta)}\|g(\theta)\|^2\mathrm{d}\theta\\ \leqslant&1+\frac{1}{\lambda}\int_{-\infty}^{t}e^{-\sigma(t-\theta)}\|g(\theta)\|^2\mathrm{d}\theta,\ \ \forall\, s<s_1.\end{aligned} \tag{3.4}$$

This ends the proof. □

Lemma 3.2. *Assume that* (H1)-(H3) *hold true. Then the process* $\{S(t,s)\}_{t\geqslant s}$ *defined by* (3.1) *has pullback* $\mathcal{D}_\sigma$*-asymptotically nullness in* ℓ^2.

Proof. By the Urylohn lemma (see e.g. [23, Theorem A.6]), there exists a smooth function $\chi(\cdot)\in C^1(\mathbb{R}_+;\mathbb{R}_+)$ satisfying

$$\begin{cases}\chi(\theta)=0, & 0\leqslant\theta\leqslant1,\\ 0\leqslant\chi(\theta)\leqslant1, & 1\leqslant\theta\leqslant2,\\ \chi(\theta)=1, & \theta\geqslant2,\\ |\chi'(\theta)|\leqslant\chi_0(\text{ positive constant}), & \theta\geqslant0.\end{cases} \tag{3.5}$$

Consider any given $\widehat{D}=\{D(\theta):\theta\in\mathbb{R}\}\in\mathcal{D}_\sigma$, $s\in\mathbb{R}$ and $u_{s^+}\in\ell^2$. Remember that $u(\cdot)=u(\cdot;s,u_{s^+})=S(t,s)u_{s^+}$ is the solution of problem (2.6)-(2.8) with the initial value u_{s^+} at initial time $s\in\mathbb{R}$. Let $M\in\mathbb{Z}_+$ and $v_k(\cdot)=\chi(\frac{|k|}{M})u_k(\cdot)$, $k\in\mathbb{Z}$. Using $v=(v_k)_{k\in\mathbb{Z}}$ to take the inner product with (2.6) gives

$$(\frac{\mathrm{d}u}{\mathrm{d}t},v)+\nu(Au,v)+\lambda(u,v)+(\tilde{f}(u),v)=(g(t),v),\ \ t>\tau,t\neq t_j,j\in\mathbb{Z}. \tag{3.6}$$

By some simple computations, we have

$$\begin{cases}(\dfrac{\mathrm{d}u}{\mathrm{d}t},v)=\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2,\\ (Au,v)=(Bu,Bv)=\displaystyle\sum_{k\in\mathbb{Z}}\chi(\frac{|k+1|}{M})(u_{k+1}-u_k)^2+\sum_{k\in\mathbb{Z}}\chi'(\frac{|\tilde{k}|}{M})\frac{u_k(u_{k+1}-u_k)}{M},\\ (\tilde{f}(u),v)=\sum_{k\in\mathbb{Z}}\chi(\dfrac{|k|}{M})f(u_k)u_k\geqslant0,\\ (g(t),v)=\displaystyle\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k(t)u_k\leqslant\frac{\lambda}{2}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2+\frac{1}{2\lambda}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(t),\end{cases} \tag{3.7}$$

where the number $|\tilde{k}|$ is located between $|k|$ and $|k+1|$. Inserting (3.7) into (3.6) and then using (3.4)-(3.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_m^2(t)+\lambda\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t)\leqslant\frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(t)+\frac{4\chi_0}{M}R_\sigma^2(t),\ \ t\neq t_j,j\in\mathbb{Z}, \tag{3.8}$$

where $t>s_1>s$ and s_1 is the pullback absorbing time as in Lemma 3.1. Now, for any $\epsilon>0$ we see from (3.4) and (H3) that there exists an $M_1=M_1(t,\epsilon)\in\mathbb{Z}_+$ such that

$$\frac{4\chi_0}{M}R_\sigma^2(t)\leqslant\epsilon^2/3,\ \ \forall\, M\geqslant M_1. \tag{3.9}$$

Set $\xi(t)=\sum\limits_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t)$ with $M\geqslant M_1$. Then for the impulsive condition of this $\xi(t)$, we have

$$\begin{aligned}\xi(t_j^+)=&\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t_j^+)=\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})\big(u_k(t_j)+\phi_{kj}(u_k(t_j))\big)^2\\ \leqslant&2\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t_j)+\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})\phi_{kj}^2(u_k(t_j))\\ \leqslant&(2+2L^2)\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t_j)=(2+2L^2)\xi(t_j),\ \ j\in\mathbb{Z}.\end{aligned} \tag{3.10}$$

$$\xi(s^+)=\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(s^+)\leqslant\|u(s^+)\|^2. \tag{3.11}$$

Applying Lemma 2.3 to (3.8)-(3.11) for $\xi(t)=\sum\limits_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t)$ with $M\geqslant M_1$ and then using (2.22), we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})u_k^2(t)\\ \leqslant&\|u_{s^+}\|^2(2+2L^2)^{i<s,t>}e^{-\lambda(t-s)}+\int_s^t(2+2L^2)^{i\langle\theta,t\rangle}e^{-\lambda(t-\theta)}(\frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(\theta)+\frac{\epsilon^2}{3})\mathrm{d}\theta\\ \leqslant&\|u_{s^+}\|^2e^{-\sigma(t-s)}+\frac{e^{-\sigma t}}{\lambda}\int_s^te^{\sigma\theta}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(\theta)\mathrm{d}\theta+\frac{\epsilon^2}{3},\ \ \forall\, t>s_1>s,\ \ M\geqslant M_1.\end{aligned} \tag{3.12}$$

From (H3) we see that for above $\epsilon>0$ there is an $M_2=M_2(\epsilon,t)\in\mathbb{Z}_+$ yields

$$\frac{e^{-\sigma t}}{\lambda}\int_s^te^{\sigma\theta}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(\theta)\mathrm{d}\theta\leqslant\frac{e^{-\sigma t}}{\lambda}\int_{-\infty}^te^{\sigma\theta}\sum_{k\in\mathbb{Z}}\chi(\frac{|k|}{M})g_k^2(\theta)\mathrm{d}\theta\leqslant\frac{\epsilon^2}{3},\ \ \forall\, M\geqslant M_2. \tag{3.13}$$

At the same time, since $u_{s^+}\in D(s)$ and $\hat D=\{D(\theta):\theta\in\mathbb{R}\}\in\mathcal{D}_\sigma$, it then follows from (3.2) that for above $\epsilon>0$ there is a $s_2=s_2(t,\epsilon,\hat D)<t$ yields

$$\|u_{s^+}\|^2e^{-\sigma(t-s)}\leqslant\epsilon^2/3,\ \ \forall\, s\leqslant s_2. \tag{3.14}$$

Picking

$$M_0=\max\{M_1,M_2\}\ \text{ and }\ \tau_0=\min\{s_1,s_2\},$$

we obtain from (3.12)-(3.14) that

$$\sup_{u_{s^+}\in D(s)}\sum_{|k|\geqslant M_0}|(S(t,s)u_{s^+})_k|^2=\sup_{u_{s^+}\in D(s)}\sum_{|k|\geqslant M_0}|u_k(t;s,u_{k,s^+})|^2\leqslant\epsilon^2,\ \ \forall\, s\leqslant s_0.$$

This concludes the proof of Lemma 3.2. □

We now have

Theorem 3.1. *Let assumptions* (H1)-(H3) *hold true. Then the process* $\{S(t,s)\}_{t\geqslant s}$ *defined by* (3.1) *possesses a pullback* $\mathcal{D}_\sigma$*-attractor (denoted by)* $\hat{\mathcal{A}}(\theta)=\{\mathcal{A}(\theta):\theta\in\mathbb{R}\}$.

Proof. Since the process $\{S(t,s)\}_{t\geqslant s}$ is continuous on ℓ^2, the result of Theorem 3.1 is obtained directly by using Lemmas 3.1, 3.2 and [40, Theorem 2.1]. □

4 Existence of the statistical solutions

In this section we first verify that for each fixed $t \in \mathbb{R}$ and $\tilde{u} \in \ell^2$ the ℓ^2-valued mapping $s \mapsto S(t,s)\tilde{u}$ belongs to $PC((-\infty,t];\ell^2)$ and is bounded on $(-\infty,t]$. Then we refine [32, Theorem 3.1] to construct a family of Borel invariant probability measures $\{m_\theta\}_{\theta\in\mathbb{R}}$ for the process $\{S(t,s)\}_{t\geqslant s}$ on ℓ^2, via the generalized Banach limit and the pullback attractor $\hat{\mathcal{A}}(\theta)$ obtained in Theorem 3.1. Further, we establish that the family of Borel invariant probability measures $\{m_\theta\}_{\theta\in\mathbb{R}}$ meets piecewise the Liouville theorem and is indeed a statistical solution of the impulsive equation (2.6).

We next will employ the notation $c_1 \lesssim c_2$ to stand for $c_1 \leqslant cc_2$ for a general constant $c > 0$ that just depends on the parameters from our problem and will not produce confusion. In addition, for a given Borel probability measure ρ_θ on ℓ^2 and a function $\psi \in C(\ell^2)$, $\int_{\ell^2} \psi(u)\mathrm{d}\rho_\theta(u)$ denotes the Bochner integral.

Lemma 4.1. *Suppose that assumptions* (H1)-(H3) *hold true. Then for each given $t \in \mathbb{R}$ and $\tilde{u} \in \ell^2$ the ℓ^2-valued mapping $s \mapsto S(t,s)\tilde{u}$ is bounded on $(-\infty,t]$.*

Proof. It is a direct consequence of Theorem 2.1. Actually, for each given $t \in \mathbb{R}$ and $\tilde{u} \in \ell^2$, we have

$$\|u(t;s,\tilde{u})\|^2 \leqslant \|\tilde{u}\|^2 + \frac{1}{\lambda}\int_{-\infty}^{t} e^{-\sigma(t-\theta)}\|g(\theta)\|^2\mathrm{d}\theta, \ \ \forall\, t > s. \tag{4.1}$$

Obviously, inequality (4.1) shows that $\|u(t;s,\tilde{u})\|^2$ is less than a constant independent of s. □

Next we establish three auxiliary lemmas concerning some type continuity of $S(t,s)\tilde{u}$ with respect to the parameters s and t.

Lemma 4.2. *Suppose that assumptions* (H1)-(H3) *hold true, and let $\tilde{u} \in \ell^2$ and $s_* \in \mathbb{R}$ be fixed. Then $\forall\, \epsilon > 0$, $\exists\, \delta = \delta(\epsilon, s_*, \tilde{u}) > 0$, small enough, yields*

$$\|S(t,s)\tilde{u} - \tilde{u}\|^2 < \epsilon, \ \ \forall\, s \in (s_*, s_* + \delta), \ \ \forall\, t \in (s, s_* + \delta). \tag{4.2}$$

Proof. Consider any given $\tilde{u} \in \mathbb{R}$ and $s_* \in \ell^2$. We assume, without loss of generality, $s_* \in (t_{j_0}, t_{j_0+1}]$ for some $j_0 \in \mathbb{Z}$. Because we will integrate (2.19) on the interval that does not contain any impulsive points from $\{t_j\}_{j\in\mathbb{Z}}$, we next split the proofs into two cases.

Case 1: $s_* \in (t_{j_0}, t_{j_0+1})$. In this case, we denote $0 < 2l = \min\{s_* - t_{j_0}, t_{j_0+1} - s_*\}$ and consider $s_* + l \geqslant t \geqslant s \geqslant s_*$.

Firstly, we prove that

$$\int_s^t \|\frac{\mathrm{d}S(\theta,\tau)\tilde{u}}{\mathrm{d}\theta}\|^2\mathrm{d}\theta \lesssim \tilde{c} = \|\tilde{u}\|^2 + \int_{s_*-l}^{s_*+l} \|g(\theta)\|^2\mathrm{d}\theta. \tag{4.3}$$

Indeed, from (2.2) and (2.6) we see that for $s \leqslant \theta \leqslant t$,

$$\|\frac{\mathrm{d}S(\theta,s)\tilde{u}}{\mathrm{d}\theta}\|^2 \lesssim \|AS(\theta,s)\tilde{u}\|^2 + \|S(\theta,s)\tilde{u}\|^2 + \|\tilde{f}(S(\theta,s)u)\|^2 + \|g(\theta)\|^2. \tag{4.4}$$

By (2.12) and (H1)

$$\|\tilde{f}(S(\theta,s)\tilde{u})\|^2 \leqslant L_f\|S(\theta,s)\tilde{u}\|^2, \tag{4.5}$$

where $L_f = \max_{\theta\in[0,L]} |f'(\theta)|$ and $L = \max_{\theta\in[s_*-l,s_*+l]} \|S(\theta,s)\tilde{u}\|^2$ are constants independent of s and t since $S(\theta,s)\tilde{u}$ is continuous for $\theta \in [s_* - l, s_* + l] \subset (t_{j_0}, t_{j_0+1})$. Inserting (4.5) into (4.4) first and then integrating the resulting inequality with respect to θ over $[s,t]$, we have

$$\int_s^t \|\frac{\mathrm{d}S(\theta,s)\tilde{u}}{\mathrm{d}\theta}\|^2\mathrm{d}\theta \lesssim (1+L_f)\int_s^t \|S(\theta,s)\tilde{u}\|^2\mathrm{d}\theta + \int_s^t \|g(\theta)\|^2\mathrm{d}\theta$$
$$\lesssim \|\tilde{u}\|^2 + \int_s^t \|g(\theta)\|^2\mathrm{d}\theta \lesssim \|\tilde{u}\|^2 + \int_{s_*-l}^{s_*+l} \|g(\theta)\|^2\mathrm{d}\theta, \tag{4.6}$$

where we have also used (2.19).

Secondly, we observe that

$$\|S(t,s)\tilde{u}-\tilde{u}\|^2 = \int_s^t \frac{\mathrm{d}\|S(\theta,s)\tilde{u}\|^2}{\mathrm{d}\theta}\mathrm{d}\theta - 2\big(S(t,s)\tilde{u}-\tilde{u},\tilde{u}\big). \tag{4.7}$$

On the one hand, again by (2.19) we have

$$\int_s^t \frac{\mathrm{d}\|S(\theta,s)\tilde{u}\|^2}{\mathrm{d}\theta}\mathrm{d}\theta \lesssim \int_s^t \|g(\theta)\|^2\mathrm{d}\theta. \tag{4.8}$$

By (H3), $g(\cdot)\in L^2_{\mathrm{loc}}(\mathbb{R};\ell^2)$, and thus there exists some $\delta' = \delta'(\epsilon, s_*, g)\in(0,l)$ yields

$$\int_s^t \frac{\mathrm{d}\|S(\theta,s)\tilde{u}\|^2}{\mathrm{d}\theta}\mathrm{d}\theta \lesssim \int_s^t \|g(\theta)\|^2\mathrm{d}\theta < \frac{\epsilon^2}{2},\ \ s_* < s \leqslant t \leqslant s_* + \delta'. \tag{4.9}$$

On the other hand, from (4.3) we conclude that the constant $\tilde{c}$ is independent of s and t. Using Cauchy's inequality and (4.3) gives

$$\Big|\big(S(\theta,s)\tilde{u}-\tilde{u},\tilde{u}\big)\Big| = \Big|\big(\int_s^t \frac{\mathrm{d}S(\theta,s)\tilde{u}}{\mathrm{d}\theta}\mathrm{d}\theta,\tilde{u}\big)\Big| \leqslant \|\tilde{u}\|\int_s^t \|\frac{\mathrm{d}S(\theta,s)\tilde{u}}{\mathrm{d}\theta}\|\mathrm{d}\theta$$
$$\leqslant \|\tilde{u}\|\Big(\int_s^t \|\frac{\mathrm{d}S(\theta,s)\tilde{u}}{\mathrm{d}\theta}\|^2\mathrm{d}\theta\Big)^{1/2}(t-s)^{1/2} \leqslant \tilde{c}^{1/2}\|\tilde{u}\|(t-s)^{1/2}, \tag{4.10}$$

which implies that there is some $\delta'' = \delta''(\epsilon, s_*, \tilde{u})\in(0,l)$ yields

$$\Big|\big(S(\theta,s)\tilde{u}-\tilde{u},\tilde{u}\big)\Big| < \frac{\epsilon^2}{2},\ \ s_* < s \leqslant t \leqslant s_* + \delta''. \tag{4.11}$$

Picking $\delta = \min\{\delta', \delta''\}$, we get (4.2) from (4.7), (4.9) and (4.11). The case $s_*\in(t_{j_0}, t_{j_0+1})$ is proved.

Case 2: $s_* = t_{j_0+1}$. In this case we denote $2l = t_{j_0+2} - t_{j_0+1}$ and consider $s_* < s \leqslant t \leqslant s_* + l$. The main difference in the proof to *case 1* is that the constants $\tilde{c}$ and L are replaced with

$$\tilde{c}' = \|\tilde{u}\|^2 + \int_{s_*}^{s_*+l}\|g(\theta)\|^2\mathrm{d}\theta \ \text{ and } \ L' = \sup_{\theta\in[s_*,s_*+l]}\|S(\theta,s)\tilde{u}\|^2,$$

respectively. Here L' is also a constant independent of s and t, since $S(\theta,s)\tilde{u}$ is continuous for $\theta\in(s_*, s_*+l]\subset(t_{j_0+1}, t_{j_0+2})$ and has right-hand limit at $\theta = s_* = t_{j_0+1}$. The remaining part of the proof is analogous to that of *case 1* and the details is omitted here. □

Similarly to Lemma 4.2, we have

Lemma 4.3. *Suppose that assumptions* (H1)-(H3) *hold true, and let* $\tilde{u}\in\ell^2$ *and* $s_*\in\mathbb{R}$ *be fixed. Then* $\forall\epsilon>0$, $\exists\delta=\delta(\epsilon,s_*,\tilde{u})>0$, *small enough, yields*

$$\|S(t,s)\tilde{u}-\tilde{u}\|^2 < \epsilon,\ \ \forall\, s\in(s_*-\delta, s_*],\ \ \forall\, t\in[s,s_*]. \tag{4.12}$$

Now, we begin to investigate some kind of continuous property of the ℓ^2-valued mapping $s\longmapsto S(t,s)\tilde{u}$ for every given $t\in\mathbb{R}$ and $\tilde{u}\in\ell^2$.

Lemma 4.4. *Suppose that assumptions* (H1)-(H3) *hold true. Then for each given $t \in \mathbb{R}$ and $\tilde{u} \in \ell^2$ the ℓ^2-valued mapping $s \mapsto S(t,s)\tilde{u}$ belongs to $PC((-\infty,t];\ell^2)$, that is*

1. *$S(t,\cdot)\tilde{u}$ is left-continuous on $(-\infty,t]$;*
2. *$S(t,\cdot)\tilde{u}$ is continuous on $(-\infty,t] \setminus \{t_j : t_j \in (-\infty,t], j \in \mathbb{Z}\}$;*
3. *$S(t,\cdot)\tilde{u}$ has right-hand limit at the impulsive points $\tau_j \in (-\infty,t]$, $j \in \mathbb{Z}$.*

Proof. Firstly, we prove item (1). Consider any given $s_* \in (-\infty,t]$. We shall establish the left-continuity of $S(t,\cdot)\tilde{u}$ at $s = s_*$. Indeed, we assume that, without loss of generality, $s_* \in (t_{j_0}, t_{j_0+1}]$ for some $j_0 \in \mathbb{Z}$. Then for any $s \in (t_{j_0}, s_*]$ we have

$$\|S(t,s)\tilde{u} - S(t,s_*)\tilde{u}\| = \|S(t,s_*)S(s_*,s)u_* - S(t,s_*)\tilde{u}\|. \tag{4.13}$$

Since, t and s_* are fixed, $S(t,s_*) : \ell^2 \mapsto \ell^2$ is continuous. The left-continuity of $S(t,\cdot)\tilde{u}$ at $s = s_*$ follows then from (4.2) and (4.13).

Secondly, we prove item (2). Without loss of generality, we just prove, in view of the result of item (1), that $S(t,\cdot)\tilde{u}$ is right-continuous on $(t_{j_0}, t_{j_0+1}) \cap (-\infty,t]$ for some $j_0 \in \mathbb{Z}$. Let $s_* \in (t_{j_0}, t_{j_0+1}) \cap (-\infty,t]$ be given and $s_* < s < t_{j_0+1} \leqslant t$. Using (2.24), we have

$$\begin{aligned}\|S(t,s_*)\tilde{u} - S(t,\tau)\tilde{u}\| =& \|S(t,s)S(s,s_*)\tilde{u} - S(t,s)\tilde{u}\| \\ \leqslant& \|S(s,s_*)\tilde{u} - \tilde{u}\| e^{-(\sigma+\lambda-\lambda_0)(t-s)}.\end{aligned} \tag{4.14}$$

The right-continuity of $S(t,\cdot)\tilde{u}$ at s_* follows from (4.14) and the fact that $S(\cdot,s_*)\tilde{u} \in C((t_{j_0}, t_{j_0+1});\ell^2)$.

Thirdly, the fact that $S(t,s)\tilde{u}$ has right-hand limit at the impulsive points $t_j \in (-\infty,t]$ is a direct outcome of Lemma 4.2 and the invariance property of the process, by using Cauchy's criterion for convergence. The proof of Lemma 4.4 is therefore complete. □

Combining Lemma 4.1 and Lemma 4.4, we declare that the process $\{S(t,s)\}_{t\geqslant s}$ possesses the so-called **PC-τ-continuity** in the following sense: for each given $t \in \mathbb{R}$ and $\tilde{u} \in \ell^2$ the ℓ^2-valued mapping $s \mapsto S(t,s)\tilde{u} \in PC((-\infty,t];\ell^2)$ and is bounded on $(-\infty,t]$.

We next will refine [32, Theorem 3.1] to construct a family of Borel invariant probability measures for the process $\{S(t,s)\}_{t\geqslant s}$ on ℓ^2, via the generalised Banach limit and the pullback attractor $\hat{\mathcal{A}}(\theta)$ assured in Theorem 3.1. The definition of generalized Banach limit can be found in [23, 32]. For any generalized Banach limit (denote by) $\mathrm{LIM}_{\theta\to+\infty}$, we have the following property (see e.g. [23, (1.38)] or [16, (2.3)])

$$|\mathrm{LIM}_{\theta\to+\infty}\varrho(\theta)| \leqslant \limsup_{\theta\to+\infty} |\varrho(\theta)|, \ \ \forall\, \varrho(\cdot) \in \mathcal{B}_+, \tag{4.15}$$

where $\mathcal{B}_+$ is the set consisting of all real-valued bounded functions on $[0,+\infty)$. Note that in our situation, we shall investigate the generalized Banach limits for $s \to -\infty$. Thus, for a fixed real function ϱ defined on the interval $(-\infty,0]$ and a fixed generalized Banach limit $\mathrm{LIM}_{\theta\to+\infty}$, we set

$$\mathrm{LIM}_{\theta\to-\infty}\varrho(\theta) = \mathrm{LIM}_{\theta\to+\infty}\varrho(-\theta). \tag{4.16}$$

From now on, we denote by $C(\ell^2)$ the set of all real-valued continuous functionals on ℓ^2.

Theorem 4.1. *Suppose that assumptions* (H1)-(H3) *hold true and $v(\cdot) : \mathbb{R} \mapsto \ell^2$ be a continuous map satisfying $v(\cdot) \in \mathcal{D}_\sigma$. Then for each generalised Banach limit $\mathrm{LIM}_{\theta\to+\infty}$ and for each $\psi \in C(\ell^2)$,* *there corresponds uniquely a family of Borel probability measures* $\{\mathsf{m}_\theta\}_{\theta\in\mathbb{R}}$ *on* ℓ^2 *and the measure* m_θ *is carried by* $\mathcal{A}(\theta)$ *and*

$$\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t \psi\big(S(t,\theta)v(\theta)\big)\mathrm{d}\theta = \int_{\mathcal{A}(t)} \psi(u)\mathrm{d}\mathsf{m}_t(u) = \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_t(u) \tag{4.17}$$

$$= \mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\int_{\ell^2} \psi\big(S(t,\theta)u\big)\mathrm{d}\mathsf{m}_\theta(\theta)\mathrm{d}\theta. \tag{4.18}$$

Furthermore, m_θ *has the following invariant property*

$$\int_{\mathcal{A}(t)} \psi(u)\mathrm{d}\mathsf{m}_t(u) = \int_{\mathcal{A}(s)} \psi\big(S(t,s)u\big)\mathrm{d}\mathsf{m}_s(u),\ \ t\geqslant s. \tag{4.19}$$

Proof. The idea of the proof is analogous to that of [32, Theorem 3.1]. The main difference is that we replace the s-continuity of $\{S(t,s)\}_{t\geqslant s}$ in [32, Theorem 3.1] with the PC-s-continuity here. Obviously, the s-continuity implies the PC-τ-continuity of the process.

Fix $\psi(\cdot)\in C(\ell^2)$ and a continuous map $v(\cdot):\mathbb{R}\mapsto\ell^2$ such that $v(\cdot)\in\mathcal{D}_\sigma$. For given $t\in\mathbb{R}$, we claim that for every compact $[t_0,t]$ the function $s\mapsto\psi(S(t,s)v(s))$ is bounded on $[t_0,t]$ with each $t_0<t$. In fact, on the one hand, from (1.4) we see that the impulsive points $\{t_j\}_{j\in\mathbb{Z}}$ belonging to the interval $[t_0,t]$ are only a finite number. We denote these impulsive points by t_{j_0+1}, t_{j_0+2}, $\cdots$, t_{j_0+N} for some $N\in\mathbb{Z}_+$. Then by Lemma 4.4, the function $s\mapsto\psi(S(t,s)v(s))$ is continuous on $[t_0,t]\backslash\{t_{j_0+1},t_{j_0+1},\cdots,t_{j_0+N}\}$, is left-continuous on $(t_0,t]$ and has right-hand limit at the points t_0, t_{j_0+1}, $\cdots$, t_{j_0+N}. Therefore, $\psi(S(t,s)v(s))$ is bounded on the compact interval $[t_0,t]$. On the other hand, from Lemma 3.1 we find that $\psi(S(t,s)v(s))$ is also bounded on the interval $(-\infty,t_0+1]$ for t_0 sufficiently large and negative, since $v(\cdot)\in\mathcal{D}_\sigma$ and $\{S(t,s)\}_{t\geqslant s}$ has pullback $\mathcal{D}_\sigma$-attracting property. Hence, we have proved that the function $\psi(S(t,s)v(s))$ is bounded on $(-\infty,t]$ and the function

$$s\longmapsto\frac{1}{t-s}\int_s^t\psi(S(t,\theta)v(\theta))\mathrm{d}\theta$$

is bounded on $(-\infty,t]$. In terms of this fact, we define

$$L(\psi)=\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t\psi(S(t,\theta)v(\theta))\mathrm{d}\theta.$$

The remainder of the proof closely follows the one of [32, Theorem 3.1] and we omit the details here. $\square$

Now, we are going to investigate the statistical solutions for equation (2.6). We first mention the class $\mathcal{T}$ consisting of test functions related to the statistical solutions. Write equation (2.6) as

$$\frac{\mathrm{d}u}{\mathrm{d}s}=\mathcal{G}(u,s):=g(s)-\nu Au-\lambda u-\tilde{f}(u),\quad s\neq t_j,\ j\in\mathbb{Z}. \tag{4.20}$$

Then $\mathcal{G}(u,s):\ell^2\times\mathbb{R}\longmapsto\ell^2$. We require that the test function $\Phi\in\mathcal{T}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(u(s))=(\Phi'(u),\mathcal{G}(u,s)),\quad s\neq t_j,\ j\in\mathbb{Z}, \tag{4.21}$$

for every solution $u(\cdot)$ of equations (2.6)-(2.8). The definition of class $\mathcal{T}$ consisting of test functions is similar with that of [43, Definition 4.2] and we omit it here.

We now specify the definition of statistical solutions for equation (4.20) and prove the existence.

Definition 4.1. *A family* $\{\rho_s\}_{s\in\mathbb{R}}$ *of Borel probability measures on* ℓ^2 *is called a statistical solution of equation* (4.20) *if* $\{\rho_s\}_{s\in\mathbb{R}}$ *satisfies:*

(a) *for every* $\psi \in C(\ell^2)$ *the mapping* $s \mapsto \int_{\ell^2} \psi(u)\mathrm{d}\rho_s(u) \in PC(\mathbb{R};\mathbb{R})$*;*

(b) *for a. e.* $s \in \mathbb{R}$*, the support of* ρ_s *is contained in* ℓ^2 *and the mapping* $u \mapsto \big(w, \mathcal{G}(u,s)\big)$ *is* ρ_s*-integrable for each* $w \in \ell^2$*. Moreover, the mapping* $s \mapsto \int_{\ell^2} \big(w, \mathcal{G}(u,s)\big)\mathrm{d}\rho_s(u)$ *belongs to* $L^1_{\mathrm{loc}}(\mathbb{R})$ *for every* $w \in \ell^2$*;*

(c) *for each* $\Phi \in \mathcal{T}$*,*

$$\int_{\ell^2} \Phi(u)\mathrm{d}\rho_s(u) - \int_{\ell^2} \Phi(u)\mathrm{d}\rho_\tau(u) = \int_\tau^s \int_{\ell^2} \big(\Phi'(u), \mathcal{G}(u,\theta)\big)\mathrm{d}\rho_\theta(u)\mathrm{d}\theta,$$

for $s, \tau \in (t_j, t_{j+1})$ *with* $s \geqslant \tau$ *and* $j \in \mathbb{Z}$*.*

Theorem 4.2. *Suppose that assumptions* (H1)-(H3) *hold true. Then* $\{\mathsf{m}_s\}_{s\in\mathbb{R}}$ *guaranteed by Theorem 4.1 is a statistical solution of equation* (4.20)*.*

Proof. We just verify that $\{\mathsf{m}_s\}_{s\in\mathbb{R}}$ guaranteed by Theorem 4.1 satisfies the conditions (a)-(c) of Definition 4.1.

Firstly, we establish item (a). Consider any $j \in \mathbb{Z}$ and any given $s_* \in (t_j, t_{j+1}]$.

In case $s_* \in (t_j, t_{j+1})$, we prove that for each $\psi \in C(\ell^2)$,

$$\lim_{s\to s^*} \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u) = \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s_*}(u). \tag{4.22}$$

In fact, from (4.17) and (4.19) we infer that

$$\int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u) - \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s_*}(u) = \int_{\mathcal{A}(s_*)} \big(\psi(S(s,s_*)u) - \psi(u)\big)\mathrm{d}\mathsf{m}_{s_*}(u) \ \text{ for } \ s > s_*. \tag{4.23}$$

Since $\|S(s,s_*)u - u\| \to 0$ as $s \to s_*^+$, $\psi \in C(\ell^2)$ and $\mathcal{A}(s_*)$ is compact in ℓ^2, equality (4.23) gives

$$\lim_{s\to s_*^+} \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u) = \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s_*}(u).$$

Similarly, we have

$$\int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s_*}(u) - \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u) = \int_{\mathcal{A}(s)} \big(\psi(S(s_*,s)u) - \psi(u)\big)\mathrm{d}\mathsf{m}_s(u) \ \text{ for } \ s < s_*. \tag{4.24}$$

Because $\|S(s_*,s)u - u\| \to 0$ as $s \to s_*^-$, $\psi \in C(\ell^2)$ and $\mathsf{m}_s(\mathcal{A}(s)) \leqslant 1$ for each $s \in \mathbb{R}$, equality (4.24) gives

$$\lim_{s\to s_*^-} \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u) = \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s_*}(u). \tag{4.25}$$

In case $s_* = t_{j+1}$, we use the same proof as (4.25) to obtain the left-continuity of $\int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u)$ at s_*. To establish the existence of $\lim_{s\to s_*^+} \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u)$, we consider $s_* < s' \leqslant s'' < t_{j+2}$ and have

$$\int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s''}(u) - \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_{s'}(u) = \int_{\mathcal{A}(s')} \big(\psi(S(s'',s')u) - \psi(u)\big)\mathrm{d}\mathsf{m}_{s'}(u). \tag{4.26}$$

Since $\psi \in C(\ell^2)$, $\mathsf{m}_{s'}(\mathcal{A}(s')) \leqslant 1$, (4.26) and Lemma 4.2 imply the existence of $\lim_{s\to s_*^+} \int_{\ell^2} \psi(u)\mathrm{d}\mathsf{m}_s(u)$. Thus item (a) is proved.

Secondly, we establish item (b). For each $s \in \mathbb{R}$ we have established that the support of m_s is contained in $\mathcal{A}(s) \subset \ell^2$. For every $u, w \in \ell^2$, we define

$$\Psi(u) = \big(w, \mathcal{G}(u, \cdot)\big). \tag{4.27}$$

Then $\Psi(\cdot) : \ell^2 \longmapsto \mathbb{R}$. Next we verify $\Psi(\cdot) \in C(\ell^2)$. Let $\tilde{u} \in \ell^2$ be given and consider $u \in \ell^2$ satisfying $\|\tilde{u} - u\| \leqslant 1$. Then from (2.2), (2.12) and (4.20) we obtain

$$\begin{aligned} |\Psi(\tilde{u}) - \Psi(u)| &= |\big(w, \mathcal{G}(\tilde{u}, \cdot) - \mathcal{G}(u, \cdot)\big)| \\ &\leqslant \nu|\big(w, A(\tilde{u} - u)\big)| + \lambda|(w, \tilde{u} - u)| + |\big(w, \tilde{f}(\tilde{u}) - \tilde{f}(u)\big)| \\ &\leqslant (4\nu + \lambda + L_f)\|w\|\|\tilde{u} - u\|, \end{aligned} \tag{4.28}$$

where L_f, as in (4.5), is a constant depending on f and $\|\tilde{u}\|$. (4.28) indicates that $\Phi(\cdot)$ given by (4.27) is continuous on ℓ^2. From (4.17) and (4.27) we see that the mapping $u \mapsto \big(w, \mathcal{G}(u, t)\big) = \Psi(u)$ is m_t-integrable for each $w \in \ell^2$. Note that we have established in item (a) that the function

$$s \mapsto \int_{\ell^2} \big(w, \mathcal{G}(u, s)\big) \mathrm{d}m_s(u) = \int_{\ell^2} \Psi(u) \mathrm{d}m_s(u)$$

is piecewise continuous on $\mathbb{R}$ and its discontinuities $\{t_j\}_{j \in \mathbb{Z}}$ are of the first-kind. Thus it lies in $L^1_{\mathrm{loc}}(\mathbb{R})$ for each $w \in \ell^2$.

Thirdly, we prove item (c). For any $\Phi \in \mathcal{T}$, we find from (4.21) that

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi(u(s)) = (\Phi'(u), G(u, s)), \quad s \neq t_j,\ j \in \mathbb{Z}.$$

Therefore, we have that

$$\Phi(u(s)) - \Phi(u(\tau)) = \int_\tau^s \big(\Phi'(u(\vartheta)), G(u(\vartheta), \vartheta)\big) \mathrm{d}\vartheta, \ \ \forall\, s, \tau \in (t_j, t_{j+1}), s \geqslant \tau, j \in \mathbb{Z}. \tag{4.29}$$

Now, for $\zeta \leqslant \tau$, we denote $u(\vartheta) = U(\vartheta, \zeta)\tilde{u}$ for $\vartheta \in [\tau, s]$ and $\tilde{u} \in \ell^2$. We use (4.29) to derive

$$\Phi(S(s, \zeta)\tilde{u}) - \Phi(S(\tau, \zeta)\tilde{u}) = \int_\tau^s \big(\Phi'(S(\vartheta, \zeta)\tilde{u}), \mathcal{G}(U(\vartheta, \zeta)\tilde{u}, \vartheta)\big) \mathrm{d}\vartheta. \tag{4.30}$$

By (4.17)-(4.18) and (4.30), we obtain by using Fubini's Theorem and some calculations that

$$\begin{aligned} &\int_{\ell^2} \Phi(u) \mathrm{d}m_s(u) - \int_{\ell^2} \Phi(u) \mathrm{d}m_\tau(u) = \int_{\mathcal{A}(s)} \Phi(u) \mathrm{d}m_s(u) - \int_{\mathcal{A}(\tau)} \Phi(u) \mathrm{d}m_\tau(u) \\ =& \mathrm{LIM}_{K \to -\infty} \frac{1}{\tau - K} \int_K^\tau \int_\tau^s \int_{\ell^2} \big(\Phi'(S(\vartheta, \zeta)\tilde{u}), \mathcal{G}(S(\vartheta, \zeta)\tilde{u}, \vartheta)\big) \mathrm{d}m_\zeta(\tilde{u}) \mathrm{d}\vartheta \mathrm{d}\zeta, \ \ t_j < \tau \leqslant s < t_{j+1}. \end{aligned}$$

Now employing (4.19), we obtain

$$\int_{\ell^2} \big(\Phi'(S(\vartheta, \zeta)\tilde{u}), \mathcal{G}(S(\vartheta, \zeta)\tilde{u}, \vartheta)\big) \mathrm{d}m_\zeta(\tilde{u}) = \int_{\ell^2} \big(\Phi'(S(\vartheta, \tau)\tilde{u}), \mathcal{G}(S(\vartheta, \tau)\tilde{u}, \vartheta)\big) \mathrm{d}m_\tau(\tilde{u}), \tag{4.31}$$

where the right-hand side of (4.31) does not depend on ζ. Hence,

$$\begin{aligned} &\int_{\mathcal{A}(s)} \Phi(u) \mathrm{d}m_s(u) - \int_{\mathcal{A}(\tau)} \Phi(u) \mathrm{d}m_\tau(u) = \int_\tau^s \int_{\ell^2} \big(\Phi'(S(\vartheta, \tau)\tilde{u}), \mathcal{G}(S(\vartheta, \tau)\tilde{u}, \vartheta)\big) \mathrm{d}m_\tau(\tilde{u}) \mathrm{d}\theta \\ =& \int_\tau^s \int_{\ell^2} \big(\Phi'(u), \mathcal{G}(u(\zeta), \zeta)\big) \mathrm{d}m_\zeta(u) \mathrm{d}\zeta, \ \ t_j < \tau \leqslant s < t_{j+1}. \end{aligned} \tag{4.32}$$

The proof of Theorem 4.2 is complete. □

Remark 4.1. *In the end of the article, we point out that, the statistical information of the discussed impulsive system does not alter with time (i.e.* $\Phi'(u(t)) = 0$*), provided that its statistical equilibrium has been reached. In this situation,* (4.32) *implies*

$$\int_{\mathcal{A}(t)} \Phi(u)\mathrm{d}m_t(u) = \int_{\mathcal{A}(s)} \Phi(u)\mathrm{d}m_s(u), \ \ t_j < s \leqslant t < t_{j+1}, \forall\, j \in \mathbb{Z}. \tag{4.33}$$

Equality (4.33) *reveals that the shape of the pullback attractor* $\mathcal{A}(\cdot)$ *could change along with the evolution of time from* s *to* t*, but the measures of* $\mathcal{A}(s)$ *and* $\mathcal{A}(t)$ *coincide with each other, that is, on each interval* (t_j, t_{j+1})*, the total measure of the attractor* $\mathcal{A}(\cdot)$ *is conservative along with the evolution of time provided that the impulsive system has attained its statistical equilibrium. This is exact the theory of Liouville Theorem in Statistical Mechanics. However, equation* (4.32) *will not always hold true on the interval containing any impulsive point, which indicates that impulses will interrupt some type of conservation of the evolutionary system. This phenomenon is in accord with our intuition to the impulsive system.*

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