p-th moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching

Tomás Caraballo
Universidad de Sevilla,
Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Facultad de Matemáticas,
c/ Tarfia s/n, 41012-Sevilla (Spain).
caraball@us.es

Lassaad Mchiri
Department of Mathematics,
Faculty of sciences of Sfax, Tunisia.
lmchiri.c@ksu.edu.sa

BELFEKI Mohsen
Department of Mathematics,
Faculty of sciences of Sfax, Tunisia.
mohsenbelfqui@yahoo.fr

Mohamed Rhaima
University of Tunis Manar,
Faculty of Sciences of Tunis,
Department of Mathematics, Tunisia.
mohamed.rhaima@fst.rnu.tn

Abstract

In this paper we focus on the p-th moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching (NSPDEwMS). By means of the Lyapunov method, we develop some sufficient conditions on the p-th moment exponential stability for NSPDEwMS. We analyze two examples to show the interest of the main results.

Keywords: Neutral stochastic pantograph differential equations, Markovian switching, Lyapunov method, p-th moment exponential stability.

1 Introduction

Stochastic differential equations (SDE) are used to model various phenomena such as physical systems, unstable stock prices, economics, biology ... SDE describe many dynamical systems, in which random effects are important and can be taken into account as random perturbations (see [2] and [11]). Many researchers have studied the stability theory for a class of stochastic differential equations and in particular stochastic delay differential equations with and without impulse effects (see [4], [7], [11], [18], [19] and [21]). Markov switched systems can be used to describe many systems subject to unpredictable fluctuation (see [15]). Neutral stochastic delay differential equations with Markovian switching can be used to model various processes and phenomena in the field of engineering , chemical and biology. Stability theory of neutral stochastic delay differential equations with Markovian switching has attracted the attention of many authors, for instance see [9], [13] and [16].

The stochastic pantograph differential equation is a kind of extension of stochastic delay differential equations (see [1], [3], [5], [6] and [12]). In recent years, as one of the most important class of stochastic delay systems, the stochastic pantograph differential equations with Markovian switching (SPDEwMS) are very well studied (see [8] and [10]). Neutral stochastic pantograph differential equations with Markovian switching (NSPDEwMS) is an important extension of SPDEwMS. They play an interesting role in industrial and mathematical problems (see [8], [10], [17] and [20]).

The stability analysis of NSPDEwMS has attracted much more attention (see [10] and [14]). We refer the reader to [1], [5],[6], [8], [12] and [14]-[20] where polynomial and amost sure exponential stability of stochastic pantograph differential equations and neutral stochastic pantograph differential equations with Markovian switching are considered. To the best of our knowledge, there is no existing result on p-th moment exponential stability of NSPDEwMS. By applying the generalized Ito formula, the classical stochastic calculus and the Lyapunov method, we investigate and give a new sufficient condition ensuring the p-th moment exponential stability for a class of NSPDEwMS.

In [10] the authors studied the same problem and prove existence, uniqueness and pathwise stability of the solutions, with general decay rate, by using an appropriate Lyapunov function satisfying some properties for the p-power of the variable which essentially implies $p \geq 2$. In our paper we prove existence and solutions and p-moment stability of solutions in the case $p > 0$. In this sense, our results complement the analysis in [10] providing sufficient conditions for the exponential stability in p-moment for any positive p.

The paper is organized as follows: In Section 2, we give some important notations and definitions. In Section 3, we establish the p-th moment exponential stability for a class of NSPDEwMS. Finally, in Section 4, we provide two illustrative examples and some numerical simulations to show our main results.

2 Preliminaries

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets. $W(t)$ is an m-dimensional Brownian motion defined on the probability space. Let $t_0 > 0$ and $C([qt_0, t_0]; \mathbb{R}^n)$ denote the family of all continuous functions φ from $[qt_0, t_0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{qt_0 \leq s \leq t_0} |\varphi(s)|$ and $|x| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^n$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $p > 0$, $L^p_{\mathcal{F}_t}([qt, t]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable, $C([qt, t]; \mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\theta) : qt \leq \theta \leq t\}$ such that $\mathbb{E}\|\varphi\|^p < \infty$.

Let $\{r(t), t \in \mathbb{R}^+ = [0, +\infty)\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with a generator $\Gamma = (\gamma_{ij})_{\mathbb{N}\times\mathbb{N}}$ given by

$$\mathbb{P}\left(r(t+\Delta) = j | r(t) = i\right) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$, while

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$.

Consider the following neutral pantograph stochastic differential equation with Markovian switching:

$$d\Big(x(t) - G\big(t, x(qt)\big)\Big) = f\big(t, x(t), x(qt), r(t)\big)dt + g\big(t, x(t), x(qt), r(t)\big)dw(t), \qquad (2.1)$$

with the initial condition $\{x(t) = qt_0 \leq t \leq t_0\} = \xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$. Let $u(t) = x(t) - G(t, x(qt))$. Here, we furthermore assume that

$$f : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^n, \quad g : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^{n\times m},$$

$$G : [t_0, +\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

We denote by $x(t; t_0, \xi)$ the solution of equation (2.1).

Let $C^{1,2}\left([qt_0, +\infty) \times \mathbb{R}^n \times S; \mathbb{R}^+\right)$ be the family of all non-negative functions $V(t, x, i)$ on $[qt_0, +\infty) \times \mathbb{R}^n \times S$, which are twice continuously differentiable with respect to x and once continuously differentiable with respect to t.

For any $(t,x,y,i) \in [qt_0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S$, $u = x - G(t,y)$, by the generalized Itô formula (see [8] and [9]) we have,

$$V(t,u(t),r(t)) = V(t_0,u(t_0),r(t_0)) + \int_{t_0}^{t} LV(s,x(s),x(qs),r(s))ds + M_t,$$

where the operator $LV(t,x,y,i) : [qt_0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}$ and the process M_t are defined respectively by

$$\begin{aligned} LV(t,x,y,i) &= V_t(t,u,i) + V_x(t,u,i)f(t,x,y,i) \\ &\quad + \frac{1}{2} trace\left(g^T(t,x,y,i)V_{xx}(t,u,i)g(t,x,y,i)\right) \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(t,u,j), \end{aligned}$$

$$M_t = \int_{t_0}^{t} V_x(s,u(s),r(s))g(s,x(s),x(qs),r(s))dW(s),$$

$$V_t = \frac{\partial V(t,x,i)}{\partial t}, \quad V_x = \left(\frac{\partial V(t,x,i)}{\partial x_1}, \ldots, \frac{\partial V(t,x,i)}{\partial x_n}\right), \quad and \;\; V_{xx} = \left(\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

3 Main results

For our purpose, we will state some assumptions which can ensure the existence and uniqueness of solution, denoted by $x(t) = x(t;t_0,\xi)$ on $t \geq t_0$, for equation

Assumption 3.1. *For each integer $d \geq 1$, there exists a positive constant k_d such that*

$$|f(t,x,y,i) - f(t,\overline{x},\overline{y},i)|^2 \vee |g(t,x,y,i) - g(t,\overline{x},\overline{y},i)|^2 \leq k_d(|x-\overline{x}|^2 + |y-\overline{y}|^2), \tag{3.1}$$

for those $x,\overline{x},y,\overline{y} \in \mathbb{R}^n$ with $|x| \vee |\overline{x}| \vee |y| \vee |\overline{y}| \leq d$ and $(t,i) \in [t_0,\infty) \times S$.

Assumption 3.2. *There exist a Lyapunov function $V \in C^{1,2}\left([qt_0,+\infty) \times \mathbb{R}^n \times S; \mathbb{R}^+\right)$ and positive constants c_1, c_2, c_3 and c_4 such that*

$$c_1|x|^p \leq V(t,x,i) \leq c_2|x|^p, \tag{3.2}$$

$$LV(t,x,y,i) \leq -\left(2+\frac{1}{q}\right)c_3|x|^p + c_4 q e^{-\frac{c_3}{c_2}(1-q)t}|y|^p \tag{3.3}$$

and

$$c_4 \leq c_3. \tag{3.4}$$

Assumption 3.3. *There exists a positive constant $L \in (0,1)$ such that for all $x, y \in \mathbb{R}^n$ and $t \geq qt_0$,*

$$|G(t,x) - G(t,y)| \leq L\,|x-y|\,. \tag{3.5}$$

For our stability purpose we need to impose the next stronger version of this assumption:

Assumption 3.4. *There exists a positive constant $L \in (0,1)$ such that for all $x, y \in \mathbb{R}^n$ and $t \geq qt_0$,*

$$|G(t,x) - G(t,y)| \leq Le^{-\frac{\delta}{p}(1-q^2)t}\,|x-y|\,, \tag{3.6}$$

where δ is a positive constant which verifies

$$q\delta \leq \frac{c_3}{c_2} \leq \delta \quad if \quad 0 < p \leq 1 \tag{3.7}$$

and

$$q\delta \leq \frac{c_3}{c_2 2^{p-1}} \leq \delta \quad if \quad p > 1. \tag{3.8}$$

Assumption 3.5. *For all $t \geq qt_0$*

$$G(t,0) = 0 \tag{3.9}$$

$$f(t,0,0,r(t)) = 0 \tag{3.10}$$

$$g(t,0,0,r(t)) = 0. \tag{3.11}$$

Remark 3.6. From (3.6) and (3.9), it yields for all $x \in \mathbb{R}^n$ and $t \geq qt_0$

$$|G(t,x)| \leq Le^{-\frac{\delta}{p}(1-q^2)t}\,|x|\,, \tag{3.12}$$

Lemma 3.7. *Let $p > 1$, $\varepsilon > 0$ and $(a,b) \in \mathbb{R}^2$. Then,*

$$|a+b|^p \leq \left(1 + \varepsilon^{\frac{1}{p-1}}\right)^{p-1}\left(|a|^p + \frac{|b|^p}{\varepsilon}\right).$$

Proof. See ([11]). □

Remark 3.8. Let $p > 1$ and $(a,b) \in \mathbb{R}^2$. By taking $\varepsilon = 1$, in Lemma 3.7, we obtain

$$|a+b|^p \leq 2^{p-1}\big(|a|^p + |b|^p\big).$$

Lemma 3.9. *Let $0 < p \leq 1$ and $(a,b) \in \mathbb{R}^2$. Then,*

$$|a+b|^p \leq (|a|^p + |b|^p)\,.$$

Proof. See ([11]). □

Theorem 3.10. *Let Assumptions 3.1, 3.2 and 3.3 hold. Then for any given initial condition data* $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$, *there exists a unique global solution* $x(\cdot) = x(\cdot, t_0, \xi)$ *to the system (2.1) on* $t \in [qt_0, \infty)$.

Proof. **First step: existence and uniqueness of the maximal solution**

By Assumption 3.1, for any given initial condition $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$, system (2.1) admits a unique maximal solution $x(t)$ defined on $[t_0, \sigma_\infty)$, where σ_∞ is the explosion time (see [13]). Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [t_0, \sigma_\infty), |x(t)| \geq k\}.$$

Since

$$\{t \in [t_0, \sigma_\infty), |x(t)| \geq k+1\} \subset \{t \in [t_0, \sigma_\infty), |x(t)| \geq k\},$$

we have

$$\tau_k \leq \tau_{k+1}.$$

Then, (τ_k) is increasing. Which allows us to define

$$\tau_\infty = \lim_{k \to \infty} \tau_k.$$

Given that

$$\tau_k \leq \sigma_\infty, \forall k \geq k_0,$$

we have

$$\tau_\infty \leq \sigma_\infty.$$

Let $i \geq 1$ and $t_0 \leq t \leq \frac{t_0}{q^i}$. By Ito's formula, we obtain

$$\begin{aligned}
&E\Big(V\big(t \wedge \tau_k, u(t \wedge \tau_k), r(t \wedge \tau_k)\big)\Big) \\
= \quad & E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + E\int_{t_0}^{t \wedge \tau_k} LV\big(s, x(s), x(qs), r(s)\big)ds \\
\leq \quad & E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + E\int_{t_0}^{t \wedge \tau_k} \big(-c_3|x(s)|^p + c_4 q|x(qs)|^p\big)ds \\
\leq \quad & E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) - c_3 E\int_{t_0}^{t \wedge \tau_k} |x(s)|^p ds + c_4 E\int_{t_0}^{t \wedge \tau_k} q|x(qs)|^p ds \qquad (3.13)
\end{aligned}$$

We have

$$\begin{aligned}\int_{t_0}^{t\wedge\tau_k} q|x(qs)|^p ds &= \int_{qt_0}^{q(t\wedge\tau_k)} |x(s)|^p ds\\ &\le \int_{qt_0}^{t\wedge\tau_k} |x(s)|^p ds\\ &= \int_{qt_0}^{t_0} |x(s)|^p ds + \int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\\ &\le (1-q)t_0\|\xi\|^p + \int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds. \qquad (3.14)\end{aligned}$$

Substituting (3.14) in (3.13), we obtain

$$\begin{aligned}&E\Big(V\big(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\big)\Big)\\ \le\ & E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + (c_4-c_3)E\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds + c_4(1-q)t_0E\big(\|\xi\|^p\big)\\ \le\ & E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + c_4(1-q)t_0E\big(\|\xi\|^p\big) \qquad (3.15)\end{aligned}$$

By (3.2), we deduce

$$c_1E\big(|u(t\wedge\tau_k)|^p\big) \le c_2E\big(|u(t_0)|^p\big) + c_4(1-q)t_0E\big(\|\xi\|^p\big) \qquad (3.16)$$

Which implies

$$E\big(|u(t\wedge\tau_k)|^p\big) \le \frac{c_2}{c_1}E\big(|u(t_0)|^p\big) + \frac{c_4}{c_1}(1-q)t_0E\big(\|\xi\|^p\big) \qquad (3.17)$$

Second step: the case $p>1$
Let $\varepsilon>0$. By Lemma 3.7, it yields

$$\begin{aligned}|x(t\wedge\tau_k)|^p &= |u(t\wedge\tau_k) + G(t\wedge\tau_k, x(q(t\wedge\tau_k)))|^p\\ &\le \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}\Big(|u(t\wedge\tau_k)|^p + \frac{|G(t\wedge\tau_k, x(q(t\wedge\tau_k)))|^p}{\varepsilon}\Big)\\ &\le \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}\Big(|u(t\wedge\tau_k)|^p + \frac{L^p}{\varepsilon}|x(q(t\wedge\tau_k))|^p\Big)\\ &\le \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}\Big(|u(t\wedge\tau_k)|^p + \frac{L^p}{\varepsilon}k^p\Big) \qquad (3.18)\end{aligned}$$

Then,

$$\frac{1}{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}|x(t\wedge\tau_k)|^p \le |u(t\wedge\tau_k)|^p + \frac{L^p}{\varepsilon}k^p \qquad (3.19)$$

It yields

$$|u(t\wedge\tau_k)|^p \geq \frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}}|x(t\wedge\tau_k)|^p - \frac{L^p}{\varepsilon}k^p \tag{3.20}$$

We obtain

$$\begin{aligned} |u(t\wedge\tau_k)|^p \geq |u(t\wedge\tau_k)|^p\chi_{\{\tau_k\leq t\}} &\geq \frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}}|x(t\wedge\tau_k)|^p\chi_{\{\tau_k\leq t\}} - \frac{L^p}{\varepsilon}k^p\chi_{\{\tau_k\leq t\}} \\ &= \frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}}|x(\tau_k)|^p\chi_{\{\tau_k\leq t\}} - \frac{L^p}{\varepsilon}k^p\chi_{\{\tau_k\leq t\}} \\ &= \frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}}k^p\chi_{\{\tau_k\leq t\}} - \frac{L^p}{\varepsilon}k^p\chi_{\{\tau_k\leq t\}} \\ &= \left(\frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon}\right)k^p\chi_{\{\tau_k\leq t\}} \end{aligned} \tag{3.21}$$

Then,

$$\begin{aligned} E(|u(t\wedge\tau_k)|^p) &\geq \left(\frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon}\right)k^pE(\chi_{\{\tau_k\leq t\}}) \\ &\geq \left(\frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon}\right)k^pP(\{\tau_k\leq t\}) \end{aligned} \tag{3.22}$$

We have

$$\frac{1}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon} = \frac{1}{\varepsilon}\left(\frac{\varepsilon}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - L^p\right).$$

and

$$\lim_{\varepsilon\to\infty}\frac{\varepsilon}{\left[1+\varepsilon^{\frac{1}{p-1}}\right]^{p-1}} - L^p = 1 - L^p > 0.$$

Then, there exists $\varepsilon_0 > 1$ large enough, such that

$$\frac{\varepsilon_0}{\left[1+\varepsilon_0^{\frac{1}{p-1}}\right]^{p-1}} - L^p > 0.$$

Hence,

$$\frac{1}{\left[1+\varepsilon_0^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon_0} > 0.$$

We deduce that, for all $k \geq k_0$, $i \geq 1$ and $t \in [t_0, \frac{t_0}{q^i}]$

$$P(\{\tau_k \leq t\}) \quad \leq \quad \frac{1}{(\frac{1}{\left[1+\varepsilon_0^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon_0})k^p} E(|u(t \wedge \tau_k)|^p) \tag{3.23}$$

Which implies by (3.17) that

$$P(\{\tau_k \leq t\}) \quad \leq \quad \frac{1}{c_1(\frac{1}{\left[1+\varepsilon_0^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon_0})k^p} \Big(c_2 E\big(|u(t_0)|^p\big) + c_4(1-q)t_0 E\big(\|\xi\|^p\big)\Big) \tag{3.24}$$

Then, for all $k \geq k_0$ and $i \geq 1$

$$P(\{\tau_k \leq \frac{t_0}{q^i}\}) \quad \leq \quad \frac{1}{c_1(\frac{1}{\left[1+\varepsilon_0^{\frac{1}{p-1}}\right]^{p-1}} - \frac{L^p}{\varepsilon_0})k^p} \Big(c_2 E\big(|u(t_0)|^p\big) + c_4(1-q)t_0 E\big(\|\xi\|^p\big)\Big) \tag{3.25}$$

By letting $k \to \infty$, it yields for all $i \geq 1$

$$P(\{\tau_\infty \leq \frac{t_0}{q^i}\}) \quad = \quad 0. \tag{3.26}$$

Third step: the case $0 < p \leq 1$
Using Lemma 3.9 to obtain

$$\begin{aligned} |x(t \wedge \tau_k)|^p &= |u(t \wedge \tau_k) + G(t \wedge \tau_k, x(q(t \wedge \tau_k)))|^p \\ &\leq |u(t \wedge \tau_k)|^p + |G(t \wedge \tau_k, x(q(t \wedge \tau_k)))|^p \\ &\leq |u(t \wedge \tau_k)|^p + L^p |x(q(t \wedge \tau_k))|^p \\ &\leq |u(t \wedge \tau_k)|^p + L^p k^p \end{aligned} \tag{3.27}$$

Then,

$$|u(t \wedge \tau_k)|^p \quad \geq \quad |x(t \wedge \tau_k)|^p - L^p k^p \tag{3.28}$$

We obtain

$$\begin{aligned} |u(t \wedge \tau_k)|^p \geq |u(t \wedge \tau_k)|^p \chi_{\{\tau_k \leq t\}} &\geq |x(t \wedge \tau_k)|^p \chi_{\{\tau_k \leq t\}} - L^p k^p \chi_{\{\tau_k \leq t\}} \\ &= |x(\tau_k)|^p \chi_{\{\tau_k \leq t\}} - L^p k^p \chi_{\{\tau_k \leq t\}} \\ &= k^p \chi_{\{\tau_k \leq t\}} - L^p k^p \chi_{\{\tau_k \leq t\}} \\ &= (1 - L^p) k^p \chi_{\{\tau_k \leq t\}} \end{aligned} \tag{3.29}$$

Then,

$$E(|u(t\wedge\tau_k)|^p) \;\geq\; (1-L^p)k^pE(\chi_{\{\tau_k\leq t\}}) \;\geq\; (1-L^p)k^pP(\{\tau_k\leq t\}) \tag{3.30}$$

We deduce that, for all $k\geq k_0$, $i\geq 1$ and $t\in[t_0,\frac{t_0}{q^i}]$

$$P(\{\tau_k\leq t\}) \;\leq\; \frac{1}{(1-L^p)k^p}E(|u(t\wedge\tau_k)|^p) \tag{3.31}$$

Which implies by (3.17) that

$$P(\{\tau_k\leq t\}) \;\leq\; \frac{1}{c_1(1-L^p)k^p}\Big(c_2E\big(|u(t_0)|^p\big)+c_4(1-q)t_0E\big(\|\xi\|^p\big)\Big) \tag{3.32}$$

Then, for all $k\geq k_0$ and $i\geq 1$

$$P(\{\tau_k\leq \frac{t_0}{q^i}\}) \;\leq\; \frac{1}{c_1(1-L^p)k^p}\Big(c_2E\big(|u(t_0)|^p\big)+c_4(1-q)t_0E\big(\|\xi\|^p\big)\Big) \tag{3.33}$$

By letting $k\to\infty$, it yields for all $i\geq 1$

$$P(\{\tau_\infty\leq \frac{t_0}{q^i}\}) \;=\; 0. \tag{3.34}$$

Fourth step: determination of σ_∞

In both cases, $p>1$ and $0<p\leq 1$, we have proved that for all $i\geq 1$

$$P(\{\tau_\infty> \frac{t_0}{q^i}\}) \;=\; 1. \tag{3.35}$$

Thus, for all $i\geq 1$

$$\tau_\infty \;>\; \frac{t_0}{q^i} \quad a.s. \tag{3.36}$$

Letting $i\to\infty$, it yields

$$\tau_\infty \;=\; \infty \quad a.s. \tag{3.37}$$

Then,

$$\sigma_\infty \;=\; \infty \quad a.s. \tag{3.38}$$

which proves Theorem 3.10. □

Remark 3.11. Notice that to prove Theorem 3.10, we can impose a weaker assumption on the Lyapunov function. Indeed, it is only necessary to assume that

$$c_1|x|^p \le V(t,x,i) \le c_2|x|^p \quad \text{and} \quad LV(t,x,y,i) \le -c_3|x|^p + c_4 q|y|^p.$$

Also, it is remarkable that our theorem completes and generalizes the corresponding one in [10] since we can cover more cases as our p only needs to be positive and not bigger than or equal 2.

Definition 3.1. The null solution of system (2.1) is said to be p−th moment exponentially stable if there exist α, $C > 0$ such that

$$\mathbb{E}\left(|x(t,t_0,\xi)|^p\right) \le Ce^{-\alpha(t-t_0)}\mathbb{E}\left(\|\xi\|^p\right)$$

for all $t_0 \in \mathbb{R}_+$, $t \ge t_0$ and $\xi \in L^p_{\mathcal{F}_t}([qt_0,t_0];\mathbb{R}^n)$.

Theorem 3.12. *Assume that Assumptions 3.1, 3.2, 3.4 and 3.5 hold. Then, for each initial function $\xi \in L^p_{\mathcal{F}_t}([qt_0,t_0];\mathbb{R}^n)$, the corresponding solution $x(\cdot) = x(\cdot,t_0,\xi)$ to system (2.1) satisfies*

$$\mathbb{E}\left(|x(t)|^p\right) \le Ce^{-\alpha(t-t_0)}\mathbb{E}\left(\|\xi\|^p\right)$$

where α and C are positive constants.

Proof. Let $x(\cdot)$ denote the solution to system (2.1) with initial value ξ. Then we define the stopping time $\tau_k = \inf\{t \ge t_0, |x(t)| \ge k\}$.

First case: $0 < p \le 1$.

For $t \ge t_0$, by Itô's formula, we obtain

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k - t_0)}V\left(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\right)\right) \\
&\quad= \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right) \\
&\qquad+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2}(s-t_0)}\left(\frac{c_3}{c_2}V\left(s,u(s),r(s)\right) + LV\left(s,x(s),x(qs),r(s)\right)\right)ds\right) \\
&\quad\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right) \\
&\qquad+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2}(s-t_0)}\left(c_3|u(s)|^p - c_3\left(2+\frac{1}{q}\right)|x(s)|^p + c_4 q e^{-\frac{c_3}{c_2}(1-q)s}|x(qs)|^p\right)ds\right)
\end{aligned} \tag{3.39}$$

Thanks to (3.12) and the fact that $L < 1$, we have

$$\begin{aligned}
|u(s)|^p &= |x(s) - G(s,x(qs))|^p \\
&\le |x(s)|^p + |G(s,x(qs))|^p \\
&\le |x(s)|^p + L^p e^{-\delta(1-q^2)s}|x(qs)|^p \\
&\le |x(s)|^p + e^{-\delta(1-q^2)s}|x(qs)|^p.
\end{aligned} \tag{3.40}$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k-t_0)}V\left(t\wedge\tau_k,u(t\wedge\tau_k),r(t\wedge\tau_k)\right)\right) \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right) \\
&+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2}(s-t_0)}\left(c_3(|x(s)|^p+e^{-\delta(1-q^2)s}|x(qs)|^p)-c_3\left(2+\frac{1}{q}\right)|x(s)|^p+c_4qe^{-\frac{c_3}{c_2}(1-q)s}|x(qs)|^p\right)ds\right).
\end{aligned}
\tag{3.41}
$$

Using (3.7),

$$
\begin{aligned}
e^{\frac{c_3}{c_2}(s-t_0)}e^{-\delta(1-q^2)s} &= e^{-\frac{c_3}{c_2}t_0}e^{(\frac{c_3}{c_2}-\delta)s}e^{q^2\delta s} \\
&\le e^{-\frac{c_3}{c_2}t_0}e^{\frac{c_3}{c_2}qs} \\
&= e^{\frac{c_3}{c_2}(qs-t_0)}.
\end{aligned}
\tag{3.42}
$$

We have also

$$
e^{\frac{c_3}{c_2}(s-t_0)}e^{-\frac{c_3}{c_2}(1-q)s}=e^{\frac{c_3}{c_2}(qs-t_0)}. \tag{3.43}
$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k-t_0)}V\left(t\wedge\tau_k,u(t\wedge\tau_k),r(t\wedge\tau_k)\right)\right) \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right)+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}\left(-c_3\left(1+\frac{1}{q}\right)e^{\frac{c_3}{c_2}(s-t_0)}|x(s)|^p+(c_3+qc_4)e^{\frac{c_3}{c_2}(qs-t_0)}|x(qs)|^p\right)ds\right) \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right)+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}\left(-c_3\left(1+\frac{1}{q}\right)+\frac{c_3}{q}+c_4\right)e^{\frac{c_3}{c_2}(s-t_0)}|x(s)|^pds\right) \\
&+\mathbb{E}\left(\int_{qt_0}^{t_0}\left(\frac{c_3}{q}+c_4\right)e^{\frac{c_3}{c_2}(s-t_0)}|x(s)|^pds\right) \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right)+c_3\left(1+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right)\int_{qt_0}^{t_0}e^{\frac{c_3}{c_2}(s-t_0)}ds \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right)+c_2\left(1+\frac{1}{q}\right)\mathbb{E}(\|\xi\|^p)\left(1-e^{-\frac{c_3}{c_2}(1-q)t_0}\right) \\
&\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right)+c_2\left(1+\frac{1}{q}\right)\mathbb{E}(\|\xi\|^p).
\end{aligned}
\tag{3.44}
$$

Thus,

$$
\begin{aligned}
\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k-t_0)}V\left(t\wedge\tau_k,u(t\wedge\tau_k),r(t\wedge\tau_k)\right)\right) &\le \mathbb{E}\left(V\left(t_0,u(t_0),r(t_0)\right)\right) \\
&+c_2\left(1+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right).
\end{aligned}
\tag{3.45}
$$

By (3.2) it follows

$$\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k-t_0)}|u(t\wedge\tau_k)|^p\right) \le \frac{1}{c_1}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t\wedge\tau_k-t_0)}V\left(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\right)\right)$$
$$\le \frac{1}{c_1}\mathbb{E}\left(V\left(t_0, u(t_0), r(t_0)\right)\right)$$
$$\le \frac{c_2}{c_1}\mathbb{E}\left(|u(t_0)|^p\right) + \frac{c_2}{c_1}\left(1+\frac{1}{q}\right)\mathbb{E}(\|\xi\|^p). \quad (3.46)$$

Letting $k\to\infty$, it yields

$$\begin{aligned}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|u(t)|^p\right) &\le \frac{c_2}{c_1}\mathbb{E}\left(|u(t_0)|^p\right) + \frac{c_2}{c_1}\left(1+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right)\\ &\le \frac{c_2}{c_1}\mathbb{E}\left|x(t_0)\right|^p) + \frac{c_2}{c_1}\mathbb{E}\left(|x(qt_0)|^p\right) + \frac{c_2}{c_1}\left(1+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right). \end{aligned} \quad (3.47)$$

Let $T>0$, for all $t_0\le t\le T$, we have

$$\begin{aligned}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right) &\le \mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|u(t)|^p\right) + \mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|G(t,x(qt))|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + \mathbb{E}\left(L^p e^{\frac{c_3}{c_2}(t-t_0)}e^{-\delta(1-q^2)t}|x(qt)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\mathbb{E}\left(e^{\frac{c_3}{c_2}t}e^{-\delta(1-q^2)t}|x(qt)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{c_2}t}e^{-\delta(1-q^2)t}|x(qt)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{qt_0\le t\le qT}\mathbb{E}\left(e^{\frac{c_3}{qc_2}t}e^{-\frac{\delta}{q}(1-q^2)t}|x(t)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{qt_0\le t\le T}\mathbb{E}\left(e^{\frac{c_3}{qc_2}t}e^{-\frac{\delta}{q}(1-q^2)t}|x(t)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{qt_0\le t\le T}\mathbb{E}\left(e^{(\frac{c_3}{c_2}-\delta)\frac{t}{q}}e^{q\delta t}|x(t)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{qt_0\le t\le T}\mathbb{E}\left(e^{q\delta t}|x(t)|^p\right)\\ &\le \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right) + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{qt_0\le t\le t_0}\mathbb{E}\left(e^{q\delta t}|x(t)|^p\right)\\ &\quad + L^p e^{-\frac{c_3}{c_2}t_0}\sup_{t_0\le t\le T}\mathbb{E}\left(e^{q\delta t}|x(t)|^p\right) \end{aligned} \quad (3.48)$$

Then,

$$
\begin{aligned}
&e^{-\frac{c_3}{c_2}t_0}\mathbb{E}\left(e^{\frac{c_3}{c_2}t}|x(t)|^p\right)\\
&\leq \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right)+L^p e^{-(\frac{c_3}{c_2}-q\delta)t_0}\mathbb{E}(\|\xi\|^p)\\
&+L^p e^{-\frac{c_3}{c_2}t_0}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{q\delta t}|x(t)|^p\right)\\
&\leq \frac{c_2}{c_1}\left(3+\frac{1}{q}\right)\mathbb{E}\left(\|\xi\|^p\right)+L^p\mathbb{E}\left(\|\xi\|^p\right)+L^p e^{-\frac{c_3}{c_2}t_0}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}t}|x(t)|^p\right)\\
&\leq \left(\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p\right)\mathbb{E}\left(\|\xi\|^p\right)+L^p e^{-\frac{c_3}{c_2}t_0}\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}t}|x(t)|^p\right).
\end{aligned}
\tag{3.49}
$$

Thus,

$$
\begin{aligned}
&\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right)\\
&\leq \left(\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p\right)\mathbb{E}\left(\|\xi\|^p\right)+L^p\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right).
\end{aligned}
\tag{3.50}
$$

Then, for all $T>t_0$, we have

$$
\sup_{t_0\leq t\leq T}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right)\leq \frac{\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p}{1-L^p}\mathbb{E}\left(\|\xi\|^p\right),
\tag{3.51}
$$

and letting $T\to\infty$,

$$
\sup_{t_0\leq t\leq \infty}\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right)\leq \frac{\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p}{1-L^p}\mathbb{E}\left(\|\xi\|^p\right).
\tag{3.52}
$$

Then, for all $t\geq t_0$

$$
\mathbb{E}\left(e^{\frac{c_3}{c_2}(t-t_0)}|x(t)|^p\right)\leq \frac{\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p}{1-L^p}\mathbb{E}\left(\|\xi\|^p\right).
\tag{3.53}
$$

Finally, we obtain

$$
\mathbb{E}\left(|x(t)|^p\right)\leq Ce^{-\frac{c_3}{c_2}(t-t_0)}\mathbb{E}\left(\|\xi\|^p\right),
\tag{3.54}
$$

where

$$
C=\frac{\left(3+\frac{1}{q}\right)\frac{c_2}{c_1}+L^p}{1-L^p}.
$$

Second case: $p>1$

For $t \geq t_0$, by Ito's formula, we obtain

$$
\begin{aligned}
& E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t\wedge\tau_k - t_0)} V\big(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\big)\Big) \\
= & \quad E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) \\
& + E\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}\Big(\frac{c_3}{c_2 2^{p-1}} V\big(s, u(s), r(s)\big) + LV\big(s, x(s), x(qs), r(s)\big)\Big) ds \\
\leq & \quad E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) \\
& + E\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}\Big(\frac{c_3}{2^{p-1}}|u(s)|^p - c_3(2+\frac{1}{q})|x(s)|^p + c_4 q e^{-\frac{c_3}{c_2}(1-q)s}|x(qs)|^p\Big) ds \quad (3.55)
\end{aligned}
$$

By (3.12) and $L < 1$, we have

$$
\begin{aligned}
|u(s)|^p &= |x(s) - G(s, x(qs)|^p \\
&\leq 2^{p-1}\big(|x(s)|^p + |G(s, x(qs)|^p\big) \\
&\leq 2^{p-1}\big(|x(s)|^p + L^p e^{-\delta(1-q^2)s}|x(qs)|^p\big) \\
&\leq 2^{p-1}\big(|x(s)|^p + e^{-\delta(1-q^2)s}|x(qs)|^p\big) \qquad (3.56)
\end{aligned}
$$

Using (3.8) to obtain

$$
\begin{aligned}
e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} e^{-\delta(1-q^2)s} &= e^{-\frac{c_3}{c_2 2^{p-1}} t_0} e^{(\frac{c_3}{c_2 2^{p-1}} - \delta)s} e^{q^2\delta s} \\
&\leq e^{-\frac{c_3}{c_2 2^{p-1}} t_0} e^{\frac{c_3}{c_2 2^p} qs} \\
&= e^{\frac{c_3}{c_2 2^p}(qs - t_0)} \qquad (3.57)
\end{aligned}
$$

As $p > 1$, then

$$
\begin{aligned}
e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} e^{-\frac{c_3}{c_2}(1-q)s} &\leq e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} e^{-\frac{c_3}{c_2 2^{p-1}}(1-q)s} \\
&= e^{\frac{c_3}{c_2 2^{p-1}}(qs-t_0)} \qquad (3.58)
\end{aligned}
$$

Substituting (3.56), (3.57) and (3.58) in (3.55),

$$
\begin{aligned}
& E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t\wedge\tau_k - t_0)} V\big(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\big)\Big) = \\
& E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) \\
& + E\int_{t_0}^{t\wedge\tau_k}\Big(- c_3(1+\frac{1}{q}) e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}|x(s)|^p + e^{\frac{c_3}{c_2 2^{p-1}}(qs-t_0)}(c_3 + qc_4)|x(qs)|^p\Big) ds \qquad (3.59)
\end{aligned}
$$

Moreover,

$$\begin{aligned}\int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(qs-t_0)}(c_3+qc_4)|x(qs)|^p ds &= \int_{qt_0}^{q(t\wedge\tau_k)} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}(\frac{c_3}{q}+c_4)|x(s)|^p ds \\ &\leq \int_{qt_0}^{t_0} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}(\frac{c_3}{q}+c_4)|x(s)|^p ds + \\ &\quad \int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)}(\frac{c_3}{q}+c_4)|x(s)|^p ds \\ &\leq (\frac{c_3}{q}+c_4)\|\xi\|^p \int_{qt_0}^{t_0} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} ds + \\ &\quad \int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} c_3(\frac{1}{q}+1)|x(s)|^p ds \\ &\leq (\frac{1}{q}+1)c_2 2^{p-1}\|\xi\|^p(1-e^{-\frac{c_3}{c_2 2^{p-1}}(1-q)t_0}) + \\ &\quad \int_{t_0}^{t\wedge\tau_k} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} c_3(\frac{1}{q}+1)|x(s)|^p ds \\ &\leq (\frac{1}{q}+1)c_2 2^{p-1}\|\xi\|^p + \\ &\quad \int_{t_0}^{q(t\wedge\tau_k)} e^{\frac{c_3}{c_2 2^{p-1}}(s-t_0)} c_3(\frac{1}{q}+1)|x(s)|^p ds \qquad (3.60)\end{aligned}$$

Then,

$$\begin{aligned}E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t\wedge\tau_k-t_0)}V\big(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\big)\Big) &\leq E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + \\ &\quad (\frac{1}{q}+1)c_2 2^{p-1}E\big(\|\xi\|^p\big). \qquad (3.61)\end{aligned}$$

By (3.2), it follows

$$\begin{aligned}&E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t\wedge\tau_k-t_0)}|u(t\wedge\tau_k)|^p\Big) \\ \leq\ & \tfrac{1}{c_1}E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t\wedge\tau_k-t_0)}V\big(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k)\big)\Big) \\ \leq\ & \tfrac{1}{c_1}E\Big(V\big(t_0, u(t_0), r(t_0)\big)\Big) + (\tfrac{1}{q}+1)\tfrac{c_2}{c_1}2^{p-1}E\big(\|\xi\|^p\big) \\ \leq\ & \tfrac{c_2}{c_1}E\Big(|u(t_0)|^p\Big) + (\tfrac{1}{q}+1)\tfrac{c_2}{c_1}2^{p-1}E\big(\|\xi\|^p\big). \qquad (3.62)\end{aligned}$$

Letting $k \to \infty$, it yields

$$\begin{aligned}
E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)}|u(t)|^p\Big) &\leq \frac{c_2}{c_1}E\Big(|u(t_0)|^p\Big) + (\frac{1}{q}+1)\frac{c_2}{c_1}2^{p-1}E\big(\|\xi\|^p\big) \\
&\leq 2^{p-1}\frac{c_2}{c_1}\Big(E\big(|x(t_0)|^p\big) + E\big(|x(qt_0)|^p\big)\Big) + \frac{c_2}{c_1}2^{p-1}(\frac{1}{q}+1)E\big(\|\xi\|^p\big) \\
&\leq \frac{c_2}{c_1}2^p E\big(\|\xi\|^p\big) + \frac{c_2}{c_1}2^{p-1}(\frac{1}{q}+1)E\big(\|\xi\|^p\big) \\
&= \frac{c_2}{c_1}2^{p-1}(\frac{1}{q}+3)E\big(\|\xi\|^p\big). \qquad (3.63)
\end{aligned}$$

Let $\varepsilon > 0$ and $T > t_0$. Using Lemma 3.7 to obtain for all $t_0 \le t \le T$

$$
\begin{aligned}
& e^{-\frac{c_3}{c_2 2^{p-1}} t_0} E\Big(e^{\frac{c_3}{c_2 2^{p-1}} t}|x(t)|^p\Big) \\
\le\ & \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)}|u(t)|^p\Big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)}|G(t,x(qt))|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} E\Big(L^p e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)} e^{-\delta(1-q^2)t}|x(qt)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} E\Big(e^{\frac{c_3}{c_2 2^{p-1}} t} e^{-\delta(1-q^2)t}|x(qt)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{t_0 \le t \le T} E\Big(e^{\frac{c_3}{c_2 2^{p-1}} t} e^{-\delta(1-q^2)t}|x(qt)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{qt_0 \le t \le qT} E\Big(e^{\frac{c_3}{q c_2 2^{p-1}} t} e^{-\frac{\delta}{q}(1-q^2)t}|x(t)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{qt_0 \le t \le T} E\Big(e^{\frac{c_3}{q c_2 2^{p-1}} t} e^{-\frac{\delta}{q}(1-q^2)t}|x(t)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + 2^{p-1} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{qt_0 \le t \le T} E\Big(e^{(\frac{c_3}{c_2 2^{p-1}}-\delta)\frac{t}{q}} e^{q\delta t}|x(t)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{qt_0 \le t \le T} E\Big(e^{q\delta t}|x(t)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{qt_0 \le t \le t_0} E\Big(e^{q\delta t}|x(t)|^p\Big) \\
& + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2 2^{p-1}} t_0} \sup_{t_0 \le t \le T} E\Big(e^{q\delta t}|x(t)|^p\Big) \\
\le\ & \frac{c_2}{c_1} 2^p \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1} (\tfrac{1}{q}+3) E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-(\frac{c_3}{c_2 2^{p-1}}-q\delta)t_0} E(\|\xi\|^p) \\
& + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon} L^p e^{-\frac{c_3}{c_2} t_0} \sup_{t_0 \le t \le T} E\Big(e^{q\delta t}|x(t)|^p\Big)
\end{aligned}
\tag{3.64}
$$

Using (3.8) to obtain

$$
\begin{aligned}
& e^{-\frac{c_3}{c_2 2^{p-1}}t_0} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}t}|x(t)|^p\Big) \\
\le\ & \tfrac{c_2}{c_1}2^p\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}(\tfrac{1}{q}+3)E\big(\|\xi\|^p\big) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon}L^p E(\|\xi\|^p) \\
& + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon}L^p e^{-\frac{c_3}{c_2 2^{p-1}}t_0} \sup_{t_0\le t\le T} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}t}|x(t)|^p\Big) \\
\le\ & \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}(\tfrac{c_2}{c_1}2^p(\tfrac{1}{q}+3)+\tfrac{L^p}{\varepsilon})E(\|\xi\|^p) + \frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon}L^p e^{-\frac{c_3}{c_2 2^{p-1}}t_0} \sup_{t_0\le t\le T} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}t}|x(t)|^p\Big)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& (1-\frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon}L^p)e^{-\frac{c_3}{c_2 2^{p-1}}t_0} \sup_{t_0\le t\le T} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}t}|x(t)|^p\Big) \\
\le\ & \big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}(\tfrac{c_2}{c_1}2^p(\tfrac{1}{q}+3)+\tfrac{L^p}{\varepsilon})E(\|\xi\|^p)
\end{aligned}
\tag{3.65}
$$

We have

$$\lim_{\varepsilon\to\infty} 1-\frac{\big[1+\varepsilon^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon}L^p = 1-L^p > 0.$$

Then, there exists $\varepsilon_1 > 0$ large enough such that

$$1-\frac{\big[1+\varepsilon_1^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon_1}L^p > 0.$$

Then, for all $T > t_0$, we have

$$\sup_{t_0\le t\le T} E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)}|x(t)|^p\Big) \ \le\ \frac{\big[1+\varepsilon_1^{\frac{1}{p-1}}\big]^{p-1}(\frac{c_2}{c_1}2^p(\frac{1}{q}+3)+\frac{L^p}{\varepsilon_1})}{1-\frac{\big[1+\varepsilon_1^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon_1}L^p}E(\|\xi\|^p) \tag{3.66}$$

Then, for all $t \ge t_0$

$$E\Big(e^{\frac{c_3}{c_2 2^{p-1}}(t-t_0)}|x(t)|^p\Big) \ \le\ \frac{\big[1+\varepsilon_1^{\frac{1}{p-1}}\big]^{p-1}(\frac{c_2}{c_1}2^p(\frac{1}{q}+3)+\frac{L^p}{\varepsilon_1})}{1-\frac{\big[1+\varepsilon_1^{\frac{1}{p-1}}\big]^{p-1}}{\varepsilon_1}L^p}E(\|\xi\|^p) \tag{3.67}$$

Finally, we obtain

$$E\Big(|x(t)|^p\Big) \ \le\ C' e^{-\frac{c_3}{c_2 2^{p-1}}(t-t_0)}E(\|\xi\|^p), \tag{3.68}$$

where $C' = \dfrac{\left[1+\varepsilon_1^{\frac{1}{p-1}}\right]^{p-1}\left(\frac{c_2}{c_1}2^p(\frac{1}{q}+3)+\frac{L^p}{\varepsilon_1}\right)}{1-\frac{\left[1+\varepsilon_1^{\frac{1}{p-1}}\right]^{p-1}}{\varepsilon_1}L^p}$

Then, in both cases, system (2.1) is p-th moment exponentially stable. □

4 Illustrative examples

In this section we will analyze two examples to illustrate the effectiveness of our abstract results.

4.1 Example 1

Let $q = \frac{1}{4}$, $S = \{1, 2\}$ and the matrix $\Gamma = (\gamma_{ij})_{1\leq i,j\leq 2}$ define by

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Consider

$$G(t, x) = \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x), \tag{4.1}$$

Using the mean value theorem to obtain

$$|\sin(x) - \sin(y)| \leq |x - y|.$$

Then,

$$|G(t, x) - G(t, y)| \leq \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}|x - y|.$$

Let

$$d\left(x(t) - G\left(t, x(qt)\right)\right) = f\left(t, x(t), x(qt), r(t)\right)dt + g\left(t, x(t), x(qt), r(t)\right)dw(t). \tag{4.2}$$

$$\begin{aligned} f(t, x, y, 1) &= -\frac{27}{4}\Big(x - \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(y)\Big), && (4.3)\\ f(t, x, y, 2) &= -13\Big(x - \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(y)\Big), && (4.4)\\ g(t, x, y, 1) &= \sqrt{2}\Big(x - \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(y)\Big), && (4.5)\\ g(t, x, y, 2) &= x - \frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(y), && (4.6) \end{aligned}$$

$$V(t, x, 1) = x^2 \quad \text{and} \quad V(t, x, 2) = \frac{1}{2}x^2. \tag{4.7}$$

We have

$$V_t(t,x,1)=0,\quad V_x(t,x,1)=2x,\quad V_{xx}(t,x,1)=2, \tag{4.8}$$

$$V_t(t,x,2)=0,\quad V_x(t,x,2)=x\quad\text{and}\quad V_{xx}(t,x,1)=1. \tag{4.9}$$

Then,

$$\begin{aligned} LV(t,x(t),x(qt),1) &= -\frac{27}{2}\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2+2\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2 \\ &\quad -\frac{1}{2}\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2 \\ &= -12\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2. \end{aligned} \tag{4.10}$$

Moreover,

$$\begin{aligned} LV(t,x(t),x(qt),2) &= -13\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2+\frac{1}{2}\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2 \\ &\quad +\frac{1}{2}\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2 \\ &= -12\Big(x(t)-\frac{1}{7}e^{-\frac{1}{2}(1-q^2)t}\sin(x(qt))\Big)^2. \end{aligned} \tag{4.11}$$

We have, for $i=1,2$

$$\begin{aligned} LV(t,x(t),x(qt),i) &= 12\Big(-x^2(t)+\frac{2}{7}e^{-\frac{1}{2}(1-q^2)t}x(t)\sin(x(qt))-\frac{1}{49}e^{-(1-q^2)t}\sin^2(x(qt))\Big) \\ &\le 12\Big(-x^2(t)+\frac{1}{2}x^2(t)+\frac{2}{49}e^{-(1-q^2)t}\sin^2(x(qt))-\frac{1}{49}e^{-(1-q^2)t}\sin^2(x(qt))\Big) \\ &= 12\Big(-\frac{1}{2}x^2(t)+\frac{1}{49}e^{-(1-q^2)t}\sin^2(x(qt))\Big) \\ &= 12\Big(-\frac{1}{12}6x^2(t)+\frac{4}{49}\frac{1}{4}e^{-(1-q^2)t}\sin^2(x(qt))\Big) \\ &\le 12\Big(-\frac{1}{12}6x^2(t)+\frac{1}{12}\frac{1}{4}e^{-(1-q^2)t}\sin^2(x(qt))\Big) \\ &\le -6x^2(t)+\frac{1}{4}e^{-(1-q^2)t}\sin^2(x(qt)) \\ &\le -6x^2(t)+\frac{1}{4}e^{-(1-q)t}\sin^2(x(qt)). \end{aligned} \tag{4.12}$$

Using

$$|\sin(x)|\le|x|,\quad \forall x\in\mathbb{R},$$

we can obtain

$$LV(t, x(t), x(qt), i) \;\; \leq \;\; -6x^2(t) + \frac{1}{4}e^{-(1-q)t}x^2(qt). \tag{4.13}$$

We deduce that assumptions 3.1, 3.2 and 3.4 hold with $\delta = 1$, $p = 2$, $q = \frac{1}{4}$, $c_1 = c_2 = c_3 = c_4 = 1$. By theorems 3.10 and 3.12, it follows that equation (4.15) admits a unique global solution and it is mean square exponentially stable.

For system (4.2), we conduct a simulation based on Euler-Maruyama scheme with step size 10^{-5}, for which we set $q = 0.25$, $t_0 = 1$ and the two initials data ξ_1, ξ_2 as a linear mapping, namely $\xi_1(t) = t^2$ and $\xi_2(t) = -\frac{1}{2}t - 1$ for all $0.25 \leq t \leq 1$. We give a sequence of computer simulations for system (4.2) as follows. Figure 1 and Figure 3 illustrates the pathwise stability by simulations of the trajectories of the solution $x(t)$ of system (4.2) with the two different initials condition ξ_1 and ξ_2. Choosing $p = 2$, $L = 0.1429$, $c_1 = c_2 = c_3 = c_4 = 1$, $C = 30.57$ and $\alpha = 1$, then the simulation result of system (4.2) show the mean square exponential stability of $x(t)$ in Figure 2 (respectively in Figure 4) with the initial condition ξ_1 (respectively ξ_2).

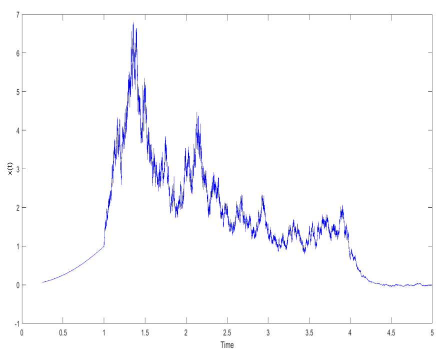


Figure 1: Simulation of trajectory of $x(t)$ of system (4.2) on the interval $[0.25, 5]$ for $\xi_1(t) = t^2$.

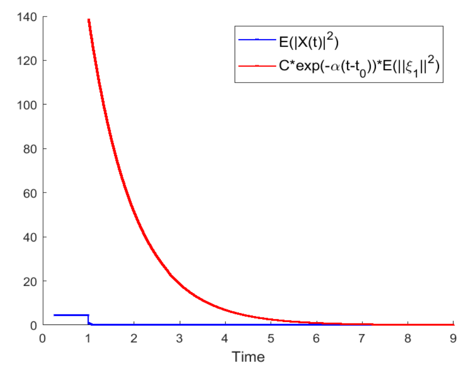


Figure 2: Mean square exponential stability of $x(t)$ of system (4.2) on the interval $[0.25, 9]$ for $\xi_1(t) = t^2$.

4.2 Example 2

Let $q = \frac{1}{6}$, $S = \{1, 2\}$ and the matrix $\Gamma = (\gamma_{ij})_{1 \leq i,j \leq 2}$ define by

$$\Gamma = \begin{pmatrix} -\gamma & \gamma \\ 1 & -1 \end{pmatrix},$$

where $\gamma > 0$. Consider

$$G(t, x) \;\; = \;\; \frac{1}{10}e^{-(1-q^2)t}x, \tag{4.14}$$

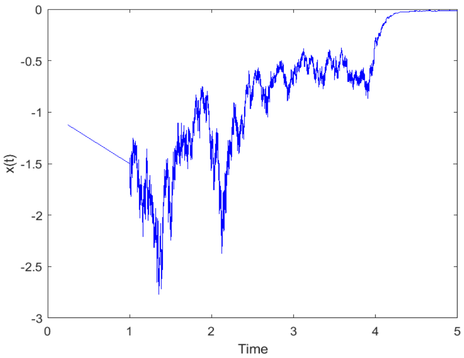


Figure 3: Simulation of trajectory of $x(t)$ of system (4.2) on the interval $[0.25, 5]$ for $\xi_2(t) = -\frac{1}{2}t - 1$.

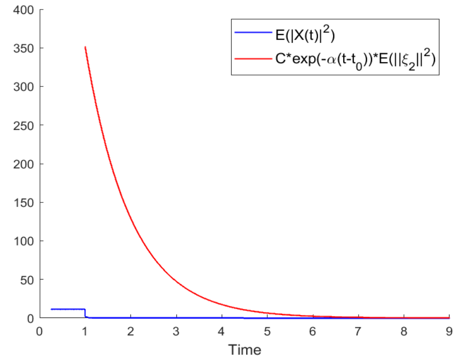


Figure 4: Mean square exponential stability of $x(t)$ of system (4.2) on the interval $[0.25, 9]$ for $\xi_2(t) = -\frac{1}{2}t - 1$.

Let

$$d\left(x(t) - G\left(t, x(qt)\right)\right) = f\Big(t, x(t), x(qt), i\Big)dt + g\Big(t, x(t), x(qt), i\Big)dw(t). \tag{4.15}$$

$$f(t,x,y,1) = -16\Big(x - \frac{1}{10}e^{-(1-q^2)t}y\Big), \tag{4.16}$$

$$f(t,x,y,2) = -33\Big(x - \frac{1}{10}e^{-(1-q^2)t}y\Big), \tag{4.17}$$

$$g(t,x,y,1) = \frac{\sqrt{\gamma}}{\sqrt{2}}\Big(x - \frac{1}{10}e^{-(1-q^2)t}y\Big), \tag{4.18}$$

$$g(t,x,y,2) = x - \frac{1}{10}e^{-(1-q^2)t}y, \tag{4.19}$$

$$V(t,x,1) = \gamma x^2 \quad \text{and} \quad V(t,x,2) = \frac{\gamma}{2}x^2. \tag{4.20}$$

We have

$$V_t(t,x,1) = 0, \quad V_x(t,x,1) = 2\gamma x, \quad V_{xx}(t,x,1) = 2\gamma, \tag{4.21}$$

$$V_t(t,x,2) = 0, \quad V_x(t,x,2) = \gamma x \quad \text{and} \quad V_{xx}(t,x,1) = \gamma. \tag{4.22}$$

Then,

$$\begin{aligned} LV(t, x(t), x(qt), 1) &= -32\gamma\Big(x(t) - \frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2 + \frac{\gamma^2}{2}\Big(x(t) - \frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2 \\ &\quad -\frac{\gamma^2}{2}\Big(x(t) - \frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2 \\ &= -32\gamma\Big(x(t) - \frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2. \end{aligned} \tag{4.23}$$

Moreover,

$$\begin{aligned} LV(t,x(t),x(qt),2) &= -33\gamma\Big(x(t)-\frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2+\frac{\gamma}{2}\Big(x(t)-\frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2 \\ &\quad +\frac{\gamma}{2}\Big(x(t)-\frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2 \\ &= -32\gamma\Big(x(t)-\frac{1}{10}e^{-(1-q^2)t}x(qt)\Big)^2. \end{aligned} \tag{4.24}$$

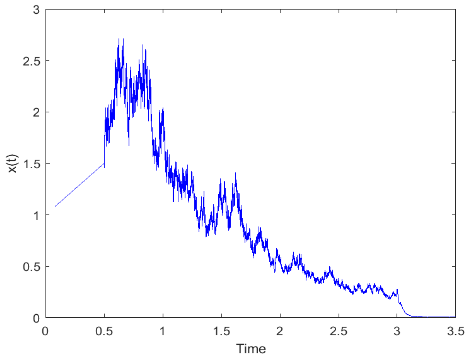
We have, for $i=1,2$

$$\begin{aligned} LV(t,x(t),x(qt),i) &= 32\gamma\Big(-x^2(t)+\frac{2}{10}e^{-(1-q^2)t}x(t)x(qt)-\frac{1}{100}e^{-2(1-q^2)t}x^2(qt)\Big) \\ &\le 32\gamma\Big(-x^2(t)+\frac{1}{2}x^2(t)+\frac{2}{100}e^{-2(1-q^2)t}x^2(qt)-\frac{1}{100}e^{-2(1-q^2)t}x^2(qt)\Big) \\ &= 32\gamma\Big(-\frac{1}{2}x^2(t)+\frac{1}{100}e^{-2(1-q^2)t}x^2(qt)\Big) \\ &= 32\gamma\Big(-\frac{1}{16}8x^2(t)+\frac{6}{100}\frac{1}{6}e^{-2(1-q^2)t}x^2(qt)\Big) \\ &\le 32\gamma\Big(-\frac{1}{16}8x^2(t)+\frac{1}{16}\frac{1}{6}e^{-2(1-q^2)t}x^2(qt)\Big) \\ &\le -2\gamma\times 8x^2(t)+2\gamma\times\frac{1}{6}e^{-2(1-q^2)t}x^2(qt) \\ &\le -2\gamma\times 8x^2(t)+2\gamma\times\frac{1}{6}e^{-2(1-q)t}x^2(qt). \end{aligned} \tag{4.25}$$

Then, for all $\gamma>0$, assumptions 3.1, 3.2 and 3.4 hold with $\delta=2$, $p=2$, $q=\frac{1}{6}$, $c_1=\frac{\gamma}{2}$, $c_2=\gamma$ and $c_3=c_4=2\gamma$. By theorems 3.10 and 3.12, we deduce that equation (4.15) admits a unique global solution and it is mean square exponentially stable.

For system (4.15), set $q=0.1667$, $t_0=0.5$ and the two initials data ξ_1, ξ_2 as a linear mapping, namely $\xi_1(t)=t+1$ and $\xi_2(t)=-\frac{1}{2}t-2$ for all $0.0833\le t\le 0.5$. Based on Euler-Maruyama scheme with step size 10^{-5}, we give a sequence of computer simulations for system (4.15) as follows. Figure 5 and Figure 7 illustrates the pathwise stability by simulations of the trajectories of the solution $x(t)$ with the two different initial condition ξ_1 and ξ_2. Choosing $p=2$, $\gamma=3$, $L=0.1$, $c_1=1.5$, $c_2=3$, $c_3=c_4=6$, $C=75.04$ and $\alpha=0.0741$, then the simulation result of system (4.15) show the mean square exponential stability of $x(t)$ in Figure 6 (respectively in Figure 8) with the initial condition ξ_1 (respectively ξ_2).

In Example 4.1 and 4.2, the simulation results clearly show that the trajectories of the corresponding stochastic systems converge rapidly to the equilibrium state for any given initial values, and verify the effectiveness of theoretical results.



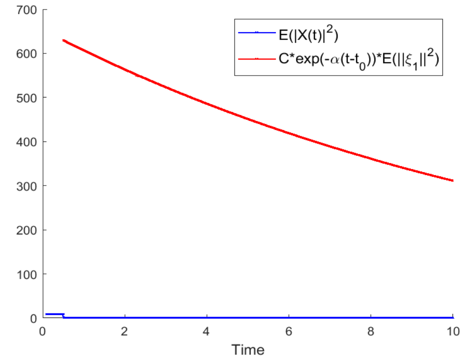
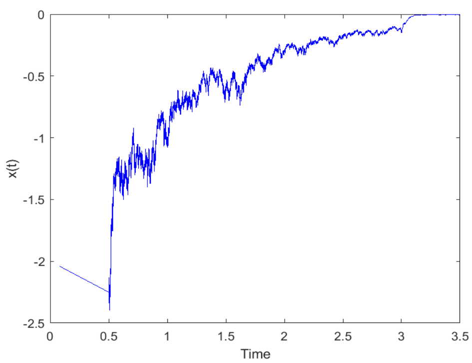


Figure 5: Simulation of trajectory of the solution $x(t)$ of system (4.15) on the interval $[\frac{1}{12}, \frac{7}{2}]$ for $\gamma = 3$ and $\xi_1(t) = t + 1$.

Figure 6: Mean square exponential stability of the solution $x(t)$ of system (4.15) on the interval $[\frac{1}{12}, 10]$ for $\gamma = 3$ and $\xi_1(t) = t + 1$.



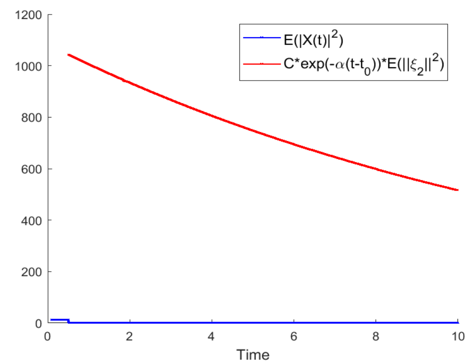


Figure 7: Simulation of trajectory of the solution $x(t)$ of system (4.15) on the interval $[\frac{1}{12}, \frac{7}{2}]$ for $\gamma = 3$ and $\xi_2(t) = -\frac{1}{2}t - 2$.

Figure 8: Mean square exponential stability of the solution $x(t)$ of system (4.15) on the interval $[\frac{1}{12}, 10]$ for $\gamma = 3$ and $\xi_2(t) = -\frac{1}{2}t - 2$.

5 Conclusion

In this paper we deal with the problem of p-th moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching. In this work, we have improved the previous work of [8] by using a new condition on Lyapunov function and the neutral term lead to the convergence of the solution exponentially to the equilibrium point in p-th moment.

Acknowledgements

The research of T. Caraballo has been supported by FEDER and the Spanish Ministerio de Ciencia, Innovación y Universidades under the project PGC2018-096540-B-I00, and Junta de Andalucía (Spain) under the FEDER projects US-1254251 and P18-FR-4509.

References

[1] J. Appleby and E. Buckwar, Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 2 (2016), 1-32.

[2] L. Arnold, Stochastic differential equations: Theory and Applications. *John Wiley, New York-London-Sydney*, 1974.

[3] C. T. H. Baker and E. Buckwar, Continuous θ-methods for the stochastic pantograph equation. *Electronic Transactions on Numerical Analysis*, Vol 11 (2000), 131-151.

[4] T. Caraballo, M. A. Hammami, L. Mchiri, On the practical global uniform asymptotic stability of stochastic differential equations. *Stochastics*, Vol 88 (2016), No. 1, 45-56.

[5] Z. Fan, M. Song and M. Liu, The αth moment stability for the stochastic pantograph equation. *Journal of Computational and Applied Mathematics*, Vol 233 (2009), 109-120.

[6] P. Guo and C. Li, Razumikhin-type technique on stability of exact and numerical solutions for the nonlinear stochastic pantograph differential equations. *BIT Numerical Mathematics*, Vol 59 (2019), 77-96.

[7] W. Hu , Q. Zhu and H. R. Karimi, Some Improved Razumikhin Stability Criteria for Impulsive Stochastic Delay Differential Systems. *IEEE Transactions on Automatic Control*, Vol 64, No. 12 (2019), 5207-5213.

[8] W. Mao, L. Hu and X. Mao, Razumikhin-type theorems on polynomial stability of hybrid stochastic systems with pantograph delay. *Discrete and continuous dynamical systems series B*, Vol 25 (2020), No. 8, 3217-3232.

[9] X. Mao and C.Yuan, Stochastic Differential Equations with Markovian Switching. *Imperial College Press*, 2006.

[10] W. Mao, L. Hu, X. Mao. Almost sure stability with general decay rate of neutral stochastic pantograph equations with Markovian switching. *Electronic Journal of Qualitative Theory of Differential Equations*, Vol 52 (2019), 1-17.

[11] X. Mao, Stochastic Differential Equations and Applications. *Ellis Horwood, Chichester*, U.K, 1997.

[12] M. Milosevic, Existence, uniqueness, almost sure polynomial stability of solution to a class of highly nonlinear pantograph stochastic differential equations and the Euler-Maruyama approximation. *Applied Mathematics and Computation*, Vol 237 (2014), 672-685.

[13] M. Obradovic and M. Milosevic, Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler-Maruyama method. *Journal of Computational and Applied Mathematics*, Vol 309 (2017), 244-266.

[14] M. Shen, W. Foi, X. Mao and S. Deng, Exponential stability of highly nonlinear neutral pantograph stochastic differential equation. *Asian Journal of Control*, Vol 22 (2020), No. 1, 1-13.

[15] B. Wang and Q. Zhu, Stability analysis of semi-Markov switched stochastic systems, *Automatica.*, Vol 94 (2018)72-80.

[16] X. Wu, W. Zhang and Y. Tang, pth Moment stability of impulsive stochastic delay differential systems with Markovian switching. *Communications in Nonlinear Science and Numerical Simulation*, Vol 18 (2013), 1870-1879.

[17] S.You, W. Mao, X. Mao and L. Hu, Analysis on exponential stability of hybrid pantograph stochastic differential equations with highly nonlinear coefficients. *Applied mathematics and computation*, Vol 263 (2015), 73-83.

[18] C. Yuan and X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching. *Automatica*, Vol 40 (2004), 343-354.

[19] T. Zhang, H. Chen, C. Yuan and T. Caraballo, On the asymptotic behavior of highly nonlinear hybrid stochastic delay differential equations. *Discrete and continuous dynamical systems series B*, Vol 24 (2019), 5355-5375.

[20] S. Zhou and M. Xue, Exponential stability for nonlinear hybrid stochastic pantograph equations and numerical approximation. *Acta Mathematica Scientia*, Vol 34 (2014), No. 4, 1254-1270.

[21] Q. Zhu, Stabilization of Stochastic Nonlinear Delay Systems With Exogenous Disturbances and the Event-Triggered Feedback Control, *IEEE Transactions on Automatic Control*, Vol 64, No. 9 (2019), 3764-3771.