h-stability in p-th moment of neutral pantograph stochastic differential equations with Markovian switching driven by Lévy noise

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Abstract

In this paper we investigate the h-stability in p-th moment of neutral pantograph stochastic differential equations with Markovian switching driven by Lévy noise. The main tool used to prove the results is the Lyapunov method. We analyze two illustrative examples to show the interest and usefulness of the main results.

Keywords: Neutral pantograph stochastic differential equations with Markovian switching, Lévy noise, h-stability, p-th moment.
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1 Introduction

Stochastic delay differential equations (SDDEs) have an important role in many branches of science, industry and economics. Such models have been used in epidemiology, biology, mechanics and finance. The stability of these stochastic delay differential equations has much interest especially for control and stabilization problems and it is still necessary to develop new methods to analyze the asymptotic properties, despite of the fact that there are many published results for the stability of ordinary stochastic differential equations and stochastic delay differential equations (see, e.g., [3], [8], [9], [14] and [15]).
Neutral stochastic differential equations with Markovian switching (NSDEMS) is a particular case of stochastic delay differential equations. In the literature, many authors have studied (NSDEMS) (see [7] and [12]).
Recently, as a special case of SDDEs, a class of nonlinear stochastic pantograph delay equations (NSPDEs) has received a great deal of attention and various studies have been carried out on the polynomial and exponential stability of NSPDEs (see [1], [2], [10], [11], [13] and [16]). One important class of stochastic delay systems, the neutral pantograph stochastic differential equations with Markovian switching (NPSDEwMS) has also been studied (see [11] and [13]). The almost sure stability with general decay rate of (NPSDEwMS) was first investigated by X. Mao et al. (see [11]). However, all equations of the above-mentioned works are driven by white noise perturbations with continuous initial data, and white noise perturbations are not always appropriate to interpret real data in a reasonable way. In real phenomena, the state of stochastic pantograph delay system may be perturbed by extreme events or abrupt impulses. A more natural mathematical framework for these phenomena takes into account other processes rather than Brownian motions. In particular, we use the Lévy noise with jumps into neutral pantograph stochastic differential equations with Markovian switching to model abrupt changes and, in particular, we generalize the results in [6] which were obtained for exponential stability and did not include Markovian switching. Nevertheless, as far as we know, there is no research on the h-stability on the stochastic case, although there are some papers on the deterministic case (see [4] and [5]). This type of stability provides new insights about the asymptotic behavior of solutions and it is well worth analyzing it.

In this paper, we will study the h-stability in p-th moment of neutral pantograph stochastic differential equations with Markovian switching driven by Lévy noise. To do this, we have structured the paper as follows. In Section 2, we introduce some basic notations and assumptions. In Section 3, we establish some sufficient conditions ensuring h-stability in p-th moment of neutral pantograph stochastic differential equations with Markovian switching driven by Lévy noise by using the Lyapunov techniques and Itô's formula. In Section 4, we analyze two illustrative examples to show our theoretical results.

2 Preliminaries and definitions

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets. $W(t)$ is an m-dimensional Brownian motion defined on the probability space. Let $t_0 > 0$, $q \in (0,1)$ and $C([qt_0, t_0]; \mathbb{R}^n)$ denote the family of the continuous functions φ from $[qt_0, t_0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{qt_0 \leq s \leq t_0} |\varphi(s)|$ and $|x| = \sqrt{x^T x}$ for any $x \in \mathbb{R}^n$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $p > 0$, $L^p_{\mathcal{F}_t}([qt, t]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable, $C([qt, t]; \mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\theta) : qt \leq \theta \leq t\}$ such that $\mathbb{E}\|\varphi\|^p < \infty$. For $a \in \mathbb{R}$, $F(a^-)$ denotes the left-hand limit of function $F(.)$ at a, i.e., $F(a^-) = \lim_{l\to 0^-} F(a + l)$.

Let $\{r(t), t \in \mathbb{R}^+ = [0, +\infty)\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with a generator $\Gamma = (\gamma_{ij})_{\mathbb{N}\times\mathbb{N}}$ given by

$$\mathbb{P}\left(r(t+\Delta) = j | r(t) = i\right) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if}\, i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if}\, i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$, while

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

We assume that the Markov chain $r(.)$ is independent of the Brownian motion $W(.)$.
Consider the following neutral pantograph stochastic differential equation with Markovian switching driven by Lévy noise:

$$d\left[x(t) - G(t, x(qt))\right] = f(t, x(t), x(qt), r(t))dt + g(t, x(t), x(qt), r(t))dW(t),$$

$$+ \int_{|z|<c} H_1(t, x(t^-), x(qt^-), r(t), z)\widetilde{N}(dt, dz) + \int_{|z|\geq c} H_2(t, x(t^-), x(qt^-), r(t), z)N(dt, dz) \quad t \geq t_0, \tag{2.1}$$

with the initial condition $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$, i.e. $x(t) := x(t; t_0, \xi) = \xi(t)$ for $qt_0 \leq t \leq t_0$. Let $u(t) = x(t) - G(t, x(qt))$. Here, we furthermore assume that

$$f : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^n, \quad g : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \longrightarrow \mathbb{R}^{n\times m},$$

$$G : [t_0, +\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

$H_i : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^d \longrightarrow \mathbb{R}^n$ (i=1, 2), and the constant $c \in (0, +\infty)$ is the maximum allowable jump size. $N(., .)$ is a Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$ with compensator $\widetilde{N}$ and intensity measure $\nu(.)$.

It is always assumed that $N(.,.)$ is independent of $W(.)$. Let $\nu(.)$ be a Lévy measure such that $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)$ and $\int_{\mathbb{R}^d\setminus\{0\}}(|z|^2 \wedge 1)\nu(dz) = \lambda < +\infty$.
Usually, the pair $(W(.), N(.,.))$ is called a Lévy noise, $\int_{|z|<c} H_1(t, x(t^-), x(qt^-), r(t), z)\widetilde{N}(dt,dz)$ is called a 'small jump' and $\int_{|z|\geq c} H_2(t, x(t^-), x(qt^-), r(t), z)N(dt,dz)$ is called 'large jump'.
We denote by $x(t;t_0,\xi)$ the solution of Eq. (2.1).

Let $C^{1,2}\left([qt_0,+\infty)\times\mathbb{R}^n\times S;\mathbb{R}^+\right)$ be the family of all non-negative functions $V(t,x,i)$ on $[qt_0,+\infty)\times\mathbb{R}^n\times S$, which are twice continuously differentiable with respect to x and once continuously differentiable with respect to t.
For any $(t,x,y,i)\in[qt_0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times S$, $u(t) = x(t) - G(t, y(t))$, with $y(t) = x(qt)$, by the generalized Itô's formula we have,

$$V(t,u(t),r(t)) = V(t_0,u(t_0),r(t_0)) + \int_{t_0}^{t} LV(s,x(s),x(qs),r(s))ds + M_t, \tag{2.2}$$

where the operator $LV(t,x,y,i) : [qt_0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^n\times S \to \mathbb{R}$ and the process M_t are defined respectively by

$$\begin{aligned} LV(t,x,y,i) &= V_t(t,u,i) + V_x(t,u,i)f(t,x,y,i) \\ &+ \frac{1}{2}trace\left(g^T(t,x,y,i)V_{xx}(t,u,i)g(t,x,y,i)\right) \\ &+ \int_{|z|<c}\big(V(t,u+H_1(t,x,y,i,z)) - V(t,u,i) - V_x(t,u,i)H_1(t,x,y,i,z)\big)\nu(dz) \\ &+ \int_{|z|\geq c}\big(V(t,u+H_2(t,x,y,i,z)) - V(t,u,i)\big)\nu(dz) \\ &+ \sum_{j=1}^{N}\gamma_{ij}V(t,u,j) \end{aligned}$$

and

$$M_t = \int_{t_0}^{t} V_x(s,u(s),r(s))g(s,x(s),x(qs),r(s))dW(s)$$

$$\begin{aligned} &+ \int_{t_0}^{t}\int_{|z|<c}\big(V(s,u(s)+H_1(s,x(s^-),x(qs^-),r(s),z)) - V(s,u(s),r(s))\big)\widetilde{N}(ds,dz), \\ &+ \int_{t_0}^{t}\int_{|z|\geq c}\big(V(s,u(s)+H_2(s,x(s^-),x(qs^-),r(s),z)) - V(s,u(s),r(s))\big)N(ds,dz) \end{aligned}$$

where

$$V_t = \frac{\partial V(t,x,i)}{\partial t}, \quad V_x = \left(\frac{\partial V(t,x,i)}{\partial x_1},\ldots,\frac{\partial V(t,x,i)}{\partial x_n}\right), \quad V_{xx} = \left(\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

For our purpose, we will state some assumptions which can ensure the existence and uniqueness of a solution $x(t) = x(t; t_0, \xi)$ on $t \geq t_0$, for equation (2.1).
$\mathcal{H}_1$: (A local Lipschitz condition): For each integer $d \geq 1$ there is an $l_d > 0$ such that

$$|f(t,x,y,i) - f(t,\overline{x},\overline{y},i)| \vee |g(t,x,y,i) - g(t,\overline{x},\overline{y},i)| \vee |H_b(t,x,y,i,z) - H_b(t,\overline{x},\overline{y},i,z)| \leq l_d\big(|x-\overline{x}| + |y-\overline{y}|\big),$$

for all $t \geq t_0$, $i \in S$ and x, $\overline{x}$, y, $\overline{y} \in \mathbb{R}^n$ with $|x| \vee |\overline{x}| \vee |y| \vee |\overline{y}| \leq d$ and $b = 1, 2$.
Besides, $f(t,0,0,i) = 0$, $g(t,0,0,i) = 0$ and $H_b(t,0,0,z) = 0$, $G(t,0) = 0$, for any $t \geq t_0$, $z \in \mathbb{R}^d$ and $b = 1, 2$.
$\mathcal{H}_2$: There exists a constant $\beta \in (0, \frac{1}{2})$ such that

$$|G(t,y) - G(t,\overline{y})| \leq \beta |y - \overline{y}|, \tag{2.3}$$

for all $t \geq t_0$ and y, $\overline{y} \in \mathbb{R}^n$.
$\mathcal{H}_3$: There exist positive numbers c_1, c_2, c_3, c_4, and p such that $c_3 \geq c_4$ and for all $(t,x,y,i) \in [qt_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S$, we have

$$c_1|x|^p \leq V(t,x,i) \leq c_2|x|^p. \tag{2.4}$$

$$LV(t,x,y,i) \leq -c_3|x|^p + c_4 q|y|^p. \tag{2.5}$$

Recall now some useful inequalities which will be used in our analysis.

Lemma 2.1. (i) *Let* $0 < p \leq 1$ *and* a, $b \in \mathbb{R}_+$. *Then*

$$(a+b)^p \leq a^p + b^p.$$

(ii) *Let* $p > 1$, $\varepsilon > 0$ *and* a, $b \in \mathbb{R}_+$. *Then*

$$(a+b)^p \leq \left(1 + \varepsilon^{\frac{1}{p-1}}\right)^{p-1} \left(a^p + \frac{b^p}{\varepsilon}\right).$$

Proof. See [9]. □

Remark 2.2. For $p > 1$, we can take $\varepsilon = 1$, then, for any a, $b \in \mathbb{R}_+$, we have

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right). \tag{2.6}$$

Theorem 2.3. *Let assumptions* $\mathcal{H}_1$-$\mathcal{H}_3$ *hold. Then for any given initial data* ξ, *there is a unique global solution* $x(t)$ *of equation* (2.1) *on* $t \in [t_0, +\infty)$.

Proof. Using Assumption $\mathcal{H}_1$, for any initial value $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$, there exists a unique maximal solution $x(\cdot) = x(\cdot; t_0, \xi)$ on $[t_0, \sigma_e)$ (see Theorem 1 in [12]), where σ_e is the explosion time. Let $k_0 > 0$ be sufficiently large such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [t_0, \sigma_e);\ |x(t)| \geq k\right\}.$$

Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \to \tau_\infty \leq \sigma_e$ a.s.. If we can show $\tau_\infty = \infty$ a.s., then we have $\sigma_e = \infty$ a.s. Therefore, we just need to show $\tau_\infty = \infty$ a.s.

Case 1: Assume $0 < p \leq 1$. By the generalized Itô formula (see, [14]) and Lemma 2.1 we have

$$\begin{aligned}
&\mathbb{E}\left(V(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k))\right)\\
&= \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + \mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} LV(s, x(s), x(qs), r(s))ds\right)\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - c_3 E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 q E\left(\int_{t_0}^{t\wedge\tau_k} |x(qs)|^p ds\right)\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - c_3 E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 E\left(\int_{qt_0}^{q(t\wedge\tau_k)} |x(s)|^p ds\right)\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - c_3 E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 E\left(\int_{qt_0}^{t\wedge\tau_k} |x(s)|^p ds\right)\\
&= \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - c_3 E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 E\left(\int_{qt_0}^{t_0} |x(s)|^p ds\right)\\
&\quad + c_4 E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right)\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - (c_3 - c_4) E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 t_0(1-q) E\|\xi\|^p\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) - (c_3 - c_4) E\left(\int_{t_0}^{t\wedge\tau_k} |x(s)|^p ds\right) + c_4 E\|\xi\|^p\\
&\leq \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_4 E\|\xi\|^p\\
&\leq c_2 \mathbb{E}\left(|u(t_0)|^p\right) + c_4 E\|\xi\|^p\\
&\leq c_2 \mathbb{E}\left(|x(t_0)|^p\right) + c_2 \beta^p \mathbb{E}\left(|x(t_0)|^p\right) + c_4 E\|\xi\|^p\\
&\leq \big(c_2(1+\beta^p) + c_4\big)\mathbb{E}\|\xi\|^p.
\end{aligned}$$

Then, by assumption $\mathcal{H}_3$,

$$\mathbb{E}|u(t\wedge\tau_k)|^p \leq \frac{1}{c_1}\big(c_2(1+\beta^p) + c_4\big)\mathbb{E}\|\xi\|^p.$$

By the definition of τ_k, we have $|x(\tau_k)| = k$ and $|x(t \wedge \tau_k)| \le k$, $|x(t \wedge \tau_k)|^p \le k^p$, including $|x(q(t \wedge \tau_k))|^p \le k^p$. For $\tau_k \le t$, we have $|x(t \wedge \tau_k)| = |x(\tau_k)| = k$, and we therefore obtain $E\big[|x(t \wedge \tau_k)|^p 1_{\tau_k \le t}\big] = k^p \mathbb{P}(\tau_k \le t)$. By the inequality

$$|x(t)|^p \le |u(t)|^p + \beta^p |x(qt)|^p,$$

we obtain

$$\begin{aligned} |u(t \wedge \tau_k)|^p &\ge |u(t \wedge \tau_k)|^p 1_{\{\tau_k \le t\}} \\ &\ge |x(t \wedge \tau_k)|^p 1_{\tau_k \le t} - \beta^p |x(q(t \wedge \tau_k))|^p 1_{\{\tau_k \le t\}} \\ &\ge k^p (1 - \beta^p) 1_{\{\tau_k \le t\}}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{P}(\tau_k \le t) &\le \frac{1}{k^p (1 - \beta^p)} \mathbb{E}|u(t \wedge \tau_k)|^p \\ &\le \frac{1}{c_1 k^p (1 - \beta^p)} \big(c_2(1 + \beta^p) + c_4\big) \mathbb{E}\|\xi\|^p. \end{aligned}$$

Letting $k \to \infty$ implies $\mathbb{P}(\tau_\infty \le t) = 0$ and, consequently, $\tau_\infty > t$ a.s. Letting $t \to \infty$, we can obtain $\tau_\infty = \infty$ a.s, which implies that there exists a global solution $x(t)$ to the system (2.1).

Case 2: Assume now $p > 1$. Proceeding as in Case 1 and using (2.6),

$$\begin{aligned} \mathbb{E}\left(V(t \wedge \tau_k, u(t \wedge \tau_k), r(t \wedge \tau_k))\right) &\le \mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_4 E\|\xi\|^p \\ &\le c_2 \mathbb{E}\left(|u(t_0)|^p\right)) + c_4 E\|\xi\|^p \\ &\le 2^{p-1} c_2 \mathbb{E}\left(|x(t_0)|^p\right) + 2^{p-1} c_2 \beta^p \mathbb{E}\left(|x(qt_0)|^p\right) + c_4 E\|\xi\|^p \\ &\le \big(2^{p-1} c_2 (1 + \beta^p) + c_4\big) \mathbb{E}\|\xi\|^p. \end{aligned}$$

From the inequality

$$|x(t)|^p \le 2^{p-1} |u(t)|^p + 2^{p-1} \beta^p |x(qt)|^p,$$

we obtain, by using the same techniques as in Case 1,

$$\begin{aligned} |u(t \wedge \tau_k)|^p &\ge |u(t \wedge \tau_k)|^p 1_{\{\tau_k \le t\}} \\ &\ge k^p \left(\frac{1}{2^{p-1}} - \beta^p\right) 1_{\{\tau_k \le t\}}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{P}(\tau_k \le t) &\le \frac{1}{k^p \left(\frac{1}{2^{p-1}} - \beta^p\right)} \mathbb{E}|u(t \wedge \tau_k)|^p \\ &\le \frac{1}{c_1 k^p \left(\frac{1}{2^{p-1}} - \beta^p\right)} \big(2^{p-1} c_2 (1 + \beta^p) + c_4\big) \mathbb{E}\|\xi\|^p. \end{aligned}$$

Letting $k \to \infty$ implies that $\mathbb{P}(\tau_\infty \le t) = 0$, and, therefore, $\tau_\infty > t$ a.s. Letting $t \to \infty$, we deduce $\tau_\infty = \infty$ a.s, which implies that there exists a global solution $x(t)$ to the system (2.1).

□

Definition 2.1. The function $h : \mathbb{R}_+ \to (0, \infty)$ is said to be an h-type function if the following conditions are satisfied:

(i) It is continuous and nondecreasing in $\mathbb{R}_+$ and continuously differentiable in $\mathbb{R}_+$.

(ii) $h(0) = 1$, $\lim_{t\to\infty} h(t) = \infty$ and $\phi = \sup_{t>0} |\frac{h'(t)}{h(t)}| < \infty$.

(iii) For any $t \ge s \ge 0$, $h(t) \le h(s)h(t-s)$.

Definition 2.2. Equation (2.1) is said to be h-stable in p-th moment, if there exist positive constants δ and c such that, for any $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$, the corresponding solution $x(\cdot; t_0, \xi)$ of (2.1) with initial value ξ satisfies

$$\mathbb{E}\left(|x(t; t_0, \xi)|^p\right) \le c\mathbb{E}\left(||\xi||^p\right) h^{-\delta}(t)h^{\delta}(t_0), \quad t \ge t_0. \tag{2.7}$$

We will impose the following condition on G to obtain the h-stability of equation (2.1).
$\mathcal{H}_4$: There exist constants $\beta \in (0, 1)$ and $\epsilon \ge 0$ such that for all y, $\overline{y} \in \mathbb{R}^n$ and $t \ge t_0$, it holds

$$|G(t, y) - G(t, \overline{y})|^p \le \beta h^{-\epsilon}((1-q)t)|y - \overline{y}|^p, \tag{2.8}$$

and $G(t, 0) = 0$.

Theorem 2.4. *Let assumptions $\mathcal{H}_1$, $\mathcal{H}_3$ and $\mathcal{H}_4$ hold except (2.5) which is replaced by*

$$LV(t, x, y, i) \le -c_3|x|^p + c_4 q h^{-\epsilon}((1-q)t)|y|^p, \tag{2.9}$$

for all $(t, x, y, i) \in [qt_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times S$, $\epsilon > 0$ and $2^{p-1}\beta < 1$, for $p > 1$.
Let $0 < \delta \le \dfrac{c_3 - c_4}{2^{p-1}c_2\phi\left(1 + \frac{\beta}{q}\right)}$, for $p > 1$, and ϕ is the positive constant which verifies condition (ii) in Definition 2.1.
Then, Equation (2.1) is h-stable in p-th moment.

Proof. Let $\xi \in L^p_{\mathcal{F}_t}([qt_0, t_0]; \mathbb{R}^n)$ be an initial function for Eq. (2.1) and denote by $x(\cdot)$ its corresponding solution. We split our analysis into two cases.

Case 1: Let $\delta \in (0, \epsilon)$ and $0 < p \leq 1$. For each $k \in \mathbb{N}^*$, define the stopping time $\tau_k = \inf\{t \geq t_0; |x(t)| \geq k\}$. For any $t \geq t_0$, by the generalized Itô formula applied to $h^\delta(t)V(t, u(t), r(t))$, Lemma 2.1, $\mathcal{H}_3$ and $\mathcal{H}_4$, we have

$$\begin{aligned}
&\mathbb{E}\left(h^\delta(t\wedge\tau_k)V(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k))\right) \\
&\quad= h^\delta(t_0)\mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right), \\
&\qquad+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)\left[\delta\frac{h^{'}(s)}{h(s)}V(s, u(s), r(s)) + LV(s, x(s), x(qs), r(s))\right]ds\right), \\
&\quad\leq h^\delta(t_0)\mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|u(s)|^p ds\right) \\
&\qquad-c_3E\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) + c_4qE\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)h^{-\epsilon}((1-q)s)|x(qs)|^p ds\right) \\
&\quad\leq h^\delta(t_0)\mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_2\phi\delta\beta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)h^{-\epsilon}((1-q)s)|x(qs)|^p ds\right) - c_3E\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_4qE\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)h^{-\epsilon}((1-q)s)|x(qs)|^p ds\right), \\
&\quad\leq h^\delta(t_0)\mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_2\phi\delta\beta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)h^{-\delta}((1-q)s)|x(qs)|^p ds\right) - c_3E\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_4qE\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)h^{-\delta}((1-q)s)|x(qs)|^p ds\right) \\
&\quad\leq h^\delta(t_0)\mathbb{E}\left(V(t_0, u(t_0), r(t_0))\right) + c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_2\phi\delta\beta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(qs)|x(qs)|^p ds\right) - c_3E\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(s)|x(s)|^p ds\right) \\
&\qquad+c_4qE\left(\int_{t_0}^{t\wedge\tau_k} h^\delta(qs)|x(qs)|^p ds\right).
\end{aligned}$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left(h^{\delta}(t\wedge\tau_k)V(t\wedge\tau_k,u(t\wedge\tau_k),r(t\wedge\tau_k))\right)\\
&\quad\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\qquad+\frac{c_2\phi\delta\beta}{q}\mathbb{E}\left(\int_{qt_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)-c_3E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\qquad+c_4E\left(\int_{qt_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\quad\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\qquad+\frac{c_2\phi\delta\beta}{q}\mathbb{E}\left(\int_{qt_0}^{t_0}h^{\delta}(s)|x(s)|^p ds\right)+\frac{c_2\phi\delta\beta}{q}\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\qquad+c_4E\left(\int_{qt_0}^{t_0}h^{\delta}(s)|x(s)|^p ds\right)+c_4E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)-c_3E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\quad\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+c_2\phi\delta\left(1+\frac{\beta}{q}\right)\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\qquad+\frac{c_2\phi\delta\beta}{q}h^{\delta}(t_0)t_0(1-q)\mathbb{E}\left(\|\xi\|^p\right)+c_4h^{\delta}(t_0)t_0(1-q)\mathbb{E}\left(\|\xi\|^p\right)\\
&\qquad+c_4E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)-c_3E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right)\\
&\quad\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+h^{\delta}(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\\
&\qquad+\left(c_2\phi\delta\left(1+\frac{\beta}{q}\right)+c_4-c_3\right)\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^p ds\right).
\end{aligned}
$$

For $\delta \le \dfrac{c_3 - c_4}{c_2\phi(1+\frac{\beta}{q})}$, by Lemma 2.1, $\mathcal{H}_3$ and $\mathcal{H}_4$, we obtain

$$\begin{aligned}
\mathbb{E}\left(h^\delta(t\wedge\tau_k)|u(t\wedge\tau_k)|^p\right) &\le \frac{1}{c_1}\mathbb{E}\left(h^\delta(t\wedge\tau_k)V(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k))\right)\\
&\le \frac{1}{c_1}h^\delta(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)\\
&+ \frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\\
&\le \frac{c_2}{c_1}h^\delta(t_0)\mathbb{E}\left(|u(t_0)|^p\right)\\
&+ \frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\\
&\le \frac{c_2}{c_1}h^\delta(t_0)\mathbb{E}\left(|x(t_0)|^p\right)+\frac{c_2}{c_1}h^\delta(t_0)\beta\mathbb{E}\left(|x(qt_0)|^p\right)\\
&+ \frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\\
&\le \frac{1}{c_1}h^\delta(t_0)\left(c_2(1+\beta)+(1-q)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right).
\end{aligned}$$

Letting $k\to\infty$, we have

$$\mathbb{E}\left(h^\delta(t)|u(t)|^p\right) \le \frac{1}{c_1}h^\delta(t_0)\left(c_2(1+\beta)+(1-q)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right). \qquad (2.10)$$

By Lemma 2.1, $\mathcal{H}_4$ and the definition of h-function, we obtain

$$\begin{aligned}
\mathbb{E}\left(h^\delta(t)|x(t)|^p\right) &\le \mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\mathbb{E}\left(h^\delta(t)h^{-\epsilon}((1-q)t)|x(qt)|^p\right),\\
&\le \mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\mathbb{E}\left(h^\delta(t)h^{-\delta}((1-q)t)|x(qt)|^p\right),\\
&\le \mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\mathbb{E}\left(h^\delta(qt)|x(qt)|^p\right).
\end{aligned}$$

Then, for any $T>0$,

$$\begin{aligned}
\sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right) &\le \sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(qt)|x(qt)|^p\right),\\
&\le \sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\sup_{qt_0\le t\le T}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right),\\
&\le \sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta\sup_{qt_0\le t\le t_0}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right)\\
&\quad +\beta\sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right),\\
&\le \sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\beta h^\delta(t_0)\mathbb{E}\left(\|\xi\|^p\right)+\beta\sup_{t_0\le t\le T}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right),
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{t_0\le t\le T}\mathbb{E}\left(h^{\delta}(t)|x(t)|^p\right) &\le \frac{1}{1-\beta}\sup_{t_0\le t\le T}\mathbb{E}\left(h^{\delta}(t)|u(t)|^p\right)+\frac{\beta}{1-\beta}h^{\delta}(t_0)\mathbb{E}\left(\|\xi\|^p\right),\\
&\le \frac{1}{c_1(1-\beta)}h^{\delta}(t_0)\left(c_2(1+\beta)+(1-q)\left(\frac{c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right)\\
&+ \frac{\beta}{1-\beta}h^{\delta}(t_0)\mathbb{E}\left(\|\xi\|^p\right),\\
&\le C_1\mathbb{E}\left(\|\xi\|^p\right)h^{\delta}(t_0).
\end{aligned}$$

where $C_1=\dfrac{1}{c_1(1-\beta)}\left(c_2(1+\beta)+(1-q)\left(\dfrac{c_2\phi\delta\beta}{q}+c_4\right)+c_1\beta\right)$.

Letting $T\to\infty$, we deduce

$$\sup_{t_0\le t<\infty}\mathbb{E}\left(h^{\delta}(t)|x(t)|^p\right)\le C_1\mathbb{E}\left(\|\xi\|^p\right)h^{\delta}(t_0). \tag{2.11}$$

This implies, for all $t\ge t_0$,

$$\mathbb{E}\left(|x(t)|^p\right)\le C_1\mathbb{E}\left(\|\xi\|^p\right)h^{-\delta}(t)h^{\delta}(t_0). \tag{2.12}$$

Case 2: Let $\delta\in(0,\epsilon)$ and $p>1$. Proceeding as in Case 1 and using (2.6), we have

$$\begin{aligned}
&\mathbb{E}\left(h^{\delta}(t\wedge\tau_k)V(t\wedge\tau_k,u(t\wedge\tau_k),r(t\wedge\tau_k))\right)\\
&= h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)\\
&\quad+\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)\left[\delta\frac{h'(s)}{h(s)}V(s,u(s),r(s))+LV(s,x(s),x(qs),r(s))\right]ds\right)\\
&\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|u(s)|^pds\right)\\
&\quad-c_3E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^pds\right)+c_4qE\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)h^{-\epsilon}((1-q)s)|x(qs)|^pds\right)\\
&\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+2^{p-1}c_2\phi\delta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^pds\right)\\
&\quad+2^{p-1}c_2\phi\delta\beta\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(qs)|x(qs)|^pds\right)-c_3E\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^pds\right)\\
&\quad+c_4qE\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(qs)|x(qs)|^pds\right)\\
&\le h^{\delta}(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)+h^{\delta}(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\\
&\quad+\left(2^{p-1}c_2\phi\delta\left(1+\frac{\beta}{q}\right)+c_4-c_3\right)\mathbb{E}\left(\int_{t_0}^{t\wedge\tau_k}h^{\delta}(s)|x(s)|^pds\right).
\end{aligned}$$

For $\delta \leq \dfrac{c_3 - c_4}{2^{p-1}c_2\phi\left(1+\frac{\beta}{q}\right)}$, by (2.6), $\mathcal{H}_3$ and $\mathcal{H}_4$, we obtain

$$\begin{aligned}
\mathbb{E}\left(h^\delta(t\wedge\tau_k)|u(t\wedge\tau_k)|^p\right) &\leq \frac{1}{c_1}\mathbb{E}\left(h^\delta(t\wedge\tau_k)V(t\wedge\tau_k, u(t\wedge\tau_k), r(t\wedge\tau_k))\right)\\
&\leq \frac{1}{c_1}h^\delta(t_0)\mathbb{E}\left(V(t_0,u(t_0),r(t_0))\right)\\
&\quad+\frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\\
&\leq \frac{c_2}{c_1}h^\delta(t_0)\mathbb{E}\left(|u(t_0)|^p\right)\\
&\quad+\frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\\
&\leq 2^{p-1}\frac{c_2}{c_1}h^\delta(t_0)\mathbb{E}\left(|x(t_0)|^p\right)+2^{p-1}\frac{c_2}{c_1}h^\delta(t_0)\beta\mathbb{E}\left(|x(qt_0)|^p\right)\\
&\quad+\frac{1}{c_1}h^\delta(t_0)(1-q)\mathbb{E}\left(\|\xi\|^p\right)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\\
&\leq \frac{1}{c_1}h^\delta(t_0)\left(2^{p-1}c_2(1+\beta)+(1-q)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right).
\end{aligned}$$

Letting now $k\to\infty$,

$$\mathbb{E}\left(h^\delta(t)|u(t)|^p\right) \leq \frac{1}{c_1}h^\delta(t_0)\left(2^{p-1}c_2(1+\beta)+(1-q)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right). \tag{2.13}$$

Then, using the same technique as in Case 1, for any $T>0$, we have

$$\begin{aligned}
\sup_{t_0\leq t\leq T}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right) &\leq \frac{2^{p-1}}{1-2^{p-1}\beta}\sup_{t_0\leq t\leq T}\mathbb{E}\left(h^\delta(t)|u(t)|^p\right)+\frac{2^{p-1}\beta}{1-2^{p-1}\beta}h^\delta(t_0)\mathbb{E}\left(\|\xi\|^p\right),\\
&\leq \frac{2^{p-1}}{c_1(1-2^{p-1}\beta)}h^\delta(t_0)\times\\
&\quad\times\left(2^{p-1}c_2(1+\beta)+(1-q)\left(\frac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)\right)\mathbb{E}\left(\|\xi\|^p\right)\\
&\quad+\frac{2^{p-1}\beta}{1-2^{p-1}\beta}h^\delta(t_0)\mathbb{E}\left(\|\xi\|^p\right)\\
&\leq C_2\mathbb{E}\left(\|\xi\|^p\right)h^\delta(t_0),
\end{aligned}$$

where $C_2 = \dfrac{2^{p-1}}{c_1(1-2^{p-1}\beta)}\left(2^{p-1}c_2(1+\beta)+(1-q)\left(\dfrac{2^{p-1}c_2\phi\delta\beta}{q}+c_4\right)+c_1\beta\right)$.
Letting $T\to\infty$, we deduce

$$\sup_{t_0\leq t<+\infty}\mathbb{E}\left(h^\delta(t)|x(t)|^p\right)\leq C_2\mathbb{E}\left(\|\xi\|^p\right)h^\delta(t_0). \tag{2.14}$$

This implies, for all $t \geq t_0$,

$$\mathbb{E}\left(|x(t)|^p\right) \leq C_2 \mathbb{E}\left(\|\xi\|^p\right) h^{-\delta}(t) h^{\delta}(t_0). \tag{2.15}$$

Therefore, for all $t \geq t_0$,

$$\mathbb{E}\left(|x(t)|^p\right) \leq C \mathbb{E}\left(\|\xi\|^p\right) h^{-\delta}(t) h^{\delta}(t_0), \tag{2.16}$$

where $C = \max\{C_1, C_2\}$. $\square$

Remark 2.5. Compared with existing results in the literature, we are considering Markovian switching and analyze h-stability of pantograph-type neutral stochastic differential equations with Lévy noise, obtaining some general results which generalize the known results (see, for instance, [6] and the references therein).

3 Examples

In this section, we will discuss some examples to show the interest of our results.

Example 3.1. *We consider the following neutral pantograph stochastic differential equation with Markovian switching driven by Lévy noise:*

$$\begin{aligned} &d\left[x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right] = f(t, x(t), x(0.5t), i)dt + g(t, x(t), x(0.5t), i)dW(t), \\ &+ \int_{|z|<1} H_1(t, x(t^-), x(0.5t^-), i, z)\widetilde{N}(dt, dz) + \int_{|z|\geq 1} H_2(t, x(t^-), x(0.5t^-), i, z)N(dt, dz) \quad t \geq t_0, \end{aligned} \tag{3.1}$$

where the initial data $x_0 = \xi \in L^p_{\mathcal{F}_t}([0.5, 1]; \mathbb{R})$, $r(1) = 1$, $z \in \mathbb{R}$ and $W(t)$ is a one dimensional Brownian motion, $N(.,.)$ is a Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R}\backslash\{0\})$ with compensator $\widetilde{N}$ and intensity measure $\nu(.)$. It is always assumed that $N(.,.)$ is independent of $W(.)$. Let $\nu(.)$ be a Lévy measure such that $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)$ and $\int_{\mathbb{R}\backslash\{0\}}(|z|^2 \wedge 1)\nu(dz) = \lambda < +\infty$. Let

$$\begin{aligned} f(t, x(t), x(0.5t), 1) &= -\frac{3}{2}\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right), \\ f(t, x(t), x(0.5t), 2) &= -\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right), \\ g(t, x(t), x(0.5t), 1) &= \frac{1}{2}\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right), \\ g(t, x(t), x(0.5t), 2) &= \frac{1}{3}\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right), \\ H_1(t, x(t), x(0.5t), 1, z) &= z\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right). \\ H_1(t, x(t), x(0.5t), 2, z) &= \frac{1}{2} z\left(x(t) - 0.3\left(\ln(e + 0.5t)\right)^{-0.5} x(0.5t)\right). \end{aligned}$$

$$H_2(t,x(t),x(0.5t),1,z) = (\sqrt{\frac{5}{2}}-1)\left(x(t)-0.3\left(\ln(e+0.5t)\right)^{-0.5}x(0.5t)\right).$$

$$H_2(t,x(t),x(0.5t),2,z) = (\sqrt{\frac{5}{8}}-1)\left(x(t)-0.3\left(\ln(e+0.5t)\right)^{-0.5}x(0.5t)\right).$$

Let $u(t) = x(t) - 0.3\left(\ln(e+0.5t)\right)^{-0.5}x(0.5t)$ *and* $\nu(dz) = \dfrac{dz}{1+z^2}$. *Let* $S = \{1,2\}$ *and the matrix* $\Gamma = (\gamma_{ij})_{1\le i,j\le 2}$ *defined by*

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

We will prove that system (3.1) is polynomially stable in mean square. To this end, let $V(t,x,i) = \psi_i x^2$, *for* $i \in S$, *where* $\psi_1 = 1$ *and* $\psi_2 = \frac{1}{2}$. *Then*

$$\frac{1}{2}|x|^2 \le V(t,x,i) \le |x|^2 \tag{3.2}$$

By the definition of LV, we have for $i = 1$

$$\begin{aligned} LV(t,x(t),x(0.5t),1) &= -3u^2(t) + \frac{1}{4}u^2(t) + u^2(t)\int_{|z|<1}\frac{z^2}{1+z^2}dz \\ &\quad + \frac{3}{2}u^2(t)\int_{|z|\ge 1}\frac{dz}{1+z^2} - \frac{1}{2}u^2(t) \\ &= \left(\frac{\pi}{4}-\frac{5}{4}\right)u^2(t) \\ &= -0.46\left(x(t)-0.3\left(\ln(e+0.5t)\right)^{-0.5}x(0.5t)\right)^2. \end{aligned}$$

For $i = 2$

$$\begin{aligned} LV(t,x(t),x(0.5t),2) &= -u^2(t) + \frac{1}{18}u^2(t) + \frac{1}{8}u^2(t)\int_{|z|<1}\frac{z^2}{1+z^2}dz \\ &\quad - \frac{3}{16}u^2(t)\int_{|z|\ge 1}\frac{dz}{1+z^2} + \frac{1}{2}u^2(t) \\ &= -\left(\frac{5\pi}{32}+\frac{7}{36}\right)u^2(t) \\ &= -0.68\left(x(t)-0.3\left(\ln(e+0.5t)\right)^{-0.5}x(0.5t)\right)^2 \end{aligned}$$

Then, using $2ab \le a^2 + b^2$, *for all* $a,\ b \in \mathbb{R}$, *we have*

$$\begin{aligned} LV(t,x(t),x(0.5t),1) &\le -0.46x^2(t) + 0.46^2x^2(t) + 0.09\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t) \\ &\quad + (-0.46)0.09\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t) \\ &\le -0.24x^2(t) + 0.05\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t) \\ &\le -0.21x^2(t) + (0.1)(0.5)\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t). \end{aligned}$$

$$
\begin{aligned}
LV(t,x(t),x(0.5t),2) &\leq -0.68x^2(t)+0.68^2x^2(t)+0.09\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t)\\
&\quad +(-0.68)0.09\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t)\\
&\leq -0.21x^2(t)+0.05\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t)\\
&= -0.21x^2(t)+(0.1)(0.5)\left(\ln(e+0.5t)\right)^{-1}x^2(0.5t).
\end{aligned}
$$

Then, for $i\in S$, we obtain

$$
LV(t,x(t),x(0.5t),i)\leq -0.21x^2(t)+(0.1)(0.5)(\ln(e+0.5t))^{-1}x^2(0.5t). \tag{3.3}
$$

Thus, the assumptions of theorem 2.4 are satisfied with $c_1=\frac{1}{2}$, $c_2=1$, $c_3=0.21$, $c_4=0.1$, $p=2$, $\beta=0.3$, $q=0.5$, $\epsilon=1$, $h(t)=\ln(e+t)$, $\phi=\frac{1}{e}$ and $\delta=0.09$. Therefore system (3.1) is h-stable in mean square.

Example 3.2. *We consider the following neutral pantograph stochastic differential equation with Markovian switching driven by Lévy noise:*

$$
\begin{aligned}
&d\left[x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right]=f(t,x(t),x(0.5t),i)dt+g(t,x(t),x(0.5t),i)dW(t),\\
&+\int_{|z|<1}H_1(t,x(t^-),x(0.5t^-),i,z)\widetilde{N}(dt,dz)+\int_{|z|\geq 1}H_2(t,x(t^-),x(0.5t^-),i,z)N(dt,dz)\quad t\geq t_0,
\end{aligned}
\tag{3.4}
$$

Where the initial data $x_0=\xi\in L^p_{\mathcal{F}_t}([0.5,1];\mathbb{R})$, $r(1)=1$, $z\in\mathbb{R}$, $W(t)$ is a one dimensional Brownian motions, $N(.,.)$ is a Poisson random measure defined on $\mathbb{R}_+\times(\mathbb{R}-\{0\})$ with compensator $\widetilde{N}$ and intensity measure $\nu(.)$. It is always assumed that $N(.,.)$ is independent of $W(.)$. Let $\nu(.)$ be a Lévy measure such that $\widetilde{N}(dt,dz)=N(dt,dz)-\nu(dz)$ and $\int_{\mathbb{R}-\{0\}}(|z|^2\wedge 1)\nu(dz)=\lambda<+\infty$. Let

$$
\begin{aligned}
f(t,x(t),x(0.5t),1)&=-\frac{7}{2}\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right),\\
f(t,x(t),x(0.5t),2)&=-\frac{1}{8}\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right),\\
g(t,x(t),x(0.5t),1)&=\frac{1}{4}\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right),\\
g(t,x(t),x(0.5t),2)&=\frac{1}{2}\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right),\\
H_1(t,x(t),x(0.5t),1,z)&=\frac{1}{2}z\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right).\\
H_1(t,x(t),x(0.5t),2,z)&=\frac{1}{3}z\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right).\\
H_2(t,x(t),x(0.5t),1,z)&=\left(x(t)-0.2\left(1+0.5t\right)^{-0.5}e^{-0.5t}x(0.5t)\right).
\end{aligned}
$$

$$H_2(t,x(t),x(0.5t),2,z) = (\frac{\sqrt{31}}{6}-1)\left(x(t)-0.2\,(1+0.5t)^{-0.5}\,e^{-0.5t}x(0.5t)\right).$$

Let $u(t) = x(t) - 0.2\,(1+0.5t)^{-0.5}\,e^{-0.5t}x(0.5t)$ *and* $\nu(dz) = \dfrac{dz}{1+z^2}$.
Let $S = \{1,2\}$ *and the matrix* $\Gamma = (\gamma_{ij})_{1\le i,j\le 2}$ *define by*

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

We will prove that system (3.4) is exponentially stable in mean square. Indeed, as in the preceding example, let $V(t,x,i) = \psi_i x^2$, *for* $i \in S$, *where* $\psi_1 = \frac{1}{2}$ *and* $\psi_2 = 1$. *Then*

$$\frac{1}{2}|x|^2 \le V(t,x,i) \le |x|^2 \tag{3.5}$$

By the definition of LV, we have for $i = 1$

$$\begin{aligned} LV(t,x(t),x(0.5t),1) &= -\frac{7}{2}u^2(t) + \frac{1}{32}u^2(t) + \frac{1}{8}u^2(t)\int_{|z|<1}\frac{z^2}{1+z^2}dz \\ &\quad +\frac{3}{2}u^2(t)\int_{|z|\ge 1}\frac{dz}{1+z^2} + \frac{1}{2}u^2(t) \\ &= \left(\frac{11\pi}{16} - \frac{87}{32}\right)u^2(t) \\ &= -0.56\left(x(t)-0.2\,(1+0.5t)^{-0.5}\,e^{-0.5t}x(0.5t)\right)^2. \end{aligned}$$

For $i = 2$,

$$\begin{aligned} LV(t,x(t),x(0.5t),2) &= -\frac{1}{4}u^2(t) + \frac{1}{4}u^2(t) + \frac{1}{9}u^2(t)\int_{|z|<1}\frac{z^2}{1+z^2}dz \\ &\quad -\frac{5}{36}u^2(t)\int_{|z|\ge 1}\frac{dz}{1+z^2} - \frac{1}{2}u^2(t) \\ &= -\left(\frac{9\pi}{72} + \frac{5}{18}\right)u^2(t) \\ &= -0.67\left(x(t)-0.2\,(1+0.5t)^{-0.5}\,e^{-0.5t}x(0.5t)\right)^2 \end{aligned}$$

Then, using $2ab \le a^2 + b^2$, *for all* $a,\, b \in \mathbb{R}$, *we have*

$$\begin{aligned} LV(t,x(t),x(0.5t),1) &\le -0.56x^2(t) + 0.56^2x^2(t) + 0.04\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t) \\ &+ (-0.56)(0.04)\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t) \\ &\le -0.24x^2(t) + 0.02\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t) \\ &= -0.24x^2(t) + (0.04)(0.5)\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t). \end{aligned}$$

$$
\begin{aligned}
LV(t,x(t),x(0.5t),2) &\leq -0.67x^2(t)+0.67^2x^2(t)+0.04\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t)\\
&\quad +(-0.67)0.04\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t)\\
&\leq -0.09x^2(t)+0.02\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t)\\
&= -0.09x^2(t)+(0.04)(0.5)\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t).
\end{aligned}
$$

Then, for $i \in S$, we obtain

$$
LV(t,x(t),x(0.5t),i) \leq -0.09x^2(t)+(0.04)(0.5)\,(1+0.5t)^{-1}\,e^{-t}x^2(0.5t). \tag{3.6}
$$

Thus the assumptions of Theorem 2.4 are satisfied with $c_1 = \frac{1}{2}$, $c_2 = 1$, $c_3 = 0.09$, $c_4 = 0.04$, $\beta = 0.2$, $p = 2$, $q = 0.5$, $\epsilon = 2$, $h(t) = (1+t)e^t$, $\phi = 2$ and $\delta = 0.008$. Therefore system (3.4) is h-stable in mean square.

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