ON A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION INVOLVING FRACTIONAL DERIVATIVE WITH MITTAG-LEFFLER KERNEL

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Abstract. In this paper, a nonlinear time-fractional Volterra equation with nonsingular Mittag-Leffler kernel in Hilbert spaces is studied. By applying the properties of Mittag-Leffler functions and the method of eigenvalue expansion, the existence of a mild solution of our problem is proved. The main tool to prove our results is the use of some Sobolev embeddings.

Keywords: Riemann-Liouville fractional derivative, Time diffusion equation, well–posedness, regularity estimates.

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1. Introduction

During the last decades, fractional calculus has become a powerful tool with accurate and successful results in modeling several complex phenomena in many various fields of science and engineering.

One of the branches of fractional calculus is the theory of fractional diffusion equations. Time-fractional diffusion equations open up great opportunities to model challenging phenomena such as long-range time memory or spatial interactions, nonlocal and local dynamics. For more details, see [1, 3, 38, 40] and the references therein. One of the modern trends in fractional calculus is the development of fractional operators with non-singular kernels. Studying new fractional derivatives with different singular or nonsingular kernels is important in order to satisfy the need for applied modeling in various fields, such as fluid mechanics, viscoelasticity, biology, physics and engineering [16, 17]. Some definitions of fractional derivatives were given based on nonsingular kernels such as the Atangana–Baleanu fractional derivatives.

Recently, many fractional models with non-singular kernel are receiving an increasing interest of many researchers with some different research directions. The special fact of Atangana–Baleanu derivative is that it possesses Mittag– Leffler function as its kernel, which is non-local as well as non-singular. The importance of fractional derivatives with non-singular kernels are particularly oriented to models of dissipative phenomena which cannot be adequately described by the classical fractional derivatives [11, 12, 13]. In [19], the authors considered a new chaotic model in two fractional operators, that is, the Caputo–Fabrizio derivative and the Atangana–Baleanu derivative. In [22], the authors considered a comparison study of bank data with different fractional operators such as Caputo, Caputo–Fabrizio and Atangana–Baleanu. They also proved that the results of the fractional Atangana–Baleanu operator is more accurate and flexible. In [23], modeling the transmission of dengue infection is introduced by using Caputo–Fabrizio (CF) and Atangana–Baleanu (AB) fractional derivatives. Until now, to the best of our knowledge, the works on analysis existence and regularity for ODEs and PDEs with Atangana–Baleanu derivative is very limited. In [18], the authors show the existence of the Keller-Segel model with Caputo and Atangana-Baleanu fractional derivative using fixed point theory. In [24], the fractional logistic model concerned with Atangana–Baleanu fractional derivative is considered.
This paper studies time fractional Volterra integro-differential equation with nonlinear source as follows

$$\mathbb{D}_t^\alpha u(x,t) + A^{\beta/2}u(x,t) = \int_0^t R(t,\tau)F(u(x,\tau))d\tau, \quad (x,t) \in (0,T)\times\Omega, \tag{1}$$

adressed with the Dirichlet boundary condition

$$u(\cdot,t)\big|_{\partial\Omega} = 0, \quad t \in (0,T), \tag{2}$$

and the initial value condition

$$u(x,0) = \phi(x), \quad x \in \Omega, \tag{3}$$

where $\mathbb{D}_t^\alpha$ is the *Atangana-Baleanu fractional derivative of order* α in Caputo sense (see Definition (2.1)). Existence and regularity for fractional diffusion equations with Caputo derivative has also been studied by several authors.

Regularity theory enables us to improve the smoothness and stability of solutions in various solution spaces and this leads to efficient ways for numerical simulations. For fractional diffusion equations with time fractional derivative in the sense of Caputo, Riemann-Liouville, etc, some authors developed and obtained interesting results. Carvalho et al. [29] established a local theory of mild solutions where A is a sectorial (nonpositive) operator. Guswanto [30] studied the existence and uniqueness of a local mild solution to a class of initial value problems for nonlinear fractional evolution equations. Besides, The existence, uniqueness and regularity of solutions are established in some previous works see, for instance, [26, 27, 28].

As for semilinear Volterra integrodifferential equations with integer order derivative, we can list some interesting works. In 1988, Heard and Rakin [31], considered the following Volterra integro-differential equation

$$\begin{cases} \partial_t u + A(t)u(t) = \int_0^t R(t,\tau)F(u(\tau))d\tau, \quad t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{4}$$

The authors studied in [32] the fractional Volterra integro-differential equation with ψ-Hilfer fractional derivative. In [33] it is proved the existence of solutions of certain kinds of nonlinear fractional integro-differential equations in Banach spaces. Rashid and Qaderi [34], established the local and global existence of mild solutions to a class of fractional integro-differential equations in an arbitrary Banach space. Rashid and Al-Omari [35] studied the local and global existence of mild solutions to a class of fractional semilinear impulsive Volterra type integro-differential evolution equations. In 2017, Gou and Li [36] studied local and global existence of mild solution for an impulsive fractional functional integro differential equation with non-compact semi-group in Banach spaces. Chen et al. [39] considered the following fractional non-autonomous integro-differential evolution equation of Volterra type in Banach space

$$\begin{cases} \partial_t^\alpha u + A(t)u(t) = \int_0^t R(t,\tau)F(u(\tau))d\tau + G(t,u(t)), \quad t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{5}$$

where ∂_t^α is the standard Caputo fractional time derivative of order $0 < \alpha \leq 1$. They first proved the local existence of mild solutions for corresponding fractional non-autonomous integro-differential evolution equation. Based on the local existence result and a piecewise extended method, they obtained a blow-up alternative result. Our new results in this paper are described as follows

- Our first goal is to establish the global existence for integro-differential evolution equation with fractional derivative. To overcome some difficulties, we introduce some weighted Lebesgue spaces. The key tool for our analysis here is the techniques on Kummer/hypergeometric function in Garrido-Atienza et al. [2, Lemma 8]. Hence, we can overcome some challenge estimates and obtain the global well-posed result.
- Next. the study on the regularity property for PDEs result in L^p Sobolev spaces is interesting and still an open direction. Until now, there are very few papers on L^p regularity for fractional evolution equation. As we know, regularity estimates for L^p spaces are key ingredients to prove the existence and uniqueness of very weak solutions of some classes of elliptic equations. The second new result in the present paper is to investigate the weighted L^p estimate for the mild solution when the initial datum ϕ belongs to L^q Sobolev space. To our knowledge, there are no previous results of this type for fractional diffusion with Atangana-Baleanu fractional derivative. The proof of our results is based on the Sobolev embedding theorems and some fixed point theorems.

The content of our paper is organized as follows. In Section 2, we recall some notation, definitions, and preliminary results regarding the solution representation and establish a definition of mild solution. In Section 3, we prove the well-posedness of a nonlinear time-fractional Volterra equation with the Atangana-Baleanu fractional derivative. In Section 4, we discuss an application of our main results to an initial value problem for a time-space Volterra equation.

2. Preliminaries

We will recall in this section some notation, definitions, and preliminary results concerning the solution representation and establish an appropriate definition of mild solution. We will structure the section in several subsections for more clarity.

2.1. **ODEs with the Atangana-Baleanu fractional derivative in Caputo's sense.** Let us first recall the following definition.

Definition 2.1. For $\varphi \in H^1(0,T)$, its Atangana-Baleanu fractional derivative in Caputo's sense of order α is given by

$$\mathbb{D}_t^\alpha \varphi(t) = \frac{\omega_\alpha}{\widehat{\alpha}} \int_0^t \frac{\partial \varphi(\tau)}{\partial \tau} E_{\alpha,1}(-\lambda_\alpha (t-\tau)^\alpha) d\tau, \tag{6}$$

where $\widehat{\alpha} = 1-\alpha$, $\omega_\alpha = \widehat{\alpha} + \alpha(\Gamma(\alpha))^{-1}$, and $\lambda_\alpha = \widehat{\alpha}^{-1}\alpha$.

Next we recall the following result.

Lemma 2.2. *Solutions of the initial value problem*

$$\mathbb{D}_t^\alpha f(t) + \kappa f(t) = G(f(t)), \; t \in (0,T), \quad f(0) = f_0, \tag{7}$$

are given by

$$f(t) = \eta E_{\alpha,1}(-\rho t^\alpha) f_0 + \frac{\widehat{\alpha}}{\omega_\alpha} \eta G(f(t)) + \mu \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}(-\rho(t-\tau)^\alpha) G(f(\tau)) d\tau,$$

where the numbers η, ρ, and μ are formulated as

$$\eta = \frac{\omega_\alpha}{\omega_\alpha + \widehat{\alpha}\kappa} \quad \rho = \frac{\alpha\kappa}{\omega_\alpha + \widehat{\alpha}\kappa}, \quad \mu = \frac{\omega_\alpha - \widehat{\alpha}}{\omega_\alpha}\eta + \frac{\widehat{\alpha}}{\omega_\alpha}\eta\rho.$$

Proof. This type of fractional differential equation involving the fractional-time derivative with nonsingular Mittag-Leffler kernel has been solved in [9] by the means of Laplace transform. □

For more details of Caputo fractional time-derivative and the fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel, we refer to recent interesting papers [42, 43, 44].

2.2. **Kummer/hypergeometric function.** Using the factor e^{-mt} with large enough number $m > 0$ plays an important role in establishing the global well-posedness. This makes the appearance of the so-called Kummer function or hypergeometric function, which is defined by

$$\mathcal{K}(a,b,m) := \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 (1-\tau)^{b-a-1} \tau^{a-1} e^{m\tau} d\tau, \quad b > a > 0, \; m \in \mathbb{C}.$$

A nice property of this function on its asymptotic behavior is the following:

$$\mathcal{K}(a,b,m) := \Gamma(b)(\Gamma(a))^{-1} e^m m^{-(b-a)} \left(1 + O\left(\frac{1}{|m|}\right)\right),$$

as can be seen in Garrido-Atienza et al. [2, Lemma 8]. For each $t > 0$, the above property can be scaled from the interval $(0,1)$ to $(0,t)$ by using a simple substitution. For $m > 0$, it may be checked that

$$\begin{aligned} \int_0^t (t-\tau)^{a-1} \tau^{b-1} e^{m\tau} d\tau &= t^{a+b-1} \int_0^t (1-\tau)^{a-1} \tau^{b-1} e^{mt\tau} d\tau \\ &= t^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \mathcal{K}(b, a+b, mt) \\ &= t^{a+b-1} \frac{e^{mt}\Gamma(a)}{(mt)^a} \left(1 + O\left(\frac{1}{mt}\right)\right) \\ &= t^{b-2} m^{-a} e^{mt} \Gamma(a) \left(t + O\left(\frac{1}{m}\right)\right). \end{aligned} \tag{8}$$

More details of this function can be found in [8].

2.3. **Solution representation.** In this paper, we consider the symmetric uniformly elliptic operator $A : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ defined by $A\varphi(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N A_{ij}(x) \frac{\partial}{\partial x_j} \varphi(x) \right) + b(x)\varphi(x)$. Here, we suppose that $A_{ij} = A_{ji}, 1 \le i,j \le N$ and there exists a constant $\Lambda > 0$ such that for all $(\xi_1, \xi_2, ..., \xi_N) \in \mathbb{R}^N$ and $\Lambda \sum_{i=1}^N \xi_i^2 \le \sum_{1 \le i,j \le N} A_{ij}(x) \xi_i \xi_j$, for $x \in \Omega \cup \partial\Omega$. Suppose, furthermore, that $A_{ij} \in C^1(\Omega \cup \partial\Omega)$, $b \in C(\Omega \cup \partial\Omega; \mathbb{R}^+)$. Then, the spectrum and the corresponding eigenvectors of A are given by $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \nearrow \infty$ and $b_1, b_2, ..., b_n, ... \subset H^2(\Omega) \cap H_0^1(\Omega)$. Note that $\{b_n\}$ forms an orthonormal basis of $L^2(\Omega)$. In order to find a formula for solutions, we firstly write equation of (1) in the n-th dimension as follows

$$\mathbb{D}_t^\alpha u_n(t) + \lambda_n^{\beta/2} u_n(t) = \int_0^t R(t,\tau) F_n(u(\cdot,\tau)) d\tau, \quad t \in (0,T),$$

which is equipped with the initial value condition $u_n(0) = \phi_n$. By applying Lemma 2.2, solutions of the above system of ODEs are given by

$$u_n(t) = \frac{\widehat{\alpha}}{\omega_\alpha} \eta_n \int_0^t R(t,\tau) F_n(u(\cdot,\tau)) d\tau + \eta_n E_{\alpha,1}(-\rho_n t^\alpha)\phi_n \\ + \mu_n \int_0^t \int_0^{t'} (t-t')^\alpha R(t',\tau) E_{\alpha,\alpha}(-\rho_n (t-t')^\alpha) F_n(u(\cdot,\tau)) d\tau dt', \tag{9}$$

where the numbers η_n, ρ_n, and μ_n are formulated by

$$\eta_n = \frac{\omega_\alpha}{\omega_\alpha + \widehat{\alpha}\lambda_n^{\beta/2}} \quad \rho_n = \frac{\alpha \lambda_n^{\beta/2}}{\omega_\alpha + \widehat{\alpha}\lambda_n^{\beta/2}}, \quad \mu_n = \frac{\omega_\alpha - \widehat{\alpha}}{\omega_\alpha}\eta_n + \frac{\widehat{\alpha}}{\omega_\alpha}\eta_n \rho_n. \tag{10}$$

Tuan, should we define F_n?

Let us recall the following relationship between the Mittag-Leffler functions $E_{\alpha,1}$, $E_{\alpha,\alpha}$ and the natural exponential function, see e.g. Section 2 in [1],

$$E_{\alpha,1}(-\rho_n t^\alpha) = \int_0^\infty \Phi_\alpha(y) \exp\big(-y t^\alpha \rho_n\big) dy, \qquad E_{\alpha,\alpha}(-\rho_n t^\alpha) = \alpha \int_0^\infty y \Phi_\alpha(y) \exp\big(-y t^\alpha \rho_n\big) dy,$$

where Φ_α denotes the Wright type function introduced by Mainardi in [41]

$$\Phi_\alpha(y) = \sum_{j=0}^\infty \frac{y^n}{n!\Gamma(1-\alpha(1+j))}, \quad y \in \mathbb{C}.$$

This function is an entire function on $\mathbb{C}$. The following classical result provides some essential relations used in this paper to obtain the main estimates.

Proposition 2.1. *For $\alpha \in (0,1)$ and $\theta > -1$. Then the following properties hold:*

$$\Phi_\alpha(y) \geq 0, \ \forall y \geq 0, \quad \text{and} \quad \int_0^\infty y^\theta \Phi_\alpha(y) dy = \frac{\Gamma(\theta+1)}{\Gamma(\theta\alpha+1)}, \ \forall \theta > -1. \tag{11}$$

This follows from (9) that

$$u_n(t) = \frac{\widehat{\alpha}}{\omega_\alpha} \eta_n \int_0^t R(t,\tau) F_n(u(\cdot,\tau)) d\tau + \int_0^\infty \Phi_\alpha(y) e^{-y t^\alpha \rho_n} \eta_n \phi_n dy \\ + \alpha \mu_n \int_0^t \int_0^{t'} \int_0^\infty y \Phi_\alpha(y) (t-t')^\alpha R(t',\tau) e^{-y(t-t')^\alpha \rho_n} F_n(u(\cdot,\tau)) dy d\tau dt', \tag{12}$$

where Φ_α is the Mainardi function or a particular Wright function, see e.g. [3, 4]. Now, let us define the following operators

$$\mathcal{B}_\rho \varphi := \sum_{j=1}^\infty \rho_n \varphi_n b_n, \quad \mathcal{B}_\eta \varphi := \sum_{j=1}^\infty \eta_n \varphi_n b_n, \quad \mathcal{B}_\mu \varphi := \sum_{j=1}^\infty \mu_n \varphi_n b_n.$$

Then, it follows from (12) that

$$u(x,t) = \frac{\widehat{\alpha}}{\omega_\alpha} \int_0^t R(t,\tau) \mathcal{B}_\eta F(u(x,\tau)) d\tau + \int_0^\infty \Phi_\alpha(y) e^{-y t^\alpha \mathcal{B}_\rho} \mathcal{B}_\eta \phi(x) dy \\ + \alpha \int_0^t \int_0^{t'} \int_0^\infty y \Phi_\alpha(y) (t-t')^\alpha R(t',\tau) e^{-y(t-t')^\alpha \mathcal{B}_\rho} \mathcal{B}_\mu F(u(x,\tau)) dy d\tau dt'. \tag{13}$$

Definition 2.3. If a function u in $L^p(0,T;L^q(\Omega))$ with some suitable numbers $p \geq 1$, $q \geq 1$ satisfies Equation (13), then it is called a mild solution of Problem (1)-(3).

3. Global existence on weighted Lebesgue space

3.1. **Fractional power operators and the solution spaces.** Firstly, we recall some literature on fractional power operators. Then, we present some useful functional spaces where the solution space will be mentioned. For each number $s \geq 0$, we define

$$\mathbb{X}^s(\Omega) := \left\{ \varphi = \sum_{n=1}^{\infty} \varphi_n b_n \in L^2(\Omega) : \sum_{n=1}^{\infty} \varphi_n^2 \lambda_n^s < \infty \right\}, \quad \varphi_n = \int_\Omega \varphi(x) b_n(x) dx.$$

Let us denote by $H^s(\Omega)$ the Sobolev-Slobodecki space $W^{s,p}(\Omega)$ when $p = 2$, and by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$. Through out of this paper, D is assumed to be smooth enough such that $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ for $0 < s < \frac{1}{2}$. This guarantees $H_0^s(\Omega) = H^s(\Omega)$. Moreover, it is well-known that

$$\mathbb{X}^s(\Omega) = \begin{cases} H_0^s(\Omega), & \text{for } 0 \leq s < \frac{1}{2}, \\ H_{00}^{1/2}(\Omega) \subsetneqq H_0^{1/2}(\Omega), & \text{for } s = \frac{1}{2}, \\ H_0^s(\Omega), & \text{for } \frac{1}{2} < s \leq 1, \\ H_0^1(\Omega) \cap H^s(\Omega), & \text{for } 1 < s \leq 2, \end{cases}$$

where we denote by $H_{00}^{1/2}(\Omega)$ the Lions–Magenes space. Let $\mathbb{X}^{-s}(\Omega)$ be the duality of $\mathbb{X}^s$ which corresponds to the dual inner product $(\cdot,\cdot)_{-s,s}$. Then, the operator $A^s : \mathbb{X}^s(\Omega) \to \mathbb{X}^{-s}v$ of fractional powers s can be defined by $A^s\varphi := \sum_{n=1}^{\infty} \beta_n^s (\varphi_n, b_n)_{-s,s} \varphi_n, \quad \forall v \in \mathbb{X}^s$. The above settings can be found in [5] (Section 3), [6] (Section 2), [7] (Section 2) and therein. In the next lemmas, we present some useful embeddings between the spaces mentioned above.

Lemma 3.1. *Let $0 \leq s \leq s' \leq 2$ and let $H^{-s}(\Omega)$ be the dual space of $H_0^s(\Omega)$. Then the following embeddings hold*

$$\mathbb{X}^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{X}^{-s}(\Omega), \tag{14}$$

and

$$\mathbb{X}^{s'}(\Omega) \hookrightarrow \mathbb{X}^s(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega) \hookrightarrow \mathbb{X}^{-s}(\Omega) \hookrightarrow \mathbb{X}^{-s'}(\Omega). \tag{15}$$

Lemma 3.2. *Given $1 \leq p, q < \infty$, $0 \leq s \leq s' < \infty$ and $s' - \frac{N}{p'} \geq s - \frac{N}{p}$. Then, there holds that*

$$W^{s',p'}(\Omega) \hookrightarrow W^{s,p}(\Omega). \tag{16}$$

Let us denote the following sets by

$$\mathcal{O}_1^+ := \left\{ (s;p) : s = \frac{N}{2},\ 1 \leq p < \infty \right\},\ \mathcal{O}_2^+ := \left\{ (s;p) : 0 \leq s < \frac{N}{2},\ 1 \leq p \leq \frac{2N}{N-2s} \right\},$$
$$\mathcal{O}_3^+ := \left\{ (s;\theta) : s > \frac{N}{2},\ \theta = s - \frac{N}{2} \right\},\ \mathcal{O}^- := \left\{ (s;p) : -\frac{N}{2} < s \leq 0,\ p \geq \frac{2N}{N-2s} \right\}.$$

As a consequence of the above lemma, we deduce that: $H^s(\Omega) \hookrightarrow L^p(\Omega)$ for $(s,p) \in \mathcal{O}_1^+ \cup \mathcal{O}_2^+$, and $H^s(\Omega) \hookrightarrow C^{0,\theta}(\Omega \cup \partial\Omega)$ for $(s,p) \in \mathcal{O}_3^+$. In contrast, $L^p(\Omega) \hookrightarrow H^s(\Omega)$ for $(s;p) \in \mathcal{O}^-$. These combine with the chain (16) to allow the following lemma.

Lemma 3.3. *a) There hold that $\mathbb{X}^s(\Omega) \hookrightarrow L^p(\Omega)$, $\quad (s;p) \in \mathcal{O}_1 \cup \mathcal{O}_2$, and $X^s(\Omega) \hookrightarrow C^{0,\theta}(\Omega \cup \partial\Omega)$, $\quad (s;p) \in \mathcal{O}_3$. Moreover,*

$$L^p(\Omega) \hookrightarrow \mathbb{X}^s(\Omega), \quad (s;p) \in \mathcal{O}^-. \tag{17}$$

According to Definition 2.3, the solution space should be $L^p(0,T;L^q(\Omega))$. In fact, since the purpose of the present paper is to investigate the global existence of mild solutions, it is necessary to narrow where we search for solutions. This is the rationale for introducing the so-called weight Lebesgue space $L^{p,\gamma,z}(0,T;L^q(\Omega))$. For given numbers $p, q \geq 1$, $\gamma > 0$, $z > 0$, this is the space containing all functions $\varphi \in L^p(0,T;L^q(\Omega))$ such that

$$L^{p,\gamma,z}(0,T;L^q(\Omega)) = \left\{ \varphi \in L^p(0,T;L^q(\Omega)), \|t^\gamma e^{-zt}\varphi\|_{L^p(0,T;L^q(\Omega))} < \infty \right\} \tag{18}$$

with the corresponding norm

$$\|\varphi\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} := \|t^\gamma e^{-zt}\varphi\|_{L^p(0,T;L^q(\Omega))} \tag{19}$$

3.2. **Global existence assertion with globally Lipschitz nonlinearity.** Our goal in this part is to establish the global existence of a mild solution of Problem (1)-(3). The following assumption will be needed throughout the paper:

- (AS) $N \geq 2$, $1 \leq p < 2$, and $q \geq 2$ such that $\frac{N}{N-\beta} < q \leq \frac{2N}{N-2\beta}$;
- (AF) Let r be satisfied $0 \leq r \leq \beta - \left(\frac{N}{2} - \frac{N}{q}\right)$. Suppose that the function $F : \mathbb{X}^0(\Omega) \to \mathbb{X}^{-r}(\Omega)$ satisfying
$$\big\|F(h_1) - F(h_2)\big\|_{\mathbb{X}^{-r}(\Omega)} \leq K_F \big\|h_1 - h_2\big\|_{\mathbb{X}^0(\Omega)}, \tag{20}$$
for all $h_1, h_2 \in \mathbb{X}^0(\Omega)$. Assume, furthermore, that $F(0) = 0$;
- (AR) Assume that $\varkappa > \frac{1-p}{p}$ and $\nu > \frac{2p-2}{p}$. Additionally, there exists a positive constant R_0 such that
$$|R(t,\tau)| \leq R_0 (t-\tau)^{\varkappa} \tau^{\nu}, \tag{21}$$
for all $0 < \tau < t < T$.

In the following theorem, we present a global existence of mild solutions to Problem (1)-(3). The word global indicates that there are no any restrictions on the time T and the Lipschitz coefficient K_F.

Theorem 3.4. *Given $0 < \alpha < 1$, $0 < \beta \leq 2$. Assume that hypothesis (AS) holds. If $\phi \in L^{\frac{Nq}{N+\beta q}}(\Omega)$ and assumptions (AF), (AR) hold, then there exists $z_0 > 0$ such that Problem (1)-(3) has only one mild solution $u \in L^{p,\gamma,z}(0,T;L^q(\Omega))$, with $0 < \gamma < \nu p + p - 1$.*

Proof. The proof is based on Banach contraction principle argument. Let us define the mapping $\mathcal{S} : L^{p,\gamma,z}(0,T;L^q(\Omega)) \to L^{p,\gamma,z}(0,T;L^q(\Omega))$ by $\mathcal{S}v = \mathcal{S}_0\phi + \mathcal{S}_1^F v + \mathcal{S}_2^F v$, for all $v \in L^{p,\gamma,z}(0,T;L^q(\Omega))$, where the terms $\mathcal{S}\phi$ and $\mathcal{S}_F v$ are given by
$$\left(\mathcal{S}_0\phi\right)(x,t) := \int_0^\infty \Phi_\alpha(y) e^{-yt^\alpha \mathcal{B}_\rho} \mathcal{B}_\eta \phi(x) dy, \ \left(\mathcal{S}_1^F v\right)(x,t) = \frac{\widehat{\alpha}}{\omega_\alpha} \int_0^t R(t,\tau) \mathcal{B}_\eta F(u(x,\tau)) d\tau,$$
$$\left(\mathcal{S}_2^F v\right)(x,t) := \alpha \int_0^t \int_0^{t'} \int_0^\infty y \Phi_\alpha(y)(t-t')^\alpha R(t',\tau) e^{-y(t-t')^\alpha \mathcal{B}_\rho} \mathcal{B}_\mu F(u(x,\tau)) dy d\tau dt'. \tag{22}$$
Now we continue to split the proof into several steps.

Step 1. Estimating the term $\mathcal{S}_0\phi$

Let us first estimate the quantity $\mathcal{B}_\eta\phi(x)$. According to the definition (10), one has
$$\omega_\alpha + \widehat{\alpha}\lambda_n^{\beta/2} = \widehat{\alpha}(1 + \lambda_n^{\beta/2}) + \alpha(\Gamma(\alpha))^{-1} \geq \widehat{\alpha}(1 + \lambda_n^{\beta/2}).$$
In addition, for each $0 \leq \xi \leq 2$, it holds that $1 + \lambda_n^{\beta/2} \geq \lambda_n^{\beta\xi/4}$. Therefore, we obtain
$$\Big\|\mathcal{B}_\eta\phi\Big\|^2_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)} = \sum_{n=1}^\infty \eta_n^2 \lambda_n^{\frac{Nq-2N}{2q}} \phi_n^2 \leq \sum_{n=1}^\infty \left(\frac{\omega_\alpha}{\omega_\alpha + \widehat{\alpha}\lambda_n^{\beta/2}}\right)^2 \lambda_n^{\frac{Nq-2N}{2q}} \phi_n^2 \leq \frac{\omega_\alpha^2}{\widehat{\alpha}^2} \sum_{n=1}^\infty \lambda_n^{\frac{N}{2} - \frac{N}{q} - \frac{\beta\xi}{2}} \phi_n^2,$$
which respectively implies
$$\|\mathcal{B}_\eta\phi\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)} \leq \frac{\omega_\alpha}{\widehat{\alpha}} \|\phi\|_{\mathbb{X}^{\frac{N}{2} - \frac{N}{q} - \frac{\beta\xi}{2}}(\Omega)}. \tag{23}$$
Moreover, for each number $0 \leq \sigma < 2$, there exists a positive constant C_σ satisfying that $e^{-yt^\alpha\rho_n} \leq C_\sigma(yt^\alpha\rho_n)^{-\sigma/2}$ for all n and all $t > 0$. We also note $\rho_n = \frac{\alpha\lambda_n^{\beta/2}}{\omega_\alpha + \widehat{\alpha}\lambda_n^{\beta/2}} \geq \frac{\alpha\lambda_1^{\beta/2}}{\omega_\alpha + \widehat{\alpha}\lambda_1^{\beta/2}} = \rho_1$, as $\lambda_n \geq \lambda_1$. Henceforth, we obtain the following estimate
$$\Big\|e^{-yt^\alpha\mathcal{B}_\rho}\mathcal{B}_\eta\phi\Big\|^2_{\mathbb{X}^{\frac{N}{2} - \frac{N}{q}}(\Omega)} = \sum_{n=1}^\infty e^{-2yt^\alpha\rho_n} \lambda_n^{\frac{Nq-2N}{2q}} \Big\langle \mathcal{B}_\eta\phi, b_n \Big\rangle^2 \leq C_\sigma^2 \sum_{n=1}^\infty (yt^\alpha\rho_n)^{-\sigma} \lambda_n^{\frac{Nq-2N}{2q}} \Big\langle \mathcal{B}_\eta\phi, b_n \Big\rangle^2,$$
which implies that
$$\Big\|e^{-yt^\alpha\mathcal{B}_\rho}\mathcal{B}_\eta\phi\Big\|_{\mathbb{X}^{\frac{N}{2} - \frac{N}{q}}(\Omega)} \leq C_\sigma \rho_1^{-\frac{\sigma}{2}} (yt^\alpha)^{-\frac{\sigma}{2}} \big\|\mathcal{B}_\eta\phi\big\|_{\mathbb{X}^{\frac{N}{2} - \frac{N}{q}}(\Omega)}. \tag{24}$$
From another point of view, it follows for $q \geq 2$ in assumption (AS) that the following Sobolev embedding
$$\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega) \hookrightarrow W_x^{N/2-N/q,2} \hookrightarrow L^q(\Omega), \tag{25}$$

holds. Therefore, there exists $M_1 > 0$ independently of x such that $\|\varphi\|_{L^q(\Omega)} \le M_1 \|\varphi\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)}$ for all $\varphi \in \mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)$. This combines with the estimates (23) and (24) to allow the following chain of estimates

$$\begin{aligned}
\left\|\mathcal{S}_0\phi\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} &= \left(\int_0^T \left(t^{\frac{\gamma}{p}} e^{-\frac{z}{p}t} \left\|\mathcal{S}_0\phi\right\|_{L^q(\Omega)}\right)^p dt\right)^{\frac{1}{p}} \\
&\le \left(\int_0^T \left(t^{\frac{\gamma}{p}} e^{-\frac{z}{p}t} \int_0^\infty \Phi_\alpha(y) \left\|e^{-yt^\alpha \mathcal{B}_\rho} \mathcal{B}_\eta \phi\right\|_{L^q(\Omega)} dy\right)^p dt\right)^{\frac{1}{p}} \\
&\le M_1 \left(\int_0^T \left(t^{\frac{\gamma}{p}} e^{-\frac{z}{p}t} \int_0^\infty \Phi_\alpha(y) \left\|e^{-yt^\alpha \mathcal{B}_\rho} \mathcal{B}_\eta \phi\right\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)} dy\right)^p dt\right)^{\frac{1}{p}} \\
&\le M_1 C_\sigma \rho_1^{-\frac{\sigma}{2}} \left(\int_0^T \left(t^{\frac{\gamma}{p}-\frac{\alpha\sigma}{2}} e^{-\frac{z}{p}t} \left(\int_0^\infty y^{-\frac{\sigma}{2}} \Phi_\alpha(y) dy\right) \left\|\mathcal{B}_\eta \phi\right\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)}\right)^p dt\right)^{\frac{1}{p}} \\
&\le \overline{M}_1 \left(\int_0^T t^{\left(\frac{\gamma}{p}-\frac{\alpha\sigma}{2}\right)p} e^{-zt} dt\right)^{\frac{1}{p}} \left\|\phi\right\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\frac{\beta\xi}{2}}(\Omega)},
\end{aligned} \tag{26}$$

where $\overline{M}_1 = M_1 C_\sigma \rho_1^{-\frac{\sigma}{2}} \frac{\Gamma\left(1-\frac{\sigma}{2}\right)}{\Gamma\left(1-\frac{\alpha\sigma}{2}\right)} \frac{\omega_\alpha}{\widehat{\alpha}}$ and we have used the fact that (see (11)) $\int_0^\infty y^{-\frac{\sigma}{2}} \Phi_\alpha(y) dy = \frac{\Gamma\left(1-\frac{\sigma}{2}\right)}{\Gamma\left(1-\frac{\alpha\sigma}{2}\right)}$. Let us take $\xi = 2$. It is easy to see that since $\frac{N}{N-\beta} < q \le \frac{2N}{N-2\beta}$, the pair $\left(\frac{N}{2} - \frac{N}{q} - \beta; q\right)$ belongs to the set $\mathcal{O}^-$ given in Lemma 3.3. Then, applying (17) of this lemma gives that the following Sobolev embedding $L^{\frac{Nq}{N+\beta q}}(\Omega) \hookrightarrow \mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega)$, namely, there exists $M_2 > 0$ independently of x such that

$$\|\varphi\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega)} \le M_2 \|\varphi\|_{L^{\frac{Nq}{N+\beta q}}(\Omega)}, \tag{27}$$

for any $\varphi \in L^{\frac{Nq}{N+\beta q}}(\Omega)$. Moreover, by taking $0 \le \sigma < \min(2(\gamma+1)/(\alpha p); 2)$, then the integral of the function $t \mapsto t^{\left(\frac{\gamma}{p}-\frac{\alpha\sigma}{2}\right)p} e^{-zt}$ on $(0,T)$ finitely exists. Consequently, we now imply from (26) that

$$\left\|\mathcal{S}_0\phi\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \le \overline{M}_1 M_2 \left(\int_0^T t^{\left(\frac{\gamma}{p}-\frac{\alpha\sigma}{2}\right)p} e^{-zt} dt\right)^{\frac{1}{p}} \left\|\phi\right\|_{L^{\frac{Nq}{N+\beta q}}(\Omega)}, \tag{28}$$

which allows us to end the Step 1.

Step 2. Estimating the term $\mathcal{S}_2^F v$

In this step, an estimation for the term $\mathcal{S}_2^F v$ will be established. For the term $\mathcal{S}_1^F v$, we shall take it up in the next step by using essential techniques in this step. Firstly, we note that the embedding $L^q(\Omega) \hookrightarrow L^2(\Omega) = \mathbb{X}^0$ holds as $q \ge 2$, i.e.,

$$\|(v_1 - v_2)(\cdot, s)\|_{L^2(\Omega)} \le M_3 \|(v_1 - v_2)(\cdot, s)\|_{L^q(\Omega)},$$

with a constant M_3 independent of x, s. On the other hand, by assumption $0 \le r \le \beta - \left(\frac{N}{2} - \frac{N}{q}\right)$ in (AF), we have $\frac{N}{2} - \frac{N}{q} - \beta \le -r$. Therefore, the following Sobolev embedding holds

$$\mathbb{X}^{-r}(\Omega) \hookrightarrow \mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega). \tag{29}$$

Then, there exists $c_F > 0$ such that $\|\varphi\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega)} \le c_F \|\varphi\|_{\mathbb{X}^{-r}(\Omega)}$ for all $\varphi \in \mathbb{X}^{-r}(\Omega)$. For the sake of simplicity let us denote $\widetilde{F}(v_1, v_2)$ the difference $F(v_1) - F(v_2)$. Then, by recalling the Sobolev embedding

$$\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega) \hookrightarrow W_x^{N/2-N/q,2} \hookrightarrow L^q(\Omega),$$

and using the analogue techniques as (23)-(24), we have

$$\begin{aligned}\left\|e^{-yt^\alpha\mathcal{B}_\rho}\mathcal{B}_\eta\widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{L^q(\Omega)} &\le M_1\left\|e^{-yt^\alpha\mathcal{B}_\rho}\mathcal{B}_\eta\widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)}\\
&\le M_1C_\sigma\rho_1^{-\frac{\sigma}{2}}(yt^\alpha)^{-\frac{\sigma}{2}}\left\|\mathcal{B}_\eta\widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{\mathbb{X}^{\frac{Nq-2N}{2q}}(\Omega)}\\
&\le \frac{M_1C_\sigma\rho_1^{-\frac{\sigma}{2}}\omega_\alpha}{\widehat{\alpha}}(yt^\alpha)^{-\frac{\sigma}{2}}\|\widetilde{F}(v_1,v_2)(\cdot,\tau)\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega)}\\
&\le \frac{M_1C_\sigma c_F\rho_1^{-\frac{\sigma}{2}}\omega_\alpha}{\widehat{\alpha}}(yt^\alpha)^{-\frac{\sigma}{2}}\|\widetilde{F}(v_1,v_2)(\cdot,\tau)\|_{\mathbb{X}^{-r}(\Omega)}\\
&\le \frac{M_1C_\sigma c_F\rho_1^{-\frac{\sigma}{2}}\omega_\alpha}{\widehat{\alpha}}K_F(yt^\alpha)^{-\frac{\sigma}{2}}\|(v_1-v_2)(\cdot,\tau)\|_{L^2(\Omega)}\\
&\le \frac{M_1M_3C_\sigma c_F\rho_1^{-\frac{\sigma}{2}}\omega_\alpha}{\widehat{\alpha}}K_F(yt^\alpha)^{-\frac{\sigma}{2}}\|(v_1-v_2)(\cdot,\tau)\|_{L^q(\Omega)},\end{aligned}\tag{30}$$

where the globally Lipschitz assumption (20) has been employed in the fourth estimate. Therefore, the difference $\mathcal{S}_Fv_1-\mathcal{S}_Fv_2$ can be estimated as follows

$$\begin{aligned}&\left\|\mathcal{S}_2^Fv_1-\mathcal{S}_2^Fv_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\\
&\le \alpha\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\int_0^t\int_0^{t'}\int_0^\infty(t-t')^\alpha R(t',\tau)\left\|e^{-y(t-t')^\alpha\mathcal{B}_\rho}\mathcal{B}_\mu\widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{L^q(\Omega)}(y\Phi_\alpha(y))\,dyd\tau dt'\right)^pdt\right)^{\frac{1}{p}}\\
&\le M_4\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\int_0^t\int_0^{t'}\int_0^\infty(t-t')^{\frac{(2-\sigma)\alpha}{2}}R(t',\tau)\left\|\widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{X^{\frac{N}{2}-\frac{N}{q}-\beta}}\left(y^{\frac{2-\sigma}{2}}\Phi_\alpha(y)\right)dyd\tau dt'\right)^pdt\right)^{\frac{1}{p}}\\
&\le M_5\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\int_0^t\int_0^{t'}(t-t')^{\frac{(2-\sigma)\alpha}{2}}R(t',\tau)\left\|(v_1-v_2)(\cdot,\tau)\right\|_{L^q(\Omega)}\left(\int_0^\infty y^{\frac{2-\sigma}{2}}\Phi_\alpha(y)dy\right)d\tau dt'\right)^pdt\right)^{\frac{1}{p}},\end{aligned}\tag{31}$$

where the above constants are $M_4:=\alpha M_1M_3C_\sigma c_F\rho_1^{-\frac{\sigma}{2}}\frac{\omega_\alpha}{\widehat{\alpha}}$, $\quad M_5=M_4K_F$. Now, it follows from Hölder's inequality applied to the dual pair (p,p_*) (which means $1/p+1/p_*=1$) that

$$\begin{aligned}&\int_0^t\int_0^{t'}(t-t')^{\frac{(2-\sigma)\alpha}{2}}R(t',\tau)\|(v_1-v_2)(\cdot,\tau)\|_{L^q(\Omega)}d\tau dt'\\
&=\int_0^t(t-t')^{\frac{(2-\sigma)\alpha}{2}}\int_0^{t'}R(t',\tau)\tau^{-\frac{\gamma}{p}}e^{\frac{z}{p}\tau}\left(\tau^{\frac{\gamma}{p}}e^{-\frac{z}{p}\tau}\|(v_1-v_2)(\cdot,\tau)\|_{L^q(\Omega)}\right)d\tau dt'\\
&\le\left(\int_0^t(t-t')^{\frac{(2-\sigma)\alpha}{2}}\left(\int_0^{t'}R^{p_*}(t',\tau)\tau^{-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau\right)^{\frac{1}{p_*}}dt'\right)\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\\
&\le R_0\left(\int_0^t(t-t')^{\frac{(2-\sigma)\alpha}{2}}\left(\int_0^{t'}(t'-\tau)^{\varkappa p_*}\tau^{\nu p_*-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau\right)^{\frac{1}{p_*}}dt'\right)\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}.\end{aligned}\tag{32}$$

Here, the integral on $(0,t')$ above can be calculated and then estimated by using the asymptotic behavior (8). In this integral, the parameters $\varkappa p_*$ and $\nu p_*-\frac{\gamma p_*}{p}$ are strictly greater than -1. Indeed, one has $\varkappa p_*=\varkappa\frac{p}{p-1}>-1$, as assumption $\varkappa>\frac{1-p}{p}$ in (AR), and $\nu p_*-\frac{\gamma p_*}{p}=\frac{\nu p-\gamma}{p-1}>\frac{\nu p-(\nu p+p-1)}{p-1}=-1$, as assumption $0<\gamma<\nu p+p-1$. Hence, it is easily checked that

$$\begin{aligned}\int_0^{t'}(t'-\tau)^{\varkappa p_*}\tau^{\nu p_*-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau&=M_6(t')^{\nu p_*-\frac{\gamma p_*}{p}-1}z^{-\varkappa p_*-1}e^{\frac{zp_*}{p}t'}\left(t'+O\left(\frac{p}{zp_*}\right)\right)\\
&\le M_7(t')^{\nu p_*-\frac{\gamma p_*}{p}-1}z^{-\varkappa p_*-1}e^{\frac{zp_*}{p}t'}\left(1+O\left(\frac{1}{z}\right)\right),\end{aligned}\tag{33}$$

where we put $M_6=(p_*/p)^{-\varkappa p_*-1}\Gamma(\varkappa p_*+1)$ and $M_7=M_6\max(T;p/p_*)$. Note that $\frac{(2-\sigma)\alpha}{2}>-1$ as $0\le\sigma<2$, and $\nu-\frac{\gamma}{p}-\frac{1}{p_*}=\frac{\nu p-\gamma-p+1}{p}>\frac{\nu p-(\nu p+p-1)-p+1}{p}>-1$, as $0<\gamma<\nu p+p-1$, $p<2$. It follows that the following integral on $(0,t)$ finitely exists. Summarily, we now obtain the following chain

$$\begin{aligned}
&\int_0^t \int_0^{t'} (t-t')^{\frac{(2-\sigma)\alpha}{2}} R(t',\tau)\|(v_1-v_2)(\cdot,\tau)\|_{L^q(\Omega)} d\tau dt' \\
&\quad \le R_0 M_7^{\frac{1}{p_*}} \left(\int_0^t (t-t')^{\frac{(2-\sigma)\alpha}{2}} (t')^{\nu-\frac{\gamma}{p}-\frac{1}{p_*}} e^{\frac{z}{p}t'} dt'\right) \left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}} z^{-\varkappa-\frac{1}{p_*}} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \\
&\quad \le R_0 M_8 \left(t^{\nu-\frac{\gamma}{p}-\frac{1}{p_*}-1} z^{-\frac{(2-\sigma)\alpha}{2}-1} e^{\frac{z}{p}t}\right) \left(t+O\left(\frac{p}{z}\right)\right) \left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}} z^{-\varkappa-\frac{1}{p_*}} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \\
&\quad \le R_0 M_9 \left(t^{\nu-\frac{\gamma}{p}-\frac{1}{p_*}-1} z^{-\frac{(2-\sigma)\alpha}{2}-1} e^{\frac{z}{p}t}\right) \left(1+O\left(\frac{1}{z}\right)\right) \left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}} z^{-\varkappa-\frac{1}{p_*}} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))},
\end{aligned}$$

where $M_8 = M_7^{\frac{1}{p_*}} (1/p)^{-\frac{(2-\sigma)\alpha}{2}-1} \Gamma\left(\frac{(2-\sigma)\alpha}{2}+1\right)$ and $M_9 = M_8 \max(T;p)$. On the other hand, as an immediate consequence of the property (11) we have

$$\int_0^\infty y^{\frac{2-\sigma}{2}} \Phi_\alpha(y) dy = \frac{\Gamma\left(\frac{2-\sigma}{2}+1\right)}{\Gamma\left(\frac{2-\sigma}{2}\alpha+1\right)}.$$

Therefore, the above arguments accordingly imply that

$$\begin{aligned}
&\left\|\mathcal{S}_2^F v_1 - \mathcal{S}_2^F v_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \\
&\le \overline{M}_\sigma \left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}} \left(\int_0^T \left(t^{\nu-\frac{1}{p_*}-1} z^{-\frac{(2-\sigma)\alpha}{2}-1}\right)^p dt\right)^{\frac{1}{p}} z^{-\varkappa-\frac{1}{p_*}} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \\
&= \overline{M}_\sigma \left(1+O\left(\frac{1}{z}\right)\right) \left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}} \left(\int_0^T t^{\nu p-\frac{p}{p_*}-p} dt\right)^{\frac{1}{p}} z^{-\frac{(2-\sigma)\alpha}{2}-1-\varkappa-\frac{1}{p_*}} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}.
\end{aligned}$$

where $\overline{M}_\sigma = R_0 M_5 M_9 \frac{\Gamma\left(\frac{2-\sigma}{2}+1\right)}{\Gamma\left(\frac{2-\sigma}{2}\alpha+1\right)} \left(1+O\left(\frac{1}{z}\right)\right)$. Here, the number $\nu p - \frac{p}{p_*} - p$ are strictly greater than -1 since $\nu > (2p-2)/p$. Indeed, it is obvious to see that

$$\nu p - \frac{p}{p_*} - p = \nu p + 1 - 2p > \frac{2p-2}{p} p + 1 - 2p = -1. \tag{34}$$

This ensures that the integral of $t \mapsto t^{\nu p - \frac{p}{p_*} - p}$ on $(0,T)$ finitely exists. Henceforth, we can conclude that there exists $z_1 > 0$ large enough satisfying

$$\left\|\mathcal{S}_2^F v_1 - \mathcal{S}_2^F v_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \le \frac{1}{4} \|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))},$$

for each given number $z \ge z_1$.

Step 3. Estimating the term $\mathcal{S}_1^F v$

Now, we also use notation $\widetilde{F}(v_1,v_2)$ instead of $F(v_1)-F(v_2)$. Recalling that the term $\mathcal{B}_\eta \widetilde{F}(v_1,v_2)$ may be proved in much the same way as (23) and (30). Thus, we have

$$\begin{aligned}
&\left\|\mathcal{S}_1^F v_1 - \mathcal{S}_1^F v_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))} \\
&\quad \le \frac{M_1 \widehat{\alpha}}{\omega_\alpha} \left(\int_0^T \left(t^{\frac{\gamma}{p}} e^{-\frac{z}{p}t} \int_0^t R(t,\tau) \left\|\mathcal{B}_\eta \widetilde{F}(v_1,v_2)(\cdot,\tau)\right\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}}(\Omega)} d\tau\right)^p dt\right)^{\frac{1}{p}} \\
&\quad \le \frac{M_1 \widehat{\alpha}}{\omega_\alpha} \left(\int_0^T \left(t^{\frac{\gamma}{p}} e^{-\frac{z}{p}t} \int_0^t R(t,\tau) \frac{\omega_\alpha}{\widehat{\alpha}} \|\widetilde{F}(v_1,v_2)(\cdot,\tau)\|_{\mathbb{X}^{\frac{N}{2}-\frac{N}{q}-\beta}(\Omega)} d\tau\right)^p dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, employing the globally Lipschitz assumption (20) and using the Hölder's inequality is an analogue of (32) lead to

$$\begin{aligned}
&\left\|\mathcal{S}_1^F v_1-\mathcal{S}_1^F v_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\\
&\quad\le K_F M_1\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\int_0^t R(t,\tau)\|(v_1-v_2)(\cdot,\tau)\|_{L^2(\Omega)}d\tau\right)^p dt\right)^{\frac{1}{p}}\\
&\quad\le M_{10}\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\int_0^t R(t,\tau)\|(v_1-v_2)(\cdot,\tau)\|_{L^q(\Omega)}d\tau\right)^p dt\right)^{\frac{1}{p}}\\
&\quad\le M_{10}\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\left(\int_0^t R^{p_*}(t,\tau)\tau^{-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau\right)^{\frac{1}{p_*}}\right)^p dt\right)^{\frac{1}{p}}\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\\
&\quad\le M_{11}\left(\int_0^T\left(t^{\frac{\gamma}{p}}e^{-\frac{z}{p}t}\left(\int_0^t (t-\tau)^{\varkappa p_*}\tau^{\nu p_*-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau\right)^{\frac{1}{p_*}}\right)^p dt\right)^{\frac{1}{p}}\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\\
&\quad\le M_{12}\left(\int_0^T t^{\nu p-\frac{p}{p_*}}dt\right)^{\frac{1}{p}}\left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}}z^{-\varkappa-\frac{1}{p_*}}\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))},
\end{aligned} \tag{35}$$

where by (33) we find that

$$\left(\int_0^t (t-\tau)^{\varkappa p_*}\tau^{\nu p_*-\frac{\gamma p_*}{p}}e^{\frac{zp_*}{p}\tau}d\tau\right)^{\frac{1}{p_*}} \le M_7^{\frac{1}{p_*}}t^{\nu-\frac{\gamma}{p}-\frac{1}{p_*}}z^{-\varkappa-\frac{1}{p_*}}e^{\frac{z}{p}t}\left(1+O\left(\frac{1}{z}\right)\right)^{\frac{1}{p_*}},$$

and $M_{10}=K_F M_1 M_3$, $M_{11}=M_{10}R_0$, $M_{12}=M_{11}M_7^{\frac{1}{p_*}}$. In the last right hand side of (35), we note that $\nu p-\frac{p}{p_*}>-1$ by (34), and so the integral of $t^{\nu p-\frac{p}{p_*}}$ on $(0,T)$ exists finitely. This implies that we can choose a positive number $z_2>0$, large enough, such that

$$\left\|\mathcal{S}_1^F v_1-\mathcal{S}_1^F v_2\right\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))}\le\frac{1}{4}\|v_1-v_2\|_{L^{p,\gamma,z}(0,T;L^q(\Omega))},$$

for each given number $z\ge z_2$. The results above show: one can find a ball $B_l\subset L^{p,\gamma,z}(0,T;L^q(\Omega))$ with center at the zero function and large enough radius l such that $\mathcal{S}$ is well-defined on B_l. Furthermore, it is also a contraction mapping for each $z\ge\max(z_1;z_2)$. Summarily, $\mathcal{S}$ has only one fixed point u in $B_l\subset L^{p,\gamma,z}(0,T;L^q(\Omega))$, which solves the equation $\mathcal{S}u=u$. Namely, u is a unique mild solution of Problem (1)-(3).

□

Remark 3.1. *The assumption (AF) accepts the case $r=0$, namely, $F:L^2(\Omega)\to L^2(\Omega)$ such that $F(0)=0$ and*

$$\left\|F(h_1)-F(h_2)\right\|_{L^2(\Omega)}\le K_F\left\|h_1-h_2\right\|_{L^2(\Omega)},$$

for all $h_1,h_2\in L^2(\Omega)$. Some direct examples of this case can be easily given as

- *Linear equation: $F(h)=ch$*
- *Sine-Gordon equation: $F(h)=c\sin(h)$.*

About more applications in the case $r>0$ will be discussed in Section 4.

Remark 3.2. *Let us discuss the special case $R(t,\tau)=\frac{1}{\Gamma(1+\varkappa)}(t-\tau)^{\varkappa}\tau^{\nu}$, $0<\tau<t<T$. By putting $G(t,u(x,t))=t^{\nu}F(u(x,t))$ and $\delta=1+\varkappa$, the right hand side of equation (1) can be rewritten as follows*

$$\int_0^t R(t,\tau)F(u(x,\tau))d\tau=\frac{1}{\Gamma(1+\varkappa)}\int_0^t(t-\tau)^{\varkappa}G(\tau,u(x,\tau))d\tau=I_t^{\delta}G(t,u(x,t)).$$

It deduces from $\varkappa>\frac{1-p}{p}$ in (AR) that $\delta>\frac{1}{p}$ (and so $\delta>1/2$). In addition, due to the assumption (AF), a suitable assumption on G should be stated that:

(AG) *Let r be satisfied $0\le r\le\beta-\left(\frac{N}{2}-\frac{N}{q}\right)$. Suppose that the function $G:(0,T)\times X^0\to X^{-r}$ satisfying $G(t,0)=0$,*

$$\left\|G(t,h_1)-G(t,h_2)\right\|_{X^{-r}}\le K_F t^{\nu}\left\|h_1-h_2\right\|_{X^0},$$

for $\nu > \frac{2p-2}{p}$ *and all* $h_1, h_2 \in X^0$*, all* $t \in [0,T]$*.*

As a consequence of Theorem 3.4, our method can be applied to establish a global existence in the space $L_t^{p,\gamma,z_0} L_x^q$ *for the following initial value problem*

$$\begin{cases} \mathbb{D}_t^\alpha u(x,t) + A^{\beta/2} u(x,t) &= I_t^\delta G(u(x,t)), \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,t) &= \phi(x), \quad (x,t) \in \Omega \times \{0\}. \end{cases}$$

where $\delta > \frac{1}{2}$ *and* G *is defined by (AG).*

4. Time-space Volterra equations

In this section, we discuss an application of our method given in Theorem 3.4 to an initial value problem for a time-space Volterra equation. We will take into account the initial value problem of finding $u = u(x,t)$, $(x,t) \in \Omega \times (0,T)$ such that

$$\mathbb{D}_t^\alpha u(x,t) + A^{\beta/2} u(x,t) = \int_0^t \int_\Omega R(t,t') Z(x,x') H(u(x',t')) dx' dt', \tag{36}$$

which concerned with the boundary and initial value conditions

$$\begin{cases} u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,t) &= \phi(x), \quad (x,t) \in \Omega \times \{0\}. \end{cases} \tag{37}$$

where ϕ is the initial value data, and the nonlinear function $H(u(x,t))$, the time and spatial kernels $R(t,t')$, $Z(x,x')$ are defined by the following hypotheses:

- $\boxed{\text{(AH)}}$ Suppose that the function $H : \mathbb{R} \to \mathbb{R}$ satisfying $H(0) = 0$, and for all $y, y' \in \mathbb{R}$

$$|H(y) - H(y')| \le K_H |y - y'|, \tag{38}$$

- $\boxed{\text{(AZ)}}$ Let q be defined by (AS) and q_* be its dual number $\left(\frac{1}{q} + \frac{1}{q_*} = 1\right)$. Assume that $s > 0$, $1 \le s' < q_*$ satisfying $\frac{s}{q} + \frac{s'}{q_*} \ge 1$. Furthermore, there exists a positive constant C such that

$$\begin{cases} \boxed{\text{(AZ1)}} : \quad \int_\Omega \left|Z(x,x')\right|^{s'} dx' \le C, \quad \text{for almost } x \in \Omega, \\ \boxed{\text{(AZ2)}} : \quad \int_\Omega \left|Z(x,x)\right|^{s} dx \le C, \quad \text{for almost } x' \in \Omega. \end{cases}$$

Theorem 4.1. *Given* $0 < \alpha < 1$, $0 < \beta \le 2$*. If the assumptions (AS), (AR), (AH), (AZ) hold and* $\phi \in L^{\frac{Nq}{N+\beta q}}(\Omega)$*, then there exists* $z_0 > 0$ *such that Problem (36)-(37) has only one mild solution* $u \in L^{p,\gamma,z}(0,T; L^q(\Omega))$*, with* $0 < \gamma < \nu p + p - 1$*.*

Proof. The main idea of the proof is to apply Theorem 3.4. It is sufficient to verify all conditions of this theorem. By comparing the models (1)-(3) and (37), it is convenient to set $F(v(x,t)) = \int_\Omega Z(x,x') H(v(x',t)) dx'$, for all $v \in L^{p,\gamma,z}(0,T; L^q(\Omega))$. In the next argument, we shall show that this function fulfills the assumption (AF) with $0 \le r < \min\left(\frac{N}{2}; \beta - \left(\frac{N}{2} - \frac{N}{q}\right)\right)$. To attain this purpose, we firstly prove that the function $F : L^q(\Omega) \to L^{\frac{qs}{q-(q-1)s'}}(\Omega)$ which corresponds to the estimate

$$\left\|F(v_1(\cdot,t)) - F(v_2(\cdot,t))\right\|_{L^{\frac{qs}{q-(q-1)s'}}(\Omega)} \le C_1 \left\|v_1(\cdot,t) - v_2(\cdot,t)\right\|_{L^q(\Omega)}, \tag{39}$$

where the constant C_1 does not depend on x, t. For simplicity of exposition, we also use notation $\widetilde{F}(v_1,v_2)(x,t) = F(v_1(x,t)) - F(v_2(x,t))$ and $v_{1,2}(x,t) = v_1(x,t) - v_2(x,t)$. It should be noted that $q - (q-1)s' > 0$ as $1 \le s' < q_*$, and $s - q + (q-1)s' > 0$ as $\frac{s}{q} + \frac{s'}{q_*} \ge 1$. Additionally, one has

$$\frac{1}{\left(\frac{qs}{q-(q-1)s'}\right)} + \frac{1}{\left(\frac{qs}{s-q+(q-1)s'}\right)} + \frac{1}{\left(\frac{q}{q-1}\right)} = 1.$$

Therefore, the inequality (39) can be proved by applying the Hölder's inequality for the numbers $\frac{qs}{q-(q-1)s'}$, $\frac{qs}{s-q+(q-1)s'}$, $\frac{q}{q-1}$. By using the assumption (AZ1), one can check the following computations

$$\begin{aligned}
&\big|\widetilde{F}(v_1,v_2)(x,t)\big|^{\frac{qs}{q-(q-1)s'}} \\
&\le M_{14}\left(\int_\Omega \big|Z(x,x')\big|\big|v_{1,2}(x',t)\big|dx'\right)^{\frac{qs}{q-(q-1)s'}} \\
&= M_{14}\left(\int_\Omega \left(\big|Z(x,x')\big|^{\frac{q-(q-1)s'}{q}}\big|v_{1,2}(x',t)\big|^{\frac{q-(q-1)s'}{s}}\right)\left(\big|v_{1,2}(x',t)\big|^{\frac{s-q+(q-1)s'}{s}}\right)\left(\big|Z(x,x')\big|^{\frac{(q-1)s'}{q}}\right)dx'\right)^{\frac{qs}{q-(q-1)s'}} \\
&\le M_{14}\left(\int_\Omega \big|Z(x,x')\big|^s\big|v_{1,2}(x',t)\big|^q dx'\right)\left(\int_\Omega \big|v_{1,2}(x',t)\big|^q dx'\right)^{\frac{s-q+(q-1)s'}{q-(q-1)s'}}\left(\int_\Omega \big|Z(x,x')\big|^{s'}dx'\right)^{\frac{(q-1)s}{q-(q-1)s'}} \\
&\le M_{15}\left(\int_\Omega \big|Z(x,x')\big|^s\big|v_{1,2}(x',t)\big|^q dx'\right)\left(\int_\Omega \big|v_{1,2}(x',t)\big|^q dx'\right)^{\frac{s-q+(q-1)s'}{q-(q-1)s'}}.
\end{aligned}$$

where we put

$$M_{14} = K_H^{qs/(q-(q-1)s')}, \quad M_{15} = M_{14}C^{(q-1)s/(q-(q-1)s')}.$$

Note that the latter integral does not depend on spatial variable (just on the time variable). Therefore, we then obtain

$$\begin{aligned}
&\big\|\widetilde{F}(v_1,v_2)(\cdot,t)\big\|_{L^{\frac{qs}{q-(q-1)s'}}(\Omega)} \\
&= \left(\int_\Omega \big|\widetilde{F}(v_1,v_2)(\cdot,t)\big|^{\frac{qs}{q-(q-1)s'}}dx\right)^{\frac{q-(q-1)s'}{qs}} \\
&\le M_{16}\left(\int_\Omega \big|v_{1,2}(x',t)\big|^q dx'\right)^{\frac{s-q+(q-1)s'}{qs}}\left(\int_\Omega\left(\int_\Omega \big|Z(x,x')\big|^s\big|v_{1,2}(x',t)\big|^q dx'\right)dx\right)^{\frac{q-(q-1)s'}{qs}},
\end{aligned}$$

where $M_{16} = M_{15}^{(q-(q-1)s')/(qs)}$. This combines with the assumption (AZ2) to deduce the following estimate after some direct calculations

$$\big\|\widetilde{F}(v_1,v_2)(\cdot,t)\big\|_{L^{\frac{qs}{q-(q-1)s'}}(\Omega)} \le M_{17}\big\|v_{1,2}(\cdot,t)\big\|_{L^q(\Omega)},$$

namely, inequality (39) has been proved. Now, it is suitable to show that F fulfills assumption (AF). Indeed, it is useful to recall that $s \ge q-(q-1)s'$ by the assumption $\frac{s}{q}+\frac{s'}{q_*}\ge 1$ in (AZ). This follows that $\frac{qs}{q-(q-1)s'} \ge q$, and hence $L^{\frac{qs}{q-(q-1)s'}}(\Omega) \hookrightarrow L^q(\Omega)$. Moreover, it is clear that $2 > \frac{2N}{N+2r}$ which deduces $L^2(\Omega) \hookrightarrow L^{\frac{2N}{N+r}}(\Omega)$. On the other hand, the pair $(-r; 2N/(N+2r))$ obviously belongs to the set $\mathcal{O}^-$, defined by Lemma 3.2. This ensures the embedding $L^{\frac{2N}{N+r}}(\Omega) \hookrightarrow \mathbb{X}^{-r}(\Omega)$. Briefly, the above arguments yield the following Sobolev embedding

$$L^{\frac{qs}{q-(q-1)s'}}(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{2N}{N+2r}}(\Omega) \hookrightarrow \mathbb{X}^{-r}(\Omega).$$

Therefore, it follows from $F : L^q(\Omega) \to L^{\frac{qs}{q-(q-1)s'}}(\Omega)$ that $F : L^q(\Omega) \to \mathbb{X}^{-r}(\Omega)$ and F also fulfills assumption (AF). By applying Theorem 3.4, Problem (36)-(37) has only one global solution u in the space $L^{p,\gamma,z}(0,T;L^q(\Omega))$. □

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