A Hopf theorem for open surfaces in product spaces

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Abstract. Hopf's theorem has been recently extended to compact genus zero surfaces with constant
mean curvature H in a product space $\mathcal{M}_k^2 \times \mathbb{R}$, where $\mathcal{M}_k^2$ is a surface with constant Gaussian curvature $k \neq 0$ [AbRo]. It also has been observed that, rather than $H =$ const., it suffices to assume that the differential dH of H is appropriately bounded [AdCT]. Here, we consider the case of simply-connected
open surfaces with boundary in $\mathcal{M}_k^2 \times \mathbb{R}$ such that dH is appropriately bounded and certain conditions on the boundary are satisfied, and show that such surfaces can all be described.

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1 Introduction

We consider surfaces Σ immersed in the product space $\mathcal{M}_k^2 \times \mathbb{R}$, where $\mathcal{M}_k^2$ is a simply-connected, 2-dimensional Riemannian manifold with constant Gaussian curvature k. Abresch and Rosenberg [AbRo] introduced a complex quadratic differential form $\mathfrak{Q}$ (see Section 2 of this paper for details) and proved that $\mathfrak{Q}$ is holomorphic if Σ is a CMC (constant mean curvature) surface; hence, if Σ is homeomorphic to a sphere, the quadratic form $\mathfrak{Q}$ vanishes. Furthermore, they classified all the CMC surfaces with vanishing $\mathfrak{Q}$.

Here we consider disk-type surfaces Σ with piece-wise regular boundary $\partial\Sigma$ and, under certain conditions, want to show that they are pieces of one of the above surfaces.

The conditions we have in mind (see Section 3, Theorem 3.1, for the precise statement) are as follows:

1) The surface is regular up to the boundary.
2) The condition $|dH| \leq h|\mathfrak{Q}|$, where h is a real continuous, non-negative function, holds in Σ and in its extensions across the boundary.
3) If $\partial\Sigma$ has corners (vertices), the number of vertices with angles $< \pi$ is at most 3.
4) The regular pieces of the boundary are curves that satisfy the equation [$\operatorname{Im}\mathfrak{Q} = 0$].

Our theorem extends to $\mathcal{M}_k^2 \times \mathbb{R}$ a result that J. Choe [Ch] proved in $\mathbb{R}^3$. In his case, condition (4) means that the regular pieces of the boundary are lines of curvature of Σ. Our proof is different from Choe's. This comes from the fact that he assumed $|dH| \equiv 0$ rather than our condition (2). It follows from $H =$ constant that the complex quadratic

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form $\mathfrak{Q} = \psi dz^2$, where z is a complex parameter for Σ, is holomorphic, that is, ψ satisfies the Cauchy-Riemann equation, whereas from our hypothesis (2) it merely follows a sort of Cauchy-Riemann inequality for ψ ([AdCT] p. 285). Thus, we must work somewhat harder. On the other hand, our proof becomes more conceptual, and, in fact, simpler.

In Section 2 we describe the complete surfaces for which $\mathfrak{Q}$ vanishes identically, classified in [AbRo], and we extend this classification to the case of open surfaces (the corresponding proof can be found in the appendix). In Section 3, we prove the above theorem, and in Section 4 we apply our result to some particular cases, where condition (4) is more transparent.

2 Preliminaries

Let $\psi : \Sigma \to \mathcal{M}^2_\kappa \times \mathbb{R}$ be an immersion with mean curvature H and second fundamental form σ.

We denote by ξ the vertical field in $\mathcal{M}^2_\kappa \times \mathbb{R}$, that is to say, $\xi = (0,1) \in T(\mathcal{M}^2_\kappa \times \mathbb{R}) = T(\mathcal{M}^2_\kappa) \times \mathbb{R}$. The tangent part of ξ to the surface will be denoted by T. If we denote by h the last coordinate of the immersion, then $T = \nabla h$.

Consider the following symmetric bilinear form on the surface,

$$(1) \qquad \tilde{\sigma}(X,Y) = 2H\sigma(X,Y) - \kappa \langle \xi, X\rangle \langle \xi, Y\rangle = 2H\sigma(X,Y) - \kappa\, dh(X)dh(Y),$$

where $\langle \cdot,\cdot\rangle$ denotes the scalar product in $\mathcal{M}^2_\kappa \times \mathbb{R}$. The $(2,0)$-part of the complexification of $\tilde{\sigma}$ is a quadratic form $\mathfrak{Q}$ that was introduced first by Abresch and Rosenberg in [AbRo]. They showed that this quadratic differential is holomorphic when the mean curvature of the surface is constant.

Given a conformal parameter $z = u + iv$ on the surface, the Abresch-Rosenberg differential is given in this parameter by $\mathfrak{Q} = Q dz^2$ where

$$(2) \qquad Q = 2H\sigma(Z,Z) - \kappa\langle Z, \xi\rangle^2 = 2H\sigma(Z,Z) - \kappa h_z^2;$$

here h_z means the partial derivative of h with respect to the complex vector field $Z := \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$.

The Codazzi equation for the surface has the following equivalent expression in terms of the Abresch-Rosenberg differential (see [FerMi] eq. (2.6)):

$$(3) \qquad Q_{\bar z} = 2pH_{\bar z} + \lambda H H_z,$$

that proves the holomorphicity of $\mathfrak{Q}$ when the mean curvature is constant (here the subscript $\bar z$ means partial derivative with respect to the complex vector field $\bar Z := \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$).

Remark 2.1. Since $\mathfrak{Q}$ is the $(2,0)$-part of the complexification of the 2-form $\tilde{\sigma}$, it is easy to show that tangent fields of curves where $\operatorname{Im}\mathfrak{Q} = 0$ correspond with directions that diagonalize $\tilde{\sigma}$. Thus, if Σ is a surface in $\mathcal{M}^2_\kappa \times \mathbb{R}$ and $\gamma \subset \mathcal{S}$ is a differentiable curve, γ satisfies $\operatorname{Im}\mathfrak{Q} = 0$ if and only if

$$\tilde{\sigma}(\gamma', J\gamma') = 0,$$

where J denotes the $\pi/2$ rotation in the tangent bundle of $\mathcal{S}$. For short, we will refer these curves as "$\tilde{\sigma}$-curvature lines".

$\mathfrak{Q}$-umbilic surfaces in product spaces

In this subsection we will describe briefly the surfaces in $\mathcal{M}^2_\kappa \times \mathbb{R}$ for which the differential $\mathfrak{Q}$ vanishes identically. For short, we will denote these surfaces by $\mathcal{S}_\kappa \subset \mathcal{M}^2_\kappa \times \mathbb{R}$.

In [AbRo] it is proved that a *complete* CMC surface in $\mathcal{M}^2_\kappa \times \mathbb{R}$ with vanishing differential $\mathfrak{Q}$ must be embedded and invariant under rotations (that is, invariant under a 1-parameter subgroup of isometries acting trivially on the factor $\mathbb{R}$). Furthermore, if $4H^2 + \kappa > 0$ then such a surface must be the rotationally invariant spheres of Hsiang and Hsiang [HsHs] and Pedrosa and Ritoré [PeRi], whereas for $4H^2 + \kappa \leq 0$ (which in particular implies $\kappa < 0$) there are three possibilities. More explicitly, they are:

(i) A convex graph over a slice $\mathcal{M}^2_\kappa \times \{\xi_0\}$, with rotational symmetry of elliptic type.
(ii) A surface of catenoidal type, also invariant under elliptic rotations.
(iii) An orbit under some 2-dimensional solvable group of $\mathrm{Iso}(\mathcal{M}^2_\kappa \times \mathbb{R})$. In this case the rotations are of parabolic type and the surfaces converge, as $4H^2 \nearrow -\kappa$, to vertical cylinders over horocycles of $\mathcal{M}^2_\kappa$.

Notice that, in particular, when $4H^2 + \kappa > 0$ the only CMC (non-minimal) surfaces with vanishing Abresch-Rosenberg differential are the rotation spheres, as it happens in the Euclidean setting with the Hopf differential. However, for $4H^2 + \kappa \leq 0$ the previous result gives that there are no CMC spheres in $\mathcal{M}^2_\kappa \times \mathbb{R}$.

Actually, the above classification is true even if we do not assume the completeness of the surfaces, as it is stated in the following theorem (see the appendix for its proof).

Theorem 2.1. *Any surface Σ in $\mathcal{M}^2_\kappa \times \mathbb{R}$ with vanishing Abresch-Rosenberg differential is part of one of the complete rotationally invariant CMC surfaces described above.*

3 A Hopf theorem for open surfaces with boundary

Throughout this section, $\mathcal{S}$ will denote a simply-connected compact surface in $\mathcal{M}^2_\kappa \times \mathbb{R}$ with piece-wise regular boundary . We will call the *vertices* of the surface the (finite) set of non-regular boundary points.

Our aim in this section is to prove the following theorem.

Theorem 3.1. *Let $\mathcal{S}$ be a simply-connected compact surface with boundary as above immersed in $\mathcal{M}^2_\kappa \times \mathbb{R}$.*

Assume that the following conditions are satisfied:

(i) *The surface is contained as an interior set in a differentiable surface $\tilde{\mathcal{S}}$ without boundary.*
(ii) *On $\tilde{\mathcal{S}}$ we have $|dH| \leq h|\mathfrak{Q}|$, where H is the mean curvature of the surface, $\mathfrak{Q}$ its Abresch-Rosenberg differential, and h is a continuous non-negative function.*
(iii) *The number of vertices in $\partial\mathcal{S}$ with angle $< \pi$ is less than or equal to* 3.

(iv) *The imaginary part of $\mathfrak{Q}$ vanishes on $\partial\mathcal{S} \setminus V$. This means that the regular pieces of the boundary of $\partial\mathcal{S}$ are integral curves of the directions that maximize or minimize the values of the real quadratic form $\tilde{\sigma}$ on the unit circle.*

Then, the surface is part of one of the rotational surfaces $\mathcal{S}_\kappa$ described in the subsection of Section 2.

The following result, due to Alencar, do Carmo and Tribuzy, will be crucial in the proof of Theorem 3.1.

Lemma 3.1 ([AdCT]). *Let f a differentiable function defined on a complex domain $U \subset \mathbb{C}$ and suppose that there exists a continuous real-valued non-negative function h such that $|f_{\bar{z}}| \leq h(z)|f(z)|$ holds on U. Then either $f \equiv 0$ in U or it has isolated zeroes. Moreover, if z_0 is a zero of f, locally around z_0, there exists an integer $k > 0$ such that*

$$f(z) = (z - z_0)^k\, g(z), \tag{4}$$

where g is a continuous function with $g(z_0) \neq 0$.

As a consequence, if the Abresch-Rosenberg differential $\mathfrak{Q}$ of a differentiable immersion into $\mathcal{M}^2_\kappa \times \mathbb{R}$ with mean curvature H satisfies $|dH| \leq h|\mathfrak{Q}|$ and does not vanish identically, then its zeroes are isolated. Moreover, if z_0 is a zero of $\mathfrak{Q}$ and we write $\mathfrak{Q} = Qdz^2$ locally around z_0, then there exists an integer $k > 0$ such that

$$Q(z) = (z - z_0)^k\, g(z), \tag{5}$$

where g is a continuous function with $g(z_0) \neq 0$.

To see that, use equation (3) to conclude from $|dH| \leq h|\mathfrak{Q}|$ that the function $Q(z)$ satisfies the condition of Lemma 3.1.

In order to prove the theorem, let us first parametrize the surface $\mathcal{S}$ in a convenient way.

We can assume that $\tilde{\mathcal{S}}$ is simply-connected and non-compact, and so we can parametrize conformally the surface as $\psi : \Sigma \to \mathcal{M}^2_\kappa \times \mathbb{R}$, where Σ is either the complex plane or the open unit disk, and $\psi(\Sigma) = \tilde{\mathcal{S}}$. Thus $\Omega := \psi^{-1}(\mathcal{S})$ is a compact planar domain bounded by regular curves meeting at the vertices at the same angles as in $\mathcal{S}$. We denote by

$$\psi_0 = \psi|_\Omega : \Omega \to \mathcal{S} \subset \mathcal{M}^2_\kappa \times \mathbb{R}$$

the corresponding conformal parametrization for $\mathcal{S}$.

On the other hand, by the Uniformization Theorem we can construct a biholomorphism

$$F : \mathcal{H}_+ \to \mathrm{Int}(\Omega) = \Omega \setminus \partial\Omega,$$

where $\mathcal{H}_+ = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$, and F extends continuously to $\partial\mathcal{H}_+ = \mathbb{R} \cup \{\infty\} \subset \overline{\mathbb{C}}$ with $F(\partial\mathcal{H}_+) = \partial\Omega$. We call V the set of points in $\partial\mathcal{H}_+$ applied into the vertices of $\partial\Omega$.

Moreover, the above map F extends conformally to any small neighbourhood of a point in $\partial\mathcal{H}_+ \setminus V$ (see [Z], Chapter VII, Theor. 10.13). The following lemma deals with the behaviour of the map F at the points in V.

Lemma 3.2. *Let $\xi_0 \in V$ be a point mapped by F in a vertex of angle $\theta \in]0, 2\pi[$. Then for ξ in a small neighbourhood of ξ_0 we have*

$$F'(\xi) = (\xi - \xi_0)^{\frac{\theta}{\pi} - 1}\, G(\xi), \tag{6}$$

where $G(\xi)$ is an analytic function non-vanishing at ξ_0.

Proof. Let U be a small semi disk centered at ξ_0 in $\overline{\mathcal{H}_+} = \mathcal{H}_+ \cup \partial\mathcal{H}_+$, and label $W = F(U)$ the corresponding neighbourhood of the vertex $F(\xi_0)$ in the planar domain Ω. Then W is conformally equivalent to an angular sector D_θ of angle θ centered at the origin. Let $\Phi : W \to D_\theta$ be the corresponding biholomorphism. Then $\Phi \circ F$ is a continuous map from the semi disk U into the angular sector D_θ which is conformal when restricted to interior points and $(\Phi \circ F)(\xi_0) = 0$. Thus it must be of the form

$$(\Phi \circ F)(\xi) = r(\xi - \xi_0)^{\theta/\pi},$$

for a suitable constant $r > 0$. Differentiating the above function we get equation (6) with $G(\xi) := \left(\frac{\pi}{r\theta}\, \Phi'(F(\xi))\right)^{-1}$, which is analytic and non-vanishing at ξ_0. $\square$

Now consider the pull-back via F of the Abresch-Rosenberg differential $\mathfrak{Q}$ of the immersion ψ, $\widehat{\mathfrak{Q}} = F^*(\mathfrak{Q})$, which is a quadratic differential in $\mathcal{H}_+$.

To be more explicit, if $\mathfrak{Q}$ is written in a conformal parameter z for the immersion ψ as $\mathfrak{Q} = Q(z)dz^2$ then the local expression of $\widehat{\mathfrak{Q}}$ in $\xi := F^{-1}(z)$ is

$$\widehat{\mathfrak{Q}} = \hat{Q}(\xi)d\xi^2, \qquad \hat{Q}(\xi) = Q(F(\xi))F'(\xi)^2. \tag{7}$$

By Lemma 3.1 we know that, if $\mathfrak{Q}$ does not vanish identically, its zeroes in Ω are isolated. Since F is a biholomorphism from $\mathcal{H}_+$ to IntΩ, the same holds for the zeroes of $\widehat{\mathfrak{Q}}$ in $\mathcal{H}_+$. Moreover, since F extends conformally to neighbourhoods of the points in $\partial\mathcal{H}_+ \setminus V$, then $\widehat{\mathfrak{Q}}$ also extends to this set and its zeroes are also isolated. In addition, Lemma 3.1 and equation (7) gives that in a neighbourhood of a zero ξ_0 of $\hat{\mathfrak{Q}}$ in $\mathcal{H}_+ \cup (\partial\mathcal{H}_+ \setminus V)$ there exists an integer $k > 0$ such that

$$\hat{Q}(\xi) = (\xi - \xi_0)^k \hat{g}(\xi), \tag{8}$$

where $\hat{g}$ is a continuous function with $\hat{g}(\xi_0) \neq 0$.

In the next lemma we study the behaviour of this quadratic differential at the points in V.

Lemma 3.3. *Let $\widehat{\mathfrak{Q}}$ be the quadratic form on $\mathcal{H}_+$ defined above and assume that $\widehat{\mathfrak{Q}}$ does not vanish identically. Take $\xi_0 \in V$ corresponding to a vertex of angle θ in Ω, and write $\widehat{\mathfrak{Q}} = \hat{Q}(\xi)d\xi^2$ locally around ξ_0. Then,*

(a) *If $\theta > \pi$, $\hat{Q}$ extends continuously to ξ_0 with $\hat{Q}(\xi_0) = 0$, and the decomposition in equation* (8) *is also valid for a suitable integer $k > 0$.*

(b) *If $\theta < \pi$, either $\hat{Q}$ approaches infinite as $\xi \to \xi_0$ and* (8) *holds with $k = -1$, or $\lim_{\xi\to\xi_0} \hat{Q}$ is finite and the situation is as in* (a).

Proof. At the points in V, equations (6) and (7) shows that in case (a) $\hat{Q}$ extends continuously with $\hat{Q}(\xi_0) = 0$ whereas in case (b) the limit exists but it could be infinite.

Moreover, by equation (7) we have that

$$\hat{Q}_{\bar{\xi}} = (Q_{\bar{z}} \circ F)\,(F')^2\,\overline{F'}.$$

Now using equation (3) and the hypothesis, we obtain

$$|Q_{\bar{z}}(z)| \le g(z)|dH| \le g(z)G(z)|Q(z)| = h(z)|Q(z)|,$$

where $h(z) = g(z)G(z)$. Thus

$$\begin{aligned}|\hat{Q}_{\bar{\xi}}(\xi)| &\le Q_{\bar{z}}(F(\xi))|F'(\xi)|F'(\xi)^2 \le h(F(\xi))|Q(F(\xi))||F'(\xi)|F'(\xi)^2\\ &= \hat{h}(F(\xi))|\hat{Q}(F(\xi))|,\end{aligned}$$

where $\hat{h}(F(\xi)) = h(F(\xi))|F'(\xi)|$. It follows that $\hat{Q}$ is in the conditions of Lemma 3.1, and this proves (a).

In case (b) if the limit of $\hat{Q}$ when $\xi \to \xi_0$ is finite we can apply the previous argument to obtain the same decomposition for a suitable integer $k \ge 0$.

If the limit of $\hat{Q}$ is infinite, then $1/\hat{Q}$ is also in the conditions of Lemma 3.1 and so there exists an integer $k \le 0$ such that equation (8) holds. On the other hand, by equations (6) and (7) we have

$$\hat{Q}(\xi) = Q(F(\xi))G(\xi)^2(\xi - \xi_0)^{\beta},$$

where $\beta = 2(\frac{\theta}{\pi} - 1) \in (-2, 2)$. Since the function $\hat{g}$ in (8) does not vanish around ξ_0, the function

$$(\xi - \xi_0)^{k-\beta} = \frac{Q(F(\xi))G(\xi)^2}{\hat{g}(\xi)}$$

must be bounded at this point, which implies that $k \ge \beta > -2$, and so the only possibility is $k = -1$. □

Now consider the line field X (possibly with singularities) given by

$$\text{(9)} \qquad \mathrm{Im}\widehat{\mathfrak{Q}} = 0.$$

Remark 3.1. The previous equation gives rise to two different line fields (possibly with singularities), that are orthogonal at the non-singular points. On the other hand, by the hypothesis (iv) in the theorem the curves in $\partial\mathcal{H}_+ \setminus V$ satisfy equation (9). Thus, we choose as X the line field which is orthogonal to these curves at the boundary.

It is then clear that X is a line field in $\mathcal{H}_+ \cup \partial\mathcal{H}_+$ with singularities at the zeroes of $\widehat{\mathfrak{Q}}$ and also at the points in $V \subset \partial\mathcal{H}_+$. Thus, as a consequence of Lemma 3.3, if $\mathfrak{Q}$ does not vanish identically on $\mathcal{S}$, then the singularities of X are isolated.

Observe also that the line field $dF(X)$ in Ω satisfies $\mathrm{Im}\mathfrak{Q} = 0$, and so its integral curves are $\tilde{\sigma}$-curvature lines of the immersion ψ (see Remark 2.1).

Let us show now that X can be reflected over $\partial\mathcal{H}_+$ to a line field (with singularities) in the whole sphere $\overline{\mathbb{C}}$. Indeed, let $p \in \partial\mathcal{H}_+$ be a non-singular point of X (in particular $p \notin V$) and α the integral curve of X passing through p. Then $F(p)$ is not a zero of $\mathfrak{Q}$ and therefore there are two orthogonal $\tilde{\sigma}$-curvature lines meeting at $F(p)$. As we saw above, one of them is $F \circ \alpha$. By hypothesis (iv), the other one is precisely the regular component of $\partial\Omega \setminus V$ containing $F(p)$. Since both directions must be orthogonal, the same holds for their corresponding images under F (recall that F is conformal except at the points in V). This means that α is orthogonal to $\partial\mathcal{H}_+$ and therefore it can be reflected across p to a differentiable curve in the whole extended plane.

Thus, we have a well defined line field (with singularities) in $\overline{\mathbb{C}}$. We will still denote by X that line field.

Proof of Theorem 3.1. We keep the notations introduced above. Reasoning by contradiction, let us assume that $\mathfrak{Q}$ does not vanish identically on $\mathcal{S}$, and so the same holds for $\widehat{\mathfrak{Q}}$.

Thus, X is a line field on the sphere $\overline{\mathbb{C}}$ with isolated singularities. The Poincaré-Hopf Theorem [Ho] states that

$$2 = \chi(\overline{\mathbb{C}}) = \sum I_{\xi_0}(X),$$

where $I_{\xi_0}(X)$ is the rotation index of X at the singularity $\xi_0 \in \overline{\mathbb{C}}$.

As we know, the singularities of X occur at the points $\xi_0 \in \mathcal{H}_+$ which are zeroes of $\mathfrak{Q}$, at their symmetric points $\bar{\xi}_0 \in \mathcal{H}_- = \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) < 0\}$, and at the points in $V \subset \partial\mathcal{H}_+ = \partial\mathcal{H}_-$. Label by $V_0 \subset V$ the set of points corresponding to vertices with angle $< \pi$ in $\mathcal{S}$.

Let us compute the rotation index of X at a singularity ξ_0. By the definition of X (see equation (9)) we have that

$$\mathrm{Arg}(\hat{Q}) + 2\mathrm{Arg}(d\xi) = \mathrm{Arg}(\hat{Q}) + 2\mathrm{Arg}(X) = m\pi,$$

where $\widehat{\mathfrak{Q}} = \hat{Q}(\xi)d\xi^2$ and m is an integer. Thus, the variation of the argument of X around the singularity is

$$\delta\mathrm{Arg}(X) = -\frac{1}{2}\delta\mathrm{Arg}(\hat{Q}).$$

By taking into account the behaviour of $\hat{Q}$ described in equation (8) and in Lemma 3.3, we have that $\delta\text{Arg}(\hat{Q}) = 2\pi k$, where k is a positive integer if $\xi_0 \in \mathcal{H}_+ \cup \mathcal{H}_- \cup (V \setminus V_0)$, and $k \geq -1$ if $\xi_0 \in V_0$. Thus the rotation index $I_{\xi_0}(X)$ of X at ξ_0 is

$$I_{\xi_0}(X) = \frac{\delta\text{Arg}(X)}{2\pi} = \frac{-k}{2}.$$

Then we have that $I_{\xi_0}(X) \leq 0$ for singularities in $\mathcal{H}_+ \cup \mathcal{H}_- \cup (V \setminus V_0)$, whereas $I_{\xi_0}(X) \leq 1/2$ for $\xi_0 \in V_0$. Thus we have

$$2 = \chi(\overline{\mathbb{C}}) \leq \sum_{p \in V_0} I_{\xi_0}(X) \leq \sum_{p \in V_0} \frac{1}{2}.$$

Now by the hypothesis (iii) in the theorem, the cardinality of V_0 is at most 3 so

$$2 \leq \sum_{p \in V_0} \frac{1}{2} \leq \frac{3}{2} < 2,$$

which leads to a contradiction and proves that the Abresch-Rosenberg differential $\mathfrak{Q}$ of $\mathcal{S}$ vanishes identically. Theorem 2.1 completes the proof of the theorem. □

4 Special cases

In some special cases, condition (iv) in Theorem 3.1 can be replaced by a more geometrical one. Our aim in the following two lemmas is to give sufficient conditions for a curve $\gamma \subset \mathcal{S}$ to be a solution of the equation that appears in condition (iv).

By definition, a *horizontal curve* on a surface $\mathcal{S}$ immersed in $\mathcal{M}_\kappa^2 \times \mathbb{R}$ is a curve contained in a horizontal slice $\mathcal{M}_\kappa^2 \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. On the other hand, a curve in $\mathcal{S}$ is said to be *vertical* if it is an integral curve of the vector field T defined in Section 2.

Lemma 4.1. *Let $\mathcal{S}$ be a surface in $\mathcal{M}_\kappa^2 \times \mathbb{R}$ and consider a differentiable curve $\gamma \subset \mathcal{S}$. Suppose that γ is either a horizontal or a vertical curvature line of $\mathcal{S}$.*

Then γ is a $\tilde{\sigma}$-curvature line of $\mathcal{S}$.

Proof. Observe that if γ is horizontal then the height function h of $\mathcal{S}$ is constant along the curve, and since $T = \nabla h$ then we have $\langle T, \gamma' \rangle = dh(\gamma') = 0$.

In case γ is vertical then $\gamma' = T$ and so $\langle J\gamma', T \rangle = 0$, where J denotes the rotation of 90º in the tangent bundle of $\mathcal{S}$.

Thus in both situations we have

$$\tilde{\sigma}(\gamma', J\gamma') = 2H\sigma(\gamma', J\gamma') - \kappa\langle T, \gamma' \rangle \langle T, J\gamma' \rangle = 2H\sigma(\gamma', J\gamma') = 0,$$

where we have used that γ' is a curvature line of the surface. □

Remark 4.1. Thanks to Joachimstahl's theorem (see for example [Spi]), the condition on γ to be a horizontal curvature line is equivalent to the fact that the surface intersect the corresponding horizontal slice $\mathcal{M}^2_\kappa \times \{t_0\}$ at a constant angle.

Lemma 4.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces in $\mathcal{M}^2_\kappa \times \mathbb{R}$ with normal vectors η_1 and η_2 respectively and the same mean curvature H. Let $\gamma \subset \mathcal{S}_1 \cap \mathcal{S}_2$ be a differentiable curve. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect along γ at a constant angle. Assume also that*

1. *If both surfaces are tangent at γ, then $\eta_1 = \eta_2$ on γ.*
2. *If the intersection between the surfaces is not tangent, then their respective angle functions satisfy $\langle \eta_1, \xi \rangle = -\langle \eta_2, \xi \rangle$.*

Then γ is a $\tilde{\sigma}$-curvature line for $\mathcal{S}_1$ if and only if it is a $\tilde{\sigma}$-curvature line for $\mathcal{S}_2$.

Proof. Let $\tilde{\sigma}_j(X,Y) = 2H\sigma_j(X,Y) - \kappa\langle X, \xi\rangle\langle Y, \xi\rangle$ the Abresch-Rosenberg 2-form on $\mathcal{S}_j$, $j = 1, 2$. Denote also by J_1 and J_2 the $\pi/2$ rotation in the tangent bundle of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.

In the first case we have that $J_1\gamma' = J_2\gamma'$ and so $\tilde{\sigma}_1(\gamma', J_1\gamma') = \tilde{\sigma}_2(\gamma', J_2\gamma')$ which gives the conclusion.

Suppose now that we are in case 2, and write

$$\eta_1 = \alpha_1 J_2\gamma' + \beta\eta_2,$$

$$\eta_2 = \alpha_2 J_1\gamma' + \beta\eta_1,$$

where $\alpha_1 = \langle \eta_2, J_1\gamma'\rangle / \langle \gamma', \gamma'\rangle$, $\alpha_2 = \langle \eta_1, J_2\gamma'\rangle / \langle \gamma', \gamma'\rangle$ and $\beta = \langle \eta_1, \eta_2\rangle$. By hypothesis, β is constant and so by differentiation we obtain

$$0 = \langle \nabla_{\gamma'}\eta_1, \eta_2\rangle + \langle \eta_1, \nabla_{\gamma'}\eta_2\rangle,$$

where ∇ is the Levi-Civita connection in $\mathcal{M}^2_\kappa \times \mathbb{R}$. Let us compute the first term in the expression of $\tilde{\sigma}_2(\gamma', J_2\gamma')$.

$$\begin{aligned} \sigma_2(\gamma', J_2\gamma') &= -\langle \nabla_{\gamma'}\eta_2, J_2\gamma'\rangle = \frac{-1}{\alpha_1}\langle \nabla_{\gamma'}\eta_2, \eta_1\rangle \\ &= \frac{1}{\alpha_1}\langle \nabla_{\gamma'}\eta_1, \eta_2\rangle = \frac{-\alpha_2}{\alpha_1}\sigma_1(\gamma', J_1\gamma'). \end{aligned}$$

Denote by T_j the tangent projection of ξ to $\mathcal{S}_j$, $T_j = \xi - \langle \xi, \eta_j\rangle\eta_j$, $j = 1, 2$, and observe that

$$\langle J_1\gamma', \xi\rangle = \langle J_1\gamma', T_1\rangle = \frac{1}{\alpha_2}\langle \eta_2, T_1\rangle = \frac{1}{\alpha_2}\big(\langle \eta_2, \xi\rangle - \langle \eta_1, \xi\rangle\langle \eta_1, \eta_2\rangle\big),$$

and analogously

$$\langle J_2\gamma', \xi\rangle = \frac{1}{\alpha_1}\big(\langle \eta_1, \xi\rangle + \langle \eta_2, \xi\rangle\langle \eta_1, \eta_2\rangle\big).$$

Now using that by hypothesis $\langle \eta_2, \xi\rangle = -\langle \eta_1, \xi\rangle$ we infer that

$$\langle J_2\gamma', \xi\rangle = \frac{-\alpha_2}{\alpha_1}\langle J_1\gamma', \xi\rangle,$$

and therefore

$$\tilde{\sigma}_2(\gamma', J_2\gamma') = \frac{-\alpha_2}{\alpha_1}\tilde{\sigma}_1(\gamma', J_1\gamma'),$$

which proves the lemma. ☐

As a result of the previous lemmas we have the following consequence of Theorem 3.1.

Corollary 4.1. *Let $\mathcal{S}$ be an immersed disk-type CMC surface in $\mathcal{M}^2_\kappa \times \mathbb{R}$ satisfying conditions* (i), (ii) *and* (iii) *in Theorem 3.1. Suppose also that the every regular component γ of $\partial\mathcal{S}$ is of one of the following types:*

1. *γ is a horizontal or a vertical curvature line of $\mathcal{S}$.*
2. *γ is a tangent intersection of $\mathcal{S}$ with a rotational surface $\mathcal{S}_\kappa$ with the same mean curvature vector.*
3. *γ is a transverse intersection at a constant angle of $\mathcal{S}$ with a rotational surface $\mathcal{S}_\kappa$ with the same mean curvature and whose angle function is opposite to the angle function of $\mathcal{S}$ along γ.*

Then, $\mathcal{S}$ is part of one of the rotational surfaces described in the subsection of Section 2.

Remark 4.2. The particular case when $\partial\mathcal{S}$ is a horizontal curvature line without singular points was also treated in [CaLi].

5 Appendix

In this section we will prove the Theorem 2.1 concerning the classification of the (not necessarily complete or CMC) surfaces with vanishing Abresh-Rosenberg differential Qdz^2. We start by fixing some notation.

Fundamental equations for surfaces in product spaces

Let $\psi : \Sigma \to \mathcal{M}^2_\kappa \times \mathbb{R}$ be a conformal immersion with mean curvature H and height function h. Let z be a conformal parameter on the surface for which the induced metric on Σ is written as $ds^2 = \lambda|dz|^2$.

Let η be the unit normal vector field to the immersion and define $u := \langle \xi, \eta \rangle$. Notice that $u^2 \leq 1$. Finally, let $p(z)dz^2$, $p := -\langle \eta_z, \psi_z \rangle$, be the classical Hopf differential of the immersion.

Then the fundamental equations of the immersion take the form (see [FerMi]):

$$(10) \quad \begin{cases} \textbf{(C.1)} & p_{\bar{z}} = \dfrac{\lambda}{2}(H_z + \kappa u h_z) \\ \textbf{(C.2)} & h_{z\bar{z}} = \dfrac{u\lambda H}{2} \\ \textbf{(C.3)} & u_z = -H h_z - \dfrac{2p}{\lambda} h_{\bar{z}} \\ \textbf{(C.4)} & \dfrac{4|h_z|^2}{\lambda} = 1 - u^2 \end{cases}$$

We call the set $\{\lambda, u, H, p, h\}$ the *fundamental data* of the surface.

Remark 5.1. **(C.1), (C.2), (C.3)** and **(C.4)** are the integrability conditions for the fundamental data of a conformal immersion in $\mathcal{M}_\kappa^2 \times \mathbb{R}$. Indeed, it can be shown that if Σ is a simply connected surface and $\{\lambda, u, H, p, h\} : \Sigma \to \mathbb{R}^+ \times [-1,1] \times \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ satisfy such conditions, there exists a unique immersed surface with the given fundamental data. For a proof and further details, see Theorem 2.3 in [FerMi].

Remark 5.2 ([GMM]). A CMC surface in $\mathcal{M}_\kappa^2 \times \mathbb{R}$ is rotationally invariant if and only there exists a conformal parameter w such that the fundamental data $\{\lambda, u, H, p, h\}$ depend only on the real part of w. Moreover, the profile curve is given by $\mathrm{Im}(w) = \text{const}$.

Proof of Theorem 2.1

Let $\varphi : \Sigma \to \mathcal{M}_\kappa^2 \times \mathbb{R}$ be a surface with vanishing Abresch-Rosenberg differential and with fundamental data $\{\lambda, u, H, p, h\}$ in a conformal parameter z on Σ.

Since $Q \equiv 0$, by [FerMi, Prop. 2.5] the mean curvature H of the surface must be constant. Also by equation (2) we obtain that

$$(11) \qquad 2Hp = \kappa h_z^2.$$

Therefore, if $H = 0$ the height function h is constant and so the surface is contained in a horizontal slice $\mathcal{M}_\kappa^2 \times \{t_0\}$, that corresponds with the example (i) in the previous classification. Thus we can assume without loss of generality that H is not zero.

Introducing (11) in **(C.3)** and using **(C.4)** we get

$$(12) \qquad u_z = -Hh_z\big(1 + \frac{\kappa}{4H^2}(1-u^2)\big).$$

Suppose first that u is constant. Then $u_z \equiv 0$ and so $u^2 = 1 + 4H^2/\kappa$, which in particular gives that $4H^2 + \kappa \leq 0$ (recall that $u^2 \leq 1$). Equation **(C.2)** takes now the form

$$h_{z\bar{z}} = \frac{2Hu}{1-u^2}|h_z|^2$$

(notice that $u^2 \neq 1$ since the surface is not minimal). If $u = 0$ then h is harmonic, and by **(C.4)** the metric is flat. Thus we have a piece of a right cylinder over a horocycle, which corresponds to the limit case in (iii).

Suppose now that u is non-zero and define the new parameter

$$s := \mathrm{Exp}\Big(\frac{-2Hu}{1-u^2}h(z)\Big).$$

From the above equation it is straightforward to check that s is harmonic with respect to z, and therefore there exists a local conformal parameter w such that $\mathrm{Re}(w) = s$. Let us call

$\Sigma' \subset \Sigma$ to an open domain where w is defined. With respect to this parameter we have $h(w) = h(s) = \frac{1-u^2}{2Hu}\log(1/s)$, and so the conformal factor of the metric is

$$\lambda = \frac{4|h_w|^2}{1-u^2} = \frac{-\kappa}{4H^2 s^2}.$$

This means that all the fundamental data $\{\lambda, u, H, p, h\}$ are well defined for $w \in \mathbb{R}^+ \times \mathbb{R} \supseteq \Sigma'$ and divergent curves on $\mathbb{R}^+ \times \mathbb{R}$ have infinite length with respect to $\lambda|dw|^2$, and hence this metric is complete on $\mathbb{R}^+ \times \mathbb{R}$. Thus, the surface can be extended (see Remark 5.1) to a complete CMC surface with vanishing Abresch-Rosenberg differential.

Finally assume that u is non-constant. Using (12) and **(C.2)**, and after some computations we have

$$u_{z\bar{z}} = \frac{-2u}{1-u^2}|u_z|^2.$$

Define $s := \operatorname{arctanh}(u)$. By the above equation $s_{z\bar{z}} = 0$ and therefore there exists a local conformal parameter defined on an open set $\Sigma' \subset \Sigma$ with $\operatorname{Re}(w) = s$. With respect to this new parameter we have that $u = \tanh(s)$, and so, in view of equation (12), we conclude that $h(w)$ depends only on s, and $h'(s) = -4H(1-u^2)/(4H^2 + \kappa(1-u^2))$. Thus we have

$$\lambda = \frac{4|h_w|^2}{1-u^2} = \frac{16H^2(1-u^2)}{(4H^2 + \kappa(1-u^2))^2}.$$

We now distinguish two cases. First, assume that $4H^2 + \kappa \leq 0$. In this case we reason as before. Indeed, observe that all the fundamental data are well-defined for w taking values in $I \times \mathbb{R}$, where $I = (\tanh^{-1}(-u_0), \tanh^{-1}(u_0)) \subsetneq \mathbb{R}$, $u_0 = \sqrt{1 + 4H^2/\kappa}$, and so the surface can be extended to a larger domain (see Remark 5.1). The above equation shows that the extended metric has the property that divergent curves in $I \times \mathbb{R}$ have infinite length, and hence is complete, which proves the lemma in case $4H^2 + \kappa \leq 0$.

To conclude suppose that $4H^2 + \kappa > 0$ and let us see that in this case the surface can be extended to a compact (and therefore complete) CMC surface. Notice first that all the fundamental data are well-defined for all the values $w \in \mathbb{C}$ and they depend only on the real part of the parameter w, which means that the surface is rotational and the profile curve is given by $\operatorname{Im}(w) = $ const. (see Remark 5.2). Moreover, all the fundamental data can be extended to $s = \pm\infty$, and $u(\pm\infty) = \pm 1$, which means that the tangent plane at the limit points is horizontal (recall that, by definition, u is the cosine of the angle between the normal map and the vertical direction).

Finally observe that the transformation $s \mapsto -s$ induces the following symmetries in the fundamental data of the surface:

$$\{\lambda, u, H, p, h\} \mapsto \{\lambda, -u, H, p, -h\},$$

this gives that the profile curve is symmetric with respect to a horizontal slice, and therefore the two end points of the surface lie in the same vertical line, which is precisely the rotation axis of the surface. As a consequence, the profile curve can be reflected over that axis giving rise to a compact surface of revolution with vanishing Abresch-Rosenberg differential, which concludes the proof □

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