EXISTENCE AND REGULARITY RESULTS FOR TERMINAL VALUE PROBLEM FOR NONLINEAR FRACTIONAL WAVE EQUATIONS

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Abstract. We consider the terminal value problem (or called final value problem, initial inverse problem, backward in time problem) of determining the initial value, in a general class of time-fractional wave equations with Caputo derivative, from a given final value. We are concerned with the existence, regularity of solutions upon the terminal value. Under several assumptions on the nonlinearity, we address and show the well-posedness (namely, the existence, uniqueness, and continuous dependence) for the terminal value problem. Some regularity results for the mild solution and its derivatives of first and fractional orders are also derived. The effectiveness of our methods are shown by applying the results to two interesting models: time fractional Ginzburg-Landau equation, and time fractional Burgers equation, where time and spatial regularity estimates are obtained.

Keywords: Fractional derivatives and integrals, Caputo fractional derivative, terminal value problem, time fractional wave equation, wellposedness, regularity estimates.

2010 MSC: 26A33, 35B65, 35R11

1. Introduction

1.1. **Statement of the problem.** As we know, the derivatives of positive integer orders of a differentiable function are determined by its properties only in an infinitesimal neighborhood of the considered point. Therefore, partial differential equations with integer-order derivatives cannot describe processes with memory. The fact that fractional calculus is a powerful tool for describing the effects of power-law memory [2]. If an integer-order derivative is replaced by a fractional one, typically Caputo, or Riemann-Liouville, Grunwald, Letnikov, Weyl derivatives, then we have time-fractional PDEs. Historically, the Riemann-Liouville and Caputo fractional derivatives are the most important ones. Time-fractional differential equations have recently become a topic of active research [3], and they also have many applications for modeling physical situations or describing a wide class of processes with memory [4], such as transport theory [5], viscoelasticity [6], rheology [7], non-Markovian stochastic processes [8], etc. In particular, the fractional diffusion equations (with respect to the time derivative of fractional order $0 < \alpha < 1$) are known to be models for sub-diffusive processes [9], and the fractional diffusion-wave/wave equations (respectively $1 < \alpha < 2$) were used for super-diffusive models of anomalous diffusion, e.g., diffusion in heterogeneous media [22].

In this paper, we consider the following fractional wave equation

$$\begin{cases} \partial_t^\alpha u(x,t) = -\mathcal{A}u(x,t) + G(t,u(x,t)), & x\in\Omega,\ 0<t\le T, \\ u(x,t) = 0, & x\in\partial\Omega, 0<t<T, \\ u_t(x,0) = 0, & x\in\Omega,\ 0<t<T, \end{cases} \tag{1}$$

where G is called a source function which will be defined later. The time fractional derivative ∂_t^α, $1 < \alpha < 2$, is understood as *the leftsided Caputo fractional derivative of order* α with respect to t, which is defined by

$$\partial_t^\alpha v(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2}{\partial s^2} v(s)(t-s)^{1-\alpha} ds,$$

whereupon Γ is the Gamma function. For $\alpha = 2$, we recover the usual time derivative of second order $\partial^2/\partial t^2$. Let us assume that Ω is a nonempty open set and possesses a Lipschitz continuous boundary in $\mathbb{R}^N$, $N \ge 1$, $T > 0$, and let $\mathcal{A}$ be a symmetric and uniformly elliptic operator on Ω defined by

$$\mathcal{A}v(x) = -\sum_{m=1}^N \frac{\partial}{\partial x_m}\left(\sum_{n=1}^N a_{mn}(x)\frac{\partial}{\partial x_n} v(x)\right) + q(x)v(x), \quad x\in\overline{\Omega},$$

where $a_{ij} \in C^1\left(\overline{\Omega}\right)$, $q \in C\left(\overline{\Omega};[0,+\infty)\right)$, and $a_{mn} = a_{nm}, 1 \le m,n \le N$. We assume also that there exists a constant $b_0 > 0$ such that, for $x \in \overline{\Omega}$, $y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$,

$$\sum_{1 \le m,n \le N} a_{mn}(x) y_m y_n \ge b_0 |y|^2.$$

This paper considers the initial inverse problem of determining the initial value $u(x,0) = u_0(x)$ from its final value $u(x,T)$. We focus to study existence, uniqueness and regularity of mild solutions of Problem (1) associated with the final value condition

$$u(x,T) = f(x), \quad x \in \Omega, \tag{2}$$

where f belongs to an appropriate space.

The study of (1) is mainly motivated by problems arising in anomalous diffusion phenomena. Anomalous diffusion and wave equations are of great interest in physics. They are frequently used for the super-diffusive models of anomalous diffusion such as diffusion in heterogeneous media. These fractional differential equations have another important issue in the probability theory related to non-Markovian diffusion processes with memory. Fractional wave equations also describe evolution processes intermediate between diffusion and wave propagation [37, 10, 6]. In [6], it has been shown that the fractional wave equation governs the propagation of mechanical diffusion-waves in viscoelastic media. Such waves are relevant in acoustics, seismology, medical imaging, etc. The physical background for a time-space fractional diffusion-wave equation can be seen in [30].

1.2. **Motivations.** If condition (2) is replaced by

$$u(x,0) = \overline{f}(x), \quad x \in \Omega, \tag{3}$$

then we have the *direct problem* or *initial value problem (IVP)* of (1). Some quasi-linear equations of the form (1) and (3) with standard time derivative ($\alpha = 2$) have been extensively studied in the published literature. The global well-posedness has been proved both in the subcritical case by Ginibre and Velo [57], and in the critical case, by Grillarkis [58], Shatah and Struwe [59, 60], and references therein.

In fractional derivative cases, such as Caputo or Riemann–Liouville derivatives, Problem (1) and (3) has been considered with $G = 0$ or $G = G(x,t)$ by some authors see, e.g. [21, 26, 27, 37, 10] and also [14, 15, 16].

Concerning Problem (1) and (3) with derivative ∂_t^α for $0 < \alpha < 1$, some authors developed and obtained interesting results. A. Carvalho et al. [45] established a local theory of mild solutions where $\mathcal{A}$ is a sectorial (nonpositive) operator. B.H. Guswanto [46] studied the existence and uniqueness of a local mild solution to a class of initial value problems for nonlinear fractional evolution equations. The existence, uniqueness, regularity, and Carleman estimates of solutions are established in some previous works (see, for instance, [20, 21, 22, 31, 33, 34]). Such research is rapidly developing, and here we do not intend to give a comprehensive list of references.

To our knowledge, the study of the initial value problem for the fractional wave equations in the nonlinear case is still limited. Recently, M. Yamamoto et. al. [22] studied Problem (1) and (3) with a linear inhomogeneous source, i.e. $G = G(x,t)$, and then further investigated local solutions with a nonlinear source. M. Warma et. al. [36] considered the existence and regularity of local and global weak solutions with a suitable growth assumption on the nonlinearity G. Very recently, the authors have studied the uniqueness of inverse problems for a fractional equation with a single measurement in [32]. Except for the works [22, 35, 36], there are very few results on Problem (1)-(2), as far as we know.

In practice, there are some physical models which are not subjected to initial value problems. Some phenomena cannot be observed at the time $t = 0$, and only can be measured at a terminal time $t = T > 0$. Hence, a final value condition appears instead of the respectively initial value one. It has great importance in engineering areas and aimed at detecting the previous state of a physical field from its present information. In a few sentences, we explain the presence of the equation $u_t(x,0) = 0$. By [44], the system (1)-(2) in the 2-dimensional case can be considered as a description for an imaging process, namely, to recover an exact picture from its blurry form. The condition $u_t(x,0) = 0$ means that the distribution does not change on the interval $(0,t_0)$ when t_0 is near zero. Therefore, the necessity of studying *terminal value problems* or *final value problems (FVPs)* or *backward problems* is out of any doubt.

The final value problem (1)-(2) with *derivatives of integer-orders* has been treated for a long time, e.g., see [17, 18, 19]. In [18], A. Carasso et. al. considered the following final value problem for the traditional wave equation (i.e. $\alpha = 2$)

$$\begin{cases} u_{tt} &= (\Delta + k)^2 u, & x \in \Omega, \quad 0 < t < T, \\ u = \Delta u &= 0, & x \in \partial\Omega, \ 0 < t < T, \\ u_t(x,0) &= g(x), & x \in \Omega, \\ u(x,T) &= f(x), & x \in \Omega, \end{cases}$$

where k is a given positive number which may equal several eigenvalues of $-\Delta$, and f, g are given functions. Up to date, little research has been done on the inverse problems of time-space fractional diffusion equations. FVPs for fractional PDEs can be roughly divided into two topics. The first one contains problems related to the ill-posedness and propose some regularization methods for approximating a sought solution. We can list some well-known results, for example, J. Jia et. al. [41], J. Liu et. al. [42], some papers of M. Yamamoto and his group see [49, 50, 51, 52, 53], B. Kaltenbacher et. al. [39, 40], W. Rundell et. al. [54, 55], J. Janno see [47, 48], etc. The second topic contains problems concerning the existence and regularity of solutions such as [26, 56]. Investigating the existence and regularity of solutions of ODE/PDE models plays an important role in both the development of the ODE/PDE theory and their applications in real-life problems. Furthermore, studying regularity helps to improve the smoothness and stability of solutions in different spaces, and hence makes numerical simulations valuable. This second topic has not been treated well in the literature.

As far as we know, there are only a few works analyzing Problem (1)-(2) and which provide existence and uniqueness results, and regularity estimates. The main difficulty in the analysis of Problem (1)-(2) and the essential difference from the traditional problems come from the nonlocality of the time-fractional derivative ∂_t^α. The major question for this work in our mind is:

What is the regularity of the corresponding solution u (output data) if the given data (input data) f, G are regular?

Our goal in this paper is to find suitable Banach spaces for the given data (f, G) in order to obtain existence and regularity results for the corresponding solution. The regularity estimates are important in the analysis of time discretization schemes for Problem (1)-(2) in the future.

The difficulties of a final value problem can be briefly described as follows (see Remark 3.1 for more details). Firstly, since the fractional derivative ∂_t^α is non-locally defined on the time interval $(0,t)$, we cannot convert a final value problem to an initial-value problem by using some substitution methods. Secondly, the formulation of mild solutions of a final value problem is more complex than the corresponding initial problem. This positively promotes us to construct new solution techniques to deal with Problem (1)-(2). Some more details can be found in Subsection 3.1, where the explicit representation of solutions relies on the eigenfunctions expansion and the Mittag-Leffler functions.

Let us describe the main results of this paper in two cases as follows. The first case is related to the properties of solutions under a globally Lipschitz (GL) assumption on the nonlinearity corresponding to two first theorems, while the second one concerns a critical nonlinearity corresponding to the third theorem. In the first theorem, we obtain the regularity results of solutions and their derivatives of first and fractional orders under the (GL) assumptions $\mathscr{H}_1$ (see page 7). The key idea is based on a Picard iteration argument and techniques to find appropriate spaces for f. Choosing spaces of f and G is a difficult and nontrivial task when we study the regularity of the solution. Although applications of our problem under $\mathscr{H}_1$ are not wide, the analysis and techniques here are helpful tools to study the next results. Moreover, the existence of a mild solution in the space L^∞ may not be obtained by considering $\mathscr{H}_1$. This can be overcome by considering the (GL) assumption $\mathscr{H}_2$ of the nonlinearity which is presented in the second theorem. The third theorem uses the contraction mapping principle to prove the existence of a mild solution in the critical case. As we know, nonlinear PDEs with critical nonlinearities are an interesting topic. We can mention [61] and references therein. Studying the initial value problem for (1) in the critical case is also a challenging problem. Therefore, investigating the regularity of the mild solution and its derivatives is very difficult.

1.3. **Outline.** The outline of this paper is as follows. In Section 2, we introduce some terminology used throughout this work. Moreover, we obtain a precise representation of solutions by using Mittag-Leffler functions. In Section 3, we investigate the well-posedness, and regularity of a mild solution to Problem (1)-(2). Three main results on the existence, uniqueness (in some suitable class of functions), regularity of the mild solution and its derivatives are proved under suitable assumptions on the terminal data and the nonlinearity. In Section 4, we apply the theoretical results to some typical fractional diffusion: Time fractional Ginzburg-Landau equation and Burgers equation. Finally, in Section 5, we provide full proofs to the main theorems established in Section 3.

2. Preliminaries

In this section we recall some properties that will be useful for the study of the well posedness of Problem (1)-(2). We start by introducing some functional spaces. Then we will recall some properties of Mittag-Leffler functions. Let the operator $\mathcal{L}$ be considered on $L^2(\Omega)$ with respect to domain $D(\mathcal{L}) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, where $L^2(\Omega)$, $W_0^{1,2}(\Omega)$, $W^{2,2}(\Omega)$ are the usual Sobolev spaces. Then the spectrum of $\mathcal{L}$ is a non-decreasing sequence of positive real numbers $\{\lambda_j\}_{j=1,2,...}$ satisfying $\lim_{j\to\infty} \lambda_j = \infty$. Moreover, there exists a positive constant $c_{\mathcal{L}}$ such that $\lambda_j \geq c_{\mathcal{L}} j^{2/d}$, for all $j \geq 1$, see [23]. Let us denote by $\{\varphi_j\}_{j=1,2,...} \subset D(\mathcal{L})$ the set of eigenfunctions of $\mathcal{L}$, i.e., $\mathcal{L}\varphi_j = \lambda_j \varphi_j$, and $\varphi_j = 0$ on $\partial\Omega$, for all $j \geq 1$. The sequence $\{\varphi_k\}_{k=1,2,...}$ forms an orthonormal basis of $L^2(\Omega)$, see e.g. [24]. For a given real number $\gamma \geq 0$, we define the Hilbert space

$$\mathbb{H}^\gamma(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\gamma} \langle v, \varphi_j \rangle^2 < \infty \right\}$$

($\langle \cdot, \cdot \rangle$ is the usual product of $L^2(\Omega)$) endowed with the norm $\|v\|_{\mathbb{H}^\gamma(\Omega)}^2 := \sum_{j=1}^{\infty} \lambda_j^{2\gamma} |\langle v, \varphi_j \rangle|^2$. We have $\mathbb{H}^0(\Omega) = L^2(\Omega)$, and $\mathbb{H}^{\frac{1}{2}}(\Omega) = W_0^{1,2}(\Omega)$. We denote by $\mathbb{H}^{-\gamma}(\Omega)$ the dual space of $\mathbb{H}^\gamma(\Omega)$ provided that the dual space of $L^2(\Omega)$ is identified with itself, e.g. see [25]. The space $\mathbb{H}^{-\gamma}(\Omega)$ is a Hilbert space with respect to the norm $\|v\|_{\mathbb{H}^{-\gamma}(\Omega)}^2 = \sum_{j=1}^{\infty} \lambda_j^{-2\gamma} \langle v, \varphi_j \rangle_{-\gamma,\gamma}^2$, for $v \in \mathbb{H}^{-\gamma}(\Omega)$ where $\langle .,. \rangle_{-\gamma,\gamma}$ is the dual product between $\mathbb{H}^{-\gamma}(\Omega)$ and $\mathbb{H}^\gamma(\Omega)$. We note that

$$\langle \tilde{v}, v \rangle_{-\gamma,\gamma} = \langle \tilde{v}, v \rangle, \quad \text{for } \tilde{v} \in L^2(\Omega), v \in \mathbb{H}^\gamma(\Omega). \tag{4}$$

By identifying $L^2(\Omega)$ with its dual space, and making use of the inclusion $\mathbb{H}^\gamma(\Omega) \hookrightarrow L^2(\Omega)$, the embedding $\mathbb{H}^\gamma(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{H}^{-\gamma}(\Omega)$ holds for $\gamma \geq 0$. Hence, it is suitable to call the space $\mathbb{H}^s(\Omega)$, $s \in \mathbb{R}$, by a Hilbert scale space. For given numbers $p \geq 1$ and $\nu \in \mathbb{R}$, let $L^p(0,T;\mathbb{H}^\nu(\Omega))$ be the space of all functions $w : (0,T) \to \mathbb{H}^\nu(\Omega)$ such that

$$\|w\|_{L^p(0,T;\mathbb{H}^\nu(\Omega))} := \left(\int_0^T \|w(t)\|_{\mathbb{H}^\nu(\Omega)}^p dt \right)^{1/p} < \infty.$$

We denote by $C([0,T];\mathbb{H}^\nu(\Omega))$ the space of all continuous functions from $[0,T]$ to $\mathbb{H}^\nu(\Omega)$ corresponding to the usual supremum norm $\|w\|_{C([0,T];\mathbb{H}^\nu(\Omega))} := \sup_{0\leq t\leq T} \|w(t)\|_{\mathbb{H}^\nu(\Omega)}$; and denote by $C^\delta([0,T];\mathbb{H}^\nu(\Omega))$, $\delta \in (0,1)$, the space of all Hölder continuous functions from $[0,T]$ to $\mathbb{H}^\nu(\Omega)$ with exponent δ, namely, $w \in C([0,T];\mathbb{H}^\nu(\Omega))$ satisfies that

$$\|w\|_{C^\delta([0,T];\mathbb{H}^\nu(\Omega))} := \sup_{0\leq t,s\leq T,\, t\neq s} \frac{\|w(t) - w(s)\|_{\mathbb{H}^\nu(\Omega)}}{|t-s|^\delta} < \infty.$$

Let us denote by $C((0,T];\mathbb{H}^\nu(\Omega))$ the set of all continuous functions which map $(0,T]$ into $\mathbb{H}^\nu(\Omega)$. For a given number $\eta > 0$, we denote by $C^\eta((0,T];\mathbb{H}^\nu(\Omega))$ the space of all functions w in $C((0,T];\mathbb{H}^\nu(\Omega))$ such that $\|w\|_{C^\eta((0,T];\mathbb{H}^\nu(\Omega))} := \sup_{0<t\leq T} t^\eta \|w(t)\|_{\mathbb{H}^\nu(\Omega)} < \infty$, see [28].

2.1. Fractional Sobolev spaces.

We recall some Sobolev embeddings as follows. Let Ω be a nonempty open set with a Lipschitz continuous boundary in $\mathbb{R}^N$, $N \geq 1$. Let us recall that the notation $W^{s,p}(\Omega)$, $s \in \{0,1,2,...\}$, $p \geq 1$, refers to the standard Sobolev one, e.g. see [1]. In the case $0 \leq s \leq 1$ is a positive real number, the intermediate space $W^{s,p}(\Omega) = \left[L^p(\Omega); W^{1,p}(\Omega)\right]_s$ can be defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(x')|}{|x-x'|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\}.$$

Since Ω is a nonempty open set and possesses a Lipschitz continuous boundary in $\mathbb{R}^N$, then the following Sobolev embedding holds

$$W^{\sigma,p}(\Omega) \hookrightarrow W^{\gamma,q}(\Omega) \qquad \text{if} \quad \begin{cases} 1 \le p, q < \infty, \\ 0 \le \gamma \le \sigma < \infty, \\ \sigma - \gamma \ge \dfrac{N}{p} - \dfrac{N}{q}. \end{cases} \tag{5}$$

By letting $p = 2$, $\gamma = 0$ in (5), one obtains that $W^{\sigma,2}(\Omega) \hookrightarrow L^q(\Omega)$ with $1 \le q < \infty$, $0 \le \sigma < \infty$, and $\sigma \ge \frac{N}{2} - \frac{N}{q}$. Henceforth, setting $0 \le \sigma < \frac{N}{2}$ infers that $1 \le q \le \frac{2N}{N-2\sigma}$. Summarily, we obtain the following embedding

$$W^{\sigma_1,2}(\Omega) \hookrightarrow L^{q_1}(\Omega) \qquad \text{if} \quad 0 \le \sigma_1 < \frac{N}{2},\ 1 \le q_1 \le \frac{2N}{N - 2\sigma_1}. \tag{6}$$

This implies $W_0^{\sigma_1,2}(\Omega) \hookrightarrow L^{q_1}(\Omega)$, and so $L^{q_1^*}(\Omega) = [L^{q_1}(\Omega)]^* \hookrightarrow [W_0^{\sigma_1,2}(\Omega)]^* = W^{-\sigma_1,2}(\Omega)$ with respect to $-\frac{N}{2} < -\sigma_1 \le 0$ and $q_1^* \ge (\frac{2N}{N-2\sigma_1})(\frac{2N}{N-2\sigma_1} - 1)^{-1} = \frac{2N}{N+2\sigma_1}$. Thus,

$$L^{q_2}(\Omega) \hookrightarrow W^{\sigma_2,2}(\Omega) \qquad \text{if} \quad -\frac{N}{2} < \sigma_2 \le 0,\ q_2 \ge \frac{2N}{N - 2\sigma_2}. \tag{7}$$

On the other hand, the Hilbert scale spaces and the fractional Sobolev space are related to each other by the following embeddings

$$\mathbb{H}^s(\Omega) \hookrightarrow W^{2s,2}(\Omega) \hookrightarrow L^2(\Omega), \quad \text{if} \quad s \ge 0. \tag{8}$$

2.2. **On the Mittag-Leffler functions.** An important function in the integral formula of solutions of many differential equations involving Caputo fractional derivatives is the Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

for $\alpha > 0$ and $\beta \in \mathbb{R}$. It will be used to represent the solution of Problem (1)-(2). We recall the following lemmas, for which their proofs can be found in many books, e.g. see [11, 12, 13]. The first one is very basic and also useful for many estimates of this paper. The second one helps us to find the derivatives of first order and fractional order α of the mild solution of problem (1)-(2). The third one is important and helps us to deal with the Mittag-Leffler function corresponding to the final time T. In the last one, we combine the first and third ones to derive some more important estimates. The proof of this lemma can be found in the Appendix. In this paper, we always consider T satisfies assumption (9) below.

Lemma 2.1. *Let $1 < \alpha < 2$ and $\beta \in \{1; \alpha\}$. Then, there exist positive constants m_α and M_α, depending only on α, such that*

$$\frac{m_\alpha}{1+t} \le |E_{\alpha,\beta}(-t)| \le \frac{M_\alpha}{1+t}, \quad \text{for all } t \ge 0.$$

Lemma 2.2. *Assume $1 < \alpha < 2$, $\lambda > 0$, and $t > 0$. Then, the following differentiation formula holds*

a) $\partial_t E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$, *and* $\partial_t(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)$;
b) $\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha)$, *and* $\partial_t^\alpha(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$.

Lemma 2.3 (see [27])**.** *Let $1 < \alpha < 2$. If the number T is large enough, then*

$$E_{\alpha,1}(-\lambda_j T^\alpha) \ne 0, \quad \text{for all } j \in \mathbb{N}, j \ge 1, \tag{9}$$

and there exist two positive constants m_α, and M_α such that

$$\frac{m_\alpha}{1 + \lambda_j T^\alpha} \le \left| E_{\alpha,1}(-\lambda_j T^\alpha) \right| \le \frac{M_\alpha}{1 + \lambda_j T^\alpha}.$$

Lemma 2.4. *Let $1 < \alpha < 2$ and $0 < \theta < 1$. For $t > 0$, and $j \in \mathbb{N}$, $j \ge 1$, there hold that*

a) $t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j z^\alpha) \le M_\alpha \lambda_j^{-\theta} t^{\alpha(1-\theta)-1}$;

b) $\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha) \le M_\alpha m_\alpha^{-1} \left(\lambda_1^{-1} + T^\alpha\right) \lambda_j^\theta t^{-\alpha(1-\theta)}$ *with* $\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha) := \dfrac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}$.

From now on, we will use $a \lesssim b$ to denote the existence of a constant $C > 0$, which may depend only on α, T such that $a \le Cb$.

3. Existence and regularity of the terminal value problem (1)-(2)

3.1. **Mild solutions.** Solutions of partial differential equations can be considered in the classical, weak, or mild sense. In this work, we will study mild solutions of FVP (1)-(2).

There are many works considering the precise formulation of mild solutions to IPVs for time fractional wave equations, such as [20, 21, 22, 29, 30, 36, 37, 38, 45], by using complex integral representations on Banach spaces or spectral representations on Hilbert scale spaces of the Mittag-Leffer operators. To study FVPs for time fractional wave equations, the precise formulation of mild solutions can be derived by using spectral representations of the inverse Mittag-Leffer operators, such as [26, 54, 52, 47, 39, 41, 42, 43]. In what follows, we state a definition of mild solutions to FVP (1)-(2) where the precise formulation can be obtained by some simple computations.

Additionally, for a given two-variables function $w = w(x,t)$, we will write $w(t)$ instead of $w(.,t)$ and understand $w(t)$ as a function of the spatial variable x.

Definition 3.1. A function u in $L^p(0,T;\mathbb{H}^\nu(\Omega))$ or $C^\eta(0,T;\mathbb{H}^\nu(\Omega))$ (with some suitable numbers $p \geq 1, \nu \geq 0$ or $\eta > 0$) is called a mild solution of Problem (1)-(2) if it satisfies the following equation

$$u(t) = \mathbf{B}_\alpha(t,T)f + \int_0^t \mathbf{P}_\alpha(t-r)G(r,u(r))dr - \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr \tag{10}$$

in the sense of $\mathbb{H}^\nu(\Omega)$, where, for $0 \leq t \leq T$, the solution operator $\mathbf{B}_\alpha$, $\mathbf{P}_\alpha$ are given by

$$\mathbf{B}_\alpha(t,T)v := \sum_{j=1}^\infty \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\langle v,\varphi_j\rangle\varphi_j, \quad \mathbf{P}_\alpha(t)v := \sum_{j=1}^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)\langle v,\varphi_j\rangle\varphi_j. \tag{11}$$

where $v = \sum_{j=1}^\infty \langle v,\varphi_j\rangle\varphi_j$.

Remark 3.1. *A mild formulation of this IVP (1), (3) is given by*

$$u(t) = \mathbf{B}_\alpha^{(0)}(t)\overline{f} + \int_0^t \mathbf{P}_\alpha(t-r)G(r,u(r))dr, \tag{12}$$

where $\mathbf{B}_\alpha^{(0)}(t)w := \sum_{j=1}^\infty E_{\alpha,1}(-\lambda_j t^\alpha)\langle w,\varphi_j\rangle\varphi_j$, *see* [20, 21, 22, 29, 30, 36, 37, 38, 45], *etc. It should be pointed out some core differences between the IVP (1), (3), and the FVP (1)-(2) for fractional wave equations as what follows*

- *The solution operator* $\mathbf{B}_\alpha(t,T)$ *is weaker than* $\mathbf{B}_\alpha^{(0)}(t)$. *Indeed, one can see that if* $v \in L^2(\Omega)$ *then* $\mathbf{B}_\alpha^{(0)}(t)v \in L^\infty(0,T;L^2(\Omega))$ *and*
$$\mathbf{B}_\alpha(t,T)v \notin L^\infty(0,T;L^2(\Omega)) \cup C([0,T];L^2(\Omega)).$$
Therefore, it is actually difficult to establish the existence of mild solutions, especially in the critical nonlinear case;
- *Mild formulation of the FVP (1)-(2) contains more terms than the IVP (1), (3). In particular, estimating the last term of (10) requires very clever techniques in acting* $\mathbf{B}_\alpha(t,T)$, $\mathbf{P}_\alpha(t-r)$ *on* $G(r,u(r))$. *In the critical nonlinear case, it is very difficult to determine where does the quantity*
$$\mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(t-r)G(r,u(r))$$
belongs?, and also how to bound this quantity such that its integration on the whole interval $(0,T)$ *is convergent?*
- *The Gronwall inequality can be applied when we estimate solutions of the IVP (1), (3). However, it cannot when we estimate solutions of the FVP (1)-(2) since (10) contains the integral on* $(0,T)$.

Hence, studying FVP (1)-(2) is a difficult task.

3.2. **Well-posedness of Problem (1)-(2) in the globally Lipschitz case.** In this section, we study the well-posedness of Problem (1)-(2), and regularity of the solution when we consider the following globally Lipschitz assumptions on G:

($\mathscr{H}_1$) The function $G : [0,T] \times \mathbb{H}^\nu(\Omega) \to \mathbb{H}^\nu(\Omega)$ satisfies $G(t,0) = 0$, and there exists a non-negative function $L_1 \in L^\infty(0,T)$ such that

$$\|G(t,w_1) - G(t,w_2)\|_{\mathbb{H}^\nu(\Omega)} \leq L_1(t)\|w_1 - w_2\|_{\mathbb{H}^\nu(\Omega)}, \tag{13}$$

for all $0 \leq t \leq T$, and $w_1, w_2 \in \mathbb{H}^\nu(\Omega)$.

($\mathscr{H}_2$) The function $G : [0,T] \times C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega)) \to H^{\nu+1}(\Omega)$ satisfies $G(t,0) = 0$, and there exists a non-negative function $L_2 \in L^\infty(0,T)$ such that

$$\|G(t,w_1) - G(t,w_2)\|_{H^{\nu+1}(\Omega)} \le L_2 \Big\| w_1 - w_2 \Big\|_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))}, \tag{14}$$

for all $0 \le t \le T$, and $w_1, w_2 \in C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))$, where

$$\Big\| w_1 - w_2 \Big\|_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))} = \|w_1 - w_2\|_{C([0,T];\mathbb{H}^\nu(\Omega))} + \|v_1 - v_2\|_{L^q(0,T;\mathbb{H}^\sigma(\Omega))},$$

and $\nu \ge 0$, $q \ge 1$, $\sigma \ge 0$.

In order to establish our main results, it is useful to note that

$$0 < \frac{\alpha - 1}{\alpha} < \frac{1}{2} < \frac{1}{\alpha} < 1, \quad \text{as} \quad 1 < \alpha < 2.$$

Besides, we recall that the Sobolev embedding $\mathbb{H}^{\nu+\theta}(\Omega) \hookrightarrow \mathbb{H}^\nu(\Omega)$ holds as $\nu \ge 0$ and $\theta \ge 0$, so there exists a positive constant $C_1(\nu,\theta)$ such that

$$\big\|v\big\|_{\mathbb{H}^\nu(\Omega)} \le C_1(\nu,\theta)\big\|v\big\|_{\mathbb{H}^{\nu+\theta}(\Omega)}, \tag{15}$$

for all $v \in \mathbb{H}^{\nu+\theta}(\Omega)$. In addition, for the reader convenience, the important constants (which may appear in some proofs) are summarily given by **(AP.4.)** in the Appendix.

The first result in Theorem 3.2 ensures the existence of a mild solution in $L^p(0,T;\mathbb{H}^\nu(\Omega))$ under appropriate assumptions on p, the final value data f, and the assumption ($\mathscr{H}_1$) on the nonlinearity G. The idea is to construct a Cauchy sequence in $L^p(0,T;\mathbb{H}^\nu(\Omega))$ which will be bounded by a power function and must converge to a mild solution of Problem (1)-(2). The solution is then bounded by the power function. After that, time continuity and spatial regularities can be consequently derived. Furthermore, we also discuss the existence of the derivatives ∂_t, ∂_t^α of the mild solution in some appropriate spaces.

Theorem 3.2. *Assume that $f \in \mathbb{H}^{\nu+\theta}(\Omega)$ and G sastisfies ($\mathscr{H}_1$) such that $\|L_1\|_{L^\infty(0,T)} \in \big(0, \mathscr{M}_1^{-1}\big)$ with $\nu \ge 0$ and θ satisfying that $\frac{\alpha-1}{\alpha} < \theta < 1$, where the constant $\mathscr{M}_1$ is given by **(AP.4)** in the Appendix. Then Problem (1)-(2) has a unique mild solution*

$$u \in L^p(0,T;\mathbb{H}^\nu(\Omega)) \cap C^{\alpha(1-\theta)}\big((0,T];\mathbb{H}^\nu(\Omega)\big),$$

for all $p \in \Big[1, \frac{1}{\alpha(1-\theta)}\Big)$, which corresponds to the estimate

$$\|u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{16}$$

The following spatial and time regularities also hold:

a) *Let θ' satisfy that $\frac{\alpha-1}{\alpha} < \theta' \le \theta$. Then $u \in L^p(0,T;\mathbb{H}^{\nu+\theta-\theta'}(\Omega))$, for all $p \in \Big[1, \frac{1}{\alpha(1-\theta')}\Big)$, which corresponds to the estimate*

$$\|u(t)\|_{\mathbb{H}^{\nu+\theta-\theta'}(\Omega)} \lesssim t^{-\alpha(1-\theta')} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

b) *Let $1 - \theta < \nu' \le 2 - \theta$. Then $u \in C^{\eta_{glo}}\Big([0,T];\mathbb{H}^{\nu-\nu'}(\Omega)\Big)$ and*

$$\big\|u(\widetilde{t}) - u(t)\big\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} \lesssim (\widetilde{t} - t)^{\eta_{glo}} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

Here η_{glo} is defined in the Appendix.

c) *Let $0 \le \nu_1 \le \min\Big\{1-\theta; \ \frac{\alpha-1}{\alpha}\Big\}$. Then $\partial_t u \in L^p(0,T;\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega))$, for all $p \in \Big[1, \frac{1}{\alpha(1-\theta-\nu_1)}\Big)$, which corresponds to the estimate*

$$\|\partial_t u(t)\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_1)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

d) *Let $\frac{\alpha-1}{\alpha} - \theta < \nu_\alpha \le \min\Big\{\frac{1}{\alpha} - \theta; \frac{\alpha-1}{\alpha}\Big\}$. Then $\partial_t^\alpha u \in L^p(0,T;\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega))$, for any $p \in \Big[1, \frac{1}{\alpha\min\{(1-\theta-\nu_\alpha);(1-\theta)\}}\Big)$, which corresponds to the estimate*

$$\|\partial_t^\alpha u(t)\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha\min\{(1-\theta-\nu_\alpha);(1-\theta)\}} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

The hidden constants (as using the notation $\lesssim$) depend only on α,ν,θ,T in the inequality (16), on $\alpha,\nu,\theta,\theta',T$ in Part a, on α,ν,θ,ν',T in Part b, on $\alpha,\nu,\theta,\nu_1,T$ in Part c, and on $\alpha,\nu,\theta,\nu_\alpha,T$ in Part d.

Remark 3.2. *We note that, inequality (16) also guarantees the continuous dependence of the solution on the final value f. In fact, if we denote by $u(f)$ and $u(\widetilde{f})$ the mild solutions of Problem (1)-(2) corresponding to the final value f and $\widetilde{f}$, then one can see that*

$$\left\|u(f)-u(\widetilde{f})\right\|_{C^{\alpha(1-\theta)}\left((0,T];\mathbb{H}^\nu(\Omega)\right)} \lesssim \|f-\widetilde{f}\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

This concludes the well-posedness of Problem (1)-(2) on $C^{\alpha(1-\theta)}\big((0,T];\mathbb{H}^\nu(\Omega)\big)$.

Proposition 3.1. *By Theorem 3.2, the smoothness of the mild solution can be summarized together as*

$$u\in\left\{\bigcup\nolimits_{1\le p<\frac{1}{\alpha(1-\theta')}} L^p(0,T;\mathbb{H}^{\nu+\theta-\theta'}(\Omega))\right\}\cap C^{\alpha(1-\theta)}\big((0,T];\mathbb{H}^\nu(\Omega)\big),$$

and

$$\begin{cases} \partial_t u\in\bigcup\nolimits_{1\le p<\frac{1}{\alpha(1-\theta-\nu_1)}} L^p(0,T;\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)),\\ \partial_t^\alpha u\in\bigcup\nolimits_{1\le p<\frac{1}{\alpha(1-\theta-\nu_\alpha)}} L^p(0,T;\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)),\end{cases}$$

where the values of the parameters are given in Theorem 3.2. Moreover, the spatial regularity in Part b shows how the best spatial regularity that the mild solution u can achieve. Then, by using some suitable Sobolev embeddings, one can derive the Gradient, Laplacian estimates for the solution on $L^q(\Omega)$ spaces.

Remark 3.3. *In fact, one can investigate the continuity of the first order derivative $\partial_t u$ which is established in Part d of the above theorem. Moreover, if the nonlinearity G is continuous in the time variable t, for instant, G verifies that*

$$\|G(t_1,v_1)-G(t_2,v_2)\|_{\mathbb{H}^\nu(\Omega)}\lesssim|t_1-t_2|^{\text{positive exponent}}+\|v_1-v_2\|_{\mathbb{H}^\nu(\Omega)},$$

then one can establish the continuity of the fractional derivative ∂_t^α of the solution.

In Theorem 3.2, under assumption $(\mathscr{H}_1)$, we do not obtain the regularity results of u in the spaces $C([0,T];\mathbb{H}^\nu(\Omega))$ or $L^\infty(0,T;\mathbb{H}^\nu(\Omega))$. The main reason is that the information at the initial time $u(0)$ does not actually exist on $\mathbb{H}^\nu(\Omega)$. To overcome this restriction, we are going to consider the existence of a mild solution in the spaces $C([0,T];\mathbb{H}^\nu(\Omega))$ or $L^\infty(0,T;\mathbb{H}^\nu(\Omega))$ by imposing the assumption $(\mathscr{H}_2)$ on the nonlineariy G. In addition, it is necessary to suppose a smoother assumption on the final value data f. In the following theorem, we will build up this existence and also a regularity result for the mild solution by using the Banach fixed-point theorem. Let us recall the fact that the embedding $\mathbb{H}^{\nu+1}(\Omega)\hookrightarrow\mathbb{H}^\sigma(\Omega)$ holds as $0\le\sigma\le\nu+1$. So, there exists a positive constant $C_2(\nu,\sigma)$ such that

$$\big\|v\big\|_{\mathbb{H}^\sigma(\Omega)}\le C_2(\nu,\sigma)\big\|v\big\|_{\mathbb{H}^{\nu+1}(\Omega)}, \tag{17}$$

for all $v\in\mathbb{H}^{\nu+1}(\Omega)$.

Theorem 3.3. *Let $\frac{\alpha-1}{\alpha}<\theta<1$, $0\le\nu\le\sigma\le\nu+1$ and $1\le q<\frac{1}{\alpha(1-\theta)}$. Assume that $f\in\mathbb{H}^{\nu+\theta+1}(\Omega)$, and G sastisfies $(\mathscr{H}_2)$ with $L_2\in\left(0,\mathscr{M}_2^{-1}\right)$ where $\mathscr{M}_2$ is given by **(AP.4)** in the Appendix. Then, Problem (1)-(2) has a unique mild solution*

$$u\in C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega)).$$

Moreover, there holds

$$\sup_{0\le t\le T}\big\|u(t)\big\|_{\mathbb{H}^\nu(\Omega)}+\left(\int_0^T\big\|u(t)\big\|^q_{\mathbb{H}^\sigma(\Omega)}dt\right)^{1/q}\lesssim\|f\|_{\mathbb{H}^{\nu+\theta+1}(\Omega)}. \tag{18}$$

Remark 3.4. *Truly, the assumptions $f\in\mathbb{H}^{\nu+1}(\Omega)$ and $(\mathscr{H}_2)$ are enough to obtain $u\in C([0,T];\mathbb{H}^\nu(\Omega))$, where $\|u(t)\|_{\mathbb{H}^{\nu+1}(\Omega)}\lesssim t^{-\alpha}$ for $t>0$. This combines with the embedding (17) that $\|u(t)\|_{\mathbb{H}^\sigma(\Omega)}\lesssim t^{-\alpha}$, for $t>0$, which does not ensure $u\in L^q(0,T;\mathbb{H}^\sigma(\Omega))$ since $t^{-\alpha}\notin L^q(0,T)$ and $1<\alpha<2$.*

Otherwise, if $f\in\mathbb{H}^{\nu+\theta+1}(\Omega)$ and G satisfies the assumption $(\mathscr{H}_2)$ as Theorem 3.3, then using the same techniques as Proof of Theorem 3.3 (see the estimates (86)-(88)) shows that

$$\|u(t)\|_{\mathbb{H}^\sigma(\Omega)}\lesssim\|u(t)\|_{\mathbb{H}^{\nu+1}(\Omega)}\lesssim t^{-\alpha(1-\theta)},\ t>0.$$

We imply $u\in L^q(0,T;\mathbb{H}^\sigma(\Omega))$ since $t^{-\alpha(1-\theta)}\in L^q(0,T)$ for $1\le q<\frac{1}{\alpha(1-\theta)}$ and $0<\alpha(1-\theta)<1$.

3.3. **Well-posedness of Problem (1)-(2) under critical nonlinearities.** The previous subsection states the results in the globally Lipschitz case, they cannot virtually be applied in many models such as time fractional Ginzburg-Landau, AllenCahn, Burgers, Navier-Stokes, Schrödinger, etc. equations. In this subsection, we state the well-posedness of Problem (1)-(2) under the critical nonlinearities case.

Theorem 3.4. *Assume that $\alpha \in (1, 2)$, $\sigma \in (-1, 0)$, $0 < \nu < 1 + \sigma$ and $s > 0$. Let ϑ such that $\vartheta \in (\nu - \sigma, 1)$ and set $\mu = \nu - \sigma$. Let ζ satisfy*

$$\zeta < \min\Big(\alpha^{-1} - (1+s)\vartheta;\ \vartheta(1-s) - \nu + \sigma\Big). \tag{19}$$

The function G satisfies the next assumption $(\mathscr{H}_3)$, that is, $G : [0, T] \times \mathbb{H}^\nu(\Omega) \longrightarrow \mathbb{H}^\sigma(\Omega)$, $G(0) = 0$ and

$$\|G(t,v_1) - G(t,v_2)\|_{\mathbb{H}^\sigma(\Omega)} \le L_3(t)\Big(1 + \|v_1\|^s_{\mathbb{H}^\nu(\Omega)} + \|v_2\|^s_{\mathbb{H}^\nu(\Omega)}\Big)\|v_1 - v_2\|_{\mathbb{H}^\nu(\Omega)}, \tag{20}$$

where L_3 satisfies that $L_3(t)t^{\alpha\zeta} \in L^\infty(0, T)$. Set

$$\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R}) := \Big\{w \in C^{\alpha\vartheta}((0, T]; \mathbb{H}^\nu(\Omega)),\ \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \le \mathcal{R}\Big\}.$$

If $f \in \mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, $f \not\equiv 0$, and $K_0T^{s\alpha\vartheta} \in \big(0, \min\big\{\frac{1}{2}\overline{\mathcal{N}_2}^{-1}; \mathcal{N}_f\big\}\big)$ with $K_0 = \|L_3(t)t^{\alpha\zeta}\|_{L^\infty(0,T)}$, where the constants are formulated by ***(AP.4)*** *in the Appendix, then Problem (1)-(2) has a unique mild solution $u \in \mathfrak{X}_{\alpha,\vartheta,\nu,T}(\widehat{\mathcal{R}})$ satisfying in addition*

$$\|u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim t^{-\alpha\vartheta}\, \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\,. \tag{21}$$

Moreover, we obtain the following spatial and time regularities

a) *Let $\vartheta \le \vartheta' \le 1$ and $\alpha\vartheta - 1 \le \beta \le \alpha\vartheta$, then $t^\beta u \in L^p(0, T; \mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega))$, for all $p \in \Big[1, \frac{1}{\alpha\vartheta-\beta}\Big)$, with respect to the estimate*

$$\big\|u(t)\big\|_{\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega)} \lesssim t^{-\alpha\vartheta'}\, \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\,.$$

b) *Let $\vartheta < \eta \le \vartheta + 1$ then $u \in C^{\eta_{cri}}([0, T]; \mathbb{H}^{\nu-\eta}(\Omega))$ and*

$$\big\|u(\widetilde{t}) - u(t)\big\|_{\mathbb{H}^{\nu-\eta}(\Omega)} \lesssim (\widetilde{t} - t)^{\eta_{cri}}\, \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\,.$$

where η_{cri} is defined in the Appendix.

The hidden constants (as using the notation $\lesssim$) depend only on $\alpha, \mu, \vartheta, \zeta, s, T$ in the inequality (21), on $\alpha, \mu, \vartheta, \vartheta', \zeta, s, T$ in Part a, and on $\alpha, \mu, \vartheta, \eta, \zeta, s, T$ in Part b.

Remark 3.5. *One can actually establish the existence of the derivatives $\partial_t u$ and ∂_t^α of the solution as follows*

i) *Assume that $\vartheta < \frac{\mu+1}{2}$ and let $\vartheta_1 \in \big[\vartheta, \frac{\nu-\sigma+1}{2}\big)$, then $\partial_t u(t) \in \mathbb{H}^{\sigma+\vartheta_1-1}(\Omega)$ for each $t > 0$ which corresponds to the estimate*

$$\|\partial_t u(t)\|_{\mathbb{H}^{\sigma+\vartheta_1-1}(\Omega)} \lesssim t^{-\alpha\left(2\vartheta_1-\mu-\frac{\alpha-1}{\alpha}\right)}\, \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\,. \tag{22}$$

ii) *Assume that $\vartheta \le \frac{2\nu-2\sigma+1}{3}$ and let $\vartheta_\alpha \in \big[\frac{\nu-\sigma+2}{3}, \frac{\nu-\sigma+5}{3}\big)$, then $\partial_t^\alpha u(t) \in \mathbb{H}^{\sigma+\vartheta_\alpha-2}(\Omega)$ for each $t > 0$ which corresponds to the estimate*

$$\|\partial_t^\alpha u(t)\|_{\mathbb{H}^{\sigma+\vartheta_\alpha-2}(\Omega)} \lesssim t^{-\max\left\{\alpha\left(\vartheta_\alpha-\frac{\mu+2}{3}\right);\alpha(2\vartheta-\mu)\right\}}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{23}$$

4. Applications

In this section we apply the theory developed in this work to some well-known equations. The classes of *time fractional Ginzburg-Landau equation* and *time fractional Burgers equation*, are studied in $L^q(\Omega)$ $(q \ge 1)$ settings via interpolationextrapolation scales and dual interpolationextrapolation scales of Sobolev spaces. We will discuss both time and spatial regularity of solutions by considering

- The time continuities of solutions on $L^q(\Omega)$ spaces with respect to the intervals $(0, T]$, $[0, T]$;
- The Gradient and Laplacian estimates for the solutions on $L^q(\Omega)$ spaces.

4.1. **Time fractional Ginzburg-Landau equation.** We discuss now an application of our methods to a final value problem for a time fractional Ginzburg-Landau equation which is stated as follows

$$\begin{cases} \partial_t^\alpha u(x,t) + \mathcal{L}u(x,t) &= \rho(t)|u(x,t)|^s u(x,t), & x \in \Omega, \quad t \in (0,T), \\ u(x,t) &= 0, & x \in \partial\Omega,\ t \in (0,T), \\ \partial_t u(x,0) &= 0, & x \in \Omega, \end{cases} \tag{24}$$

associated with the final value data (2) and where $s > 0$ is a given number.

The Ginzburg-Landau equations were introduced to describe the behavior of superconductors [62]. Their fractional generalizations have been suggested by [63], and some models and applications of time fractional generalizations can be found in [64] and references therein. We also refer the reader to [65, 66] for complex Ginzburg-Landau equations.

In the following result, Theorem 3.4 will be applied to obtain a mild solution of Problem (24),(2), where time and spatial regularity estimates in two cases $0 < s < \frac{4}{N}$ and $s \geq \frac{4}{N}$.

Theorem 4.1. *Assume that* $2 \leq N \leq 4$.

a) ***The case*** $0 < s < \dfrac{4}{N}$***:*** *Let the numbers ϱ, ν, σ, μ, ϑ, ϑ' respectively satisfy that*

$$\varrho \in \left(0, \frac{Ns}{8}\right], \ \nu \in \left[\frac{N}{4} - \frac{\varrho}{s}, \frac{N}{4}\right), \ \mu \in \left(\mu_0, \frac{N+4}{8}\right), \ \vartheta \in \left(\mu, \frac{N+4}{8}\right), \ \vartheta' \in \left[\vartheta + \frac{4-N}{8}, 1\right),$$

where $\mu = \nu - \sigma$ and $\mu_0 := \max\left\{\nu; s\left(\frac{N}{4} - \nu\right)\right\}$. If $f \in \mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, and $\rho(t) \leq C_\rho t^b$ with $b > -\min\left\{\frac{1}{\alpha} - (1+s)\vartheta; (\vartheta - \mu) - s\vartheta\right\}$ and C_ρ is small enough, then Problem (24) has a unique mild solution u such that

(Time regularity) *$u \in C^{\alpha\vartheta}((0,T]; L^4(\Omega)) \cap C^{\eta_{cri}}([0,T]; \mathbb{H}^{\nu-\eta}(\Omega))$ where $\vartheta < \eta \leq \vartheta + 1$. This solution satisfies the estimate*

$$t^{\alpha\vartheta} \|u(t)\|_{L^4(\Omega)} \mathbf{1}_{t>0} + \frac{\|u(t+\gamma) - u(t)\|_{\mathbb{H}^{\nu-\eta}(\Omega)}}{\gamma^{\eta_{cri}}} \mathbf{1}_{t\geq 0} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{25}$$

(Spatial regularity) *For each $t > 0$, $u(t)$ belongs to $W^{1,\frac{4N}{3N-8\nu}}(\Omega)$ and verifies the estimate*

$$t^{\alpha\vartheta'} \left\|\nabla u(t)\right\|_{L^{\frac{4N}{3N-8\nu}}(\Omega)} + t^{\alpha\vartheta'} \left\|(-\Delta)^{\vartheta'-\vartheta} u(t)\right\|_{L^4(\Omega)} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{26}$$

b) ***The case*** $s \geq \dfrac{4}{N}$***:*** *Let the numbers ν, σ, μ, ϑ, ϑ' respectively satisfy that*

$$\nu \in \left[\frac{N}{4} - \frac{1}{2s}, \frac{N}{4}\right), \ \mu \in \left(\mu_0, \frac{N+4}{8}\right), \ \vartheta \in \left(\mu, \frac{N+4}{8}\right), \ \vartheta' \in \left[\vartheta + \frac{4-N}{8}, 1\right),$$

whereupon $\mu = \nu - \sigma$ and $\mu_0 := \max\left\{\nu; s\left(\frac{N}{4} - \nu\right)\right\}$. If $f \in \mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, and $\rho(t) \leq C_\rho t^b$ such that $b > -\min\left\{\frac{1}{\alpha} - (1+s)\vartheta; (\vartheta - \mu) - s\vartheta\right\}$ and C_ρ is small enough, then Problem (24) has a unique mild solution u such that

(Time regularity) *$u \in C^{\alpha\vartheta}((0,T]; L^{Ns}(\Omega)) \cap C^{\eta_{cri}}([0,T]; \mathbb{H}^{\nu-\eta}(\Omega))$ where $\vartheta < \eta \leq \vartheta + 1$. This solution satisfies the estimate*

$$t^{\alpha\vartheta} \|u(t)\|_{L^{Ns}(\Omega)} \mathbf{1}_{t>0} + \frac{\|u(t+\gamma) - u(t)\|_{\mathbb{H}^{\nu-\eta}(\Omega)}}{\gamma^{\eta_{cri}}} \mathbf{1}_{t\geq 0} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{27}$$

(Spatial regularity) *For each $t > 0$, $u(t)$ belongs to $W^{1,\frac{4N}{3N-8\nu}}(\Omega)$ and verifies the estimate*

$$t^{\alpha\vartheta'} \left\|\nabla u(t)\right\|_{L^{\frac{4N}{3N-8\nu}}(\Omega)} + t^{\alpha\vartheta'} \left\|(-\Delta)^{\vartheta'-\vartheta} u(t)\right\|_{L^{Ns}(\Omega)} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{28}$$

Proof. a) This proof will be based on applying and improving Theorem 3.4 actually. We firstly exhibit some explanations justifying that the assumptions in this part are suitable.

- $0 < \varrho < \dfrac{1}{2}$ since $\dfrac{Ns}{8} < \dfrac{N\left(\frac{4}{N}\right)}{8} = \dfrac{1}{2}$ as the assumption $s < \dfrac{4}{N}$;
- $0 < \varrho \leq \dfrac{3N-4}{8} s$ since $\dfrac{Ns}{8} \leq \dfrac{Ns}{\left(\frac{8N}{3N-4}\right)} = \dfrac{3N-4}{8}$ by the fact that $8 \geq \dfrac{8N}{3N-4}$;

- $\nu \geq \dfrac{N}{8}$ since $\nu \geq \dfrac{N}{4} - \dfrac{\varrho}{s} \geq \dfrac{N}{4} - \dfrac{N}{8} = \dfrac{N}{8}$ as the assumption $\nu \geq \dfrac{N}{4} - \dfrac{\varrho}{s}$ and $\varrho \leq \dfrac{Ns}{8}$.
- The interval $\left(\mu_0, \dfrac{N+4}{8}\right)$ is not really empty. Indeed, it is easy to see from $\nu < \dfrac{N}{4}$ and $\dfrac{N}{4} < \dfrac{N+4}{8}$ that $\nu < \dfrac{N+4}{8}$. Moreover, we have
$$s\left(\frac{N}{4} - \nu\right) \leq s\left(\frac{N}{4} - \left(\frac{N}{4} - \frac{\varrho}{s}\right)\right) = \varrho < \frac{1}{2} < \frac{N+4}{8},$$
by using assumption $\nu > \dfrac{N}{4} - \dfrac{\varrho}{s}$ and noting that $\varrho < \dfrac{1}{2}$.
- The interval $\left[\vartheta + \dfrac{4-N}{8}, 1\right)$ is also not empty as $\vartheta < \dfrac{N+4}{8}$.
- The number σ belongs to $\left(-\dfrac{N}{4}, 0\right)$ since $\sigma = \nu - \mu \leq \mu_0 - \mu < 0$, and furthermore
$$\nu - \mu > \frac{N}{4} - \frac{\varrho}{s} - \frac{N+4}{8} \geq \frac{N}{4} - \frac{3N-4}{8} - \frac{N+4}{8} = -\frac{N}{4},$$
by using the assumption $\nu \geq \dfrac{N}{4} - \dfrac{\varrho}{s}$ and the fact that $\varrho < \dfrac{3N-4}{8}s$.

Secondly, we obtain some important Sobolev embeddings which help to establish the existence of a mild solution. By applying the embeddings (7)-(8), and using the dualities $[\mathbb{H}^{-\sigma}(\Omega)]^* = \mathbb{H}^{\sigma}(\Omega)$, $\left[W^{-2\sigma,2}(\Omega)\right]^* = W^{2\sigma,2}(\Omega)$, one can see that

- The Sobolev embedding $L^{\frac{2N}{N-4\sigma}}(\Omega) \hookrightarrow W^{2\sigma,2}(\Omega)$ holds as $-\dfrac{N}{2} < 2\sigma < 0$, $\dfrac{2N}{N-4\sigma} = \dfrac{2N}{N-2(2\sigma)}$;
- The Sobolev embedding $\mathbb{H}^{-\sigma}(\Omega) \hookrightarrow W^{-2\sigma,2}(\Omega)$ holds as $-\sigma > 0$, which consequently implies $W^{2\sigma,2}(\Omega) \hookrightarrow \mathbb{H}^{\sigma}(\Omega)$.

As a consequence of the above embeddings, we obtain the following Sobolev one
$$L^{\frac{2N}{N-4\sigma}}(\Omega) \hookrightarrow \mathbb{H}^{\sigma}(\Omega). \tag{29}$$
By the assumption $s\left(\frac{N}{4} - \nu\right) \leq \mu_0 < \mu$, we have
$$\frac{2N(1+s)}{N-4\sigma} = \frac{2N(1+s)}{N-4(\nu-\mu)} \leq \frac{2N(1+s)}{N-4\nu+4s\left(\frac{N}{4}-\nu\right)} = \frac{2N}{N-4\nu}.$$
Therefore, using (7) yields that
$$W^{2\nu,2}(\Omega) \hookrightarrow L^{\frac{2N(1+s)}{N-4\sigma}}(\Omega) \quad \text{since} \quad 0 \leq 2\nu < \frac{N}{2}, \ \frac{2N(1+s)}{N-4\sigma} \leq \frac{2N}{N-4\nu}.$$
Besides, using (8) invokes that $\mathbb{H}^{\nu}(\Omega) \hookrightarrow W^{2\nu,2}(\Omega)$, as $\nu \geq 0$, which consequently infers the embedding
$$\mathbb{H}^{\nu}(\Omega) \hookrightarrow L^{\frac{2N(1+s)}{N-4\sigma}}(\Omega). \tag{30}$$

Thirdly, let us set the nonlinearity $G(t,v) := \rho(t)|v|^s v$, and show that G satisfies $(\mathscr{H}_3)$. Indeed, it is obvious that $|G(t,v_1) - G(t,v_2)|$ is pointwise bounded by $(1+s)\left(|v_1|^s + |v_2|^s\right)|v_1 - v_2|$, and so one can derive the following chain of estimates
$$\begin{aligned}
\|G(t,v_1) - G(t,v_2)\|_{\mathbb{H}^{\sigma}(\Omega)} &\lesssim \|G(v_1) - G(v_2)\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)} \\
&\lesssim \rho(t)\left[\left\||v_1|^s|v_1 - v_2|\right\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)} + \left\||v_2|^s|v_1 - v_2|\right\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)}\right] \\
&\lesssim \rho(t)\left(\|v_1\|^s_{L^{\frac{2N(1+s)}{N-4\sigma}}(\Omega)} + \|v_2\|^s_{L^{\frac{2N(1+s)}{N-4\sigma}}(\Omega)}\right)\|v_1 - v_2\|_{L^{\frac{2N(1+s)}{N-4\sigma}}(\Omega)} \\
&\lesssim \rho(t)\left(\|v_1\|^s_{\mathbb{H}^{\nu}(\Omega)} + \|v_2\|^s_{\mathbb{H}^{\nu}(\Omega)}\right)\|v_1 - v_2\|_{\mathbb{H}^{\nu}(\Omega)},
\end{aligned}$$
where the embedding (29) has been used in the first estimate, the pointwise boundedness in the second estimate, the Hölder inequality in the third one, and the embedding (30) in the last estimate. Therefore, we can take the Lipschitz coefficient in the form $K(t) = K_{\mathrm{Gin}}\rho(t)$ with some positive constant K_{Gin}.

Furthermore, the assumption $b > -\min\left\{\frac{1}{\alpha} - (1+s)\vartheta; (\vartheta-\mu) - s\vartheta\right\}$ ensures that there always exists a real constant ζ such that

$$-b < \zeta < \min\left\{\frac{1}{\alpha} - (1+s)\vartheta;\ (\vartheta-\mu) - s\vartheta\right\},$$

then one derives $K(t) \le C_\rho K_{\mathrm{Gin}} t^{-\alpha\zeta} t^{\alpha(b+\zeta)} \le K_0 t^{-\alpha\zeta}$, where $K_0 = C_\rho K_{\mathrm{Gin}} T^{\alpha(b+\zeta)}$. We conclude that ζ and G satisfy assumptions of Theorem 3.4. It is obvious that all assumptions of this theorem also fulfill the assumptions of Theorem 3.4. Thus, applying Theorem 3.4 invokes that Problem (24) has a unique mild solution $u \in C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega)) \cap C^{\eta_{cri}}([0,T];\mathbb{H}^{\nu-\eta}(\Omega))$ with C_ρ small enough. Now, the assumption $\nu \ge \dfrac{N}{4} - \dfrac{\varrho}{s}$ implies that

$$\frac{2N}{N-4\nu} \ge \frac{2N}{N-4\left(\frac{N}{4}-\frac{\varrho}{s}\right)} = \frac{2N}{\left(4\frac{\varrho}{s}\right)} \ge \frac{2N}{\left(4\frac{N}{8}\right)} = 4, \tag{31}$$

where $\dfrac{\varrho}{s} \le \dfrac{N}{8}$. Hence, as Ω is a bounded domain, we have that $L^{\frac{2N}{N-4\nu}}(\Omega) \hookrightarrow L^4(\Omega)$. Besides, applying the embedding (6) again combined with the above embedding allow that

$$W^{2\nu,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-4\nu}}(\Omega) \hookrightarrow L^4(\Omega),$$

where we note that $0 < 2\nu < \frac{N}{2}$, $\frac{2N}{N-4\nu} = \frac{2N}{N-2(2\nu)}$. Therefore, we deduce that

$$u \in C^{\alpha\vartheta}((0,T];L^4(\Omega)) \cap C^{\eta_{cri}}\left([0,T];\mathbb{H}^{\nu-\eta}(\Omega)\right)$$

where $\vartheta < \eta \le \vartheta + 1$ as in Part b of Theorem 3.4 and

$$t^{\alpha\vartheta}\left\|u(t)\right\|_{L^4(\Omega)} \mathbf{1}_{t>0} + \frac{\left\|u(t+\gamma)-u(t)\right\|_{\mathbb{H}^{\nu-\eta}(\Omega)}}{\gamma^{\eta_{cri}}} \mathbf{1}_{t\ge 0} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}.$$

This shows inequality (34). Finally, we need to prove inequality (26). Indeed, we have

$$\vartheta' - \vartheta \ge \frac{4-N}{8} \quad \text{as} \quad \vartheta' \in \left[\vartheta + \frac{4-N}{8}, 1\right), \tag{32}$$

which associates with $\nu \ge \dfrac{N}{8}$ that $\nu + (\vartheta'-\vartheta) \ge \dfrac{N}{8} + \dfrac{4-N}{8} = \dfrac{1}{2}$. Therefore, we obtain

- The Sobolev embedding $\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega) \hookrightarrow W^{2\nu+2(\vartheta'-\vartheta),2}(\Omega)$ holds as $\nu + (\vartheta'-\vartheta) > 0$.
- The Sobolev embedding $W^{2\nu+2(\vartheta'-\vartheta),2}(\Omega) \hookrightarrow W^{1,\frac{4N}{3N-8\nu}}(\Omega)$ holds by using the embedding (5) as $\dfrac{4N}{3N-8\nu} \ge 1$ and $2\nu + 2(\vartheta'-\vartheta) \ge 1$, where we note from (32) that

$$2\nu + 2(\vartheta'-\vartheta) - 1 \ge 2\nu + 2\left(\frac{4-N}{8}\right) - 1 = 2\nu - \frac{N}{4} = \frac{N}{2} - \frac{N}{\left(\frac{4N}{3N-8\nu}\right)}.$$

Two above embeddings consequently infer that the Sobolev embedding $\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega) \hookrightarrow W^{1,\frac{4N}{3N-8\nu}}(\Omega)$ holds. Hence, we deduce from Part a of Theorem 3.4 that $u(t) \in W^{1,\frac{4N}{3N-8\nu}}(\Omega)$ with respect to the estimate

$$t^{\alpha\vartheta'}\left\|\nabla u(t)\right\|_{L^{\frac{4N}{3N-8\nu}}(\Omega)} + t^{\alpha\vartheta'}\left\|(-\Delta)^{\vartheta'-\vartheta}u(t)\right\|_{L^4(\Omega)} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)},$$

which finalizes the proof of Part a of this theorem.

b) We note from Part a that the number ϱ belongs to the interval $\left(0,\frac{1}{2}\right)$. In this part, we try to extend the method in Part a with $\varrho = \frac{1}{2}$. It is important to explain the similarities and differences between the numbers in this part from Part a as follows

- $\nu \ge \dfrac{N}{8}$ since $\nu \ge \dfrac{N}{4} - \dfrac{1}{2s} \ge \dfrac{N}{4} - \dfrac{1}{2\left(\frac{4}{N}\right)} = \dfrac{N}{8}$ by employing $\nu \ge \dfrac{N}{4} - \dfrac{1}{2s}$ and $s \le \dfrac{4}{N}$.
- The interval $\left(\mu_0, \dfrac{N+4}{8}\right)$ is not really empty since

$$s\left(\frac{N}{4}-\nu\right) \le s\left(\frac{N}{4} - \left(\frac{N}{4} - \frac{1}{2s}\right)\right) = \frac{1}{2} < \frac{N+4}{8}.$$

- The number σ belongs to $\left(-\dfrac{N}{4}, 0\right)$ since

$$\nu - \mu > \frac{N}{4} - \frac{1}{2s} - \frac{N+4}{8} \geq \frac{N}{4} - \frac{1}{2\left(\frac{4}{N}\right)} - \frac{N+4}{8} = -\frac{1}{2} \geq -\frac{N}{4},$$

by also employing $\nu \geq \dfrac{N}{4} - \dfrac{1}{2s}$ and $s \leq \dfrac{4}{N}$.

By using the same methods as Part a, one can establish the existence and uniqueness of a mild solution u to Problem (24) in $C^{\alpha\vartheta}((0,T];\mathbb{H}^{\nu}(\Omega)) \cap C^{\eta_{cri}}([0,T];\mathbb{H}^{\nu-\eta}(\Omega))$ with C_ρ is small enough. Next, inequality (31) can be modified as

$$\frac{2N}{N-4\nu} \geq \frac{2N}{N-4\left(\frac{N}{4}-\frac{1}{2s}\right)} = Ns.$$

Hence, we obtain the Sobolev embedding

$$W^{2\nu,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-4\nu}}(\Omega) \hookrightarrow L^{Ns}(\Omega),$$

which implies inequality (27). Moreover, we also have $\nu + (\vartheta' - \vartheta) \geq \frac{1}{2}$ by noting the assumption $\vartheta' \in \left[\vartheta + \frac{4-N}{8}, 1\right)$ and the fact that $\nu \geq \frac{N}{8}$. Then, we obtain the Sobolev embedding

$$\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega) \hookrightarrow W^{1,\frac{4N}{3N-8\nu}}(\Omega),$$

and inequality (28) also holds. We finally complete the proof. □

Remark 4.1. *In fact, considering $N \notin \{2;3;4\}$ for time fractional Ginzburg-Landau equations, the problem is not easy. The main reasons are: requirements for the numbers $\nu, \mu, \vartheta, \vartheta'$ and applying Sobolev embeddings. For instance, in the assumptions on $\varrho, \nu, \mu, \vartheta, \vartheta'$, we need $(N+4)/8 \leq 1$, which holds for $N \leq 4$; we need $8 \geq \frac{8N}{3N-4}$ in the beginning of Proof of Theorem 4.1, which holds for $N \geq 2$.*

4.2. **Time fractional Burgers equation.** In this subsection, we deal with a terminal value problem for a *time fractional Burgers equation* which is given by

$$\begin{cases} \partial_t^\alpha u(x,t) + \rho(t)(u\cdot\nabla)u(x,t) = \Delta u(x,t), & x \in \Omega,\ 0<t<T, \\ u(x,t) = 0, & 0<t<T, \\ \partial_t u(x,0) = 0, & x \in \Omega, \end{cases} \tag{33}$$

associated with the final value data (2). Here f and ρ are given functions, and the operator $\mathcal{A}$ is $-\Delta$ which acts on $L^2(\Omega)$ with its domain $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$.

H. Bateman first introduced Burgers equation [67]. J.M. Burgers used it to model turbulence [68], which has been commonly referred to as Burgers equation. This equation also appears when investigating flow through a shock wave propagating in a viscous fluid [69]. We refer the reader to [70, 71] and references therein for the standard Burgers equations, and to [72, 73] with time derivatives of fractional orders.

In the following theorem, we will apply Theorem 3.4 to obtain a mild solution of Problem (33), and then obtain the spatial regularity with an $L^q(\Omega)$-estimate for ∇u and a $\mathbb{H}^\nu(\Omega)$-estimate for $(-\Delta)^{\vartheta'-\vartheta}u$.

Theorem 4.2. *Assume that $3 \leq N \leq 4$ (N is the dimension of Ω). Let the numbers ν, σ, μ, ϑ, ϑ' respectively satisfy that*

$$\nu \in \left[\frac{1}{2}, \frac{N}{4}\right), \quad \mu \in \left[\mu_2, \frac{N+4}{8}\right), \ \vartheta \in \left(\mu, \frac{N+4}{8}\right), \ \vartheta' \in \left[\vartheta + \frac{4-N}{8}, 1\right),$$

where $\mu = \nu - \sigma$ and $\mu_2 := \max\left\{\nu; \frac{N+2}{4} - \nu\right\}$. If $\rho(t) \leq C_\rho t^b$ with $b > -\min\left\{-\mu; \frac{1}{\alpha} - 2\vartheta\right\}$, $f \in \mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, and C_ρ is small enough. Then Problem (33) has a unique mild solution u such that

a) ***(Time regularity)*** *Let $\vartheta < \eta \leq \vartheta + 1$. Then we have*

$$u \in C^{\alpha\vartheta}((0,T];L^{\frac{2N}{4\mu-2}}(\Omega)) \cap C^{\eta_{cri}}\left([0,T];\mathbb{H}^{-\eta}(\Omega)\right),$$

and time regularity result for u holds

$$t^{\alpha\vartheta}\|u(t)\|_{L^{\frac{2N}{4\mu-2}}(\Omega)} + \frac{\|u(t+\gamma)-u(t)\|_{\mathbb{H}^{\nu-\eta}(\Omega)}}{\gamma^{\eta_{cri}}} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \tag{34}$$

b) ***(Spatial regularity)*** *For each $t > 0$, $u(t)$ belongs to $W^{1,\frac{4N}{3N-4}}(\Omega)$ and satisfies the following estimate*

$$t^{\alpha\vartheta'}\left\|\nabla u(t)\right\|_{L^{\frac{4N}{3N-4}}(\Omega)} + t^{\alpha\vartheta'}\left\|(-\Delta)^{\vartheta'-\vartheta}u(t)\right\|_{\mathbb{H}^{\nu}(\Omega)} \lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}. \quad (35)$$

Proof. In order to prove this theorem, we will apply Theorem 3.4, and then improve the time and spatial regularities of the mild solution. Let us set $G(t,v) = -\rho(t)(v\cdot\nabla)v$ and show that G satisfies assumption $(\mathscr{H}_3)$ corresponding to $s = 1$. Firstly, we analyze the values of the numbers ν, σ, μ, ϑ, ϑ' as follows

- The interval $\left[\mu_2, \dfrac{N+4}{8}\right)$ is not empty since $\nu < \dfrac{N+4}{8}$ as $\nu < \dfrac{N}{4} \leq \dfrac{N+4}{8}$ (here $N \leq 4$), and

$$\frac{N+2}{4} - \nu < \frac{N+2}{4} - \frac{1}{2} < \frac{N+2}{4} - \frac{N}{8} = \frac{N+4}{8}.$$

- The numbers $\dfrac{N-4\sigma}{4\mu-2}$, $\dfrac{N-4\sigma}{N+2-4\nu}$ are greater than 1. Indeed, $\mu \geq \mu_2 \geq \nu \geq \dfrac{1}{2}$, and

$$\begin{aligned}\frac{N-4\sigma}{4\mu-2} &= \frac{N+2-4\nu}{4\mu-2} + 1 > \frac{N+2-4\left(\frac{N}{4}\right)}{4\mu-2} + 1 > 1;\\ \frac{N-4\sigma}{N+2-4\nu} &= \frac{4\mu-2}{N+2-4\nu} + 1 > 1.\end{aligned}$$

  Moreover, these are the dual numbers of each other.

Therewith, one can obtain the following chains of the Sobolev embeddings by applying (5), (6), and (7). Indeed, we have

- The Sobolev embedding $L^{\frac{2N}{N-4\sigma}}(\Omega) \hookrightarrow W^{2\sigma,2}(\Omega)$ holds as $-\dfrac{N}{2} < 2\sigma \leq 0$, $\dfrac{2N}{N-4\sigma} = \dfrac{2N}{N-2(2\sigma)}$, and $W^{2\sigma,2}(\Omega) \hookrightarrow \mathbb{H}^{\sigma}(\Omega)$ as $\sigma \leq 0$, and so that

$$L^{\frac{2N}{N-4\sigma}}(\Omega) \hookrightarrow W^{2\sigma,2}(\Omega) \hookrightarrow \mathbb{H}^{\sigma}(\Omega). \quad (36)$$

- The Sobolev embedding $\mathbb{H}^{\nu}(\Omega) \hookrightarrow W^{2\nu,2}(\Omega)$ holds as $\nu \geq 0$, and $W^{2\nu,2}(\Omega) \hookrightarrow W^{1,\frac{2N}{N+2-4\nu}}(\Omega)$ as $\nu \geq \dfrac{1}{2}$, $2\nu - 1 = \dfrac{N}{2} - N/\left(\frac{2N}{N+2-4\nu}\right)$ which implies the following Sobolev embedding

$$\mathbb{H}^{\nu}(\Omega) \hookrightarrow W^{2\nu,2}(\Omega) \hookrightarrow W^{1,\frac{2N}{N+2-4\nu}}(\Omega). \quad (37)$$

- The Sobolev embedding $W^{2\nu,2}(\Omega) \hookrightarrow L^{\frac{2N}{4\mu-2}}(\Omega)$ holds as $0 < 2\nu < \dfrac{N}{2}$, $1 \leq \dfrac{2N}{4\mu-2} \leq \dfrac{2N}{N-4\nu}$ ($\nu < \mu$), and henceforth

$$\mathbb{H}^{\nu}(\Omega) \hookrightarrow W^{2\nu,2}(\Omega) \hookrightarrow L^{\frac{2N}{4\mu-2}}(\Omega). \quad (38)$$

- The Sobolev embedding $\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega) \hookrightarrow W^{2\nu+2(\vartheta'-\vartheta),2}(\Omega) \hookrightarrow W^{2-\frac{N}{4},2}(\Omega)$ holds since $\nu + (\vartheta'-\vartheta) \geq \dfrac{1}{2} + \dfrac{4-N}{8} = 1 - \dfrac{N}{8}$ by using the assumption $\vartheta' \in \left[\vartheta + \dfrac{4-N}{8}, 1\right)$. In addition, $W^{2-\frac{N}{4},2}(\Omega) \hookrightarrow W^{1,\frac{4N}{3N-4}}(\Omega)$ as $2 - \dfrac{N}{4} \geq 1$, $2 - \dfrac{N}{4} - 1 = \frac{N}{2} - N/\left(\dfrac{4N}{3N-4}\right)$. Therefore, we obtain the following Sobolev embedding

$$\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega) \hookrightarrow W^{2\nu+2(\vartheta'-\vartheta),2}(\Omega) \hookrightarrow W^{2-\frac{N}{4},2}(\Omega) \hookrightarrow W^{1,\frac{4N}{3N-4}}(\Omega). \quad (39)$$

On account of the above embeddings and the Hölder inequality, we deduce the following chain of estimates

$$\begin{aligned}
\|G(v_1)-G(v_2)\|_{\mathbb{H}^\sigma(\Omega)} &\lesssim \rho(t)\Big(\|(v_1\cdot\nabla)(v_1-v_2)\|_{\mathbb{H}^\sigma(\Omega)}+\|((v_1-v_2)\cdot\nabla)v_2\|_{\mathbb{H}^\sigma(\Omega)}\Big)\\
&\lesssim \rho(t)\left(\|(v_1\cdot\nabla)(v_1-v_2)\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)}+\|((v_1-v_2)\cdot\nabla)v_2\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)}\right)\\
&\lesssim \rho(t)\left(\|v_1\|_{L^{\frac{2N}{N-4\sigma}\frac{N-4\sigma}{4\mu-2}}(\Omega)}\|\nabla(v_1-v_2)\|_{L^{\frac{2N}{N-4\sigma}\frac{N-4\sigma}{N+2-4\nu}}(\Omega)}\right.\\
&\qquad\left.+\|v_1-v_2\|_{L^{\frac{2N}{N-4\sigma}\frac{N-4\sigma}{4\mu-2}}(\Omega)}\|\nabla v_2\|_{L^{\frac{2N}{N-4\sigma}\frac{N-4\sigma}{N+2-4\nu}}(\Omega)}\right)\\
&= \rho(t)\left(\|v_1\|_{L^{\frac{2N}{4\mu-2}}(\Omega)}\|\nabla(v_1-v_2)\|_{L^{\frac{2N}{N+2-4\nu}}(\Omega)}\right.\\
&\qquad\left.+\|v_1-v_2\|_{L^{\frac{2N}{4\mu-2}}(\Omega)}\|\nabla v_2\|_{L^{\frac{2N}{N+2-4\nu}}(\Omega)}\right)\\
&\lesssim \rho(t)\Big(\|v_1\|_{\mathbb{H}^\nu(\Omega)}+\|v_2\|_{\mathbb{H}^\nu(\Omega)}\Big)\|v_1-v_2\|_{\mathbb{H}^\nu(\Omega)},
\end{aligned}$$

where the chain (36) has been used in the first estimate, the triangle inequality in the second estimate, the Hölder inequality with the dual numbers $\frac{N-4\sigma}{4\mu-2}$, $\frac{N-4\sigma}{N+2-4\nu}$ in the third one, the chains (37) and (38) in the last one. This means that G is really a critical nonlinearity from $\mathbb{H}^\nu(\Omega)$ to $\mathbb{H}^\sigma(\Omega)$ with respect to $s=1$ and $\mathfrak{N}(v_1,v_2)=\|v_1\|_{\mathbb{H}^\nu(\Omega)}+\|v_2\|_{\mathbb{H}^\nu(\Omega)}$. Furthermore, we can write $K(t)=\rho(t)K_{\mathrm{Bur}}$ with some positive constant K_{Bur}. Let us take ζ satisfying that

$$-b<\zeta<\min\left\{-\mu;\frac{1}{\alpha}-2\vartheta\right\},$$

then one has $K(t)\le K_0t^{-\alpha\zeta}$, where $K_0=C_\rho K_{\mathrm{Bur}}T^{\alpha(b+\zeta)}$. Due to the above arguments, we consequently conclude that G fulfills assumption $(\mathscr{H}_3)$. One can check that all numbers in this theorem obviously satisfy the assumptions of Theorem (3.4). Hence, we can apply Theorem (3.4) and the chain (38) ensures that Problem (33) has a unique mild solution

$$u\in C^{\alpha\vartheta}((0,T];L^{\frac{2N}{4\mu-2}}(\Omega))\cap C^{\eta_{cri}}\left([0,T];\mathbb{H}^{\nu-\eta}(\Omega)\right),$$

with C_ρ being small enough. Besides, the boundedness (21) and Part b of Theorem (3.4) can be combined to imply the following estimate

$$t^{\alpha\vartheta}\|u(t)\|_{L^{\frac{2N}{4\mu-2}}(\Omega)}+\frac{\|u(t+\gamma)-u(t)\|_{\mathbb{H}^{\nu-\eta}(\Omega)}}{\gamma^{\eta_{cri}}}\lesssim\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)},$$

i.e., inequality (34) is easily obtained. We now prove the spatial regularity. Indeed, Part a of Theorem (3.4) can be rewritten as $(-\Delta)^{\vartheta'-\vartheta}u(t)\in\mathbb{H}^\nu(\Omega)$ with respect to the estimate

$$\left\|(-\Delta)^{\vartheta'-\vartheta}u(t)\right\|_{L^{\frac{2N}{4\mu-2}}(\Omega)}\lesssim\left\|(-\Delta)^{\vartheta'-\vartheta}u(t)\right\|_{\mathbb{H}^\nu(\Omega)}\lesssim t^{-\alpha\vartheta'}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}.$$

On the other hand, by using the chain (39), we deduce that

$$\left\|\nabla u(t)\right\|_{L^{\frac{4N}{3N-4}}(\Omega)}\lesssim\left\|(-\Delta)^{\vartheta'-\vartheta}u(t)\right\|_{\mathbb{H}^\nu(\Omega)}\lesssim t^{-\alpha\vartheta'}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)},$$

which implies inequality (35). □

Remark 4.2. *Considering $N\notin\{2;3;4\}$ for time fractional Burgers equation is not an easy task by the following reasons: requirements for the numbers $\nu,\mu,\vartheta,\vartheta'$ and applying Sobolev embeddings. In the future, we will develop our method to deal with the cases $N\notin\{2;3;4\}$.*

5. Proof of Theorem 3.2, Theorem 3.3, and Theorem 3.4

In this section, we provide full proofs for Theorem 3.2, Theorem 3.3, and Theorem 3.4. In Subsection 5.1, we prove Theorem 3.2 by using some new techniques of the Picard approximation method. In Subsection 5.2, we show Theorem 3.3 by applying Banach fixed point theorem. And we end the section by proving Theorem 3.4 in Subsection 5.3. For the sake of convenience, some important constants, which will be used in the proofs, will be listed in part **(AP.4)** of the Appendix.

5.1. **Proof of Theorem 3.2.** Let us begin with the proof of Theorem 3.2 by using Picard's approximation method. We construct a Picard sequence defined by Lemma 5.1. With some appropriate assumptions, we will bound the sequence by a power function. Then, we can prove it is a Cauchy sequence in a Banach space as Lemma 5.2. Now, we consider two following lemmas.

Lemma 5.1. *Let the Picard sequence* $\{w^{(k)}\}_{k=1,2,...}$ *be defined by* $w^{(1)}(t) = f$, *and*

$$w^{(k+1)}(t) = \mathbf{B}_\alpha(t,T)f + \int_0^t \mathbf{P}_\alpha(t-r)G(r,w^{(k)}(r))dr - \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,w^{(k)}(r))dr, \quad 0 \le t \le T. \tag{40}$$

Then, for all $t>0$, $k \in \mathbb{N}$, $k \ge 1$, *it holds*

$$\left\| w^{(k)}(t) \right\|_{\mathbb{H}^\nu(\Omega)} \le \mathcal{N}_1 t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\theta+\nu}(\Omega)}. \tag{41}$$

where $\mathcal{N}_1$ *is given by* ***(AP.4.)*** *in the Appendix.*

Lemma 5.2. *Let* $\left\{u^{(k)}\right\}_{k=1,2,...}$ *be the sequence defined by Lemma 5.1, then it is a bounded and Cauchy sequence in the Banach space* $L^p(0,T;\mathbb{H}^\nu(\Omega))$ *with* $p \in \left[1, \frac{1}{\alpha(1-\theta)}\right)$.

Proof of Lemma 5.1. Let us consider the case $k=1$. Firstly, $\|f\|_{\mathbb{H}^\nu(\Omega)}$ is bounded by $C_1(\nu,\theta)\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}$ upon the embedding (15). Furthermore, it is straightforward from **(AP.4.)** in the Appendix that $\mathcal{N}_1 \ge C_1(\nu,\theta)t^{\alpha(1-\theta)}$ by noting the number $\alpha(1-\theta)$ is contained in the interval (0,1). These easily imply the desired inequality (41) for $k=1$. Assume that (41) holds for $k=n$. This means that

$$\left\| w^{(n)}(t) \right\|_{\mathbb{H}^\nu(\Omega)} \le \mathcal{N}_1 t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{42}$$

We show that (41) holds for $k = n+1$. Thanks to definition (11), using the fact that $\{\varphi_j\}$ is an orthonormal basis of $L^2(\Omega)$, and then using Lemma 2.4, one arrives at

$$\begin{aligned}
\left\| \mathbf{B}_\alpha(t,T)f \right\|_{\mathbb{H}^\nu(\Omega)} &\le M_\alpha m_\alpha^{-1} T^{\alpha(1-\theta)} (T^{\alpha\theta} + \lambda_1^{-\theta}) t^{-\alpha(1-\theta)} \left(\sum_{j=1}^\infty \lambda_j^{2\nu} \langle f, \varphi_j \rangle^2 \right)^{1/2} \\
&= M_\alpha m_\alpha^{-1} T^{\alpha(1-\theta)} \left(T^{\alpha\theta} + \lambda_1^{-\theta}\right) t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^\nu(\Omega)} \\
&\le M_\alpha m_\alpha^{-1} T^{\alpha(1-\theta)} \left(T^{\alpha\theta} + \lambda_1^{-\theta}\right) t^{-\alpha(1-\theta)} C_1(\nu,\theta) \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},
\end{aligned}$$

where we have used the Sobolev embedding $\mathbb{H}^{\nu+\theta}(\Omega) \hookrightarrow \mathbb{H}^\nu(\Omega)$. Next, let us estimate the integrals by using assumption $(\mathscr{H}_1)$. The idea is to try to bound them by the convergent improper integrals. Indeed, one can show that

$$\begin{aligned}
\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,w^{(n)}(r))dr \right\|_{\mathbb{H}^\nu(\Omega)} &\le \int_0^t \left\| \sum_{j=1}^\infty (t-r)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha) G_j(r,w^{(n)}(r))\varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\le M_\alpha \lambda_1^{-\theta} \int_0^t (t-r)^{\alpha(1-\theta)-1} \left\| G(r,w^{(n)}(r)) \right\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\le \|L_1\|_{L^\infty(0,T)} M_\alpha \lambda_1^{-\theta} \int_0^t (t-r)^{\alpha(1-\theta)-1} \left\| w^{(n)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr, \quad (43)
\end{aligned}$$

where we denote $G_j(r,w^{(n)}(r)) = \langle G(r,w^{(n)}(r)), \varphi_j \rangle$. On the other hand, the quantity $\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha)(T-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)$ is obviously bounded by $\mathcal{M}_1(T-r)^{\alpha(1-\theta)-1}t^{-\alpha(1-\theta)}$ due to Lemma 2.4. Here, the constant $\mathcal{M}_1$ is given by **(AP.4)** in the Appendix. We then obtain the following estimate

$$\begin{aligned}
&\left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,w^{(n)}(r))dr \right\|_{\mathbb{H}^\nu(\Omega)} \\
&\le \int_0^T \left\| \sum_{j=1}^\infty (T-r)^{\alpha-1} \mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha) E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) G_j(r,w^{(n)}(r))\varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\le \|L_1\|_{L^\infty(0,T)} \mathcal{M}_1 t^{-\alpha(1-\theta)} \int_0^T (T-r)^{\alpha(1-\theta)-1} \left\| w^{(n)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr, \quad (44)
\end{aligned}$$

where we note that the constant $\mathcal{M}_1$ is given by **(AP.4.)** in the Appendix. According to the above inequalities, we need to estimate the integral $\int_0^t (t-r)^{\alpha(1-\theta)-1} \left\| w^{(n)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr$. To do this, we will apply the inductive hypothesis (42). Moreover, by also using the fact that $1 \leq T^{\alpha(1-\theta)} t^{-\alpha(1-\theta)}$ for all $0 < t \leq T$, and $\int_0^t (t-r)^{\alpha(1-\theta)-1} r^{-\alpha(1-\theta)} dr$ is equal to $\pi / \sin(\pi\alpha(1-\theta))$, we consequently obtain the following estimates

$$\int_0^t (t-r)^{\alpha(1-\theta)-1} \left\| w^{(n)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr \leq \mathcal{N}_1 \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)} \int_0^t (t-r)^{\alpha(1-\theta)-1} r^{-\alpha(1-\theta)} dr$$
$$\leq \mathcal{N}_1 \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)} \frac{\pi T^{\alpha(1-\theta)}}{\sin(\pi\alpha(1-\theta))} t^{-\alpha(1-\theta)}. \tag{45}$$

Here, in the last equality, we use **(AP.1.)** in the Appendix. By similar arguments as above, we obtain

$$\left\| \int_0^t \mathbf{P}_\alpha(t-r) G(r, w^{(n)}(r)) dr + \int_0^T \mathbf{B}_\alpha(t,T) \mathbf{P}_\alpha(T-r) G(r, w^{(n)}(r)) dr \right\|_{\mathbb{H}^\nu(\Omega)}$$
$$\leq \|L_1\|_{L^\infty(0,T)} \mathscr{M}_1 \mathcal{N}_1 t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{46}$$

From some preceding estimates and by some simple computations, we can find that

$$\left\| w^{(n+1)}(t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq \left\| \mathbf{B}_\alpha(t,T) f \right\|_{\mathbb{H}^\nu(\Omega)} + \left\| \int_0^t \mathbf{P}_\alpha(t-r) G(r, w^{(n)}(r)) dr \right\|_{\mathbb{H}^\nu(\Omega)}$$
$$+ \left\| \int_0^T \mathbf{B}_\alpha(t,T) \mathbf{P}_\alpha(T-r) G(r, w^{(n)}(r)) dr \right\|_{\mathbb{H}^\nu(\Omega)}$$
$$\leq \Big(\mathcal{M}_1 M_\alpha^{-1} C_1(\nu,\theta) + \|L_1\|_{L^\infty(0,T)} \mathscr{M}_1 \mathcal{N}_1 \Big) t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}$$
$$= \mathcal{N}_1 t^{-\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

By the induction method, we deduce that (42) holds for any $k = n \in \mathbb{N}$, $n \geq 1$. □

Proof of Lemma 5.2. Since $p \in [1, \frac{1}{\alpha(1-\theta)})$, we know that the function $t \mapsto t^{-\alpha(1-\theta)}$ is $L^p(0,T)$-integrable which implies that $\{u^{(n)}\}$ is a bounded sequence in $L^p(0,T;\mathbb{H}^\nu(\Omega))$. Hence, it is necessary to prove $\{u^{(n)}\}$ is a Cauchy sequence. By using the notation $\mathbf{u}^{(n,k)} := u^{(n+k)} - u^{(n)}$ and using triangle inequality, one has the following estimate

$$\left\| \mathbf{u}^{(n+1,k)}(t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq \|L_1\|_{L^\infty(0,T)} M_\alpha \lambda_1^{-\theta} \int_0^t (t-r)^{\alpha(1-\theta)-1} \left\| \mathbf{u}^{(n,k)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr$$
$$+ \|L_1\|_{L^\infty(0,T)} \mathcal{M}_1 t^{-\alpha(1-\theta)} \int_0^T (T-r)^{\alpha(1-\theta)-1} \left\| \mathbf{u}^{(n,k)}(r) \right\|_{\mathbb{H}^\nu(\Omega)} dr.$$

Similarly to the proof of Lemma 5.1, one can use the inductive hypothesis to estimate the above right hand side. Then by iterating the same computations in Lemma 5.1, we can bound $\|\mathbf{u}^{(n+1,k)}(t)\|_{\mathbb{H}^\nu(\Omega)}$ by the quantity $2\mathcal{N}_1 (\|L_1\|_{L^\infty(0,T)} \mathscr{M}_1)^n t^{-\alpha(1-\theta)}$. Summarily, one can obtain the following conclusion by the induction method

$$\left\| \mathbf{u}^{(n,k)}(t) \right\|_{\mathbb{H}^\nu(\Omega)} \leq 2\mathcal{N}_1 \Big(\|L_1\|_{L^\infty(0,T)} \mathscr{M}_1 \Big)^{n-1} t^{-\alpha(1-\theta)}, \tag{47}$$

which completed the proof by letting $n \to \infty$. □

Proof of Theorem 3.2. We firstly prove the existence of a mild solution u in the space $L^p(0,T;\mathbb{H}^\nu(\Omega))$, and then obtain its continuity in the following steps (1 and 2). After that, we will present the proofs the Parts a - d in the sequel.

Step 1: Prove the existence of a mild solution u in the space $L^p(0,T;\mathbb{H}^\nu(\Omega))$: Since $L^p(0,T;\mathbb{H}^\nu(\Omega))$ is a Banach space and $\{u^{(n)}\}_{n=1,2,\ldots}$ is a Cauchy sequence in $L^p(0,T;\mathbb{H}^\nu(\Omega))$. Thanks to Lemmas 5.1, 5.2, we deduce that there exists a function $u \in L^p(0,T;\mathbb{H}^\nu(\Omega))$ such that $\lim_{n\to+\infty} u^{(n)} = u$. Now, we show that u is a mild solution of the problem by showing that $u = \overline{\mathbf{J}}u$, where

$$\overline{\mathbf{J}}u(t) := \mathbf{B}_\alpha(t,T) f + \int_0^t \mathbf{P}_\alpha(t-r) G(r, u(r)) dr - \int_0^T \mathbf{B}_\alpha(t,T) \mathbf{P}_\alpha(T-r) G(r, u(r)) dr. \tag{48}$$

Since the sequence $\{u^{(n)}\}$ converges to u in $L^p(0,T;\mathbb{H}^\nu(\Omega))$-norm, there exists a sub-sequence $\{u^{(n_m)}\}$ that point-wise converges to u, i.e., $u^{(n_m)}(t) \longrightarrow u(t)$ in $\mathbb{H}^\nu(\Omega)$-norm for almost everywhere t in $(0,T)$. Let us denote by $\mathbf{v}^{(n_m)} := u^{(n_m)} - u$. This fact and taking $k \to \infty$ in the estimate (47) allow us to obtain

$$\left\|\mathbf{v}^{(n_m)}\right\|_{\mathbb{H}^\nu(\Omega)} \le 2\mathcal{N}_1\Big(\|L_1\|_{L^\infty(0,T)}\mathscr{M}_1\Big)^{n_m-1} t^{-\alpha(1-\theta)}, \tag{49}$$

for almost everywhere t in $(0,T)$. Moreover, we note from Lemma 3.3 that this sub-sequence is also bounded by the power function $t \mapsto t^{-\alpha(1-\theta)}$. These help us to apply dominated convergence theorem as follows. Indeed, it is obvious that the quantities

$$\int_0^t (t-r)^{\alpha(1-\theta)-1}\left\|\mathbf{v}^{(n_m)}\right\|_{\mathbb{H}^\nu(\Omega)}dr, \quad t^{-\alpha(1-\theta)}\int_0^T (T-r)^{\alpha(1-\theta)-1}\left\|\mathbf{v}^{(n_m)}\right\|_{\mathbb{H}^\nu(\Omega)}dr,$$

point-wise converge to zero as (49), and are bounded by $L^p(0,T)$-integrable functions. Thus, the same computations as (43) show

$$\begin{aligned}
&\left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,u^{(n_m)}(r)) - G(r,u(r))\Big)dr\right\|^p_{L^p(0,T;\mathbb{H}^\nu(\Omega))}\\
&\quad = \int_0^T \left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,u^{(n_m)}(r)) - G(r,u(r))\Big)dr\right\|^p_{\mathbb{H}^\nu(\Omega)} dt\\
&\quad \le \int_0^T \left\{\int_0^t (t-r)^{\alpha(1-\theta)-1}\left\|\mathbf{v}^{(n_m)}\right\|_{\mathbb{H}^\nu(\Omega)}dr\right\}^p dt,
\end{aligned} \tag{50}$$

and by reasoning similarly as in (44), we have the following bound

$$\begin{aligned}
&\left\|\int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\Big(G(r,u^{(n_m)}(r)) - G(r,u(r))\Big)dr\right\|^p_{L^p(0,T;\mathbb{H}^\nu(\Omega))}\\
&\quad = \int_0^T \left\|\int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\Big(G(r,u^{(n_m)}(r)) - G(r,u(r))\Big)dr\right\|^p_{\mathbb{H}^\nu(\Omega)} dt\\
&\quad \le \int_0^T \left\{t^{-\alpha(1-\theta)}\int_0^T (T-r)^{\alpha(1-\theta)-1}\left\|\mathbf{v}^{(n_m)}\right\|_{\mathbb{H}^\nu(\Omega)}dr\right\}^p dt.
\end{aligned} \tag{51}$$

The right hand-side of (50) and (51) tend to zero when n_m goes to positive infinity. The above arguments conclude that u satisfies $u = \overline{\mathbf{J}}u$, and thus is a mild solution of Problem (1)-(2) in $L^p(0,T;\mathbb{H}^\nu(\Omega))$. By taking the limit of the left hand side of (41) (with respect to $\{u^{(n_m)}\}$), we obtain

$$\|u\|_{L^p(0,T;\mathbb{H}^\nu(\Omega))} \lesssim \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{52}$$

Step 2: Prove $u \in C^{\alpha(1-\theta)}\big((0,T];\mathbb{H}^\nu(\Omega)\big)$: We need to estimate $u(\widetilde{t}) - u(t)$ in $\mathbb{H}^\nu(\Omega)$ norm, for all $0 < t \le \widetilde{t} \le T$. By the formulation (10), the triangle inequality yields that

$$\begin{aligned}
\left\|u(\widetilde{t}) - u(t)\right\|_{\mathbb{H}^\nu(\Omega)} &\le \left\|\Big(\mathbf{B}_\alpha(\widetilde{t},T) - \mathbf{B}_\alpha(t,T)\Big)f\right\|_{\mathbb{H}^\nu(\Omega)}\\
&\quad + \left\|\int_0^t \Big(\mathbf{P}_\alpha(\widetilde{t}-r) - \mathbf{P}_\alpha(t-r)\Big)G(r,u(r))dr\right\|_{\mathbb{H}^\nu(\Omega)}\\
&\quad + \left\|\int_t^{\widetilde{t}} \mathbf{P}_\alpha(\widetilde{t}-r)G(r,u(r))dr\right\|_{\mathbb{H}^\nu(\Omega)}\\
&\quad + \left\|\int_0^T \Big(\mathbf{B}_\alpha(\widetilde{t},T) - \mathbf{B}_\alpha(t,T)\Big)\mathbf{P}_\alpha(T-r)G(r,u(r))dr\right\|_{\mathbb{H}^\nu(\Omega)}\\
&:= \|\mathfrak{M}_1\|_{\mathbb{H}^\nu(\Omega)} + \|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} + \|\mathfrak{M}_3\|_{\mathbb{H}^\nu(\Omega)} + \|\mathfrak{M}_4\|_{\mathbb{H}^\nu(\Omega)}.
\end{aligned} \tag{53}$$

In what follows, we will estimate the terms $\mathfrak{M}_j$ for $1 \le j \le 4$.

Estimate of $\mathfrak{M}_1$: Using the fact that $\partial_t E_{\alpha,1}(-\lambda_j t^\alpha) = -\lambda_j t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)$, we find that

$$\frac{E_{\alpha,1}(-\lambda_j\widetilde{t}^\alpha) - E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} = \int_t^{\widetilde{t}} -\lambda_j r^{\alpha-1}\frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}dr.$$

The latter inequality leads to

$$\begin{aligned}\left\|\Big(\mathbf{B}_\alpha(\widetilde{t},T)-\mathbf{B}_\alpha(t,T)\Big)f\right\|_{\mathbb{H}^\nu(\Omega)} &\lesssim \int_t^{\widetilde{t}}\left\|\sum_{j\in\mathbb{N}}\lambda_j r^{\alpha-1}\frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}f_j\varphi_j\right\|_{\mathbb{H}^\nu(\Omega)}dr\\ &\lesssim \left(\int_t^{\widetilde{t}} r^{\alpha(\theta-1)-1}dr\right)\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}\\ &= \frac{\widetilde{t}^{\alpha(1-\theta)}-t^{\alpha(1-\theta)}}{\alpha(1-\theta)t^{\alpha(1-\theta)}\widetilde{t}^{\alpha(1-\theta)}}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},\end{aligned}\tag{54}$$

where we have used the fact that

$$\lambda_j r^{\alpha-1}\frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \lesssim \lambda_j r^{\alpha-1}\frac{1+\lambda_j T^\alpha}{1+(\lambda_j r^\alpha)^2} \lesssim \lambda_j^\theta r^{\alpha(\theta-1)-1}.\tag{55}$$

By noting that $0<\alpha(1-\theta)<1$, we now have $\widetilde{t}^{\alpha(1-\theta)}-t^{\alpha(1-\theta)}\le(\widetilde{t}-t)^{\alpha(1-\theta)}$ and furthermore $\alpha(1-\theta)t^{\alpha(1-\theta)}\widetilde{t}^{\alpha(1-\theta)}\ge\alpha(1-\theta)t^{2\alpha(1-\theta)}$. Consequently, thanks to the estimate (54), we arrive at

$$\|\mathfrak{M}_1\|_{\mathbb{H}^\nu(\Omega)}\lesssim t^{-2\alpha(1-\theta)}(\widetilde{t}-t)^{\alpha(1-\theta)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.\tag{56}$$

Estimate of $\mathfrak{M}_2$: It follows by differentiating $\partial_\rho\Big(\rho^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j\rho^\alpha)\Big)=\rho^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_j\rho^\alpha)$, for all $\rho>0$ (see Lemma 2.2), that

$$\begin{aligned}&\int_0^t\left((\widetilde{t}-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(\widetilde{t}-r)^\alpha)-(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)\right)G_j(r,u(r))dr\\ &\quad=\int_0^t\int_{t-r}^{\widetilde{t}-r}\rho^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_j\rho^\alpha)G_j(r,u(r))d\rho dr.\end{aligned}\tag{57}$$

This implies that

$$\begin{aligned}\|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} &\lesssim \int_0^t\int_{t-r}^{\widetilde{t}-r}\rho^{\alpha-2}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}d\rho dr\\ &\lesssim (\widetilde{t}-t)^{\alpha-1}\int_0^t\|u(r)\|_{\mathbb{H}^\nu(\Omega)}dr\lesssim(\widetilde{t}-t)^{\alpha-1}\|u\|_{L^p(0,T;\mathbb{H}^\nu(\Omega))}\\ &\lesssim(\widetilde{t}-t)^{\alpha-1}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},\end{aligned}\tag{58}$$

where we note that

$$\int_{t-r}^{\widetilde{t}-r}\rho^{\alpha-2}d\rho=\frac{(\widetilde{t}-r)^{\alpha-1}-(t-r)^{\alpha-1}}{\alpha-1}\le\frac{(\widetilde{t}-t)^{\alpha-1}}{\alpha-1},$$
$$\rho^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_j\rho^\alpha)\lesssim\rho^{\alpha-2}\frac{1}{1+\lambda_j\rho^\alpha}\lesssim\rho^{\alpha-2}.$$

Estimate of $\mathfrak{M}_3$: For all $t<r<\widetilde{t}$, using the inequality $|E_{\alpha,\alpha}(-\lambda_j(\widetilde{t}-r)^\alpha)|\le M_\alpha$ and the fact that $(\widetilde{t}-r)^{\alpha-1}\le(\widetilde{t}-t)^{\alpha-1}$ show

$$\left\|\mathfrak{M}_3\right\|_{\mathbb{H}^\nu(\Omega)}\le M_\alpha(\widetilde{t}-t)^{\alpha-1}\int_t^{\widetilde{t}}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}dr\lesssim(\widetilde{t}-t)^{\alpha-1}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.\tag{59}$$

Estimate of $\mathfrak{M}_4$: Using (55), we obtain

$$\begin{aligned}&\left\|\sum_{j=1}^\infty(T-r)^{\alpha-1}\lambda_j\rho^{\alpha-1}\frac{E_{\alpha,\alpha}(-\lambda_j\rho^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)G_j(r,u(r))\varphi_j\right\|_{\mathbb{H}^\nu(\Omega)}\\ &\qquad\lesssim(T-r)^{\alpha(1-\theta)-1}\rho^{\alpha(1-\theta)-1}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}.\end{aligned}$$

By using the fact that $\widetilde{t}^{\alpha(1-\theta)} - t^{\alpha(1-\theta)} \le (\widetilde{t} - t)^{\alpha(1-\theta)}$, and $t^{\alpha(1-\theta)}\widetilde{t}^{\alpha(1-\theta)} \ge t^{2\alpha(1-\theta)}$, we derive that

$$
\begin{aligned}
\|\mathfrak{M}_4\|_{\mathbb{H}^\nu(\Omega)} &\lesssim \int_0^T \int_t^{\widetilde{t}} (T-r)^{\alpha(1-\theta)-1} \xi^{\alpha(\theta-1)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} d\xi dr \\
&\lesssim \frac{\widetilde{t}^{\alpha(1-\theta)} - t^{\alpha(1-\theta)}}{\alpha(1-\theta)t^{\alpha(1-\theta)}\widetilde{t}^{\alpha(1-\theta)}} \int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\lesssim t^{-2\alpha(1-\theta)}(\widetilde{t}-t)^{\alpha(1-\theta)} \int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr. \qquad (60)
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(u(r))\|_{\mathbb{H}^\nu(\Omega)} &\lesssim \int_0^T (T-r)^{\alpha(1-\theta)-1} \|u(r)\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\lesssim \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)} \int_0^T (T-r)^{\alpha(1-\theta)-1} r^{-\alpha(1-\theta)} dr \\
&\lesssim \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.
\end{aligned}
$$

The latter estimate together with (60) lead to

$$
\|\mathfrak{M}_4\|_{\mathbb{H}^\nu(\Omega)} \lesssim t^{-2\alpha(1-\theta)}(\widetilde{t}-t)^{\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \qquad (61)
$$

<u>*Obtaining the estimate for $u(\widetilde{t}) - u(t)$:*</u> Combining (53), (56), (58), (59), (61) leads us to

$$
\|u(\widetilde{t}) - u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim \left[t^{-2\alpha(1-\theta)}(\widetilde{t}-t)^{\alpha(1-\theta)} + (\widetilde{t})^{\alpha-1}\right] \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},
$$

which implies that $u \in C^{\alpha(1-\theta)}\big((0,T];\mathbb{H}^\nu(\Omega)\big)$, and the inequality (16) can be obtained by using (52) .

In what follows, we carry out the proofs of Part (a), (b), (c), (d).

<u>*Part (a).*</u> *Prove $u \in L^p(0,T;\mathbb{H}^{\nu+\theta-\theta'}(\Omega))$, for any $p \in \left[1, \frac{1}{\alpha(1-\theta')}\right)$.*

Lemma 2.4 yields the estimate

$$
\begin{aligned}
\left\|\mathbf{B}_\alpha(t,T)f\right\|^2_{\mathbb{H}^{\nu+\theta-\theta'}(\Omega)} &= \sum_{j=1}^\infty \lambda_j^{2\nu+2\theta-2\theta'} \left|\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha)\right|^2 \langle f,\varphi_j\rangle^2 \\
&\lesssim t^{-2\alpha(1-\theta')} \sum_{j=1}^\infty \lambda_j^{2\nu+2\theta-2\theta'} \lambda_j^{2\theta'} \langle f,\varphi_j\rangle^2. \qquad (62)
\end{aligned}
$$

Therefore, $\mathbf{B}_\alpha(t,T)f \in L^p(0,T;\mathbb{H}^{\nu+\theta-\theta'}(\Omega))$. Further, we infer from assumption $(\mathscr{H}_1)$ and the estimate $(t-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha) \lesssim \lambda_j^{-(\theta-\theta')}(t-r)^{\alpha(1-(\theta-\theta'))-1}$ as in Lemma 2.1

$$
\begin{aligned}
\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,u(r))dr \right\|_{\mathbb{H}^{\nu+\theta-\theta'}(\Omega)} &\lesssim \int_0^t (t-r)^{\alpha(1-\theta+\theta')-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\lesssim \int_0^t (t-r)^{\alpha(1-\theta+\theta')-1} r^{-\alpha(1-\theta)} dr \lesssim t^{-\alpha(1-\theta')}, \qquad (63)
\end{aligned}
$$

where we used the definition of the Beta function as **(AP.1.)** in the Appendix, and the fact that $t^{\alpha\theta'} = t^{-\alpha(1-\theta')}t^\alpha \lesssim t^{-\alpha(1-\theta')}$. Now, we proceed to estimate the last term of u. Lemma 3.2 yields that

$$
\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha)(T-r)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha) \lesssim \lambda_j^{\theta'-\theta} t^{-\alpha(1-\theta')}(T-r)^{\alpha(1-\theta)-1}.
$$

This invokes from assumption $(\mathscr{H}_1)$ that

$$\left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr \right\|_{\mathbb{H}^{\nu+\theta-\theta'}(\Omega)}$$
$$\lesssim t^{-\alpha(1-\theta')} \int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr$$
$$\lesssim t^{-\alpha(1-\theta')} \int_0^T (T-r)^{\alpha(1-\theta)-1} r^{-\alpha(1-\theta)} dr \lesssim t^{-\alpha(1-\theta')}. \tag{64}$$

Taking the above estimates (62), (63), (64) together, we imply that $u \in L^p(0,T;\mathbb{H}^{\nu+\theta-\theta'}(\Omega))$, for all $p \in \left[1, \frac{1}{\alpha(1-\theta')}\right)$, and complete this part.

<u>*Part (b).*</u> *Show that* $u \in C\left([0,T];\mathbb{H}^{\nu-\nu'}(\Omega)\right)$.

Let t and $\widetilde{t}$ be such that $0 \le t < \widetilde{t} \le T$. By using equation (53) and the embedding $\mathbb{H}^\nu(\Omega) \hookrightarrow \mathbb{H}^{\nu-\nu'}(\Omega)$, we obtain

$$\|u(\widetilde{t}) - u(t)\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$$
$$\le \|\mathfrak{M}_1\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} + \|\mathfrak{M}_2\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} + \|\mathfrak{M}_3\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} + \|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$$
$$\lesssim \|\mathfrak{M}_1\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} + |\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} + \|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} + \|\mathfrak{M}_3\|_{\mathbb{H}^\nu(\Omega)}. \tag{65}$$

For the right-hand side of (53), we thus need to estimate the terms $\|\mathfrak{M}_1\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$, $\|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$. We now continue to consider the following estimates.

Estimate $\|\mathfrak{M}_1\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$: It is easy to show that

$$\|\mathfrak{M}_1\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} \lesssim \frac{(\widetilde{t}-t)^{\alpha(\theta+\nu'-1)}}{\alpha(\theta+\nu'-1)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

Estimate $\|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)}$: By a similar argument as in (61), we obtain

$$\|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} \lesssim \int_0^T \int_t^{\widetilde{t}} (T-r)^{\alpha(1-\theta)-1} \rho^{\alpha(\theta+\nu'-1)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} d\rho dr. \tag{66}$$

Since $0 < \alpha(\theta+\nu'-1) < 2$, we split it into the two following cases:
Case 1. If $0 < \alpha(\theta+\nu'-1) \le 1$ then apply $(a+b)^\sigma \le a^\sigma + b^\sigma,\ a,b \ge 0, 0 < \sigma < 1$, we have

$$(\widetilde{t})^{\alpha(\theta+\nu'-1)} - t^{\alpha(\theta+\nu'-1)} \le (\widetilde{t}-t)^{\alpha(\theta+\nu'-1)}. \tag{67}$$

From two latter observations, we find that

$$\|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} \lesssim \int_0^T \int_t^{\widetilde{t}} (T-r)^{\alpha(1-\theta)-1} \rho^{\alpha(\theta+\nu'-1)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} d\rho dr$$
$$\lesssim \frac{(\widetilde{t})^{\alpha(\theta+\nu'-1)} - t^{\alpha(\theta+\nu'-1)}}{\alpha(\theta+\nu'-1)} \int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr$$
$$\lesssim h^{\alpha(\theta+\nu'-1)} \int_0^T (T-r)^{\alpha(1-\theta)-1} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} dr$$
$$\lesssim h^{\alpha(\theta+\nu'-1)} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{68}$$

Case 2. If $1 < \alpha(\theta+\nu'-1) \le 2$ then, since $0 \le t \le \widetilde{t} \le T$, we have

$$(\widetilde{t})^{\alpha(\theta+\nu'-1)} - t^{\alpha(\theta+\nu'-1)}$$
$$= \left[(\widetilde{t})^{\alpha(\theta+\nu'-1)} - t^{\alpha(\theta+\nu'-1)-1}(\widetilde{t})\right] + \left[t^{\alpha(\theta+\nu'-1)-1}\widetilde{t} - t^{\alpha(\theta+\nu'-1)}\right]$$
$$= \widetilde{t}\left[(\widetilde{t})^{\alpha(\theta+\nu'-1)-1} - t^{\alpha(\theta+\nu'-1)-1}\right] + (\widetilde{t}-t)t^{\alpha(\theta+\nu'-1)-1}$$
$$\le \max\left(T;\ T^{\alpha(\theta+\nu'-1)-1}\right)\left[(\widetilde{t}-t)^{\alpha(\theta+\nu'-1)-1} + (\widetilde{t}-t)\right]. \tag{69}$$

Therefore

$$\|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\nu'}(\Omega)} \lesssim \left\{ \begin{array}{l} (\widetilde{t}-t)^{\alpha(\theta+\nu'-1)}\mathbf{1}_{0<\alpha(\theta+\nu'-1)\le 1} \\ \left((\widetilde{t}-t)^{\alpha(\theta+\nu'-1)-1}+\gamma\right)\mathbf{1}_{1<\alpha(\theta+\nu'-1)<2} \end{array} \right\} \|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{70}$$

Collecting the results (65), (66),(58), (59), (68) and (70), we deduce that $u \in C([0,T];\mathbb{H}^{\nu-\nu'}(\Omega))$ and finish the desired inequality.

Part (c). Show that

$$\|\partial_t u(t)\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

In order to establish the result, let us define the following projection operator, for any $v=\sum_{j=1}^{\infty}\langle v,\varphi_j\rangle\varphi_j$ and any $M>0$,

$$\mathcal{P}_M v := \sum_{j=1}^{M}\langle v,\varphi_j\rangle\varphi_j$$

and the two following operator

$$\mathscr{D}_{1,\alpha}(t,T)v := \sum_{j=1}^{\infty}\frac{-\lambda_j t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\langle v,\varphi_j\rangle\varphi_j, \tag{71}$$

$$\mathscr{D}_{2,\alpha}(t)v := \sum_{j=1}^{\infty} t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_j t^\alpha)\langle v,\varphi_j\rangle\varphi_j. \tag{72}$$

Noting that $\mathcal{P}_M$ has finite rank, we have the following equality after some simple computations

$$\begin{aligned}\partial_t\mathcal{P}_M u(t) = \mathscr{D}_{1,\alpha}(t,T)\mathcal{P}_M f &+ \int_0^t \mathscr{D}_{2,\alpha}(t-r)\mathcal{P}_M G(r,u(r))dr \\ &- \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)\mathcal{P}_M G(r,u(r))dr.\end{aligned} \tag{73}$$

One can infer from $0\le\nu_1\le 1-\theta$ that $0\le\nu_1<\dfrac{2\alpha-1}{\alpha}-\theta$. Hence, this can be associated with $\dfrac{\alpha-1}{\alpha}<\theta<1$ that $1<\theta+\nu_1+\dfrac{1}{\alpha}<2$, and this implies that

$$\begin{aligned} t^{\alpha-1}\left|\frac{E_{\alpha,\alpha}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\right| &\lesssim t^{\alpha-1}\left(\frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha}\right)^{2-\theta-\nu_1-\frac{1}{\alpha}}\left(\frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha}\right)^{\frac{1}{\alpha}+\nu_1+\theta-1} \\ &\lesssim t^{-\alpha(1-\theta-\nu_1)}\lambda_j^{\frac{1}{\alpha}+\nu_1+\theta-1}. \end{aligned} \tag{74}$$

It is easy to see that

$$\begin{aligned}\left\|\mathscr{D}_{1,\alpha}(t,T)\Big(\mathcal{P}_{M'}-\mathcal{P}_M\Big)f\right\|^2_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} &= \sum_{j=M+1}^{M'}\lambda_j^{2\nu-2\nu_1-\frac{2}{\alpha}}\left|\frac{-\lambda_j t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\langle f,\varphi_j\rangle\right|^2 \\ &\lesssim t^{-2\alpha(1-\theta-\nu_1)}\sum_{j=M+1}^{M'}\lambda_j^{2\nu+2\theta}\left|\langle f,\varphi_j\rangle\right|^2.\end{aligned} \tag{75}$$

On the other hand,

$$\begin{aligned}&\left\|\int_0^t \mathscr{D}_{2,\alpha}(t-r)\Big(\mathcal{P}_{M'}-\mathcal{P}_M\Big)G(r,u(r))dr\right\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \\ &\qquad\lesssim \int_0^t (t-r)^{\alpha-2}\left(\sum_{j=M+1}^{M'}\lambda_j^{2\nu-2\nu_1-\frac{2}{\alpha}}G_j^2(r,u(r))\right)^{1/2}dr \\ &\qquad\lesssim \int_0^t (t-r)^{\alpha-2}\left(\sum_{j=M+1}^{M'}\lambda_j^{2\nu}G_j^2(r,u(r))\right)^{1/2}dr.\end{aligned} \tag{76}$$

Now, let us estimate the third term on the right hand side of (73). We see that

$$\left\| \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)\Big(\mathcal{P}_{M'}-\mathcal{P}_M\Big)G(r,u(r))dr \right\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)}$$

$$\lesssim \int_0^T (T-r)^{\alpha-1}\left(\sum_{j=M+1}^{M'} \lambda_j^{2\nu-2\nu_1-\frac{2}{\alpha}} \left| t^{-\alpha(1-\theta-\nu_1)}\lambda_j^{\frac{1}{\alpha}+\nu_1+\theta}\lambda_j^{-\theta}(T-r)^{-\alpha\theta}G_j(r,v(r))\right|^2 \right)^{\frac{1}{2}} dr$$

$$\lesssim t^{-\alpha(1-\theta-\nu_1)}\int_0^T (T-r)^{\alpha(1-\theta)-1}\left(\sum_{j=M+1}^{M'} \lambda_j^{2\nu}G_j^2(r,v(r))\right)^{\frac{1}{2}} dr, \tag{77}$$

where we have used the estimates $\Big|E_{\alpha,\alpha}(-\lambda_j(T-r)^\alpha)\Big| \lesssim \lambda_j^{-\theta}(T-r)^{-\alpha\theta}$, and

$$\left| \lambda_j^{\nu-\nu_1-\frac{1}{\alpha}} \frac{-\lambda_j t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} E_{\alpha,\alpha}(-\lambda_j(T-\tau)^\alpha)G_j(v(r)) \right|$$

$$\lesssim (T-r)^{-\alpha\theta}t^{-\alpha(1-\theta)}\lambda_j^{\nu-\nu_1}\Big|G_j(r,v(r))\Big|. \tag{78}$$

Applying the Lebesgue dominated convergence theorem, we deduce that three terms

$$\mathscr{D}_{1,\alpha}(t,T)\mathcal{P}_M f, \quad \int_0^t \mathscr{D}_{2,\alpha}(t-r)\mathcal{P}_M G(r,u(r))dr, \quad \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)\mathcal{P}_M G(r,u(r))dr$$

are Cauchy sequences in the space $\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)$. Then, we obtain three convergences in the space $\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)$ as follows

$$\lim_{M\to\infty} \mathscr{D}_{1,\alpha}(t,T)\mathcal{P}_M f = \mathscr{D}_{1,\alpha}(t,T)f,$$
$$\lim_{M\to\infty} \int_0^t \mathscr{D}_{2,\alpha}(t-r)\mathcal{P}_M G(r,u(r))dr = \int_0^t \mathscr{D}_{2,\alpha}(t-r)G(r,u(r))dr,$$
$$\lim_{M\to\infty} \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)\mathcal{P}_M G(r,u(r))dr = \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr. \tag{79}$$

The above equality implies that $\partial_t\mathcal{P}_M u(t)$ consequently converges to $\partial_t u(t)$ in $\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)$. Further, the following estimates also hold

$$\|\mathscr{D}_{1,\alpha}(t,T)f\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},$$

and

$$\left\| \int_0^t \mathscr{D}_{2,\alpha}(t-r)G(r,u(r))dr \right\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)}$$
$$\lesssim \int_0^t (t-r)^{\alpha-2}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}dr$$
$$\lesssim \|f\|_{\mathbb{H}^{\nu+\theta}}\int_0^t (t-r)^{\alpha-2}r^{-\alpha(1-\theta)}dr \lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}},$$

$$\left\| \int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr \right\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)}$$
$$\lesssim t^{-\alpha(1-\theta-\nu_1)}\int_0^T (T-r)^{\alpha(1-\theta)-1}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}dr$$
$$\lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}\int_0^T (T-r)^{\alpha(1-\theta)-1}r^{-\alpha(1-\theta)}dr$$
$$\lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

Here, we note that $\int_0^t (t-r)^{\alpha-2}r^{-\alpha(1-\theta)}dr \lesssim t^{\alpha\theta-1}$ and $t^{\alpha\theta-1} = t^{-\alpha(1-\theta-\nu_1)}t^{\alpha\left(1-\nu_1-\frac{1}{\alpha}\right)} \lesssim t^{-\alpha(1-\theta-\nu_1)}$ in the second estimate by using **(AP.1.)** in the Appendix and noting that $1-\nu_1-\dfrac{1}{\alpha} \geq 0$ as $\nu_1 \leq \dfrac{\alpha-1}{\alpha}$.

Consolidating all the above arguments, we obtain that

$$\Big\|\partial_t u(t)\Big\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \le \Big\|\mathscr{D}_{1,\alpha}(t,T)f\Big\|_{\mathbb{H}^{\mu}(\Omega)} + \Bigg\|\int_0^T \mathscr{D}_{1,\alpha}(t,T)\mathscr{P}_\alpha(T-r)G(r,u(r))dr\Bigg\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)}$$
$$+\Bigg\|\int_0^t \mathscr{D}_{2,\alpha}(t-r)G(r,u(r))dr\Bigg\|_{\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_1)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

Since $\nu_1 \le 1-\theta$ and $1 < \theta+\nu_1+\dfrac{1}{\alpha}$, these straightforwardly imply that $0 < \alpha(1-\theta-\nu_1) < 1$ and $\partial_t u \in L^p(0,T;\mathbb{H}^{\nu-\nu_1-\frac{1}{\alpha}}(\Omega))$, for all $p \in \Big[1, \frac{1}{\alpha(1-\theta-\nu_1)}\Big)$. This completes Part c.

<u>Part (d).</u> *Show that*

$$\|\partial_t^\alpha u(t)\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha\min\{(1-\theta-\nu_\alpha);(1-\theta)\}}\|f\|_{\mathbb{H}^{\nu+\theta}}.$$

To study the fractional derivative of order α of the mild solution u, let us consider the following operators given by

$$\mathscr{D}_{3,\alpha}(t,T)w := \sum_{j=1}^\infty \frac{-\lambda_j E_{\alpha,\alpha}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\langle w,\varphi_j\rangle\varphi_j,$$
$$\mathscr{D}_{4,\alpha}(t)w := -\sum_{j=1}^\infty \lambda_j t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)\langle w,\varphi_j\rangle\varphi_j.$$

By applying the projection $\Big(\mathcal{P}_{M'}-\mathcal{P}_M\Big)$ to the solution u, and then calculating the fractional differentiation ∂_t^α

$$\partial_t^\alpha \mathcal{P}_M u(t) = \mathscr{D}_{3,\alpha}(t,T)\mathcal{P}_M f + \int_0^t \mathscr{D}_{4,\alpha}(t-r)\mathcal{P}_M G(r,u(r))dr$$
$$-\int_0^T \mathscr{D}_{3,\alpha}(t,T)\mathscr{P}_\alpha(T-r)\mathcal{P}_M G(r,u(r))dr + \mathcal{P}_M G(t,u(t)). \tag{80}$$

By using the fact that $\dfrac{1}{\alpha}-\theta < \dfrac{2\alpha-1}{\alpha}-\theta$, it follows from the assumption $\dfrac{\alpha-1}{\alpha}-\theta < \nu_\alpha \le \dfrac{1}{\alpha}-\theta$ that $\dfrac{\alpha-1}{\alpha}-\theta < \nu_\alpha < \dfrac{2\alpha-1}{\alpha}-\theta$. Thus, we find that $1 < \theta+\nu_\alpha+\dfrac{1}{\alpha} < 2$. Therewith, the same techniques as (75) invoke that $\mathscr{D}_{3,\alpha}(t,T)f$ exists in the space $\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)$ if $f \in \mathbb{H}^{\nu+\theta}(\Omega)$, and

$$\|\mathscr{D}_{3,\alpha}(t,T)f\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha\left(\frac{1}{\alpha}-\nu_\alpha-\theta\right)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{81}$$

The proof for integrals $\int_0^t \mathscr{D}_{4,\alpha}(t-r)G(r,u(r))dr$, and $\int_0^T \mathscr{D}_{3,\alpha}(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr$ in the space $\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)$ can be done by using the same argument of (76) and (77) by using assumption $(\mathscr{H}_1)$ and the argument of Cauchy sequences. Aside from the above existence results, we can also verify the following estimates

$$\Bigg\|\int_0^t \mathscr{D}_{4,\alpha}(t-r)G(r,u(r))dr\Bigg\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim \int_0^t (t-r)^{\alpha-2}\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}dr$$
$$\lesssim t^{-\alpha\left(\frac{1}{\alpha}-\nu_\alpha-\theta\right)}t^{\alpha\left(1-\nu_\alpha-\frac{1}{\alpha}\right)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}$$
$$\lesssim t^{-\alpha\left(\frac{1}{\alpha}-\nu_\alpha-\theta\right)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}, \tag{82}$$

where $1-\nu_\alpha-\dfrac{1}{\alpha} \ge 0$ as $\nu_\alpha \le \dfrac{\alpha-1}{\alpha}$, and by a similar argument, we obtain

$$\Bigg\|\int_0^T \mathscr{D}_{3,\alpha}(t,T)\mathbf{P}_\alpha(T-r)G(r,u(r))dr\Bigg\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_\alpha)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}. \tag{83}$$

Henceforth, we find that $\Big\|\partial_t^\alpha u(t)\Big\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)} \lesssim t^{-\alpha(1-\theta-\nu_\alpha)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}$. Since $\nu_\alpha \le \dfrac{\alpha-1}{\alpha}$, we have $\alpha(1-\theta-\nu_\alpha) \ge \alpha\left(1-\theta-\frac{\alpha-1}{\alpha}\right) = \frac{1}{\alpha}-\theta > \frac{\alpha-1}{\alpha}-\theta > 0$. In addition, it can be deduced from

$\dfrac{\alpha-1}{\alpha}-\theta<\nu_\alpha$ that $\alpha(1-\theta-\nu_\alpha)<1$. Hence, we straightforwardly infer that $0<\alpha(1-\theta-\nu_\alpha)<1$. Moreover, it results from $\nu_\alpha>\dfrac{\alpha-1}{\alpha}-\theta$ that $\nu-\nu_\alpha-\dfrac{1}{\alpha}<\nu-(1-\theta)<\nu$, and the Sobolev embedding $\mathbb{H}^\nu(\Omega)\hookrightarrow\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)$ holds. This implies that

$$\|G(t,u(t))\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)}\lesssim\|G(t,u(t))\|_{\mathbb{H}^\nu(\Omega)}\lesssim t^{-\alpha(1-\theta)}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)}.$$

Combining the above inequalities finally shows that

$$\|\partial_t^\alpha u(t)\|_{\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega)}\lesssim t^{-\alpha\min\{(1-\theta-\nu_\alpha);(1-\theta)\}}\|f\|_{\mathbb{H}^{\nu+\theta}(\Omega)},$$

and $\partial_t^\alpha u\in L^p(0,T;\mathbb{H}^{\nu-\nu_\alpha-\frac{1}{\alpha}}(\Omega))$, for all $p\in\left[1,\frac{1}{\alpha\min\{(1-\theta-\nu_\alpha);(1-\theta)\}}\right)$. The proof is accomplished. ☐

Remark 5.1. *Let us explain the uniqueness of the mild solution in Theorem 3.2. For this purpose, we assume that u_1,u_2 are two mild solutions of Problem (1)-(2), which satisfy that $u_1=\overline{\mathbf{J}}u_1$ and $u_2=\overline{\mathbf{J}}u_2$. We will show $u_1\equiv u_2$. Indeed, it follows from $u_1,u_2\in C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))$ that*

$$\|u_1(r)-u_2(r)\|_{\mathbb{H}^\nu(\Omega)}\leq\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}r^{-\alpha(1-\theta)},\quad\forall r\in(0,T].\tag{84}$$

Furthermore, by using the analogous arguments as in (43), (44), one can directly obtain the following chain

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{\mathbb{H}^\nu(\Omega)}&\leq\|L_1\|_{L^\infty(0,T)}M_\alpha\lambda_1^{-\theta}\int_0^t(t-r)^{\alpha(1-\theta)-1}\|u_1(r)-u_2(r)\|_{\mathbb{H}^\nu(\Omega)}dr\\&+\|L_1\|_{L^\infty(0,T)}\mathcal{M}_1t^{-\alpha(1-\theta)}\int_0^T(T-r)^{\alpha(1-\theta)-1}\|u_1(r)-u_2(r)\|_{\mathbb{H}^\nu(\Omega)}dr\\&\leq\|L_1\|_{L^\infty(0,T)}\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}M_\alpha\lambda_1^{-\theta}\frac{\pi}{\sin(\pi\alpha(1-\theta))}\\&+\|L_1\|_{L^\infty(0,T)}\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}\mathcal{M}_1t^{-\alpha(1-\theta)}\frac{\pi}{\sin(\pi\alpha(1-\theta))},\end{aligned}$$

*where we have used (84) and the part **(AP.1.)** of Appendix in the last estimate. Therefore, multiplying two sides of the above estimates by $t^{\alpha(1-\theta)}$ and then taking the supremum with respect to $t\in(0,T]$ give*

$$\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}\leq\|L_1\|_{L^\infty(0,T)}\mathscr{M}_1\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}.$$

Since $\|L_1\|_{L^\infty(0,T)}\in\left(0,\mathscr{M}_1^{-1}\right)$, it follows that $\|u_1-u_2\|_{C^{\alpha(1-\theta)}((0,T];\mathbb{H}^\nu(\Omega))}=0$. Consequently, $u_1\equiv u_2$.

5.2. **Proof of Theorem 3.3.** The proof of Theorem 3.3 relies on a contraction mapping principle. In order to prove this, we first prove the following Lemma

Lemma 5.3. *Let us pick $\dfrac{\alpha-1}{\alpha}<\theta<1$, $0\leq\nu\leq\sigma\leq\nu+1$ and $1\leq q<\dfrac{1}{\alpha(1-\theta)}$. Assume that $f\in\mathbb{H}^{\nu+\theta+1}(\Omega)$ and G sastisfies $(\mathscr{H}_2)$ with $\|L_2\|_{L^\infty(0,T)}\in\left(0,\mathscr{M}_2^{-1}\right)$. Set*

$$\mathscr{T}v(t):=\mathbf{B}_\alpha(t,T)f+\int_0^t\mathbf{P}_\alpha(t-r)G(r,v(r))dr-\int_0^T\mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,v(r))dr.\tag{85}$$

Then, for any $v\in C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))$, it holds that

$$\mathscr{T}v\in C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega)).$$

Proof of Lemma 5.3. We split this proof into the following steps.

<u>*Prove $\mathscr{T}v\in C([0,T];\mathbb{H}^\nu(\Omega))$:*</u> Namely, we need to estimate the norm $\|\mathscr{T}v(\widetilde{t})-\mathscr{T}v(t)\|_{\mathbb{H}^\nu(\Omega)}$ for all $0\leq t<\widetilde{t}\leq T$. For more convenience, we will use notation $\mathfrak{M}_j$, $1\leq j\leq4$ as (53) again. However, the estimates for $\mathfrak{M}_j$ in Step 2 of the proof of Theorem (3.2) will be modified suitably to fit the assumptions of f and G in this theorem. Indeed, a slight modification of the techniques in the estimates (54) and

(55) invokes that

$$\|\mathfrak{M}_1\|_{\mathbb{H}^\nu(\Omega)} \leq \int_t^{\widetilde{t}} \left\| \sum_{j=1}^{\infty} \lambda_j r^{\alpha-1} \frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} f_j \varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} d\tau$$
$$\lesssim \left(\int_t^{\widetilde{t}} r^{\alpha\theta-1} dr \right) \|f\|_{\mathbb{H}^{\nu+\theta+1}(\Omega)}$$
$$\lesssim \left\{ \begin{array}{l} (\widetilde{t}-t)^{\alpha\theta} \mathbf{1}_{0<\alpha\theta\leq 1} \\ \left((\widetilde{t}-t)^{\alpha\theta-1} + \gamma \right) \mathbf{1}_{1<\alpha\theta<2} \end{array} \right\} \|f\|_{\mathbb{H}^{\nu+\theta+1}(\Omega)},$$

where we note that $\frac{1-\theta}{2}$ belongs to $(0,1)$, and it notes that $0 < \alpha - 1 < \alpha\theta < \alpha < 2$ as $\frac{\alpha-1}{\alpha} < \theta < 1$. Next, estimates for the terms $\mathfrak{M}_j$, $2 \leq j \leq 4$ will be based on assumption $(\mathscr{H}_2)$ of the nonlinearity G. We see that

$$\|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} \lesssim \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha-2} \|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)} d\rho dr$$
$$\lesssim \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha-2} \|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)} d\rho dr$$
$$\lesssim \|v\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap \mathscr{L}^q(0,T;\mathbb{H}^\sigma(\Omega))} \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha-2} d\rho dr,$$

where the norm $\|G(r,u(r))\|_{\mathbb{H}^\nu(\Omega)}$ is certainly $\lesssim$-bounded by $\|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)}$ due to the embedding $\mathbb{H}^{\nu+1}(\Omega) \hookrightarrow \mathbb{H}^\nu(\Omega)$. Observe from the above estimate that the last right hand side clearly tends to zero as $\widetilde{t}$ tends to t. Hence, the preceding estimate implies the continuity of the term $\mathfrak{M}_2$ on $\mathbb{H}^\nu(\Omega)$. In addition, the continuity of the term $\mathfrak{M}_3$ is obvious by using similar arguments as in (59) and assumption $(\mathscr{H}_2)$. Precisely,

$$\|\mathfrak{M}_3\|_{\mathbb{H}^\nu(\Omega)} \lesssim \int_t^{\widetilde{t}} \|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)} dr \lesssim (\widetilde{t}-t) \|v\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))}.$$

Finally, we consider the term $\|\mathfrak{M}_4\|_{\mathbb{H}^\nu(\Omega)}$. The idea is to combine similar arguments as in Step 5 and the modification in the above estimates for $\mathfrak{M}_1$. Here, the maximum of the spatial smoothness of G should be estimated in the space $\mathbb{H}^{\nu+1}(\Omega)$. Indeed, the following chain of the estimates can be checked

$$\|\mathfrak{M}_4\|_{\mathbb{H}^\nu(\Omega)}$$
$$\leq \int_0^T \int_{t_1}^{t_2} \left(\sum_{j=1}^{\infty} \left| \lambda_j^\nu (T-\tau)^{\alpha-1} \lambda_j \rho^{\alpha-1} \mathscr{E}_{\alpha,T}(-\lambda_j \rho^\alpha) E_{\alpha,\alpha}(-\lambda_j (T-r)^\alpha) G_j(r,u(r)) \right|^2 \right)^{\frac{1}{2}} d\rho dr$$
$$\lesssim \int_0^T \int_{t_1}^{t_2} (T-r)^{\alpha(1-\theta)-1} \rho^{\alpha\theta-1} \|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)} d\rho dr$$
$$\lesssim \|v\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))} \left[(t_2)^{\alpha\theta} - (t_1)^{\alpha\theta} \right]$$
$$\lesssim \|v\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))} \left\{ \begin{array}{l} (t_2-t_1)^{\alpha\theta} \mathbf{1}_{0<\alpha\theta\leq 1} \\ \left((t_2-t_1)^{\alpha\theta-1} + (t_2-t_1) \right) \mathbf{1}_{1<\alpha\theta<2} \end{array} \right\}.$$

The preceding estimates lead to $\mathscr{T}v \in C([0,T];\mathbb{H}^\nu(\Omega))$.

Prove $\mathscr{T}v \in L^q(0,T;\mathbb{H}^\sigma(\Omega))$: We observe that

$$\left\| \mathbf{B}_\alpha(t,T)f \right\|_{\mathbb{H}^\sigma(\Omega)}^2 = \sum_{j=1}^{\infty} \lambda_j^{2\sigma} \left| \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \right|^2 \langle f, \varphi_j \rangle^2 \lesssim t^{-2\alpha(1-\theta)} \sum_{j=1}^{\infty} \lambda_j^{2\theta+2\sigma} \langle f, \varphi_j \rangle^2$$
$$\lesssim t^{-2\alpha(1-\theta)} \sum_{j=1}^{\infty} \lambda_j^{2\theta+2\nu+2} \langle f, \varphi_j \rangle^2 = t^{-2\alpha(1-\theta)} \|f\|_{\mathbb{H}^{\nu+\theta+1}}^2. \tag{86}$$

Thus, $\mathbf{B}_\alpha(t,T)f \in L^q(0,T;\mathbb{H}^\sigma(\Omega))$ since $t^{-\alpha(1-\theta)} \in L^q(0,T;\mathbb{R})$.

Next, we estimate the second term of $\mathcal{J}v$ where we will bound the operator norm of $\mathbf{P}_\alpha(t-r)$ on $\mathbb{H}^\sigma(\Omega)$ by $M_\alpha(t-r)^{\alpha-1}$, and then we estimate $\|G(r,v(r))\|_{\mathbb{H}^\sigma(\Omega)}$ by $\|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)}$ upon assumption

($\mathscr{H}_2$) and the embedding $\mathbb{H}^{\nu+1}(\Omega) \hookrightarrow \mathbb{H}^{\sigma}(\Omega)$ as $0 \le \sigma \le \nu+1$, see (17). Precisely, these arguments can be performed as follows

$$\begin{aligned}
&\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,v(r))dr \right\|^q_{L^q(0,T;\mathbb{H}^\sigma(\Omega))} \\
&\quad = \int_0^T \left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,v(r))dr \right\|^q_{\mathbb{H}^\sigma(\Omega)} dt \\
&\quad \le \big(C_2(\nu,\sigma)M_\alpha\big)^q \int_0^T \left(\int_0^t (t-r)^{\alpha-1} \|G(r,v(r))\|_{\mathbb{H}^{\nu+1}(\Omega)} dr \right)^q dt \\
&\quad \le \Big(\|L_2\|_{L^\infty(0,T)} \overline{\mathcal{M}}_1 \Big)^q \|v\|^q_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))},
\end{aligned} \tag{87}$$

where the constant $\overline{\mathcal{M}}_1$ is given by **(AP.4)** in the Appendix.

Next, we will estimate the $L^q(0,T;\mathbb{H}^\sigma(\Omega))$-norm of the last term of $\mathcal{J}v$. The idea is to combine estimates for the operators $\mathbf{B}_\alpha(t,T)$ and $\mathbf{P}_\alpha(T-r)$. We can estimate that the operator $\mathbf{B}_\alpha(t,T)$ maps $\mathbb{H}^{\sigma+\theta}(\Omega)$ into $\mathbb{H}^\sigma(\Omega)$, and the operator $\mathbf{P}_\alpha(T-r)$ maps $\mathbb{H}^\sigma(\Omega)$ into $\mathbb{H}^{\sigma+\theta}(\Omega)$. Therefore, by using assumption ($\mathscr{H}_2$), this term can be estimated on the space $\mathbb{H}^\sigma(\Omega)$. In the technical aspect, we also note that the assumption $1-1/(\alpha q) < \theta < 1$ guarantees that $(\alpha-1)/\alpha < \theta < 1$, and so $\alpha(1-\theta) \in (0,1)$. Moreover, the power function $t^{-\alpha(1-\theta)q}$ is clearly integrable on $(0,T)$. Indeed, one can show the following chain of estimates

$$\begin{aligned}
&\left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,v(r))dr \right\|^q_{L^q(0,T;\mathbb{H}^\sigma(\Omega))} \\
&\quad = \int_0^T \left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,v(r))dr \right\|^q_{\mathbb{H}^\sigma(\Omega)} dt \\
&\quad \le \mathcal{M}^q_{\alpha,T,\theta} \int_0^T t^{-\alpha(1-\theta)q} \left(\int_0^T \|\mathbf{P}_\alpha(T-r)G(r,v(r))\|_{\mathbb{H}^{\sigma+\theta}(\Omega)} dr \right)^q dt \\
&\quad \le \overline{\mathcal{M}}^q_{\alpha,T,\theta} \int_0^T t^{-\alpha(1-\theta)q} \left(\int_0^T (T-\tau)^{\alpha(1-\theta)-1} \|G(r,v(r))\|_{\mathbb{H}^\sigma(\Omega)} dr \right)^q dt \\
&\quad \le \Big(\|L_2\|_{L^\infty(0,T)} \overline{\mathcal{M}}_2 \Big)^q \|v\|^q_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))},
\end{aligned} \tag{88}$$

where $\mathcal{M}_{\alpha,T,\theta} := T^{\alpha(1-\theta)}(T^{\alpha\theta} + \lambda_1^{-\theta})$, $\overline{\mathcal{M}}_{\alpha,T,\theta} = \mathcal{M}_{\alpha,T,\theta} M_\alpha$, and the constant $\overline{\mathcal{M}}_2$ is given by **(AP.4)** in the Appendix. A collection of the derived estimates (86), (87), (88), reveals $\mathscr{T}v \in L^q(0,T;\mathbb{H}^\sigma(\Omega))$. Finally, we wrap up the proof. □

Proof of Theorem 3.3. In order to show that Problem (1)-(2) has a unique mild solution, we will prove the operator $\mathscr{Q}$ has a unique fixed point in $C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))$. The proof is based on the Banach contraction principle. We have

$$\begin{aligned}
&\left\| \mathscr{T}v_1 - \mathscr{T}v_2 \right\|_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))} \\
&= \|\mathscr{T}v_1 - \mathscr{T}v_2\|_{L^q(0,T;\mathbb{H}^\sigma(\Omega))} + \|\mathscr{T}v_1 - \mathscr{T}v_2\|_{C([0,T];\mathbb{H}^\nu(\Omega))} \\
&:= \mathfrak{M}_5 + \mathfrak{M}_6.
\end{aligned} \tag{89}$$

To estimate $\mathfrak{M}_5$, we apply the previous results in estimating (87) and (88) to obtain

$$\begin{aligned}
\mathfrak{M}_5 \le& \left\| \int_0^t \mathbf{P}_\alpha(t-r)\left(G(r,v_1(r)) - G(r,v_2(r))\right)dr \right\|_{L^q(0,T;\mathbb{H}^\sigma(\Omega))} \\
&+ \left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\left(G(r,v_1(r)) - G(r,v_2(r))\right)dr \right\|_{L^q(0,T;\mathbb{H}^\sigma(\Omega))} \\
\le& \|L_2\|_{L^\infty(0,T)} \big(\overline{\mathcal{M}}_1 + \overline{\mathcal{M}}_2\big) \|v_1 - v_2\|_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))}.
\end{aligned} \tag{90}$$

On the other hand, to bound the term $\mathfrak{M}_6$, we estimate the operator norm of $\mathbf{P}_\alpha(t-r)$ acting on $\mathbb{H}^\nu(\Omega)$ by $M_\alpha(t-r)^{\alpha-1}$, and of $\mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)$ acting from $\mathbb{H}^\nu(\Omega)$ to $\mathbb{H}^{\nu+1}(\Omega)$ by $M_\alpha^2 m_\alpha^{-1}(T^\alpha + \lambda_1^{-1})$. By applying the embedding $\mathbb{H}^\nu(\Omega) \hookrightarrow \mathbb{H}^{\nu+1}(\Omega)$ and assumption $(\mathscr{H}_2)$ we can deduce the following estimates

$$\begin{aligned}
\mathfrak{M}_6 &\leq \left\| \int_0^t \mathbf{P}_\alpha(t-r)\left(G(r,v_1(r)) - G(r,v_2(r))\right) dr \right\|_{C([0,T];\mathbb{H}^\nu(\Omega))} \\
&\quad + \left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\left(G(v_1(r)) - G(v_2(r))\right) dr \right\|_{C([0,T];\mathbb{H}^\nu(\Omega))} \\
&\leq M_\alpha \sup_{0\leq t\leq T} \left(\int_0^t (t-r)^{\alpha-1} \left\| G(r,v_1(r)) - G(r,v_2(r)) \right\|_{H^\nu(\Omega)} dr \right) \\
&\quad + \frac{M_\alpha^2}{m_\alpha}(T^\alpha + \lambda_1^{-1}) \int_0^T (T-r)^{\alpha-1} \left\| G(r,v_1(r)) - G(r,v_2(r)) \right\|_{\mathbb{H}^{\nu+1}(\Omega)} dr \\
&\leq \|L_2\|_{L^\infty(0,T)} \overline{\mathcal{M}}_3 \left\| v_1 - v_2 \right\|_{C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))}. \qquad (91)
\end{aligned}$$

A collection of the estimates (89), (90), (91) implies that

$$\left\| \mathcal{J}v_1 - \mathcal{J}v_2 \right\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))} \leq \|L_2\|_{L^\infty(0,T)} \mathscr{M}_2 \left\| v_1 - v_2 \right\|_{C([0,T];\mathbb{H}^\nu(\Omega))\cap L^q(0,T;\mathbb{H}^\sigma(\Omega))}.$$

Since $\|L_2\|_{L^\infty(0,T)}\mathscr{M}_2 < 1$, we conclude that $\mathcal{J}$ is a contraction in $C([0,T];\mathbb{H}^\nu(\Omega)) \cap L^q(0,T;\mathbb{H}^\sigma(\Omega))$ which ensures the existence and uniqueness of a fixed point. The desired inequality is easy to obtain. Hence, we finalize the proof. □

5.3. **Proof of Theorem 3.4.** To start with, let us prove the following lemmas.

Lemma 5.4. *Assume that all assumptions of Theorem 3.4 are fulfilled.*

a) *For $t > 0$, and $\mathcal{N}_2$ given by **(AP.4.)** in the Appendix, we have*

$$\|\mathbf{B}_\alpha(t,T)f\|_{\mathbb{H}^\nu(\Omega)} \leq \mathcal{N}_2 t^{-\alpha\vartheta} \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}, \qquad (92)$$

Moreover, the following convergence holds

$$\mathbf{B}_\alpha(\widetilde{t},T)f \xrightarrow{\widetilde{t}\to t} \mathbf{B}_\alpha(t,T)f \quad in \quad \mathbb{H}^\nu(\Omega). \qquad (93)$$

b) *For $t > 0$, $w \in \mathfrak{X}_{\alpha,\vartheta,\nu,T}$, and $\mathscr{N}_2$ given by **(AP.4)** in the Appendix, it follows*

$$\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr \right\|_{\mathbb{H}^\nu(\Omega)} \leq \mathscr{N}_2 K_0 \big(T^{s\alpha\vartheta} + \mathcal{R}^s\big) t^{-\alpha\vartheta} \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))}. \qquad (94)$$

Moreover, the following convergence holds

$$\int_0^{\widetilde{t}} \mathbf{P}_\alpha(\widetilde{t}-r)G(r,w(r))dr \xrightarrow{\widetilde{t}\to t} \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr \quad in \quad \mathbb{H}^\nu(\Omega). \qquad (95)$$

c) *For $t > 0$, $w \in \mathfrak{X}_{\alpha,\vartheta,\nu,T}$, it holds*

$$\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr \right\|_{\mathbb{H}^{\nu+1-\vartheta}(\Omega)} \leq \mathscr{N}_2 K_0 \big(T^{s\alpha\vartheta} + \mathcal{R}^s\big) t^{-\alpha\vartheta} \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))}. \qquad (96)$$

Proof. Part (a).
By applying the first part of Lemma 2.4, we obtain

$$\left\| \mathbf{B}_\alpha(t,T)f \right\|^2_{\mathbb{H}^\nu(\Omega)} = \sum_{j=1}^\infty \left| \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \right|^2 f_j^2 \leq \mathcal{N}_2^2 t^{-2\alpha\vartheta} \|f\|^2_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)},$$

where $\mathcal{N}_2$ is given by **(AP.4.)** in the Appendix. This directly implies inequality (92). Let us proceed to prove the convergence (93). By the fact that $E_{\alpha,\alpha}(-z) \lesssim (1+z^2)^{-1}$ for all $z \geq 0$, see e.g. [11, 12, 13], one can apply the same techniques as in (55) to show the following inequalities

$$\left| \frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \right| \lesssim \frac{1+\lambda_j T^\alpha}{[1+(\lambda_j r^\alpha)^2]^{1-\frac{1-\vartheta}{2}}} \lesssim \lambda_j \left[(\lambda_j r^\alpha)^2\right]^{\frac{1-\vartheta}{2}-1}, \qquad (97)$$

where $0 < (1-\vartheta)/2 < (1-\mu)/2 < 1$. Hence, we derive that

$$\left\|\mathbf{B}_\alpha(\widetilde{t},T)f - \mathbf{B}_\alpha(t,T)f\right\|_{\mathbb{H}^\nu(\Omega)} \le \int_t^{\widetilde{t}} r^{\alpha-1}\left\|\sum_{j=1}^\infty \lambda_j \frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} f_j\varphi_j\right\|_{\mathbb{H}^\nu(\Omega)} dr,$$

$$\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\int_t^{\widetilde{t}} r^{-\alpha\vartheta-1}dr.$$

Since the integral in the above inequality tends to zero as t approaches $\widetilde{t}$ from the right, we obtain (93) and finish the proof of Part (a).

Part (b).

We divide this proof into two parts as follows.

Step 1. Prove inequality (94). It follows from $E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha) \le M_\alpha \lambda_j^{-\mu}(t-r)^{-\alpha\mu}$ that

$$\begin{aligned}&\left\|\int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr\right\|_{\mathbb{H}^\nu(\Omega)}\\ &\qquad \le \int_0^t (t-r)^{\alpha-1}\left\|\sum_{j=1}^\infty E_{\alpha,\alpha}(-\lambda_j(t-r)^\alpha)G_j(w(r))\varphi_j\right\|_{\mathbb{H}^\nu(\Omega)} dr\\ &\qquad \le M_\alpha\int_0^t (t-r)^{\alpha(1-\mu)-1}\left\|G(r,w(r))\right\|_{\mathbb{H}^\sigma(\Omega)}dr, \end{aligned} \tag{98}$$

where $\sigma = \nu-\mu$. Since $w\in\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$, we see that $\|w(r)\|_{\mathbb{H}^\nu(\Omega)} \le \mathcal{R}r^{-\alpha\vartheta}$. Thus, we have in view of (20) that $\|G(r,w(r))\|_{\mathbb{H}^\sigma(\Omega)} \le L_3(r)\Big(1+\|w(r)\|^s_{\mathbb{H}^\nu(\Omega)}\Big)\|w(r)\|_{\mathbb{H}^\nu(\Omega)}$. It follows from (98) that

$$\left\|\int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr\right\|_{\mathbb{H}^\nu(\Omega)} \le M_\alpha\|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))}\widehat{L}_3(t) \tag{99}$$

where

$$\widehat{L}_3(t) := \int_0^t (t-r)^{\alpha(1-\mu)-1}\Big(r^{-\alpha\vartheta}+\mathcal{R}^s r^{-(1+s)\alpha\vartheta}\Big)L_3(r)dr. \tag{100}$$

Our next purpose is to find an upper bound of $\widehat{L}_3(t)$. In order to control this term, we observe from $0<r<T$ that $r^{-\alpha\vartheta} \le T^{s\alpha\vartheta}r^{-(1+s)\alpha\vartheta}$, and from $K_0 = \|L_3(t)t^{\alpha\zeta}\|_{L^\infty(0,T)}$ that $L_3(r) \le K_0 r^{-\alpha\zeta}$ which yield the following estimates

$$\begin{aligned}\widehat{L}_3(t) &\le \big(T^{s\alpha\vartheta}+\mathcal{R}^s\big)\int_0^t (t-r)^{\alpha(1-\mu)-1}r^{-(1+s)\alpha\vartheta}L_3(r)dr\\ &\le K_0\big(T^{s\alpha\vartheta}+\mathcal{R}^s\big)\int_0^t (t-r)^{\alpha(1-\mu)-1}r^{-\alpha((1+s)\vartheta+\zeta)}dr.\end{aligned}$$

By noting $\min\{\alpha(1-\mu)-1;\ -\alpha((1+s)\vartheta+\zeta)\} > -1$ as $0<\mu<1$, $\zeta < \alpha^{-1}-(1+s)\vartheta$, and using **(AP.1.)** in the Appendix, we find that

$$\int_0^t (t-r)^{\alpha(1-\mu)-1}r^{-\alpha((1+s)\vartheta+\zeta)}dr \le \mathscr{N}_1 t^{\alpha((1-\mu)-(1+s)\vartheta-\zeta)}.$$

This implies that

$$\widehat{L}_3(t) \le K_0\big(T^{s\alpha\vartheta}+\mathcal{R}^s\big)\mathscr{N}_1 T^{\alpha((1-\mu)-s\vartheta-\zeta)}\ t^{-\alpha\vartheta}, \tag{101}$$

where we have noted that $\zeta \le (1-\mu)-s\vartheta$ since $\zeta < \alpha^{-1}-\vartheta-s\vartheta \le (1-\mu)-s\vartheta$ as $\alpha^{-1}<1$ and $\vartheta>\mu$. The latter estimate together with (99) and (100) imply that

$$\left\|\int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr\right\|_{\mathbb{H}^\nu(\Omega)} \le \mathscr{N}_2 K_0\big(T^{s\alpha\vartheta}+\mathcal{R}^s\big)t^{-\alpha\vartheta}\big\|w\big\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))},$$

where we recall that $\mathscr{N}_2$ is given by **(AP.4)** in the Appendix.

Step 2. Show that (95) holds. By dealing with $\|G(r, w(r))\|_{\mathbb{H}^\sigma(\Omega)}$ using the same arguments in Step 1, we derive that

$$\begin{aligned}
\|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} &\leq \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha-2} \left\| \sum_{j=1}^{\infty} E_{\alpha,\alpha-1}(-\lambda_j \rho^\alpha) G_j(r, u(r))\varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} d\rho dr \\
&\lesssim \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha(1-\mu)-2} \|G(r, u(r))\|_{\mathbb{H}^\sigma(\Omega)} d\rho dr \\
&\lesssim \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha(1-\mu)-2} \left(\mathcal{R} r^{-\alpha\vartheta} + \mathcal{R}^{1+s} r^{-(1+s)\alpha\vartheta}\right) L_3(r) d\rho dr \\
&\lesssim \left| \int_0^t \left((\widetilde{t}-r)^{\alpha(1-\mu)-1} - (t-r)^{\alpha(1-\mu)-1} \right) r^{-\alpha((1+s)\vartheta+\zeta)} dr \right|,
\end{aligned}$$

where $\mathfrak{M}_2$ is formulated by (53). By the fact that $\alpha(1-\mu) > 0$ and $1 - \alpha\left((1+s)\vartheta + \zeta\right) > 0$ and using **(AP.2.)** in the Appendix, we know that the right hand-side of the latter inequality tends to zero, as $\widetilde{t}$ approaches t. Hence, $\|\mathfrak{M}_2\|_{\mathbb{H}^\nu(\Omega)} \xrightarrow{\widetilde{t}\to t} 0$. Now, in the same way as above, we obtain

$$\begin{aligned}
\|\mathfrak{M}_3\|_{\mathbb{H}^\nu(\Omega)} &\leq \int_t^{\widetilde{t}} (\widetilde{t}-r)^{\alpha-1} \left\| \sum_{j=1}^{\infty} E_{\alpha,\alpha}(-\lambda_j (\widetilde{t}-\tau)^\alpha) G_j(r, u(r))\varphi_j \right\|_{\mathbb{H}^\nu(\Omega)} dr \\
&\lesssim \int_t^{\widetilde{t}} (\widetilde{t}-r)^{\alpha(1-\mu)-1} \|G(r, u(r))\|_{\mathbb{H}^\sigma(\Omega)} dr \\
&\lesssim \int_t^{\widetilde{t}} (\widetilde{t}-r)^{\alpha(1-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr,
\end{aligned} \tag{102}$$

where $\mathfrak{M}_3$ is formulated by (53). From that $(\widetilde{t}-r)^{\alpha(1-\vartheta)} \leq (\widetilde{t}-t)^{\alpha(1-\vartheta)}$ as $t \leq r \leq \widetilde{t}$, we bound the right hand-side of (102) as follows

$$\begin{aligned}
\text{(RHS) of (102)} &\leq (\widetilde{t}-t)^{\alpha(1-\vartheta)} \int_0^{\widetilde{t}} (\widetilde{t}-r)^{\alpha(\vartheta-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr \\
&\lesssim (\widetilde{t}-t)^{\alpha(1-\vartheta)} \int_0^{\widetilde{t}} (\widetilde{t}-r)^{\alpha(\vartheta-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr,
\end{aligned}$$

Noting that $\alpha(\vartheta-\mu) > 0$ and $1 - \alpha\left((1+s)\vartheta + \zeta\right) > 0$, we ensure that $\int_0^{\widetilde{t}} (\widetilde{t}-r)^{\alpha(\vartheta-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr$ is convergent. The above observations imply that $\|\mathfrak{M}_3\|_{\mathbb{H}^\nu(\Omega)} \xrightarrow{\widetilde{t}\to t} 0$. Since

$$\int_0^{\widetilde{t}} \mathbf{P}_\alpha(\widetilde{t}-r) G(r, w(r)) dr - \int_0^t \mathbf{P}_\alpha(t-r) G(r, w(r)) dr = \mathfrak{M}_2 + \mathfrak{M}_3,$$

we finish this step.

Part (c). In view of $0 \leq 1 + [(\nu-\sigma) - \vartheta] \leq 1$, one can see that

$$\begin{aligned}
&\left\| \int_0^t \mathbf{P}_\alpha(t-r) G(r, w(r)) dr \right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&\qquad \leq M_\alpha \int_0^t (t-r)^{\alpha(\vartheta-\mu)-1} \|G(r, w(r))\|_{\mathbb{H}^\sigma(\Omega)} dr \\
&\qquad \leq M_\alpha K_0 (T^{s\alpha\vartheta} + \mathcal{R}^s) \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \int_0^t (t-r)^{\alpha(\vartheta-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr \\
&\qquad \leq M_\alpha K_0 (T^{s\alpha\vartheta} + \mathcal{R}^s) \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \mathscr{N}_1 t^{\alpha((\vartheta-\mu)-(1+s)\vartheta-\zeta)} \\
&\qquad \leq \mathscr{N}_2 K_0 \big(T^{s\alpha\vartheta} + \mathcal{R}^s\big) t^{-\alpha\vartheta} \|w\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))},
\end{aligned} \tag{103}$$

where we also recall that $\mathscr{N}_2$ is given by **(AP.4)** in the Appendix. This completes the proof. □

Proof of Theorem 3.4. The proof will be based on a contraction mapping theorem on a Banach space. For this purpose, let us define the mapping

$$\mathscr{Q} : \mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R}) \longrightarrow \mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$$

given by

$$\mathscr{Q}w = \mathbf{B}_\alpha(t,T)f + \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr - \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,w(r))dr. \tag{104}$$

Since $f \in \mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, the convergence (93) in Part a of Lemma 5.4 yields that the first term of $\mathscr{Q}$ is time-continuous for all $0 < t \le T$. The estimate (92) means that this term belongs to $C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))$. Similarly, we observe from G satisfying assumption $(\mathscr{H}_3)$ and the estimate (94), the convergence (95) in in Part b of Lemma 5.4 that the second term of $\mathscr{Q}$ belongs to $C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))$. On the other hand, using Part c of Lemma 5.4 shows that the integral $\int_0^T \mathbf{P}_\alpha(T-r)G(r,w(r))dr$ belongs to $\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$, so we deduce from Part a of Lemma 5.4 that

$$\mathbf{B}_\alpha(t,T)\int_0^T \mathbf{P}_\alpha(T-r)G(r,w(r))dr \text{ belongs to } C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega)). \tag{105}$$

Therefore, the last term of $\mathscr{Q}$ also belongs to $C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))$.

Prove $\mathscr{Q}$ maps $\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$ into itself: Indeed, let $w^\dagger$, $w^\ddagger$ belong to the space $\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$, then using the formula (104) we can obtain the following chain of estimates

$$\begin{aligned}
t^{\alpha\vartheta} &\left\|\mathscr{Q}w^\dagger(t) - \mathscr{Q}w^\ddagger(t)\right\|_{\mathbb{H}^\nu(\Omega)} \\
&\le t^{\alpha\vartheta}\left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr\right\|_{\mathbb{H}^\nu(\Omega)} \\
&\quad + t^{\alpha\vartheta}\left\|\mathbf{B}_\alpha(t,T)\int_0^T \mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr\right\|_{\mathbb{H}^\nu(\Omega)} \\
&\le \mathscr{N}_2 K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\left\|w^\dagger - w^\ddagger\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \\
&\quad + \mathscr{N}_2\left\|\int_0^T \mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr\right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&\le \overline{\mathscr{N}_2}K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\|w^\dagger - w^\ddagger\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))},
\end{aligned}$$

where on the right hand-side of (104), we have used the inequalities (92) of Lemma 5.4, (94) of Lemma 5.4 in the first estimate, and the inequality (96) of Lemma 5.4 in the second estimate. This implies that

$$\left\|\mathscr{Q}w^\dagger - \mathscr{Q}w^\ddagger\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \le \overline{\mathscr{N}_2}K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\|w^\dagger - w^\ddagger\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))}. \tag{106}$$

By letting $w^\ddagger = 0$ into the latter equality and noting that $\mathscr{Q}\overline{w}^\ddagger(t) = \mathbf{B}_\alpha(t,T)f$ if $w^\ddagger = 0$, we derive

$$\left\|\mathscr{Q}w^\dagger - \mathbf{B}_\alpha(t,T)f\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \le \overline{\mathscr{N}_2}K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\|w^\dagger\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))}.$$

From (92) and using the triangle inequality, we know that

$$\begin{aligned}
\left\|\mathscr{Q}w^\dagger\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} &\le \left\|\mathscr{Q}w^\dagger - \mathbf{B}_\alpha(t,T)f\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} + \sup_{0\le t\le T} t^{\alpha\vartheta}\|\mathbf{B}_\alpha(t,T)f\|_{\mathbb{H}^\nu(\Omega)} \\
&\le \overline{\mathscr{N}_2}K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\|w^\dagger\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} + \mathscr{N}_2\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}.
\end{aligned}$$

Since $w^\dagger \in \mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$, we have $\|w^\dagger\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \le \mathcal{R}$. This implies that

$$\left\|\mathscr{Q}w^\dagger\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} \le \underbrace{\overline{\mathscr{N}_2}K_0(T^{s\alpha\vartheta} + \mathcal{R}^s)\mathcal{R} + \mathscr{N}_2\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}}_{:=\pi(\mathcal{R})}. \tag{107}$$

Due to the assumption $K_0T^{s\alpha\vartheta} \in \big(0, \min\big\{\frac{1}{2}\overline{\mathscr{N}_2}^{-1}; \mathcal{N}_f\big\}\big)$, we now show that there exists $0 < \overline{\mathcal{R}} < \widehat{\mathcal{R}}$ which is a solution to the equation $\pi(\mathcal{R}) = \mathcal{R}$, where we denote by the constant

$$\widehat{\mathcal{R}} := \left(\frac{1 - \overline{\mathscr{N}_2}K_0T^{s\alpha\vartheta}}{(1+s)\overline{\mathscr{N}_2}K_0}\right)^{1/s}.$$

We note that the function $\mathcal{R} \mapsto \widehat{\pi}(\mathcal{R}) := \pi(\mathcal{R}) - \mathcal{R}$ is continuous on $(0, \widehat{\mathcal{R}})$ with the values $\widehat{\pi}(0) = \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}$ and

$$
\begin{aligned}
\widehat{\pi}(\widehat{\mathcal{R}}) &= \overline{\mathcal{N}_2} K_0 (T^{s\alpha\vartheta} + \widehat{\mathcal{R}}^s)\widehat{\mathcal{R}} + \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} - \widehat{\mathcal{R}} \\
&= \Big(\overline{\mathcal{N}_2} K_0 \widehat{\mathcal{R}}^s - (1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta})\Big)\widehat{\mathcal{R}} + \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&= \left(1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}\right)\left(\frac{1}{1+s} - 1\right)\widehat{\mathcal{R}} + \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&= \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} - \frac{s}{1+s}\left(1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}\right)\widehat{\mathcal{R}} \\
&= \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} - \frac{s}{1+s}\frac{\left(1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}\right)^{1+1/s}}{(1+s)^{1/s}(\overline{\mathcal{N}_2} K_0)^{1/s}} \\
&< \mathcal{N}_2 \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \left(1 - \frac{s}{1+s}\frac{(1/2)^{1+1/s}}{(1+s)^{1/s} s}(2(1+s))^{1+1/s}\right) \\
&= 0,
\end{aligned}
$$

where we note that $1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta} > \frac{1}{2}$. Therefore, there exists $0 < \overline{\mathcal{R}} < \widehat{\mathcal{R}}$ such that $\pi(\mathcal{R}) = \mathcal{R}$. So it follows from (107) that $\mathscr{Q}$ maps $\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$ into itself.

Prove $\mathscr{Q}$ is a contraction mapping, then establish the existence of the mild solution: We note that

$$
\begin{aligned}
\overline{\mathcal{N}_2} K_0 (T^{s\alpha\vartheta} + \widehat{\mathcal{R}}^s) &= \overline{\mathcal{N}_2} K_0 \left(T^{s\alpha\vartheta} + \frac{1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}}{(1+s)\overline{\mathcal{N}_2} K_0}\right) \\
&= \frac{1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}}{1+s} - \left(1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}\right) + 1 \\
&= 1 - \frac{s}{1+s}\left(1 - \overline{\mathcal{N}_2} K_0 T^{s\alpha\vartheta}\right) < \frac{2+s}{2+2s}.
\end{aligned}
$$

Hence, we can deduce from (106) that

$$
\begin{aligned}
\left\|\mathscr{Q} w^{\dagger} - \mathscr{Q} w^{\ddagger}\right\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^{\nu}(\Omega))} &\le \overline{\mathcal{N}_2} K_0 (T^{s\alpha\vartheta} + \widehat{\mathcal{R}}^s) \|w^{\dagger} - w^{\ddagger}\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^{\nu}(\Omega))} \\
&\le \frac{2+s}{2+2s} \|w^{\dagger} - w^{\ddagger}\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^{\nu}(\Omega))}.
\end{aligned}
$$

We imply that $\mathscr{Q}$ is a contraction mapping on $\mathfrak{X}_{\alpha,\vartheta,\nu,T}(\mathcal{R})$ which has a unique fixed point u in this space. This fixed point is the unique mild solution of Problem (1)-(2). In addition, inequality (21) can be easily obtained. The remaining of the proof is split into the following steps.

Part a) Show that $u \in L^p(0, T; \mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega))$ for all $1 \le p < \frac{1}{\alpha\vartheta'}$:

It is easy to see the estimate $\|\mathbf{B}_\alpha(t, T) f\|_{\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega)} \lesssim t^{-\alpha\vartheta'} \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}$ for all $t > 0$. Moreover, by applying Lemma 5.4, we obtain

$$
\begin{aligned}
&\left\|\mathbf{B}_\alpha(t, T)\int_0^T \mathbf{P}_\alpha(T-r) G(r, u(r)) dr\right\|_{\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega)} \\
&\qquad \lesssim t^{-\alpha\vartheta'} \left\|\int_0^T \mathbf{P}_\alpha(T-r) G(r, u(r)) dr\right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \lesssim t^{-\alpha\vartheta'} \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}.
\end{aligned}
$$

On the other hand, it follows from $\nu + (\vartheta' - \vartheta) \le \nu + (1 - \vartheta)$ that the Sobolev embedding $\mathbb{H}^{\nu+(1-\vartheta)}(\Omega) \hookrightarrow \mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega)$ holds. Hence, we can infer from Lemma 5.4 that

$$
\begin{aligned}
\left\|\int_0^t \mathbf{P}_\alpha(t-r) G(r, u(r)) dr\right\|_{\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega)} &\lesssim \left\|\int_0^t \mathbf{P}_\alpha(t-r) G(r, u(r)) dr\right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&\lesssim t^{-\alpha\vartheta} \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \lesssim t^{-\alpha\vartheta'} \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}.
\end{aligned}
$$

Summarily, the solution $u \in L^p(0,T;\mathbb{H}^{\nu+(\vartheta'-\vartheta)}(\Omega))$ for all $1 \le p < \dfrac{1}{\alpha\vartheta'}$ since $t^{-\alpha\vartheta'}$ clearly belongs to $L^p(0,T;\mathbb{R})$ for all $1 \le p < \dfrac{1}{\alpha\vartheta'}$. The proof is finalized.

Part b) Show that $u \in C([0,T];\mathbb{H}^{\nu-\eta}(\Omega))$: Let t, t' such that $0 \le t \le \widetilde{t} \le T$. Our purpose here is to find an upper bound of the norm $\left\| u(\widetilde{t}) - u(t) \right\|_{\mathbb{H}^{\nu-\eta}(\Omega)}$. Since $\vartheta < \eta \le \vartheta + 1$ and $0 < \vartheta < 1$, the number $\dfrac{1+\vartheta-\eta}{2}$ consequently belongs to $[0,1]$. Hence, replacing $1 - \dfrac{1-\vartheta}{2}$ by $\dfrac{1+\vartheta-\eta}{2}$ helps to improve the inequalities (97). Indeed, we have

$$\left| \frac{E_{\alpha,\alpha}(-\lambda_j r^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \right| \lesssim r^{\alpha(\eta-\vartheta-1)} \lambda_j^{\eta-\vartheta}. \tag{108}$$

As a consequence of the above inequality, we have

$$\begin{aligned} \left\| \mathbf{B}_\alpha(\widetilde{t},T)f - \mathbf{B}_\alpha(t,T)f \right\|_{\mathbb{H}^{\nu-\eta}(\Omega)} &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \int_t^{\widetilde{t}} r^{\alpha(\eta-\vartheta)-1} dr \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \left\{ \begin{array}{l} (\widetilde{t}-t)^{\alpha(\eta-\vartheta)} \mathbf{1}_{0<\alpha(\eta-\vartheta)\le 1} \\ \left((\widetilde{t}-t)^{\alpha(\eta-\vartheta)-1} + \gamma \right) \mathbf{1}_{1<\alpha(\eta-\vartheta)<2} \end{array} \right\}, \end{aligned}$$

where the number $\alpha(\eta-\vartheta)$ belongs to the interval $(0,2)$. Employing Lemma 5.4 allows that

$$\begin{aligned} \|\mathfrak{M}_4\|_{\mathbb{H}^{\nu-\eta}(\Omega)} &\lesssim \left\| \int_0^T \mathscr{P}_\alpha(T-r) G(r,u(r)) dr \right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \left\{ \begin{array}{l} (\widetilde{t}-t)^{\alpha(\eta-\vartheta)} \mathbf{1}_{0<\alpha(\eta-\vartheta)\le 1} \\ \left((\widetilde{t}-t)^{\alpha(\eta-\vartheta)-1} + \gamma \right) \mathbf{1}_{1<\alpha(\eta-\vartheta)<2} \end{array} \right\} \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \left\{ \begin{array}{l} (\widetilde{t}-t)^{\alpha(\eta-\vartheta)} \mathbf{1}_{0<\alpha(\eta-\vartheta)\le 1} \\ \left((\widetilde{t}-t)^{\alpha(\eta-\vartheta)-1} + \gamma \right) \mathbf{1}_{1<\alpha(\eta-\vartheta)<2} \end{array} \right\}, \end{aligned}$$

provided that notation $\mathfrak{M}_4$ is given by (53). Now, let us consider the terms $\mathfrak{M}_2$ and $\mathfrak{M}_3$. It indicates from $\mu < \vartheta$ and $\vartheta < \eta$ that $0 \le -\sigma \le \eta - \nu$, and it results $\mathbb{H}^\sigma(\Omega) \hookrightarrow \mathbb{H}^{\nu-\eta}(\Omega)$. This suggests to estimate the term $\mathfrak{M}_2$. In actual fact, we have

$$\begin{aligned} \|\widetilde{t}\|_{\mathbb{H}^{\nu-\eta}(\Omega)} &\lesssim \int_0^t \int_{t-r}^{t'-r} \rho^{\alpha-2} \|G(r,u(r))\|_{\mathbb{H}^\sigma(\Omega)} d\rho dr \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \int_0^t \int_{t-r}^{\widetilde{t}-r} \rho^{\alpha-2} \left(r^{-\alpha\vartheta} + r^{-(1+s)\alpha\vartheta} \right) \mathfrak{L}_3(r) d\rho dr \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} (\widetilde{t}-t)^{\alpha-1} \int_0^t \left(r^{-\alpha(\vartheta+\zeta)} + r^{-\alpha((1+s)\vartheta+\zeta)} \right) dr, \end{aligned}$$

provided that $L_3(t) \le K_0 t^{-\alpha\zeta}$ as G satisfies (20). Since $\zeta < \dfrac{1}{\alpha} - (1+s)\vartheta$, we derive that the integral on the right-hand side of the previous expression is convergent. Hence, we obtain immediately the estimate

$$\|\mathfrak{M}_2\|_{\mathbb{H}^{\nu-\eta}(\Omega)} \lesssim \|u\|_{C^{\alpha\vartheta}((0,T];\mathbb{H}^\nu(\Omega))} (\widetilde{t}-t)^{\alpha-1}$$

and from the local property of G as in (20), we find that

$$\begin{aligned} \|\mathfrak{M}_3\|_{\mathbb{H}^{\nu-\eta}(\Omega)} &\lesssim \int_t^{\widetilde{t}} (\widetilde{t}-\tau)^{\alpha-1} \|G(r,u(r))\|_{\mathbb{H}^\sigma(\Omega)} dr \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \int_t^{\widetilde{t}} (\widetilde{t}-r)^{\alpha-1} \left(r^{-\alpha\vartheta} + r^{-(1+s)\alpha\vartheta} \right) \mathfrak{L}_3(r) dr \\ &\lesssim \|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \int_t^{\widetilde{t}} (\widetilde{t}-r)^{\alpha-1} \left(r^{-\alpha(\vartheta+\zeta)} + r^{-\alpha((1+s)\vartheta+\zeta)} \right) dr \lesssim (\widetilde{t}-t). \end{aligned}$$

The above explanations imply $u \in C([0,T];\mathbb{H}^{\nu-\eta}(\Omega))$. $\square$

6. Discussion and remark on global well-posedness

In this section, our main purpose is to establish the global existence and uniqueness of a mild solution to Problem (1)-(2) by considering critical nonlinearities. We focus on the nonlinearities that appear in application models, such as time fractional Ginzburg-Landau, Allen–Cahn, Burgers, Navier-Stokes, Schrödinger equations, etc. By observing the applications in Subsections 4.1 and 4.2, the nonlinearities

$$G(t,v)=\rho(t)|v|^s v, \quad \text{or} \quad G(t,v)=-\rho(t)(v\cdot\nabla)v,$$

fulfill the following locally Lipschitz continuity

$$\|G(t,v_1)-G(t,v_2)\|_{\mathbb{H}^{\sigma}(\Omega)} \le L_4(t)\Big(\|v_1\|^s_{\mathbb{H}^{\nu}(\Omega)}+\|v_2\|^s_{\mathbb{H}^{\nu}(\Omega)}\Big)\|v_1-v_2\|_{\mathbb{H}^{\nu}(\Omega)}, \tag{109}$$

see the proofs of Theorems 4.1 and 4.2.

Recall that Theorem 3.4 had considered the space $C^{\alpha\vartheta}((0,T];\mathbb{H}^{\nu}(\Omega))$ corresponding to the weighted function $t^{\alpha\vartheta}$. In order to establish a global existence, we take inspiration from replacing $t^{\alpha\vartheta}$ by another weighted function, which includes a suitable parameter m. For each $m>0$, we set

$$\Phi_m(t) := \frac{te^{\frac{t}{m}}}{e^{\frac{t}{m}}-1}, \quad t>0,$$

and define the following time weighted space

$$\mathbf{X}_{\nu,\vartheta}[\Phi_m] := \Big\{ w\in L^{\infty}_{loc}(0,T;\mathbb{H}^{\nu}(\Omega)) \;\Big|\; \|w\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} := \Big\| t^{\alpha\vartheta}\Phi_m(t)\|w(t)\|_{\mathbb{H}^{\nu}(\Omega)}\Big\|_{L^{\infty}(0,T)} <\infty\Big\}.$$

Theorem 6.1. *Assume that $\alpha\in(1,2)$, $\sigma\in(-1,0)$, $0<\nu<1+\sigma$ and $s>0$. Let ϑ such that $\vartheta\in(\nu-\sigma,1)$ and set $\mu=\nu-\sigma$. Let ζ satisfy*

$$\zeta<\min\Big(\alpha^{-1}-(1+s)\vartheta;\ \vartheta(1-s)-\nu+\sigma\Big). \tag{110}$$

The function $G:[0,T]\times\mathbb{H}^{\nu}(\Omega)\longrightarrow\mathbb{H}^{\sigma}(\Omega)$ satisfies $G(0)=0$ and (109) with $L_4(t)t^{\alpha\zeta}\in L^{\infty}(0,T)$. Let $f\in\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)$. There exists $m>0$, $\kappa>0$ such that if $\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\le\kappa$ then Problem (1)-(2) has a unique mild solution

$$u\in\mathbf{X}_{\nu,\vartheta}[\Phi_m]\cap L^p(0,T;\mathbb{H}^{\nu}(\Omega)),$$

for $p\in[1,1/\alpha\vartheta)$, which corresponds to the estimate

$$\|u(t)\|_{\mathbb{H}^{\nu}(\Omega)} \lesssim \frac{1}{t^{\alpha\vartheta}\sqrt{m+t}}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}, \ \text{a.e. } t\in(0,T). \tag{111}$$

In order to prove Theorem 6.1, we need the following lemma. Suppose that L_4 is defined by the above theorem. Then, the following improper integral can be estimated similarly as $\widehat{L}_3(t)$ (see definition (100) and the estimate (101))

$$\begin{aligned}\widehat{L}_4(t) &:= \int_0^t (t-r)^{\alpha(1-\mu)-1}r^{-(1+s)\alpha\vartheta}L_4(r)dr\\ &\le \widetilde{K}_0\int_0^t (t-r)^{\alpha(1-\mu)-1}r^{-\alpha((1+s)\vartheta+\zeta)}dr,\ \widetilde{K}_0:=\|L_4(t)t^{\alpha\zeta}\|_{L^{\infty}(0,T)}\\ &\le \widetilde{K}_0\mathscr{N}_1 t^{\alpha((1-\mu)-(1+s)\vartheta-\zeta)} \le \widetilde{K}_0\mathscr{N}_1 T^{\alpha((1-\mu)-s\vartheta-\zeta)}t^{-\alpha\vartheta},\end{aligned} \tag{112}$$

for each $t>0$. This argument will help to estimate the nonlinear function G. Furthermore, the operators $\mathbf{B}_\alpha(t,T)$, $\mathbf{P}_\alpha(t)$ can be estimated analogously as the proof of Lemma 5.4. In Lemma 6.2 below, these tools will be used to establish relevant Lipschitz continuities for solution terms on a closed ball $\mathcal{U}_{\mathcal{R}}$ of the space $\mathbf{X}_{\nu,\vartheta}[\Phi_m]$.

Lemma 6.2. *Assume that all assumptions of Theorem 6.1 are fulfilled. Let $\mathcal{U}_{\mathcal{R}}$ be the closed ball in $\mathbf{X}_{\nu,\vartheta}[\Phi_m]$ centered at zero with radius $\mathcal{R}>0$. There exists a constant $m_0>0$ such that*

a) *For all $w^{\dagger},w^{\ddagger}\in\mathcal{U}_{\mathcal{R}}$ and $m\ge m_0$,*

$$\left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,w^{\dagger}(r))-G(r,w^{\ddagger}(r))\Big)dr\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \le \mathscr{N}_3\frac{\mathcal{R}^s}{m^s}\left\|w^{\dagger}-w^{\ddagger}\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]},$$

where $\mathscr{N}_3:=4M_\alpha\widetilde{K}_0\mathscr{N}_1T^{\alpha((1-\mu)-s\vartheta-\zeta)}$;

b) *For all* $w^\dagger, w^\ddagger \in \mathcal{U}_\mathcal{R}$ *and* $m \geq m_0$,

$$\left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr \right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq \mathscr{N}_4 \frac{\mathcal{R}^s}{m^s} \left\| w^\dagger - w^\ddagger \right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]},$$

where $\mathscr{N}_4 := 4\mathcal{N}_2 M_\alpha \widetilde{K}_0 \mathscr{N}_1 T^{\alpha((\vartheta-\mu)-(1+s)\vartheta-\zeta)}$.

Proof. We firstly prove Part a. According to the definition of the space $\mathbf{X}_{\nu,\vartheta}[\Phi_m]$, we can see that: if a function $w \in \mathbf{X}_{\nu,\vartheta}[\Phi_m]$, then $\|w(r)\|_{\mathbb{H}^\nu(\Omega)} \leq (r^{-\alpha\vartheta}/\Phi_m(r))\|w\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]}$ for almost everywhere $r \in (0,T)$. Therefore, by using assumption (109), we can estimate the difference $G(r,w^\dagger(r)) - G(r,w^\ddagger(r))$ in $\mathbb{H}^\sigma(\Omega)$ as follows

$$\begin{aligned}\left\|G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\right\|_{\mathbb{H}^\sigma(\Omega)} &\leq L_4(r)\Big(\|w^\dagger(r)\|^s_{\mathbb{H}^\nu(\Omega)} + \|w^\ddagger(r)\|^s_{\mathbb{H}^\nu(\Omega)}\Big)\|w^\dagger(r) - w^\ddagger(r)\|_{\mathbb{H}^\nu(\Omega)} \\ &\leq L_4(r)\left(\left(\frac{\mathcal{R}r^{-\alpha\vartheta}}{\Phi_m(r)}\right)^s + \left(\frac{\mathcal{R}r^{-\alpha\vartheta}}{\Phi_m(r)}\right)^s\right)\frac{r^{-\alpha\vartheta}}{\Phi_m(r)}\left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]},\end{aligned}$$

for all $w_\dagger, w_\ddagger \in \mathcal{U}_\mathcal{R}$. Moreover, it is helpful to note that the function $y \mapsto ye^y - e^y + 1$ is increasing on the interval $(0,\infty)$ since $\partial(ye^y - e^y + 1)/\partial y = ye^y \geq 0$ on $(0,\infty)$. This implies the inequality $ye^y - e^y + 1 \geq 0$ for all $y \in (0,\infty)$, which ensures that $e^y - 1 \leq ye^y$ for all $y \in (0,\infty)$. Subsequently,

$$\frac{1}{\Phi_m(r)} = \frac{e^{\frac{r}{m}} - 1}{re^{\frac{r}{m}}} \leq \frac{1}{m}, \quad \forall r > 0,\ m > 0.$$

Due to the above arguments and using the same way as (98), we obtain the following chain

$$\begin{aligned}&\left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \\ &\quad= \operatorname*{ess\,sup}_{0\leq t\leq T} t^{\alpha\vartheta}\Phi_m(t)\left\|\int_0^t \mathbf{P}_\alpha(t-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr\right\|_{\mathbb{H}^\nu(\Omega)} \\ &\quad\leq \operatorname*{ess\,sup}_{0\leq t\leq T} t^{\alpha\vartheta}\Phi_m(t)\int_0^t (t-r)^{\alpha(1-\mu)-1}M_\alpha\left\|G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\right\|_{\mathbb{H}^\sigma(\Omega)}dr \\ &\quad\leq \frac{2M_\alpha\mathcal{R}^s}{m^s}\left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]}\left(\operatorname*{ess\,sup}_{0\leq t\leq T}\frac{t^{\alpha\vartheta}\Phi_m(t)}{m}\widehat{L}_4(t)\right) \\ &\quad\leq \frac{2M_\alpha\mathcal{R}^s}{m^s}\left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]}\left(\operatorname*{ess\,sup}_{0\leq t\leq T}\frac{\Phi_m(t)}{m}\right)\widetilde{K}_0\mathscr{N}_1 T^{\alpha((1-\mu)-s\vartheta-\zeta)},\end{aligned}$$

where (112) has been used with $\widetilde{K}_0 := \|L_4(t)t^{\alpha\zeta}\|_{L^\infty(0,T)}$. Let us consider the latter essential supremum. By simple computations, the derivative of first order $\partial\Phi_m/\partial t$ is equal to the product of $e^{t/m}/(e^{t/m}-1)^2$ and $e^{t/m} - 1 - t/m$. Hence, $\partial\Phi_m/\partial t > 0$ for all $t > 0$, and so $\Phi_m(t)$ is increasing on $(0,T]$. This argument yields that the supermum $\operatorname{ess\,sup}_{0\leq t\leq T}\Phi_m(t)$ is bounded by $\Phi_m(T)$. On the other hand, one can observe the following limit

$$\lim_{m\to+\infty}\operatorname*{ess\,sup}_{0\leq t\leq T}\frac{\Phi_m(t)}{m} \leq \lim_{m\to+\infty}\frac{\Phi_m(T)}{m} = \lim_{m\to+\infty}\frac{T}{m}\frac{e^{\frac{T}{m}}}{e^{\frac{T}{m}} - 1} = 1. \tag{113}$$

Therefore, there exists a positive constant m_0 such that

$$\operatorname*{ess\,sup}_{0\leq t\leq T}\frac{\Phi_m(t)}{m} \leq 2, \quad \forall m \geq m_0. \tag{114}$$

This shows Part a.

Now, we proceed to prove Part b. By applying the estimate (92) in Part a of Lemma 5.4 and using the same argument as in the estimate (103), one can see that

$$\begin{aligned}
&\left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr \right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \\
&\quad = \operatorname*{ess\,sup}_{0\le t\le T} t^{\alpha\vartheta}\Phi_m(t) \left\| \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr \right\|_{\mathbb{H}^\nu(\Omega)} \\
&\quad \le \operatorname*{ess\,sup}_{0\le t\le T} \mathscr{N}_2\Phi_m(t) \left\| \int_0^T \mathbf{P}_\alpha(T-r)\Big(G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\Big)dr \right\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \\
&\quad \le \operatorname*{ess\,sup}_{0\le t\le T} \mathscr{N}_2\Phi_m(t) \int_0^T (T-r)^{\alpha(\vartheta-\mu)-1} M_\alpha \left\|G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\right\|_{\mathbb{H}^\sigma(\Omega)} dr.
\end{aligned}$$

The norm of the difference $G(r,w^\dagger(r)) - G(r,w^\ddagger(r))$ in $\mathbb{H}^\sigma(\Omega)$ can be dealt as Part a. Moreover, we recall that $\alpha(\vartheta-\mu) > 0$ and $1-\alpha((1+s)\vartheta+\zeta) > 0$, see Proof of Part b of Lemma 5.4. Hence, the improper integrals below can be estimated similarly as (112). In summary, we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T} \mathscr{N}_2\Phi_m(t) \int_0^T (T-r)^{\alpha(\vartheta-\mu)-1} M_\alpha \left\|G(r,w^\dagger(r)) - G(r,w^\ddagger(r))\right\|_{\mathbb{H}^\sigma(\Omega)} dr \\
&\quad \le \frac{2\mathscr{N}_2 M_\alpha \mathcal{R}^s}{m^s} \left(\operatorname*{ess\,sup}_{0\le t\le T} \frac{\Phi_m(t)}{m} \right) \left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \left(\int_0^T (T-r)^{\alpha(\vartheta-\mu)-1} r^{-(1+s)\alpha\vartheta} L_4(r)dr \right) \\
&\quad \le \frac{2\mathscr{N}_2 M_\alpha \mathcal{R}^s}{m^s} \left(\operatorname*{ess\,sup}_{0\le t\le T} \frac{\Phi_m(t)}{m} \right) \left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \widetilde{K}_0 \left(\int_0^T (T-r)^{\alpha(\vartheta-\mu)-1} r^{-\alpha((1+s)\vartheta+\zeta)} dr \right) \\
&\quad \le \frac{2\mathscr{N}_2 M_\alpha \mathcal{R}^s}{m^s} \left(\operatorname*{ess\,sup}_{0\le t\le T} \frac{\Phi_m(t)}{m} \right) \left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \widetilde{K}_0 \mathscr{N}_1 T^{\alpha((\vartheta-\mu)-(1+s)\vartheta-\zeta)}.
\end{aligned}$$

Part b is then proved by the argument (114). □

Next, we shall use Lemma 6.2 to obtain Theorem 6.1, where the contraction mapping principle is suitably used to prove the existence and uniqueness. In fact, a small norm assumption on f is required.

Proof of Theorem 6.1. The limit in (113) subsequently ensures that one can find a constant m satisfying $(T/m)e^{\frac{T}{m}}/(e^{\frac{T}{m}}-1) \le 2$ and $m \ge m_0$, where m_0 is the constant obtained by Lemma 6.2. In the following arguments, we set $\mathcal{R} := 4\mathscr{N}_2 m\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}$. Let us use the notation $\mathcal{U}_\mathcal{R}$ as Lemma 6.2, and define the mapping $\widetilde{\mathscr{Q}} : \mathcal{U}_\mathcal{R} \to \mathcal{U}_\mathcal{R}$ by

$$\widetilde{\mathscr{Q}}w := \mathbf{B}_\alpha(t,T)f + \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr - \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,w(r))dr, \tag{115}$$

for $w \in \mathcal{U}_\mathcal{R}$. It is necessary to prove that $\widetilde{\mathscr{Q}}$ possesses a unique fixed point in $\mathcal{U}_\mathcal{R}$.

Firstly, we will show that $\widetilde{\mathscr{Q}}$ is well-defined on $\mathcal{U}_\mathcal{R}$, namely, $\widetilde{\mathscr{Q}}w \in \mathcal{U}_\mathcal{R}$ for all $w \in \mathcal{U}_\mathcal{R}$. Let us consider the first term in the right hand side of (115). By applying the estimate (92) in Part a of Lemma 5.4, we can see that

$$\begin{aligned}
\|\mathbf{B}_\alpha(\cdot,T)f\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} &= \operatorname*{ess\,sup}_{0\le t\le T} t^{\alpha\vartheta}\Phi_m(t)\|\mathbf{B}_\alpha(t,T)f\|_{\mathbb{H}^\nu(\Omega)} \\
&\le \operatorname*{ess\,sup}_{0\le t\le T} t^{\alpha\vartheta}\Phi_m(t)\big(\mathscr{N}_2 t^{-\alpha\vartheta}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}\big) \le 2\mathscr{N}_2 m\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)},
\end{aligned}$$

where the essential supremum $\operatorname{ess\,sup}_{0\le t\le T}\Phi_m(t)$ is bounded by $\Phi_m(T)$, and $\Phi_m(T) \le 2m$ for $m \ge m_0$ as (114) (see the proof of Part a of Lemma 6.2). In addition, the two last terms in the right hand side of (115) can be bounded by making use of Lemma 6.2 with respect to $w^\dagger = w$ and $w^\ddagger \equiv 0$. Indeed,

$$\left\| \int_0^t \mathbf{P}_\alpha(t-r)G(r,w(r))dr - \int_0^T \mathbf{B}_\alpha(t,T)\mathbf{P}_\alpha(T-r)G(r,w(r))dr \right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \le \mathscr{N}_5 \frac{\mathcal{R}^s}{m^s}\|w\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]},$$

where $\mathscr{N}_5 := \mathscr{N}_3 + \mathscr{N}_4$.

Let us set $\kappa := (4\mathscr{N}_2)^{-1}(2\mathscr{N}_5)^{-1/s}$ and assume that $\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} \le \kappa$. Then, the radius $\mathcal{R}$ is bounded by $m(2\mathscr{N}_5)^{-1/s}$. This consequently implies $\mathscr{N}_5\mathcal{R}^s/m^s \le 1/2$. Henceforth, by combining the above estimates, we derive

$$\|\widetilde{\mathscr{Q}}w\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq 2\mathscr{N}_2 m\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)} + \frac{\mathscr{N}_5\mathcal{R}^s}{m^s}\|w\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq \frac{\mathcal{R}}{2}+\frac{\mathcal{R}}{2} = \mathcal{R}.$$

We conclude that the mapping $\widetilde{\mathscr{Q}}$ is well-defined on $\mathcal{U}_{\mathcal{R}}$.

Secondly, $\widetilde{\mathscr{Q}}$ is obviously a contraction mapping on $\mathcal{U}_{\mathcal{R}}$. Indeed, for all $w^\dagger, w^\ddagger \in \mathcal{U}_{\mathcal{R}}$, making use of Lemma 6.2 gives that

$$\|\widetilde{\mathscr{Q}}w^\dagger - \widetilde{\mathscr{Q}}w^\ddagger\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq \frac{\mathscr{N}_5\mathcal{R}^s}{m^s}\left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq \frac{1}{2}\left\|w^\dagger - w^\ddagger\right\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]}.$$

Due to the contraction mapping principle, this reads that $\widetilde{\mathscr{Q}}$ possesses a unique fixed point $u \in \mathcal{U}_{\mathcal{R}}$. Besides, we also have $\|u\|_{\mathbf{X}_{\nu,\vartheta}[\Phi_m]} \leq 4\mathscr{N}_2 m\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}$, which deduces

$$\|u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim \frac{t^{-\alpha\vartheta}}{\Phi_m(t)}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}, \ \textit{a.e. } t \in (0,T).$$

In order to obtain property (111), we will prove the following inequality $\frac{e^a-1}{ae^a} \leq \frac{1}{\sqrt{1+a}}$, for all $a > 0$. For this purpose, we set $h(a) = ((e^a-1)/e^a)\sqrt{1+a} - a$, for $a \geq 0$. By making some direct computations, one can check the following differentiation

$$\frac{\partial h}{\partial a} = \frac{2a+1+e^a(1-2\sqrt{1+a})}{2e^a\sqrt{1+a}}, \quad \text{and} \quad \frac{\partial^2 h}{\partial a^2} = -\frac{4a^2+4a+e^a-1}{4e^a(1+a)\sqrt{1+a}},$$

It is obvious that $\partial^2 h/\partial a^2 \leq 0$ for all $a > 0$. Subsequently, $\partial h/\partial a$ is decreasing on $\mathbb{R}_+$, and so $\partial h/\partial a \leq 0$ for all $a > 0$. This deduces that $h(a) \leq 0$. Summarily, $\frac{e^a-1}{ae^a} \leq \frac{1}{\sqrt{1+a}}$, for all $a > 0$, which shows that $1/\Phi_m(t) \leq 1/(m\sqrt{1+t/m})$, and so

$$\|u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim \frac{t^{-\alpha\vartheta}}{\sqrt{m+t}}\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}, \ \textit{a.e. } t \in (0,T).$$

Note that, the latter inequality also yields that $\|u(t)\|_{\mathbb{H}^\nu(\Omega)} \lesssim t^{-\alpha\vartheta}$, which ensures $u \in L^p(0,T;\mathbb{H}^\nu(\Omega))$ for $p \in [1, 1/\alpha\vartheta)$. We complete the proof. □

Remark 6.1. *Upon the above proof, we just need to choose m_0 such that $\frac{T}{m_0}\frac{e^{T/m_0}}{e^{T/m_0}-1} \leq 2$, which requires $T/m_0 < 1.594...$ Of course, we can take $m_0 = T$. Therefore, it can be allowed to take $m = T$ in Lemma 6.2. Besides, the constant $\kappa := (4\mathscr{N}_2)^{-1}(2\mathscr{N}_5)^{-1/s}$ does not depend on m.*

On the other hand, by using the same techniques as in Theorem 3.4, one can also construct a suitable time continuity and spatial regularity for the solution.

Appendix

(AP.1.) A singular integral. It is useful to recall some basic properties of a singular integral. For given $z_1 > 0$, $z_2 > 0$, and $0 \leq a < b \leq T$, we denote by

$$\mathcal{K}(z_1, z_2, a, b) := \int_a^b (b-\tau)^{z_1-1}(\tau-a)^{z_2-1}d\tau = (b-a)^{z_1+z_2-1}\mathbf{B}(z_1,z_2), \tag{116}$$

where $\mathbf{B}$ is the Beta function, $\mathbf{B}(z_1,z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt$. Moreover, a special case of the Beta function is $\mathbf{B}(z, 1-z) = \pi/\sin(\pi z)$, see e.g. [11, 12, 13].

(AP.2.) A useful limit. For $a > 0$, $b > 0$, $t > 0$, $h > 0$, the following convergence holds

$$\int_0^t (t+h-r)^{a-1}r^{b-1}dr \xrightarrow{h\to 0^+} \int_0^t (t-r)^{a-1}r^{b-1}dr.$$

Indeed, it can be proved by noting that $\displaystyle\int_0^t (t-r)^{a-1}r^{b-1}dr = t^{a+b-1}\mathbf{B}(a,b)$ and

$$\int_0^t (t+h-r)^{a-1}r^{b-1}dr = (t+h)^{a+b-1}\int_0^{t/(t+h)}(1-s)^{a-1}s^{b-1}ds \xrightarrow{h\to 0^+} t^{a+b-1}\mathbf{B}(a,b).$$

(AP.3.) Proof of Lemma 2.4. The first inequality is estimated as follows

$$z^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j z^\alpha) \le M_\alpha z^{\alpha-1}\frac{1}{1+\lambda_j z^\alpha} = M_\alpha z^{\alpha-1}\Big(\frac{1}{1+\lambda_j z^\alpha}\Big)^\theta\Big(\frac{1}{1+\lambda_j z^\alpha}\Big)^{1-\theta} \le M_\alpha \lambda_j^{-\theta} z^{\alpha(1-\theta)-1},$$

and the second inequality is showed as follows

$$\begin{aligned}\mathscr{E}_{\alpha,T}(-\lambda_j t^\alpha) = \left|\frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)}\right| &\le \frac{M_\alpha}{m_\alpha}\left(\frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha}\right)^{1-\theta}\left(\frac{1+\lambda_j T^\alpha}{1+\lambda_j t^\alpha}\right)^{\theta}\\ &\le \frac{M_\alpha}{m_\alpha} T^{\alpha(1-\theta)}(1+\lambda_j T^\alpha)^\theta t^{-\alpha(1-\theta)}\\ &\le M_\alpha m_\alpha^{-1} T^{\alpha(1-\theta)}\Big(T^{\alpha\theta}+\lambda_1^{-\theta}\Big)\lambda_j^\theta t^{-\alpha(1-\theta)}.\end{aligned}$$

The proof is completed.

(AP.4.) List of constants. Here, we list some important constants appeared in this paper, where some of them contain the constant $C_1(\nu,\theta)$, $C_2(\nu,\sigma)$ in the embeddings (15) and (17). These constants cannot be omitted in some proofs of this paper.

$$\begin{cases}
\mathcal{M}_1 = \mathcal{M}_1(\alpha,\theta,T) := M_\alpha^2 m_\alpha^{-1} T^{\alpha(1-\theta)}\Big(T^{\alpha\theta}+\lambda_1^{-\theta}\Big),\\
\mathcal{M}_2 = \mathcal{M}_2(\alpha,\theta,T) := \dfrac{M_\alpha^2}{m_\alpha}\dfrac{T^{\alpha(1-\theta)}}{\alpha^2\theta(1-\theta)}(T^\alpha+\lambda_1^{-1}),\\
\mathscr{M}_1 = \mathscr{M}_1(\alpha,\theta,T) := \dfrac{\pi M_\alpha \lambda_1^{-\theta}T^{\alpha(1-\theta)}+\pi\mathcal{M}_1}{\sin\Big(\pi\alpha(1-\theta)\Big)},\\
\overline{\mathcal{M}}_1 = \overline{\mathcal{M}}_1(q,\alpha,\nu,\sigma,T) := C_2(\nu,\sigma)M_\alpha\dfrac{T^{\alpha+1/q}}{\alpha(\alpha q+1)^{1/q}},\\
\overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_2(q,\alpha,\theta,T) := \dfrac{T^{\alpha(1-\theta)}(T^{\alpha\theta+1/q}+\lambda_1^{-\theta})}{(1-\alpha(1-\theta)q)^{1/q}\alpha(1-\theta)},\\
\overline{\mathcal{M}}_3 = \overline{\mathcal{M}}_3(\alpha,T) := \dfrac{M_\alpha T^\alpha}{\alpha}\left(1+\dfrac{M_\alpha}{m_\alpha}(T^\alpha+\lambda_1^{-1})\right),\\
\mathscr{M}_2 = \mathscr{M}_2(q,\alpha,\nu,\sigma,T) := \|L_2\|_{L^\infty(0,T)}\displaystyle\sum_{1\le j\le 3}\overline{\mathcal{M}}_j,\\
\mathcal{N}_1 = \mathcal{N}_1(\alpha,\theta,\nu,T) := \mathcal{M}_1 M_\alpha^{-1}C_1(\nu,\theta)\Big(1-\|L_1\|_{L^\infty(0,T)}\mathscr{M}_1\Big)^{-1},\\
\mathcal{N}_2 = \mathcal{N}_2(\alpha,\vartheta,T) := M_\alpha m_\alpha^{-1}T^{\alpha\vartheta}\Big(T^{\alpha(1-\vartheta)}+\lambda_1^{\vartheta-1}\Big),\\
\mathscr{N}_1 = \mathscr{N}_1(\alpha,\mu,\vartheta,\zeta) := \max\Big\{\mathbf{B}(\alpha z_j;1-\alpha(1+s)\vartheta-\alpha\zeta), j=1,2\Big\},\ \begin{cases}z_1=\vartheta-\mu,\\ z_2=1-\mu,\end{cases}\\
\mathscr{N}_2 = \mathscr{N}_2(\alpha,\mu,\vartheta,\zeta,s,T) := M_\alpha\mathscr{N}_1\max\Big\{T^{\alpha(z_j-s\vartheta-\zeta)}, j=1,2\Big\},\\
\overline{\mathscr{N}_2} = \overline{\mathscr{N}_2}(\alpha,\mu,\vartheta,\zeta,s,T) := \mathscr{N}_2(1+\mathcal{N}_2T^{-\alpha\vartheta}),\\
\mathcal{N}_f = \mathcal{N}_f(\alpha,\nu,\vartheta,T,s) := \left(\dfrac{s}{\mathcal{N}_2\|f\|_{\mathbb{H}^{\nu+(1-\vartheta)}(\Omega)}}\right)^s\dfrac{1}{(2+2s)^{1+s}},\\
\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\alpha,\mu,\vartheta,\zeta,s,T) := \left(\dfrac{1-\overline{\mathscr{N}_2}K_0T^{s\alpha\vartheta}}{(1+s)\overline{\mathscr{N}_2}K_0}\right)^{1/s},\\
\eta_{glo} = \eta_{glo}(\alpha,\theta,\nu') := \left\{\begin{array}{l}\min\{\alpha(\theta+\nu'-1);\alpha-1\}\,\mathbf{1}_{0<\alpha(\theta+\nu'-1)\le 1}\\ \min\{\alpha(\theta+\nu'-1)-1;\alpha-1\}\,\mathbf{1}_{1<\alpha(\theta+\nu'-1)<2}\end{array}\right\},\\
\eta_{cri} = \eta_{glo}(\alpha,\eta,\vartheta) := \left\{\begin{array}{l}\min\{\alpha(\eta-\vartheta);\alpha-1\}\,\mathbf{1}_{0<\alpha(\eta-\vartheta)\le 1}\\ \min\{\alpha(\eta-\vartheta)-1;\alpha-1\}\,\mathbf{1}_{1<\alpha(\eta-\vartheta)<2}\end{array}\right\}.
\end{cases}$$

Acknowledgements. The first author is supported by Vietnam National Foundation for Science and Technology Development under grant number 101.02-2019.09. The research of T.C. has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-096540-B-I00, and Consejería de Economía y Conocimiento (Junta de Andalucía) under the FEDER project US-1254251.

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