Proceedings of XIII Fall Workshop on
Geometry and Physics
Murcia, September 20–22, 2004
Publ. de la RSME, Vol. 9 (2005), 149–154

Periodic maximal graphs in the Lorentz-Minkowki space $\mathbb{L}^3$

Isabel Fernández[1] and Francisco J. López[2]

[1]isafer@ugr.es [2]fjlopez@ugr.es

Departamento de Geometría y Topología
Universidad de Granada

Abstract. We study maximal graphs in the Lorentz-Minkowski space $\mathbb{L}^3$ invariant under a discrete group of isometries and having a finite number of singularities in its fundamental piece. We also give a method to construct them, based on the Weierstrass representation for maximal surfaces.

Keywords: Maximal surfaces, conelike singularities, periodic surfaces.

2000 Mathematics Subject Classification: 53C50, 53C42, 53A10

1. Introduction

Maximal surfaces in a Lorentzian manifold are spacelike surfaces with zero mean curvature. In the Lorentz-Minkowski space $\mathbb{L}^3$ these surfaces arise as local maxima for the area functional associated to variations of the surface by spacelike surfaces. Also, maximal graphs in $\mathbb{L}^3$ are the solutions for a quasi-linear elliptic differential equation, and therefore a maximum principle for them is satisfied. As in the case of minimal surfaces in the Euclidean space, maximal surfaces have a conformal representation (*Weierstrass representation*) in terms of meromorphic data on a Riemann surface.

A classical result by Calabi [1] asserts that the unique complete maximal surfaces in $\mathbb{L}^3$ are the spacelike planes. However, if we allow the existence of singularities, there is a vast theory of complete maximal surfaces, see for example [10], [2], [4], [5], [6]. In this paper we focus our attention on isolated embedded singularities of maximal surfaces, also called *conelike singularities* (see [7]). If in adittion the surface is complete or proper, it turns out that it is a graph over any spacelike plane of $\mathbb{L}^3$.

We say that a surface is *periodic* if it is invariant under a group of isometries G of $\mathbb{L}^3$ acting properly and freely on $\mathbb{L}^3$. This paper develops the main results obtained by the authors in [3] for periodic maximal surfaces in the embedded case. In concrete, we show that the group G contains a finite index subgroup G_0 which is a group of translations of rank 0 (that is, $G_0 = \{Id\}$), 1 (singly periodic surfaces), or 2 (doubly periodic surfaces). We also use the Weierstrass representation of maximal surfaces to give a recipe recovering these surfaces.

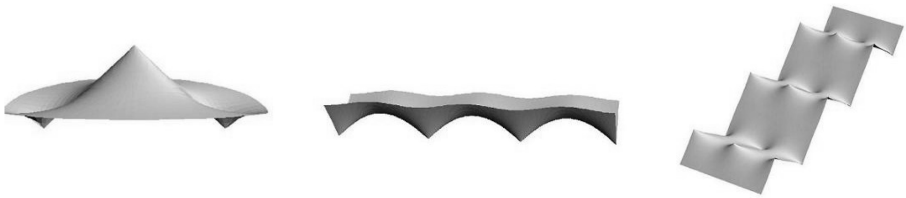

Figure 1: Examples of maximal graphs with isolated singularities

2. Preliminaries

2.1. Spacelike immersion with isolated singularities

Through this paper $\mathbb{L}^3$ will denote the 3-dimensional Lorentz-Minkowski space, that is $\mathbb{L}^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, and $\mathcal{M}$ a differentiable surface.

An immersion $X : \mathcal{M} \longrightarrow \mathbb{L}^3$ is said to be *spacelike* if for any $p \in \mathcal{M}$, the tangent plane $T_p\mathcal{M}$ with the induced metric is spacelike, that is to say, the induced metric on $\mathcal{M}$ is Riemannian. This metric induces a conformal structure on $\mathcal{M}$, and so it becomes in a Riemann surface.

Let $F \subset \mathcal{M}$ be a *discrete closed* subset of a differentiable surface $\mathcal{M}$ and ds^2 a Riemannian metric in $\mathcal{M} - F$. Take a point $q \in F$, an open disk $\mathcal{D}(q)$ in $\mathcal{M}$ such that $\mathcal{D}(q) \cap F = \{q\}$ and an isothermal parameter z for ds^2 on $\mathcal{D}(q) - \{q\}$. Then write $ds^2 = h|dz|^2$, where $h(w) > 0$ for any $w \in z(\mathcal{D}(q) - \{q\})$. By definition, the Riemannian metric ds^2 is *singular* at q if for any disk $\mathcal{D}(q)$ and any parameter z as above, the limit $\lim_{p \to q} h(z(p))$ vanishes (as a matter of fact, it suffices to check this condition just for one disc and conformal parameter). The metric ds^2 is said to be singular at F if it is singular at any point of F. In this case, $(\mathcal{M}, ds^2)$ is said to be a Riemannian surface with isolated singularities and F is the singular set of $(\mathcal{M}, ds^2)$.

Definition 1. Let $X : \mathcal{M} \to \mathbb{L}^3$ be a continuous map. Suppose there is a discrete closed $F \subset \mathcal{M}$ subset such that $X|_{\mathcal{M}-F}$ is a spacelike immersion and $(\mathcal{M}, ds^2)$ is a Riemannian surface with isolated singularities in F, where ds^2 is the metric induced by X.

Then, X is said to be a spacelike immersion with (isolated) singularities at F, and $X(\mathcal{M})$ a spacelike surface with (isolated) singularities at $X(F)$.

The following lemma describes the behavior of a spacelike immersion around an isolated singularity.

Lemma 1 ([3])**.** *Let $X : \mathcal{M} \to \mathbb{L}^3$ be a spacelike immersion with isolated singularities and Π a spacelike plane. Label $\pi : \mathbb{L}^3 \to \Pi$ as the Lorentzian orthogonal projection.*

Then, $h := \pi \circ X$ is a branched local homeomorphism and its branch points correspond to the locally non embedded singularities of X.

As a consequence, if X is an embedding locally around the singular points and is proper, then $X(\mathcal{M})$ is a graph over any spacelike plane (in particular X is an embedding). The same conclusion holds if we replace proper by complete.

2.2. Maximal surfaces

A maximal immersion $X : \mathcal{M} \to \mathbb{L}^3$ is a spacelike immersion with vanishing mean curvature. The notion of maximal immersion with (isolated) singularities is defined analogously.

If $X : \mathcal{M} \to \mathbb{L}^3$ is a (everywhere regular) maximal immersion, it is known that there exist a meromorphic map g with $|g| \neq 1$ and a holomorphic 1-form ϕ_3 defined on the Riemann surface $\mathcal{M}$ satisfying that the vectorial 1-form $\Phi = (\phi_1, \phi_2, \phi_3) := (\frac{i}{2}(\frac{1}{g} - g)\phi_3, \frac{-1}{2}(\frac{1}{g} + g)\phi_3, \phi_3)$ is holomorphic, non vanishing and without real periods in $\mathcal{M}$. Moreover, up to a translation X is given by $X(p) = \operatorname{Re} \int_{p_0}^{p} \Phi$, where p_0 is an arbitrary point.

Either the pair (g, ϕ_3) or the vectorial 1-form Φ is called *the Weierstrass representation* of the maximal immersion X.

As mentioned before, we will focus our attention in embedded surfaces, and therefore, we will consider only embedded singularities. For a more general treatment of non-embedded maximal surfaces with isolated singularities see [3]. As a consequence of Lemma 1, any proper (or complete) maximal surface with isolated embedded singularities is a graph over any spacelike plane.

The behaviour of a maximal immersion around embedded isolated singularities is well known (see for example [4], [7]). As a matter of fact, if $\mathcal{D}$ is a disc around such a singularity p, then $\mathcal{D} \setminus \{p\}$ is conformally equivalent to an annulus $\mathcal{A}$, and the Gauss map of the immersion becomes lightlike at the boundary component of $\mathcal{A}$ corresponding to the singularity. Moreover, around $X(p)$ the surface $X(\mathcal{M})$ is asymptotic to a component of the light cone at $X(p)$. For this reason, isolated embedded singularities of maximal surfaces are also called *conelike singularities*.

Definition 2. We say that a maximal graph with isolated singularities $X : \mathcal{M} \to \mathbb{L}^3$ is G-periodic if $X(\mathcal{M})$ is invariant under a discrete subgroup G of isometries acting freely and properly of $\mathbb{L}^3$. We say that X is singly (resp. doubly) periodic if G is a group of translations of rank one (resp. two).

If in addition the quotient of the singular set of X under the relation induced by G is finite we say that X is *of finite type.*

Let $X : \mathcal{M} \to \mathbb{L}^3$ be a maximal graph of finite type and label $F = \{p_\alpha : \alpha \in \Lambda\} \subset \mathcal{M}$ as its singular set. Taking into account the local behavior around the singularities described above and the results about the Koebe uniformization given in [8], we can deduce that the Riemann surface $\mathcal{M} \setminus F$ is biholomorphic to a circular domain $\mathbb{C} \setminus \cup_{\alpha\in\Lambda} D_\alpha$, where D_α are pairwise disjoint closed discs in $\overline{\mathbb{C}}$ whose boundaries $\gamma_\alpha := \partial(D_\alpha)$, $\alpha \in \Lambda$, correspond to the singularities. In this setting, we label

$$\mathcal{M}_0 := \mathbb{C} \setminus \cup_{\alpha\in\Lambda} \mathrm{Int}(D_\alpha) \tag{1}$$

and we refer to it as the *conformal support* of X. The conformal reparameterization $X_0 : \mathbb{C} \setminus \cup_{\alpha\in\Lambda} D_\alpha \to \mathbb{L}^3$ extends to $\mathcal{M}_0$ by putting $X_0(\gamma_\alpha) = X(p_\alpha)$.

3. Main Results

Theorem 2 ([3])**.** *Let $X : \mathcal{M} \to \mathbb{L}^3$ be a G-periodic maximal graph of finite type. Then the subgroup G_0 of G consisting of the positive and orthochronous (that is, preserving $\mathbb{H}^2_+$) isometries of G, which is a finite index subgroup of G, is either the identity or a group of spacelike translations of rank 1 or 2.*

Our aim now is to describe the global behavior of the G-periodic maximal graphs of finite type when G is one of the three groups given in the above theorem in terms of its Weierstrass data.

Thus, let $X : \mathcal{M} \to \mathbb{L}^3$ be as in the statement of the theorem, and consider its conformal support $\mathcal{M}_0$ and the conformal reparameterization $X_0 : \mathcal{M}_0 \to \mathbb{L}^3$ (see Equation (1)). Since the isometries in G preserves the singular set we can regard G as group of transformations in $\mathcal{M}_0$. So, we can consider the induced immersion $\hat{X}_0 : \hat{\mathcal{M}}_0 = \mathcal{M}_0/G \to \mathbb{L}^3/G$. It follows that $\hat{X}_0$ is a complete maximal immersion with a finite number of singularities. Moreover, since $\mathcal{M}$ is simply connected, $\hat{X}_0(\hat{\mathcal{M}}_0)$ is an embedded surface in $\mathbb{L}^3/G$. If in addition G is a translational group, the Weierstrass data of X_0 can be also pushed out to $\hat{\mathcal{M}}_0$.

Observe that the Riemann surface with boundary $\hat{\mathcal{M}}_0 = \mathcal{M}_0/G$ is biholomorphic to $\Sigma \setminus \cup_{j=0}^k \mathrm{Int}(D_j)$, where D_j are pairwise disjoint closed discs and

$\Sigma = \mathbb{C}$ if $G = \{Id\}$, $\Sigma = \mathbb{C}^*$ in the singly periodic case, and Σ is a torus in the doubly periodic case.

In order to use the tools we need for our purposes is useful to work with boundaryless surfaces, for this reason we introduce the notion of the *double surface* of the conformal support. This surface is nothing but the quotient of $\hat{\mathcal{M}}_0 \cup \hat{\mathcal{M}}_0^*$ by identifying their boundary components, $\partial(\hat{\mathcal{M}}_0) \equiv \partial(\hat{\mathcal{M}}_0^*)$, where $\hat{\mathcal{M}}_0^*$ is the mirror surface associated to $\hat{\mathcal{M}}_0$ (see [9] for more details). It follows that the Weierstrass data Φ can be extended holomorphically to the double surface $\mathfrak{S}$ and satisfy $J^*(\Phi) = -\overline{\Phi}$, where $J : \mathfrak{S} \to \mathfrak{S}$ is the mirror involution, that maps each point of $\hat{\mathcal{M}}_0$ into its mirror image and vice versa (observe that the fixed point set of J coincides with $\partial(\hat{\mathcal{M}}_0)$).

Theorem 3 ([3])**.** *Let $\overline{\mathfrak{S}}$ be a compact Riemann surface of genus $k \geq 0$ and $J : \overline{\mathfrak{S}} \to \overline{\mathfrak{S}}$ be an antiholomorphic involution having $k+1$ pairwise disjoint Jordan curves of fixed points $\gamma_0, \ldots, \gamma_k$. Suppose also that $\overline{\mathfrak{S}} \setminus \cup_{j=0}^k \gamma_j$ has two connected components, namely Ω and $J(\Omega)$, any one of them homeomorphic (and so biholomorphic) to a circular domain*[1] *in the extended complex plane $\overline{\mathbb{C}}$ or in a torus $\mathbb{T}$.*

Consider a meromorphic vectorial 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined on $\overline{\mathfrak{S}}$, non vanishing, with $J^*(\Phi) = -\overline{\Phi}$ and having poles at $F_\infty \cup J(F_\infty)$, where $F_\infty \subset \Omega$ consists of one (and in this case the poles are double) or two points (and in this case the poles are simple) if $\Omega \subset \overline{\mathbb{C}}$ and $F_\infty = \emptyset$ if $\Omega \subset \mathbb{T}$.

Finally, label G as the group of translations of vectors $\{Re \int_\gamma \Phi \ : \ \gamma \in H_1(\Omega_0, \mathbb{Z})\}$, where Ω_0 is the quotient surface obtained from $\overline{\Omega} \setminus F_\infty$ by identifying each component γ_j of $\partial(\Omega)$ to a point $q_j \notin \Omega$, $j = 0, \ldots, k$ ($q_j \neq q_h$ for $j \neq h$).

Then, G has rank 0 (if F_∞ has 1 point), 1 (if F_∞ contains 2 points) or 2 ($F_\infty = \emptyset$), and the map

$$\hat{X}_0 : \overline{\Omega} \setminus F_\infty \to \mathbb{L}^3/G, \quad \hat{X}_0 = Re(\int \Phi),$$

is well defined and provides a complete maximal surface with $k+1$ singular points (namely $\hat{X}_0(\gamma_j)$, $j = 0, \ldots, k$) whose lifting to $\mathbb{L}^3$ is a G-periodic maximal graph with $k+1$ singular points in its fundamental piece.

Conversely, any such surface can be obtained in this way.

From the behaviour of the Weierstrass data described above we can deduce the asymptotic behaviour of the lifted periodic surface in $\mathbb{L}^3$. It turns out that if $G = \{Id\}$ the surface is asymptotic at infinity to either half catenoid or a spacelike plane, and in the singly periodic case the surface is asymptotic to

[1] that is, an open domain bounded by analytical circles

two spacelike half planes. In the doubly periodic case the resulting surface is contained in a slab (see [4], [3] for a detailed proof).

Examples of surfaces constructed using the above representation (for example, the surfaces in Figure 1) can be found in [3].

Acknowledgments

This work has been partially supported by Spanish MEC-FEDER grant number MTM2004-00160.

References

[1] E. Calabi. Examples of the Bernstein problem for some nonlinear equations. *Proc. Symp. Pure Math.*, vol. 15 (1970), 223-230.

[2] F.J.M. Estudillo and A. Romero. Generalized maximal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$, *Math. Proc. Camb. Phil. Soc.* 111 (1992), 515-524.

[3] I. Fernández and F.J. López. Periodic Maximal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$, ArXiV e-print math.DG/0412461v1.

[4] I. Fernández, F.J. López and R. Souam. The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, to appear in *Mathematische Annalen*.

[5] A.A. Klyachin. Description of the set of singular entire solutions of the maximal surface equation, *Sbornik Mathematics* 194 (2003), 1035-1054.

[6] V.A. Klyachin and V.M. Miklyukov. Geometric structures of tubes and bands of zero mean curvature in Minkowski space, *Annales Academia Scientiarum Fennicae Mathematica* 28 (2003) 239-270.

[7] O. Kobayashi. Maximal surfaces with conelike singularities, *J. Math. Soc. Japan* 36 (1984), 609–617.

[8] Z.-X. He and O. Schramm. Fixed points, Koebe uniformization and circle packings, *Ann. of Math.* 2 **137** (1993), 369-406.

[9] K. Strebel. *Quadratic Differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 5. Springer-Verlag, Berlin, 1984.

[10] M. Umehara and K. Yamada. Maximal surfaces with singularities in Minkowski space, to appear in *Hokkaido Mathematical Journal*.