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Saddle-node bifurcation of canard limit cycles in piecewise linear systems

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Abstract

We study saddle-node bifurcations of canard limit cycles in PWL systems by using singular perturbation theory tools. We distinguish two cases: the subcritical and the supercritical. In the subcritical case, we find saddle-node bifurcations of canard cycles both with head and without head. Moreover, we detect a transition between them. In the supercritical case, we find situations with two saddle-node bifurcations, which take place exponentially close in the parameter space; one of headless canards and another of canards with head. There, three canard cycles can coexist.

1. Introduction

The classical canard explosion is a phenomenon that occurs in limit cycles of planar slow-fast systems. It was discovered and analyzed by Benoit et al. in 1981 [2] in the Van der Pol oscillator and consists on the fast transition, by changing one parameter of the system, from a small amplitude Hopf-like limit cycle to a relaxation oscillation cycle.

The analysis of the slow-fast dynamics is done by using tools from Geometric Singular Perturbation Theory. The main idea consist on reconstructing the global behavior by splitting and then joining, in a suitable way, the fast and slow dynamics. Under hyperbolicity conditions, Fenichel Theorem describes the existence of invariant slow manifolds close to compact parts of the fast nullcline and also describes the stability properties of these slow manifolds [5]. However, when the fast nullcline folds, which is the case in the canard phenomenon, normal hyperbolicity is lost, and Fenichel Theorem cannot be applied. Different sophisticated techniques have been developped in order to analyze this behavior around the fold, such as, for instance, the blow up technique [7].

On the other hand, some authors have analyzed the possibility of reproducing the canard phenomenon in systems more amenable to study, such as, piecewise linear (PWL) systems. Even when some dynamical aspects of the slow-fast behavior had been observed in PWL systems, it has taken some time to understand the way of reproducing the slow-fast dynamics properly, see [4] and references therein.

In [6], the authors reproduced part of the canard explosion phenomenon in the PWL context, in particular the one involving hyperbolic headless canards. Here we present the main results obtained in [3], where we consider an extension of the system analyzed in [6], which allows for the existence of both canards with and without head and both, hyperbolic and non-hyperbolic canard cycles. In particular, the system is able to reproduce saddle-node bifurcations of canard limit cycles.

The obtained results in [3] are comparable with those obtained for smooth vector fields, by Krupa and Szmolyan in [7]. Furthermore, we have found new scenarios that, as far as we are concerned, had not been previously reported in the smooth framework. Surprisingly, we find situations where two saddle-node bifurcations of canard cycles take place, one of headless canards and another one of canards with head. In such a case, we show the coexistence of three canard limit cycles.

The outline of this work is given as follows. First, in Section 2, we review the canard explosion and saddle-node canard cycles in the smooth case. Second, in Section 3, we introduce the PWL systems which we focus on and we present the Main Results. Finally, Section 4 is devoted to present some conclusions.

2. Background on canard cycles: canard explosion.

Canard solutions take place in planar differential slow-fast systems, that is, systems of the form [2, 7],

$$\left\{ \begin{aligned} \varepsilon\dot{x} &= f(x, y, a, \varepsilon), \\ \dot{y} &= g(x, y, a, \varepsilon), \end{aligned} \right. \tag{2.1}$$

where $f, g \in C^r, r \geq 3, a \in \mathbb{R}, 0 < \varepsilon \ll 1$ and the dot denotes the derivative with respect to the temporal variable τ. After the rescaling in time $t = \tau/\varepsilon$, system (2.1) writes as

$$\left\{ \begin{aligned} x' &= f(x, y, a, \varepsilon), \\ y' &= \varepsilon g(x, y, a, \varepsilon), \end{aligned} \right. \tag{2.2}$$

where the prime denotes the derivative with respect to the fast time t. Systems (2.1) and (2.2) are equivalent through the identity when $\varepsilon > 0$, but they have not the same limit for $\varepsilon = 0$. In fact, the limit of system (2.1), called *slow subsystem*, is a semi-explicit Differential Algebraic Equation (DAE), where the relation between the variables is given by

$$S = \{(x, y) : f(x, y, a, 0) = 0\}.$$

Assuming that $f_y(x, y, a, 0) \neq 0$ it follows that S is the graph of a differentiable function $y = \varphi_a(x)$, and the DAE reduces to the differential equation

$$f_x(x, \varphi_a(x), a, 0)\dot{x} = -f_y(x, \varphi_a(x), a, 0)g(x, \varphi_a(x), a, 0), \tag{2.3}$$

which is called the *reduced equation*. On the other hand, the limit for $\varepsilon = 0$ of system (2.2), called *fast subsystem*, is a differential equation having S as the locus of every equilibrium point. From here S is called the *critical manifold*.

Canard cycles develop along a branch born at a Hopf bifurcation, at $a = a_H$, and the canard explosion takes place at a value which is at a distance of $O(\varepsilon)$ from the a_H. This means that very close to the bifurcation point a_H, before the explosion, the cycles have the characteristics of typical Hopf cycles. This Hopf bifurcation arises only for $\varepsilon > 0$ and is usually known as a singular Hopf bifurcation [1].

The existence of saddle-node bifurcation of canard cycles in the smooth framework has been analyzed in [7]. There, the authors consider two different cases, depending whether the Hopf bifurcation where the cycle is born is supercritical or subcritical. Thus, after proving the existence of the maximal canard, they distinguish two different scenarios:

- *Supercritical case*: In Theorem 3.3, authors state the existence of a family of periodic orbits. These periodic orbits can be stable Hopf-type limit cycles, canard limit cycles or relaxation oscillations. To analyze the stability of the canard limit cycles, they use the *way in-way out function* $R(s)$, which is the limit of the integral of the divergence along the slow manifolds when $\varepsilon \to 0$. In Theorem 3.4, assuming that this function is negative, the authors state that the canard limit cycles of the family are stable.
- *Subcritical case*: In Theorem 3.5, authors state the existence of other family of periodic orbits. The orbits of that family can be unstable Hopf-type limit cycles, canard limit cycles or relaxation oscillations. Again, to analyze the stability of canard cycles, they use the *way in-way out function* $R(s)$. In Theorem 3.6, assuming that this function has exactly one simple zero at $s = s_{lp,0}$, the authors state that there exists a function $s_{lp}(\sqrt{\varepsilon})$ having limiting point at $s_{lp,0}$ when $\varepsilon \to 0$, such that canard limit cycles are unstable for $s < s_{lp}(\sqrt{\varepsilon})$ and stable for $s > s_{lp}(\sqrt{\varepsilon})$.

3. Statement of the piecewise linear system and Main Results.

In this section, first we introduce the family of PWL differential systems considered in [3] and after that we stay the main results, whose proofs can be consulted there.

The class of planar differential systems considered in [3] reads,

$$\begin{cases} x' = y - f(x, a, k, m, \varepsilon), \\ y' = \varepsilon(a - x), \end{cases} \tag{3.1}$$

where the prime denotes the derivative with respect to the time t, $(x, y)^T \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$, and the x-nullcline is defined by the graph of the continuous PWL function with four segments given by

$$f(x, a, k, m, \varepsilon) = \begin{cases} x + 1 - k(\sqrt{\varepsilon} - 1) - m(\sqrt{\varepsilon} + a), & \text{if } x < -1 \\ -k(x + \sqrt{\varepsilon}) - m(\sqrt{\varepsilon} + a), & \text{if } -1 < x \leq -\sqrt{\varepsilon}, \\ m(x - a), & \text{if } |x| \leq \sqrt{\varepsilon}, \\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a), & \text{if } x > \sqrt{\varepsilon}, \end{cases} \tag{3.2}$$

with $k > 0$, $a \in \mathbb{R}$ and $|m| < 2\sqrt{\varepsilon}$.

Note that, the phase space is splitted into four regions: the lateral half-planes $LL = \{(x, y) : x \leq -1\}$ and $R = \{(x, y) : x \geq \sqrt{\varepsilon}\}$, and the central bands $L = \{(x, y) : -1 \leq x \leq -\sqrt{\varepsilon}\}$ and $C = \{(x, y) : |x| \leq \sqrt{\varepsilon}\}$. Restricted to any of these regions, the vector field is linear.

We proceed now to present the main results in [3]. These results concern to the existence of a one parameter family of canard limit cycles in the PWL system (3.1)-(3.2), and to the description about how this family organizes along a curve in the plane (x, a), where x is the width of the canard limit cycle and a is the parameter value. The results also provide information about the stability of the limit cycles.

In the first result in [3], it is assured that the starting point of the curve organizing the family of limit cycles exhibited by system (3.1)-(3.2) takes place at a Hopf-like bifurcation. At this bifurcation a limit cycle appears after the change of stability of the singular point, just like in the Hopf bifurcation. The difference between both kind of bifurcations is the relation between the amplitude of the limit cycle and the bifurcation value, this relation is linear in the Hopf-like bifurcation and a square root in the Hopf bifurcation.

Next theorem, which we include subsequently, is devoted to the existence of the maximal canard trajectory, that is, a trajectory connecting the attracting and the repelling branches of the slow manifold.

Theorem 3.1 *Set $m = \pm\sqrt{\varepsilon}$. There exist a value $\varepsilon_0 > 0$ and a function $a = \tilde{a}(k, \varepsilon; m)$, analytic as a function of $(k, \sqrt{\varepsilon})$, defined in the open set $U = (0, +\infty) \times (0, \varepsilon_0)$ and such that, for $(k, \varepsilon) \in U$, a solution of system* (3.1)-(3.2) *starting in the attracting branch of the slow manifold, μ_R, connects to the repelling branch of the slow manifold, μ_L, if and only if $a = \tilde{a}(k, \varepsilon; m)$. In such case, the time of flight of the transition is $\tau_C(k, \varepsilon; m) > 0$. First terms of the expansions of $\tilde{a}(k, \varepsilon; m)$ and $\tau_C(k, \varepsilon; m)$ are given as follows,*

$$\tilde{a}(k,\varepsilon;m) = \begin{cases} \dfrac{e^{\frac{\pi}{\sqrt{3}}}-1}{e^{\frac{\pi}{\sqrt{3}}}+1}\sqrt{\varepsilon} - \dfrac{e^{\frac{\pi}{\sqrt{3}}}}{\left(e^{\frac{\pi}{\sqrt{3}}}+1\right)^2}\left(\dfrac{1-k^2}{k^2}\right)\varepsilon^{3/2} + O(\varepsilon^2), & \text{if } m = -\sqrt{\varepsilon}, \\ -\dfrac{e^{\frac{\pi}{\sqrt{3}}}-1}{e^{\frac{\pi}{\sqrt{3}}}+1}\sqrt{\varepsilon} - \dfrac{e^{\frac{\pi}{\sqrt{3}}}}{\left(e^{\frac{\pi}{\sqrt{3}}}+1\right)^2}\left(\dfrac{1-k^2}{k^2}\right)\varepsilon^{3/2} + O(\varepsilon^2), & \text{if } m = \sqrt{\varepsilon}, \end{cases} \tag{3.3}$$

and

$$\tau_C(k,\varepsilon;m) = \begin{cases} \dfrac{2\pi}{\sqrt{3}}\dfrac{1}{\sqrt{\varepsilon}} - \dfrac{1+k}{k} - \dfrac{1-k^2}{2k^2}\sqrt{\varepsilon} + O(\varepsilon), & \text{if } m = -\sqrt{\varepsilon}, \\ \dfrac{2\pi}{\sqrt{3}}\dfrac{1}{\sqrt{\varepsilon}} - \dfrac{1+k}{k} + \dfrac{1-k^2}{2k^2}\sqrt{\varepsilon} + O(\varepsilon), & \text{if } m = \sqrt{\varepsilon}. \end{cases} \tag{3.4}$$

The existence of the maximal canard trajectory, together with the divergence of the flow in a neighborhood of the slow manifold, provide the arguments used in [3] to prove the following result about the existence of canard cycles of any suitable width. To state the result in a proper way we introduce the following values

$$x_r = -(1+k) + k\sqrt{\varepsilon} - \lambda_L^s(\sqrt{\varepsilon}+a), \quad x_s = -\sqrt{\varepsilon} - \lambda_L^s(\sqrt{\varepsilon}+a). \tag{3.5}$$

These values correspond with the end points of the interval such that limit cycles having width contained in (x_r, x_s) are canard limit cycles. In fact, limit cycles having width $x < x_r$ are relaxation oscillations whereas limit cycles having width $x > x_s$ are still under the effect of the Hopf-like bifurcation.

Theorem 3.2 *Fix ε_0 sufficiently small and set $m = \pm\sqrt{\varepsilon}$. There exists a function $a = \hat{a}(k, \varepsilon, x_0; m)$, C^∞ function of $(k, \sqrt{\varepsilon}, x_0)$, defined in the open set $U = (0, +\infty) \times (0, \varepsilon_0) \times (x_r, x_s)$, fulfilling*

$$\begin{array}{ll} |\hat{a}(k,\varepsilon,x_0;m) - \tilde{a}(k,\varepsilon;m)| \approx |x_0| e^{-\frac{x_0}{\varepsilon^{3/2}}} & x_0 \in [-1, x_s), \\ |\hat{a}(k,\varepsilon,x_0;m) - \tilde{a}(k,\varepsilon;m)| \approx |x_0 - x_r| e^{-\frac{x_0 - x_r}{\varepsilon}} & x_0 \in (x_r, -1), \end{array}$$

with $\tilde{a}(k, \varepsilon; m)$ the function defined in Theorem 3.1*, and such that, for $(k, \varepsilon, x_0) \in U$ and $a = \hat{a}(k, \varepsilon, x_0; m)$ system* (3.1)-(3.2) *possesses a canard limit cycle, Γ_{x_0}, passing through $(x_0, f(x_0))$. The canard limit cycle is headless if $x_0 \in (-1, x_s)$ and with head if $x_0 \in (x_r, -1)$.*

Previous result describes the canard explosion taking place in the PWL framework. There, it can be observed that the slope of the explosion is different before and after the maximal canard.

In the following result, the stability of the canard limit cycles obtained in the previous theorem is established. The results are divided into two theorems, depending on whether the Hopf-like bifurcation is supercritical or subcritical.

Theorem 3.3 *Set $\varepsilon > 0$ small enough, $m = -\sqrt{\varepsilon}$, $x_0 \in (x_r, x_u) \cup [-1, x_s)$ and $a = \hat{a}(k, \varepsilon, x_0; m)$. Let Γ_{x_0} be the canard limit cycle of system* (3.1)-(3.2) *whose existence has been proved in Theorem* 3.2*. The following statements hold*:

a) *For $k \le 1$, the canard limit cycle Γ_{x_0} is hyperbolic and stable.*

b) *For $k > 1$, there exist exactly two values $x_1 \in (-1, x_s)$ and $x_2 \in (x_r, x_u)$ such that the canard limit cycle Γ_{x_0} is hyperbolic and stable if $x_0 \in (x_r, x_2) \cup (x_1, x_s)$, hyperbolic and unstable if $x_0 \in (x_2, x_u) \cup (-1, x_1)$, and a saddle-node canard cycle if $x_0 = x_1$ and $x_0 = x_2$.*

Theorem 3.4 *Set $\varepsilon > 0$ small enough, $m = \sqrt{\varepsilon}$, $x_0 \in (x_r, x_u) \cup [-1, x_s)$ and $a = \hat{a}(k, \varepsilon, x_0; m)$. Let Γ_{x_0} be the canard limit cycle of system* (3.1)-(3.2) *whose existence has been proved in Theorem* 3.2. *The following statements hold*:

a) *For $k < 1$, there exists exactly one value $x_1 \in (-1, x_s)$ such that Γ_{x_0} is an hyperbolic limit cycle, if $x_0 \in (x_r, x_u) \cup (-1, x_s) \setminus \{x_1\}$, and a saddle-node canard cycle, if $x_0 = x_1$. Moreover, Γ_{x_0} is stable if $x_0 < x_1$ and unstable if $x_0 > x_1$.*

b) *For $k = 1$, the canard limit cycle Γ_{x_0} is hyperbolic and stable if $x_0 \in (x_r, x_u)$ and hyperbolic and unstable if $x_0 \in (-1, x_s)$.*

c) *For $k > 1$, there exists exactly one value $x_2 \in (x_r, x_u)$ such that Γ_{x_0} is hyperbolic, if $x_0 \in (x_r, x_u) \cup (-1, x_s) \setminus \{x_2\}$, and a saddle-node canard cycle, if $x_0 = x_2$. Moreover, Γ_{x_0} is stable if $x_0 < x_2$ and unstable if $x_0 > x_2$.*

Subsequently, in the last main result, it is stated that for every width between the smallest canard cycle and the relaxation oscillation cycle, that is for every $x_0 \in (x_r, x_u) \cup [-1, x_s)$, there exist values of the parameters such that system (3.1)-(3.2) exhibits a saddle-node canard limit cycle Γ_{x_0} of width x_0.

Theorem 3.5 *Consider system* (3.1)-(3.2) *with $m = -\sqrt{\varepsilon}$ or $m = \sqrt{\varepsilon}$. For each $x_0 \in (x_r, x_u) \cup (-1, x_s)$, there exists a value ε_0 and a function $k_{x_0}(\varepsilon)$ defined for $\varepsilon \in (0, \varepsilon_0)$, such that system* (3.1)-(3.2) *with parameters $k = k_{x_0}(\varepsilon)$ and $a = \hat{a}(k_{x_0}(\varepsilon), \varepsilon, x_0; m)$ exhibits the saddle-node canard Γ_{x_0} whose existence has been stated in Theorem* 3.3 *for $m = -\sqrt{\varepsilon}$ and in Theorem* 3.4 *for $m = \sqrt{\varepsilon}$, respectively.*

4. Conclusions.

In [3], we have analyzed the existence of saddle-node bifurcation of canard cycles in PWL systems. We have revised in the PWL context the known results in the smooth framework [7]. Let us point out the similarities and differences that we have found:

Canard cycles in [7] develop along a branch born at a Hopf bifurcation, at $a = a_H$, and the canard explosion takes place at a value which is at a distance of $O(\varepsilon)$ from the a_H. In the PWL context, we have checked that the canard explosion takes place at a value which is at a distance of $O(\sqrt{\varepsilon})$ from the a_H.

In the *Supercritical case, $m = -\sqrt{\varepsilon}$*: System (3.1)-(3.2) is able to reproduce the dynamics in the smooth case with $k \leq 1$, that is, the existence of a family of stable canard cycles. By letting k increase, we have found new scenarios that have not been reported in the smooth framework. Specifically, when $k > 1$, we find situations where two saddle-node bifurcations of canard cycles take place, one of headless canards and another one of canards with head. In this case, three canard limit cycles can coexist.

In the *Subcritical case, $m = \sqrt{\varepsilon}$*: In this case, system (3.1)-(3.2) can reproduce the dynamics in the smooth case, with the benefit that in the PWL case we can control the different behaviors that appear in an easier way. Concretely, we have proved the existence of saddle-node bifurcation of headless canards for $k < 1$, and of canards with head for $k > 1$.

It has been stated in Theorem 3.5 that in both subcritical and supercritical cases, for every height between the smallest canard cycle and the relaxation oscillation cycle there exist parameters k and ε such that a saddle-node canard limit cycle with this height exists.

The use of this simpler family of slow-fast systems to reproduce canard dynamics bring us some information which could be interesting when revisiting the smooth context. In particular, conditions $k < 1$ and $k > 1$ organizing the dynamics in the main results, suggest the importance of the ratio between the slopes of the fast nullcline in order to exhibit or not saddle-node canard cycles with head. Bearing this in mind, we believe that only saddle-node canard cycles with head can appear when the slope of the repelling branch of the critical manifold is larger than the slope of the attracting branches of the critical manifold. As this is not the case in the Van der Pol system, we can expect only headless saddle-node canard cycles there.

Last, we would like to point out that some quantitative information obtained in [3] could be relevant for applications. For instance, we highlight the period of the canard cycles and the location of the saddle-node canards in terms of the parameter. Finally, the dependence between the height of a canard cycle and the bifurcation parameter a at which it appears could be approximated from the estimation $|\tilde{a} - \hat{a}|$ appearing in Theorem 3.2.

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