THE CLASSIFICATION OF NATURALLY GRADED p-FILIFORM LEIBNIZ ALGEBRAS

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In the present article the classification of n-dimensional naturally graded p-filiform ($1 \leq p \leq n-4$) Leibniz algebras is obtained. A splitting of the set of naturally graded Leibniz algebras into the families of Lie and non Lie Leibniz algebras by means of characteristic sequences (isomorphism invariants) is proved.

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1. INTRODUCTION

Intensive investigations of Lie algebras lead to consideration of more general objects—Mal'cev algebras, binary Lie algebras, Lie superalgebras, Leibniz algebras, and others.

Leibniz algebras present a "noncommutative" (to be more precise, a "non-antisymmetric") analogue of Lie algebras [8], as algebras which satisfy the following Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

It should be noted that, if a Leibniz algebra satisfies the identity $[x, x] = 0$, then the Leibniz and the Jacobi identities coincide. Therefore, Leibniz algebras are a "non-antisymmetric" generalization of Lie algebras.

Many articles, including [2, 5, 6, 9], were devoted to the investigation of cohomological and structural properties of Leibniz algebras.

The well-known natural gradations of nilpotent Lie and Leibniz algebras are very helpful in investigations of properties of those algebras in the general case

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without restriction on the gradation. Indeed, we can always choose a homogeneous basis and thus the gradation allows to obtain more explicit conditions for the structural constants. Moreover, such gradation is useful for the investigation of cohomologies for the algebras considered, because it induces corresponding gradation of the group of cohomologies. A similar approach was considered in [7, 10], and others.

Let L be an arbitrary n-dimensional Leibniz algebra, and let $\{e_1, \dots, e_n\}$ be a basis of the algebra L. Then the table of multiplication on the algebra is defined by the products of the basic elements, namely, as $[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^k e_k$, where γ_{ij}^k are the structural constants. Thus the problem of classification of algebras can be reduced to the problem of finding a description of the structure constants up to a nondegenerate basis transformation.

From the Leibniz identity, we have the following polynomial equalities for the structural constants:

$$\sum_{l=1}^{n}(\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0, \qquad 1 \le i, j, k, \quad m \le n$$

But the straightforward description of structural constants is somewhat cumbersome and, therefore, one has to apply different methods of investigation. In the present article, we have used the classification methods based on the properties of operators of right multiplications of basic elements of algebra.

Since the description of all nilpotent Leibniz algebras is an unsolvable task (even in case of Lie algebras), we reduce our discussion to naturally graded case, with some restriction on their characteristic sequences.

Recall that the classification of naturally graded p-filiform $(0 \le p \le 2)$ Leibniz algebras and naturally graded p-filiform Lie algebras have been obtained in works [2–4]. In this article, we continue the study of complex finite dimensional nilpotent Leibniz algebras. Namely, we classify the n-dimensional complex naturally graded p-filiform $(1 \le p \le n-4)$ Leibniz algebras. Moreover, a splitting of the set of naturally graded Leibniz algebras into the families of Lie and non-Lie Leibniz algebras by means of characteristic sequence is proved.

Throughout the article, all spaces and algebras are considered over the field of complex numbers. For convenience, we omit the products which are equal to zero. Also we shall not consider the algebras which are the direct sum of algebras of less dimensions (such algebras are called split algebras).

2. PLELIMINARIES

Definition 2.1 ([8]). A vector space L over a field F with a multiplication $[-,-] : L \otimes L \to L$ is called a Leibniz algebra *if it satisfies the identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

For the examples of Leibniz algebras we refer to articles [5, 8, 9]. $\mathcal{L}(x, y, z)$ denotes the polynomial

$$\mathcal{L}(x, y, z) = [x, [y, z]] - [[x, y], z] + [[x, z], y].$$

It is obvious that Leibniz algebras are determined by the identity $\mathcal{L}(x, y, z) = 0$.

Given an arbitrary Leibniz algebra L, we define the lower central series

$$L^1 = L, \qquad L^{k+1} = [L^k, L], \qquad k \geq 1.$$

Definition 2.2. An algebra L is called *nilpotent* if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal such number s is called the index of nilpotency or nilindex.

For an element x of the Leibniz algebra L, we define the operator of right multiplication $R_x : L \to L$ as $R_x(y) = [y, x]$.

Let x be an element from the set $L \backslash L^2$ such that R_x is a nilpotent operator. Denote by $C(x) = (n_1, n_2, \ldots, n_k)$ the decreasing sequence which consists of the dimensions of the Jordan blocks of the R_x. On the set of such sequences, we consider the lexicographic order, i.e., $C(x) = (n_1, n_2, \ldots, n_k) \leq C(y) = (m_1, m_2, \ldots, m_s)$ means that there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

Definition 2.3. The sequence $C(L) = \max_{x \in L \backslash L^2} C(x)$ is called the *characteristic sequence* of the Leibniz algebra L.

As in the Lie algebras case, it can be shown that the characteristic sequence is an invariant isomorphism [3].

Definition 2.4. A Leibniz algebra L is called *p-filiform* if $C(L) = (n - p, \underbrace{1, \ldots, 1}_{p})$, where $p \geq 0$.

Note that the above definition agrees with the definition of p-filiform Lie algebras when $p > 0$ [3]. Since in the case of Lie algebras there no singly-generated algebra, the notion of 0-filiform algebra for Lie algebras has no sense.

Definition 2.5. The set $\mathcal{R}(L) = \{x \in L : [y, x] = 0 \text{ for all } y \in L\}$ is called the *right annihilator of the algebra L*.

Definition 2.6. Given an n-dimensional p-filiform Leibniz algebra L, put $L_i = L^i/L^{i+1}$, $1 \leq i \leq n - p$, and $grL = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-p}$. Then $[L_i, L_j] \subseteq L_{i+j}$, and we obtain the graded algebra grL. If grL and L are isomorphic, in notation $grL \cong L$, we say that *L is naturally graded.*

For convenience we shall use the expression “graded algebra” instead of “naturally graded algebra.”

We now present the list of the graded p-filiform $(1 \leq p \leq n - 1)$ Lie algebras [3].

$\mathcal{L}(n, r_1, r_2, \ldots, r_{p-1})$
(r_j odd, $1 \leq j \leq p - 1, 3 \leq r_1 < r_2 < \cdots < r_{p-1} \leq n - p$)

$$\begin{cases} [x_0, x_i] = x_{i+1} & 1 \leq i \leq n - p - 1 \\ [x_i, x_{r_j - i}] = (-1)^{i-1} y_j & 1 \leq i \leq \frac{r_j - 1}{2}, \qquad 1 \leq j \leq p - 1 \end{cases}$$

$\mathcal{Q}(n, r_1, r_2, \ldots, r_{p-1})$
(r_j odd, $1 \le j \le p-1$, $3 \le r_1 < r_2 < \cdots < r_{p-1} \le n-p-2$, $n-p$ odd)

$$\begin{cases} [x_0, x_i] = x_{i+1} & 1 \le i \le n-p-1 \\ [x_i, x_{r_j - i}] = (-1)^{i-1} y_j & 1 \le i \le \frac{r_j - 1}{2}, \qquad 1 \le j \le p-1 \\ [x_i, x_{n-p-i}] = (-1)^{i-1} x_{n-p} & 1 \le i \le \dfrac{n-p-1}{2} \end{cases}$$

$\tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-1)$
(r_j odd, $1 \le j \le p-2$, $3 \le r_1 < r_2 < \cdots < r_{p-2} \le n-p-3$, $n-p$ even)

$$\begin{cases} [x_0, x_i] = x_{i+1} & 1 \le i \le n-p-1 \\ [x_i, x_{r_j - i}] = (-1)^{i-1} y_j & 1 \le i \le \dfrac{r_j - 1}{2}, \ 1 \le j \le p-2 \\ [x_i, x_{n-p-1-i}] = (-1)^{i-1} (x_{n-p-1} + y_{p-1}) & 1 \le i \le \dfrac{n-p-2}{2} \\ [x_i, x_{n-p-i}] = (-1)^{i-1} \dfrac{(n-p-2i)}{2} x_{n-p} & 1 \le i \le \dfrac{n-p-2}{2} \\ [x_1, y_{p-1}] = \dfrac{(p+2-n)}{2} x_{n-p} & \end{cases}$$

$\tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-2)$
(r_j odd, $1 \le j \le p-2$, $3 \le r_1 < r_2 < \cdots < r_{p-2} \le n-p-4$, $n-p$ odd)

$$\begin{cases} [x_0, x_i] = x_{i+1} & 1 \le i \le n-p-1 \\ [x_i, x_{r_j - i}] = (-1)^{i-1} y_j & 1 \le i \le \dfrac{r_j - 1}{2}, \quad 1 \le j \le p-2 \\ [x_i, x_{n-p-2-i}] = (-1)^{i-1} (x_{n-p-2} + y_{p-1}) & 1 \le i \le \dfrac{n-p-3}{2} \\ [x_i, x_{n-p-1-i}] = (-1)^{i-1} \dfrac{(n-p-1-2i)}{2} x_{n-p-1} & 1 \le i \le \dfrac{n-p-3}{2} \\ [x_i, x_{n-p-i}] = (-1)^{i} (i-1) \dfrac{(n-p-1-i)}{2} x_{n-p} & 2 \le i \le \dfrac{n-p-1}{2} \\ [x_i, y_{p-1}] = \dfrac{(p+3-n)}{2} x_{n-p-2+i} & 1 \le i \le 2, \end{cases}$$

where laws of the algebras are expressed in the basis $\{x_0, x_1, \ldots, x_{n-p}, y_1, y_2, \ldots, y_{p-1}\}$.

The following theorems present the classification of p-filiform ($0 \le p \le 2$) Leibniz algebras.

Theorem 2.1 ([2]). *Any n-dimensional zero-filiform Leibniz algebra is isomorphic to the following algebra*:

$$L_n^0 : [e_i, e_1] = e_{i+1}, \qquad 1 \le i \le n-1.$$

We recall that 1-filiform Lie algebras (respectively, 2-filiform Lie algebras) are called filiform Lie algebras (respectively, quasi-filiform Lie algebras). In the following theorem, we summarize the results of [2, 10].

Theorem 2.2. *Any nonsplit n-dimensional graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$L_n^1 = \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1 \end{cases}$$

$$L_n^2 = \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n-1 \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1} e_n & 2 \le i \le n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ *for even n and* $\alpha = 0$ *for odd n.*

Theorem 2.3 [4]. *Any nonsplit n-dimensional* $(n \ge 6)$ *graded 2-filiform non-Lie Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_1, e_{n-1}] = e_2 + e_n, & \\ [e_i, e_{n-1}] = e_{i+1}, & 2 \le i \le n-3 \end{cases} \qquad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-3 \\ [e_1, e_{n-1}] = e_n. & \end{cases}$$

Remark 2.1. The description up to isomorphism of 5-dimensional 2-filiform Leibniz algebras can be found in [1, 4].

3. NATURALLY GRADED *p*-FILIFORM LEIBNIZ ALGEBRAS

Let L be a graded p-filiform n-dimensional Leibniz algebra. Then there exists a basis $\{e_1, e_2, \dots, e_{n-p}, f_1, \dots, f_p\}$ such that $e_1 \in L^2$ and $C(e_1) = (n-p, \underbrace{1, \dots, 1}_{p-nimes})$. By the definition of characteristic sequence the operator R_{e_1} in the Jordan form has one block J_{n-p} of size $n-p$ and p blocks J_1 (where $J_1 = \{0\}$) of size one.

The possible forms for the operator R_{e_1} are the following:

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_{n-p} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \dots, \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & J_{n-p} \end{pmatrix}.$$

By a shift of the basic elements, it is easy to prove that all cases when the Jordan block J_{n-p} is placed in a position from the first, are mutually isomorphic cases. Thus, we can reduce the study to the following two possibilities of Jordan form of the matrix R_{e_1}:

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}, \qquad \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_{n-p} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}.$$

Definition 3.1. A p-filiform Leibniz algebra L is called *of the first type (type I)* if the operator R_{e_1} has the form

$$\begin{pmatrix} J_{n-p} & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & J_1 \end{pmatrix}$$

and *of the second type (type II)* in the other case.

It should be noted that a p-filiform Leibniz algebra of the first type is non-Lie algebra. Indeed, if the Leibniz algebra is of the first type, then $[e_1, e_1] = e_2$, which contradicts the identity $[x, x] = 0$. It is easy to see that the algebras L_n^0 and L_n^1 are Leibniz algebras of the first type.

3.1. Classification of Graded *p*-Filiform Leibniz Algebras of the Type I

Let L be a p-filiform Leibniz algebra of the first type. Then there exists a basis $\{e_1, \dots, e_{n-p}, f_1, \dots, f_p\}$ (so called the adapted basis) such that

$$\begin{aligned} &[e_i, e_1] = e_{i+1}, && 1 \le i \le n-p-1, \\ &[f_j, e_1] = 0, && 1 \le j \le p. \end{aligned}$$

From these products, we have

$$\langle e_1 \rangle \subseteq L_1, \qquad \langle e_2 \rangle \subseteq L_2, \dots, \langle e_{n-p} \rangle \subseteq L_{n-p}.$$

However, we do not have information about the elements $\{f_1, f_2, \dots, f_p\}$.

Let us denote by $r_1, r_2, \dots, r_p$, the position of the basic elements $f_1, f_2, \dots, f_p$, respectively, in natural gradation, i.e., $f_i \in L_{r_i}$ for $1 \le i \le p$. Without loss of generality, one can suppose that $1 \le r_1 \le r_2 \le \cdots \le r_p \le n-p$. It should be noted that $\{e_2, e_3, \dots, e_{n-p}\} \subseteq \mathcal{R}(L)$. For p-filiform Leibniz algebras, the following theorem holds.

Theorem 3.1 ([4]). *Let L be a graded p-filiform Leibniz algebra of the type I. Then $r_s \le s$ for any $s \in \{1, 2, \dots, p\}$.*

To prove the main result of this subsection, we need the next lemmas.

Lemma 3.1. *An arbitrary p-filiform Leibniz algebra of the type I satisfying the property $r_i = 1$ for $1 \le i \le p$ is split.*

Proof. Let L be an algebra which satisfy the conditions of the lemma. Then $f_i \in L_1$ for $1 \le i \le p$ and

$$L_1 = \langle e_1, f_1, \dots, f_p \rangle, \qquad L_2 = \langle e_2 \rangle, \dots, L_{n-p} = \langle e_{n-p} \rangle$$

Introduce the notations

$$[e_1, f_i] = \alpha_i e_2, \qquad [f_j, f_i] = \beta_{i,j} e_2 \qquad 1 \le i, j \le p.$$

Now, consider the equations

$$[[f_j, f_i], e_1] = [f_j, [f_j, e_1]] + [[f_j, e_1], f_i] = 0.$$

Since $[[f_j, f_i], e_1] = \beta_{i,j} e_3$, we have $\beta_{i,j} = 0$.

Let us suppose that $\alpha_{i_0} \neq 0$ for some $i_0 \in \{1, \dots, p\}$ (otherwise, evidently, we get split algebra). Taking the transformation of basis

$$\begin{aligned} &e_i' = e_i, \qquad 1 \le i \le n-p, \qquad f_1' = \frac{1}{\alpha_{i_0}} f_{i_0}, \\ &f_j' = \alpha_{i_0} f_j - \alpha_j f_{i_0}, \qquad 2 \le j \le p, \quad j \neq i_0, \\ &f_{i_0}' = \alpha_{i_0} f_1 - \alpha_1 f_{i_0}, \end{aligned}$$

we can assume that $[e_1, f_1] = e_2, [e_1, f_i] = 0, 2 \le i \le p$. Applying the Leibniz identity, it is not difficult to see that

$$\begin{aligned} &[e_i, f_1] = e_{i+1}, \qquad 2 \le i \le n-p-1 \\ &[e_i, f_j] = 0, \qquad\quad 2 \le i \le n-p-1, \qquad 2 \le j \le p. \end{aligned}$$

Hence $L = \{e_1, e_2, \dots, e_{n-p}, f_1\} \oplus \{f_2\} \oplus \cdots \oplus \{f_p\}$, i.e. we obtain a split algebra. $\square$

Lemma 3.2. *Let for n-dimensional graded p-filiform Leibniz algebra L $(n - p \ge 4)$ of the type I, the condition*

$$1 = r_1 = \cdots = r_k < r_{k+1}, \qquad 1 \le k \le p-1$$

holds. Then

$$\begin{aligned} &L_1 = \langle e_1, f_1, f_2 \dots, f_k \rangle, \qquad L_2 = \langle e_2, f_{k+1}, f_{k+2} \dots, f_p \rangle, \\ &L_3 = \langle e_3 \rangle, \dots, L_{n-p} = \langle e_{n-p} \rangle. \end{aligned}$$

Proof. Let L be an algebra of the type I, which satisfies the condition of the lemma. Then $f_i \in L_1$ for $1 \le i \le k$ and

$$L_1 = \langle e_1, f_1, \dots, f_k \rangle, \qquad L_2 \supset \langle e_2 \rangle, \dots, L_{n-p} \supset \langle e_{n-p} \rangle.$$

Introduce the notations

$$[e_1, f_j] = \alpha_j e_2 + \sum_{s=1}^{p-k} \beta_{j,s} f_{k+s}, \qquad 1 \le j \le k.$$

From the equalities $[[e_i, f_j], e_1] = [e_{i+1}, f_j]$, we get $[e_i, f_j] = \alpha_j e_{i+1}$ for $2 \le i \le n - p - 1$, $1 \le j \le k$. Since $n - p \ge 4$, by equalities $[[f_i, f_j], e_1] = 0$, we can conclude $[f_i, f_j] = \gamma_{ij}^1 f_{k+1} + \cdots + \gamma_{ij}^{p-k} f_p$ for $1 \le i, j \le k$

We may assume that $[f_i, f_1] = 0$ for $1 \le i \le k$. Indeed, otherwise, if we consider the element $Ae_1 + Bf_1$ with sufficiently large value of A and $B \neq 0$ fixed, then $rank(R_{Ae_1+Bf_1}) > n - p$. Therefore, $C(Ae_1 + Bf_1)$ would be larger than $(n - p, 1, \ldots, 1)$, i.e., we get a contradiction. Applying similar arguments, we obtain $[f_i, f_j] = 0$ for $1 \le i, j \le k$.

Thus, we have the following products:

$$\begin{cases} [e_i, e_1] = e_{i+1} & 1 \le i \le n - p - 1 \\ [e_1, f_j] = \alpha_j e_2 + \sum\limits_{s=1}^{p-k} \beta_{j,s} f_{k+s} & 1 \le j \le k \\ [e_i, f_j] = \alpha_j e_{i+1} & 2 \le i \le n - p - 1, \quad 1 \le j \le k \\ [e_{n-p}, f_j] = [f_j, e_1] = 0 & 1 \le j \le p \\ [f_i, f_j] = 0 & 1 \le i, j \le k. \end{cases} \qquad (*)$$

Since $[f_i, f_j] = [f_i, e_1] = 0$ for $1 \le i, j \le k$, one can assume $L_2 = \langle [e_1, e_1], [e_1, f_i] \rangle$, i.e., $dim(L_2) \le k + 1$.

Let us suppose $f_{k+1}, f_{k+2}, \ldots, f_{k+s}$ lie in L_2 for $1 \le s \le p - k$. Then applying similar arguments as for $[f_i, f_1]$, we obtain $[f_{k+i}, f_j] = 0$ for $1 \le i \le s$, $1 \le j \le k$. Now using (*) we conclude $L_3 =< e_3 >$ and $k + s = p$.

Thus,

$$L_1 = \langle e_1, f_1, f_2 \ldots, f_k \rangle, \qquad L_2 = \langle e_2, f_{k+1}, f_{k+2} \ldots, f_p \rangle,$$
$$L_3 = \langle e_3 \rangle, \ldots, L_{n-p} = \langle e_{n-p} \rangle.$$

□

The following theorem presents the classification of graded nonsplit p-filiform Leibniz algebras $(n - p \ge 4)$ of the type I.

Theorem 3.2. *Any n-dimensional graded nonsplit p-filiform Leibniz algebra $(n - p \ge 4)$ of the type I is isomorphic to one of the following non-isomorphic algebras*:

p is even

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n - p - 1 \\ [e_1, f_j] = f_{\frac{p}{2}+j}, & 1 \le j \le \frac{p}{2} \end{cases}$$

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-p-1 \\ [e_1, f_1] = e_2 + f_{\frac{p}{2}+1}, & \\ [e_i, f_1] = e_{i+1}, & 2 \leq i \leq n-p-1 \\ [e_1, f_j] = f_{\frac{p}{2}+j}, & 2 \leq j \leq \dfrac{p}{2} \end{cases}$$

p is odd

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-p-1 \\ [e_1, f_j] = f_{\lfloor \frac{p}{2} \rfloor + 1 + j}, & 1 \leq j \leq \left\lfloor \dfrac{p}{2} \right\rfloor \\ [e_i, f_{\lfloor \frac{p}{2} \rfloor + 1}] = e_{i+1}, & 1 \leq i \leq n-p-1. \end{cases}$$

Proof. Due to Lemmas 3.1 and 3.2, we have

$$L_1 = \langle e_1, f_1, f_2 \ldots, f_k \rangle, \qquad L_2 = \langle e_2, f_{k+1}, f_{k+2} \ldots, f_p \rangle$$

and

$$\begin{cases} [e_i, e_1] = e_{i+1} & 1 \leq i \leq n-p-1 \\ [e_1, f_j] = \alpha_j e_2 + \sum\limits_{s=1}^{p-k} \beta_{j,s} f_{k+s} & 1 \leq j \leq k \\ [e_i, f_j] = \alpha_j e_{i+1} & 2 \leq i \leq n-p-1, \quad 1 \leq j \leq k \\ [e_{n-p}, f_j] = [f_j, e_1] = 0 & 1 \leq j \leq p \\ [f_i, f_j] = 0 & 1 \leq i, j \leq k, \\ [f_{k+i}, f_j] = 0 & 1 \leq i \leq s \leq p-k, \quad 1 \leq j \leq k. \end{cases} \tag{**}$$

Consider the case when p is even. If $k > \frac{p}{2}$, then putting $f'_k = a_1 f_1 + a_2 f_2 + \cdots + a_k f_k$ for appropriate values of $a_1, a_2, \ldots, a_k$, we obtain $[e_1, f'_k] = 0$, but is lead that algebra L is split. In case $k < \frac{p}{2}$, it is clearly that algebra L is also split. Thus, we have $k = \frac{p}{2}$.

If $\beta_{j,1} = 0$ for $1 \leq j \leq \frac{p}{2}$, then $f_{\frac{p}{2}+1} \notin L_2$, that is a contradiction to the condition $L_2 =< e_2, f_{\frac{p}{2}+1}, f_{\frac{p}{2}+2} \ldots, f_p >$. Hence, $\beta_{j_0,1} \neq 0$ for some $j_0 \in \{1, \ldots, \frac{p}{2}\}$. Making the change $f'_1 = f_{j_0}$, $f'_{j_0} = f_1$, and $f'_{\frac{p}{2}+1} = \sum_{s=1}^{\frac{p}{2}} \beta_{j_0,s} f_{\frac{p}{2}+s}$, we obtain

$$[e_1, f'_1] = \alpha'_1 e_2 + \sum_{s=1}^{\frac{p}{2}} \beta'_{1,s} f_{\frac{p}{2}+s} = \alpha'_1 e_2 + f'_{\frac{p}{2}+1},$$

where $\alpha'_1 = \alpha_{j_0}$, $\beta'_{1,s} = \beta_{j_0,s}$.

Taking the transformation of the basis

$$e'_i = e_i, \quad 1 \leq i \leq n-p, \qquad f'_j = f_j - \beta_{j,1} f_1, \quad 2 \leq j \leq \frac{p}{2},$$

we may assume $\beta_{j,1} = 0$ for $2 \leq j \leq \frac{p}{2}$.

It should be noted that $\beta_{j,2} = 0$ for all $1 \leq j \leq \frac{p}{2}$ implies $f_{\frac{p}{2}+2} \notin L_2$, article is a contradiction to the condition that is a contradiction to the condition $L_2 =< e_2, f_{\frac{p}{2}+1}, f_{\frac{p}{2}+2} \dots, f_p >$.

Let $\beta_{t,2} \neq 0$ for some $t \in \{1, 2, \dots, \frac{p}{2}\}$. Then using similar argumentations as in case $\beta_{j_0,1} \neq 0$, we obtain $\beta_{j,2} = 0$, $3 \leq j \leq \frac{p}{2}$. Continuing as above, we obtain $\beta_{j,s} = 0$, $s+1 \leq j \leq \frac{p}{2}$, and $\beta_{j,j} = 1$ $1 \leq j \leq \frac{p}{2}$.

Below we summarize obtained products

$$\begin{array}{ll}
[e_i, e_1] = e_{i+1} & 1 \leq i \leq n-p-1 \\
[e_1, f_j] = \alpha_j e_2 + f_{\frac{p}{2}+j} & 1 \leq j \leq \frac{p}{2} \\
[e_i, f_j] = \alpha_j e_{i+1} & 2 \leq i \leq n-p-1, \quad 1 \leq j \leq \frac{p}{2}.
\end{array}$$

Note that in Leibniz algebra L the elements of the form $[x, x]$, $[x, y] + [y, x]$ lie in $\mathcal{R}(L)$ and $\mathcal{R}(L)$ is ideal of the algebra. Therefore basic elements e_i $(2 \leq i \leq n-p)$ and f_j $(\frac{p}{2}+1 \leq j \leq p)$ lie in right annihilator. Taking into account the condition $n - p \geq 4$, we conclude that $[f_i, f_j] = 0$, $1 \leq i, j \leq p$.

Consider now the possible cases. If $\alpha_j = 0$ for all for some $j \in \{1, 2, \dots, \frac{p}{2}\}$, then we get the first algebra of list in the theorem.

Let us suppose that $\alpha_t \neq 0$, for some $t \in \{1, 2, \dots, \frac{p}{2}\}$. Then, if necessary, applying the following change of basis:

$$\begin{array}{lll}
f'_1 = \dfrac{1}{\alpha_t} f_t, & f'_{\frac{p}{2}+1} = \dfrac{1}{\alpha_t} f_{\frac{p}{2}+t}, & \\
f'_j = \alpha_t f_j - \alpha_j f_t, & f'_{\frac{p}{2}+j} = \alpha_t f_{\frac{p}{2}+j} - \alpha_j f_{\frac{p}{2}+t} & 2 \leq j \leq \frac{p}{2}, \quad j \neq t, \\
f'_t = \alpha_t f_1 - \alpha_1 f_t, & f'_{\frac{p}{2}+t} = \alpha_t f_{\frac{p}{2}+1} - \alpha_1 f_{\frac{p}{2}+t}, &
\end{array}$$

we can assume $\alpha_1 = 1$ and $\alpha_j = 0$, $2 \leq j \leq \frac{p}{2}$. Thus, we get the second algebra of the list in the theorem.

Let us suppose now that p is odd. Since $dim(L_2) \leq k+1$, then $k \geq \lfloor \frac{p}{2} \rfloor + 1$. In case when $k > \lfloor \frac{p}{2} \rfloor + 1$ by the similar way as in case of p even, we obtain a split algebra. Hence $k = \lfloor \frac{p}{2} \rfloor + 1$.

If in (**) $\alpha_j = 0$ for $1 \leq j \leq \lfloor \frac{p}{2} \rfloor + 1$, then algebra L is split.

If $\alpha_j \neq 0$ for some $j \in \{1, \dots, \lfloor \frac{p}{2} \rfloor + 1\}$, then putting $f'_{\lfloor \frac{p}{2} \rfloor+1} = a_1 f_1 + a_2 f_2 + \dots + a_{\lfloor \frac{p}{2} \rfloor+1} f_{\lfloor \frac{p}{2} \rfloor+1}$ we can choose the parameters $a_1, a_2, \dots, a_{\lfloor \frac{p}{2} \rfloor+1}$ such that $[e_1, f'_{\lfloor \frac{p}{2} \rfloor+1}] = e_2$. In a similar way as to the case of p even, we obtain

$$\begin{array}{ll}
[e_i, e_1] = e_{i+1} & 1 \leq i \leq n-p-1 \\
[e_1, f_j] = \alpha_j e_2 + f_{\lfloor \frac{p}{2} \rfloor+1+j} & 1 \leq j \leq \lfloor \frac{p}{2} \rfloor \\
[e_i, f_j] = \alpha_j e_{i+1} & 2 \leq i \leq n-p-1, \quad 1 \leq j \leq \lfloor \frac{p}{2} \rfloor \\
[e_i, f_{\lfloor \frac{p}{2} \rfloor+1}] = e_{i+1} & 1 \leq i \leq n-p-1
\end{array}$$

Due to the change of basis

$$f'_j = f_j - \alpha_j f_{\lfloor \frac{p}{2} \rfloor+1}, \qquad 1 \leq j \leq \lfloor \frac{p}{2} \rfloor,$$

we can assume that $\alpha_j = 0$ for $1 \leq j \leq \lfloor \frac{p}{2} \rfloor$, which leads to the getting of the third algebra of the list of the theorem. $\square$

3.2. Classification of Graded *p*-Filiform Leibniz Algebras of the Type II

Let L be an n-dimensional naturally graded p-filiform Leibniz algebra of type II, and let $\{e_1, \ldots, e_{n-p}, f_1, \ldots, f_p\}$ be an adapted basis, i.e., we have the products

$$
\begin{aligned}
&[e_1, e_1] = 0, \\
&[e_i, e_1] = e_{i+1}, \qquad 2 \leq i \leq n-p-1, \\
&[e_{n-p}, e_1] = f_1, \\
&[f_j, e_1] = 0, \qquad 1 \leq j \leq p.
\end{aligned}
$$

Obviously,

$$L_1 \supseteq \langle e_1, e_2 \rangle, \quad L_2 \supseteq \langle e_3 \rangle, \ldots, \quad L_{n-p-1} \supseteq \langle e_{n-p} \rangle, \quad L_{n-p} \supseteq \langle f_1 \rangle.$$

Here $e_2 \in L_{r_1}$, (i.e., $r_1 = 1$) and $f_j \in L_{r_j}, 2 \leq j \leq p$. Without loss of generality, one can assume $1 = r_1 \leq r_2 \leq r_3 \leq \cdots \leq r_p$.

Lemma 3.3. *Let L be a complex n-dimensional graded p-filiform Leibniz algebra $(n - p \geq 3)$ of the type II with the property $r_i = 1, 1 \leq i \leq p$. Then L is a Lie algebra.*

Proof. Let an algebra L satisfy to the conditions of the lemma. Then

$$L_1 = \langle e_1, e_2, f_2, \ldots, f_p \rangle, \quad L_2 = \langle e_3 \rangle, \ldots, \quad L_{n-p-1} = \langle e_{n-p} \rangle, \quad L_{n-p} = \langle f_1 \rangle.$$

Similar to the case of type I, we can obtain $[f_i, f_j] = 0$, $1 \leq i, j \leq p$.

Let us introduce the notations

$$
\begin{cases}
[e_1, e_2] = \alpha e_3, & \\
[e_2, e_2] = \theta e_3, & \\
[e_1, f_j] = \beta_j e_3, & 2 \leq j \leq p, \\
[e_2, f_j] = \gamma_j e_3, & 2 \leq j \leq p, \\
[f_j, e_2] = \delta_j e_3, & 2 \leq j \leq p.
\end{cases}
$$

From the Leibniz identity, namely,

$$
\begin{aligned}
\mathcal{L}(e_1, e_i, e_1) &= \mathcal{L}(e_1, f_j, e_1) = \mathcal{L}(e_i, f_j, e_1) \\
&= \mathcal{L}(e_1, f_i, e_2) = \mathcal{L}(e_1, e_2, e_2) = \mathcal{L}(e_1, e_1, e_2) = 0,
\end{aligned}
$$

we can derive,

$$
\begin{aligned}
&[e_1, e_i] = \alpha e_{i+1}, \qquad 2 \leq i \leq n-p-1, \beta_j = 0, \qquad 1 \leq j \leq p, \\
&[e_i, f_j] = \gamma_j e_{i+1}, \qquad [e_{n-p}, f_j] = \gamma_j f_1, \qquad 2 \leq i \leq n-p-1, \qquad 2 \leq j \leq p, \\
&\quad -\alpha\gamma_j = \alpha\delta_j, \qquad 2 \leq j \leq p, \qquad \theta\alpha = 0, \qquad \alpha^2 + \alpha = 0.
\end{aligned}
$$

If $\alpha = -1$, then $\theta = 0$, $\gamma_i = -\delta_i$, and L is a Lie algebra. If $\alpha = 0$, then putting $e_2' = e_2 - \theta e_1$ we can suppose $\theta = 0$. For the element $y_1 = Ae_1 + e_2$ with sufficiently large value A and appropriate transformation of basis we obtain that operator R_{y_1} in the Jordan form has in the first position the block J_{n-p}, i.e., the assumption $\alpha = 0$ contradicts the conditions of the lemma. $\square$

Lemma 3.4. *For n-dimensional graded p-filiform Leibniz algebra L $(n - p \geq 4)$ of the type II, the condition assume*

$$1 = r_1 = \cdots = r_k < r_{k+1}, \qquad 1 \leq k \leq p - 1.$$

Then in the algebra L, we have products

$$\begin{cases} [e_1, e_1] = 0, \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - p - 1, \\ [e_{n-p}, e_1] = f_1, \\ [f_j, e_1] = 0, & 1 \leq j \leq p, \\ [e_1, e_i] = \alpha e_{i+1}, & 2 \leq i \leq n - p - 1, \\ [e_1, e_{n-p}] = \alpha f_1, \\ [e_1, f_1] = 0, \\ [e_2, e_2] = \lambda e_3, \end{cases} \qquad (***)$$

where $\alpha(\alpha + 1) = 0$ and $\alpha\lambda = 0$.

Proof. Let algebra L satisfy the conditions of the lemma. Then, we have $e_2, f_i \in L_1$, $1 \leq i \leq k$, and

$$L_1 = \langle e_1, e_2, f_2, \ldots, f_k \rangle, \quad L_i \supseteq \langle e_{i+1} \rangle, \quad 2 \leq i \leq n - p - 1, \quad L_{n-p} \supseteq \langle f_1 \rangle.$$

Without loss of generality, one can assume that $[e_1, e_2] = \alpha e_3 + \beta f_{k+1}$ (because, if $[e_1, e_2] = \alpha e_3 + (*)f_{k+1} + (*)f_{k+2} + \cdots + (*)f_p$, then we can put $\beta f_{k+1}' = (*)f_{k+1} + (*)f_{k+2} + \cdots + (*)f_p$).

If $\beta \neq 0$, then for the element $e_1' = Ae_1 + Be_2$, we have

$$\begin{aligned} [e_1', e_1'] &= AB\beta f_{k+1} + B^2[e_2, e_2], \\ [e_2', e_1'] &= Ae_3 + B[e_2, e_2], \\ [e_3', e_1'] &= Ae_4 + B[e_3, e_2], \\ &\vdots \\ [e_{n-p}', e_1'] &= Af_1 + B[e_{n-p}, e_2], \\ [f_1', e_1'] &= 0, \\ &\vdots \\ [f_k', e_1'] &= B[f_k, e_2]. \end{aligned}$$

Choosing A sufficiently large and $B \neq 0$ fixed, we can conclude that the operator $R_{e_1'}$ has *rank* at least $n-p$, i.e., we have a contradiction with the condition $C(L) = (n-p, 1, \ldots, 1)$. Therefore, $\beta = 0$.

Repeating the above argument for the elements $Ae_1 + Bf_i$, $Be_1 + Af_i$, for $2 \leq i \leq k$, and $Be_1 + Ae_2$, we obtain

$$[e_1, f_j] = 0, \quad 1 \leq j \leq k, \quad [f_i, f_j] = 0, \quad 1 \leq i, j \leq k, \quad [e_2, e_2] = \lambda e_3,$$

for some λ.

From the equalities

$$[e_1, [e_i, e_1]] = [[e_1, e_i], e_1] \qquad \text{for } 2 \leq i \leq n-p, [e_1, [e_1, e_2]] = -[[e_1, e_2], e_1],$$

we obtain

$$\begin{aligned} &[e_1, e_i] = \alpha e_{i+1}, && 2 \leq i \leq n-p-1, \\ &[e_1, e_{n-p}] = \alpha f_1, && \\ &[e_1, f_1] = 0, && \end{aligned}$$

where $\alpha(\alpha + 1) = 0$.

The equality $\lambda\alpha = 0$ follows from $0 = [e_1, [e_2, e_2]] = \lambda[e_1, e_3] = \lambda\alpha e_4$ (in case of $n-p=3$ here instead e_4 will be f_1). $\square$

In the following theorem, we clarify situation in the case of type II.

Theorem 3.3. *Let L be a complex n-dimensional graded p-filiform Leibniz algebra of type II with $n-p \geq 3$. Then L is a Lie algebra.*

Proof. Due to Lemmas 3.3 and 3.4, we should consider the possible cases for α in (***). Let us suppose that $\alpha = -1$. Then $\lambda = 0$. The following expression may be proved in much the similar way as in Lemma 3.4:

$$[f_j, e_2] = \alpha_{j-1} e_3 + \sum_{s=1}^{j-1} \gamma_{s,j} f_{k+s}, \qquad 2 \leq j \leq k.$$

Further, we apply the analogous argument as in Lemma 3.4 for the element $Ae_1 + Be_2$ to obtain that $\gamma_{ij} = 0$, $1 \leq i < j \leq k$.

Summarizing,

$$\begin{aligned} &[e_i, e_1] = e_{i+1}, && 2 \leq i \leq n-p-1, \\ &[e_{n-p}, e_1] = f_1, && \\ &[e_1, e_i] = -e_{i+1}, && 2 \leq i \leq n-p-1, \\ &[e_1, e_{n-p}] = -f_1, && \\ &[f_j, e_2] = \alpha_{j-1} e_3 && 2 \leq j \leq k. \end{aligned}$$

Now, using the equality $[f_j, [e_i, e_1]] = [[f_j, e_i], e_1]$ with $2 \leq j \leq k$ and $2 \leq i \leq n - p - 1$, we obtain

$$\begin{array}{lll} [f_j, e_i] = \alpha_{j-1} e_{i+1}, & 2 \leq j \leq k, & 2 \leq i \leq n-p-1, \\ [f_j, e_{n-p}] = \alpha_{j-1} f_1, & 2 \leq j \leq k, & \\ [f_j, f_1] = 0, & 2 \leq j \leq k. & \end{array}$$

Introduce the notation $[e_2, f_2] = \eta_1 e_3 + \lambda_{1,2} f_{k+1}$. Then from $[[e_i, e_1], f_2] = [[e_i, f_2], e_1]$ with $2 \leq i \leq n - p - 1$, one gets

$$\begin{array}{ll} [e_i, f_2] = \eta_1 e_{i+1}, & 3 \leq i \leq n-p-1, \\ [e_{n-p}, f_2] = \eta_1 f_1, & \\ [f_1, f_2] = 0. & \end{array}$$

Condition $\alpha_1 = -\eta_1$ follows from $[e_1, [f_2, e_2]] = -[[e_1, e_2], f_2]$.

If we apply for the element $Ae_1 + Be_2 + Cf_2$ with sufficiently large value of A and $B, C \neq 0$ fixed, by the above argument, we obtain a contradiction with the condition $\lambda_{1,2} \neq 0$. Thus, $\lambda_{1,2} = 0$.

Repeating in a similar way, we can suppose that

$$[e_2, f_j] = \eta_{j-1} e_3, \qquad 3 \leq j \leq k$$

and by using the Leibniz identity we get that $\eta_{j-1} = -\alpha_{j-1}$ for $3 \leq j \leq k$. It is easy to obtain (by the Leibniz identity) the following products:

$$\begin{array}{lll} [e_i, f_j] = -\alpha_{j-1} e_{i+1}, & 2 \leq i \leq n-p-1, & 2 \leq j \leq k, \\ [e_{n-p}, f_j] = -\alpha_{j-1} f_1, & & 2 \leq j \leq k, \\ [f_1, f_j] = 0, & & 2 \leq j \leq k. \end{array}$$

Summarizing all products, we conclude that $L_2 = \langle e_3 \rangle$ and

$$\begin{array}{lll} [e_1, e_i] = -[e_i, e_1], & 1 \leq i \leq n-p, & \\ [e_2, e_i] = -[e_i, e_2], & 1 \leq i \leq n-p, & \\ [e_1, f_j] = [f_j, e_1] = 0, & 1 \leq j \leq p, & \\ [e_i, f_j] = -[f_j, e_i], & 1 \leq i \leq n-p, & 1 \leq j \leq k. \end{array}$$

From the chain of equalities

$$\begin{aligned} [e_i, e_j] &= [e_i, [e_{j-1}, e_1]] = [[e_i, e_{j-1}], e_1] - [[e_i, e_1], e_{j-1}] \\ &= -[e_1, [e_i, e_{j-1}]] + [[e_1, e_i], e_{j-1}] = -([[e_1, e_i], e_{j-1}] - [[e_1, e_{j-1}], e_i]) \\ &\quad + [[e_1, e_i], e_{j-1}] = [[e_1, e_{j-1}], e_i] = -[e_j, e_i], \end{aligned}$$

we deduce $[e_i, e_j] = -[e_j, e_i]$, $1 \leq i, j \leq n - p$.

Since $[e_{r_k+1}, f_j] = -\alpha_{j-1} e_{r_k+2}$ $(2 \leq j \leq k)$, then we may suppose that $f_{k+1} = [e_{r_k+1}, e_2] + \alpha e_{r_k+1}$.

Consider now the chain of equalities

$$
\begin{aligned}
[f_{k+1}, e_1] &= [[e_{r_k+1}, e_2] + \alpha e_{r_k+1}, e_1] = [[e_{r_k+1}, e_2], e_1] + \alpha[e_{r_k+1}, e_1] \\
&= [[e_{r_k+1}, e_1], e_2] - [e_{r_k+1}, [e_1, e_2]] + \alpha[e_{r_k+2}, e_1] = -[[e_1, e_{r_k+1}], e_2] \\
&\quad + [[e_1, e_2], e_{r_k+1}] - \alpha[e_1, e_{r_k+1}] = -[e_1, f_{k+1}].
\end{aligned}
$$

Similarly, one can prove $[f_{k+1}, e_2] = -[e_2, f_{k+1}]$.
The equations

$$
\begin{aligned}
[f_{k+1}, f_i] &= [[e_{r_k+1}, e_2] + \alpha e_{r_k+1}, f_i] = [[e_{r_k+1}, e_2], f_i] + \alpha[e_{r_k+1}, f_i] \\
&= [[e_{r_k+1}, f_i], e_2] - [e_{r_k+1}, [f_i, e_2]] + \alpha[e_{r_k+2}, f_i] \\
&= -[[f_i, e_{r_k+1}], e_2] + [[f_i, e_2], e_{r_k+1}] - \alpha[f_i, e_{r_k+1}] = -[f_i, f_{k+1}]
\end{aligned}
$$

give $[f_{k+1}, f_i] = -[f_i, f_{k+1}]$.

If $f_{k+j} \in L_{r_{k+j}}$ for $1 \leq j \leq p-k$, then we can suppose $f_{k+j} = [e_{r_{k+j}}, e_2] + \sum_{t \in L_{r_{k+j}-1}} a_t[f_t, e_2] + b_j e_{r_{k+j}+1}$ for some $a_t, b_j \in \mathbb{C}$. Using induction on j, we obtain

$$
\begin{aligned}
&[f_{k+j}, e_1] = -[e_1, f_{k+j}], \qquad 1 \leq j \leq p-k, \\
&[f_{k+j}, e_2] = -[e_2, f_{k+j}], \qquad 1 \leq j \leq p-k, \\
&[f_{k+j}, f_i] = -[f_i, f_{k+j}], \qquad 1 \leq j \leq p-k, \quad 1 \leq i \leq k.
\end{aligned}
$$

Thus, we prove $[x, y] = -[y, x]$ for every $x \in L$, $y \in L_1$. Using the above methods, we obtain that $[x, y] = -[y, x]$ for any $x, y \in L$, i.e., L is a Lie algebra.

Let us suppose $\alpha = 0$. Then from (***), we have $[e_1, e_j] = 0$ for $2 \leq j \leq n-p$, $[e_1, f_1] = 0$, and $[e_2, e_2] = \lambda e_3$. If we make the change $e_2' = e_2 - \lambda e_1$, we can assume $\lambda = 0$.

Applying the above argument for the element $Ae_1 + e_2$ with sufficiently large value of A, we obtain a contradiction with the supposition $\alpha = 0$. $\square$

The last theorem completes the classification of naturally graded p-filiform Leibniz algebras in an each dimension. In particular, the classification of naturally graded p-filiform non-Lie Leibniz algebras $n - p \geq 4$ is present in the list of Theorem 3.2. Moreover, it should be noted that despite complement of the set of Lie algebras in the set of Leibniz algebras forms a Zariski open set (it is well-known that an open set in Zariski topology is big set), the list of naturally graded p-filiform non-Lie Leibniz algebras $n - p \geq 4$ is simpler than in Lie algebras case.

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