Some cohomologically rigid solvable Leibniz algebras[1]

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Abstract. In this paper we describe solvable Leibniz algebras whose quotient algebra by one-dimensional ideal is a Lie algebra with rank equal to the length of the characteristic sequence of its nilpotent radical. We prove that such Leibniz algebra is unique and centerless. Also it is proved that the first and the second cohomology groups of the algebra with coefficients in itself is trivial.

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1. Introduction

Leibniz algebras are characterized as algebras whose the right multiplication operators are derivations, it is a generalization of Lie algebra, while for a Leibniz algebra to be a Lie algebra it suffices to add the condition that the operators of right and left multiplications alternate. Leibniz algebras have been introduced by Loday in [23] as algebras satisfying the (right) Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

During the last decades the theory of Leibniz algebras has been actively studied. Some (co)homology and deformation properties; results on various types of decompositions; structure of solvable and nilpotent Leibniz algebras; classifications of some classes of graded nilpotent Leibniz algebras were obtained in numerous papers devoted to Leibniz algebras, see, for example, [4, 7, 8, 11–13, 15, 16, 18, 20, 22, 24] and reference therein.

In fact, many results on Lie algebras have been extended to the Leibniz algebra case. For instance, an analogue of Levi's theorem for the case of Leibniz algebras asserts that Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie subalgebra [7]. Therefore, the description of finite-dimensional Leibniz algebras shifts to the study of solvable Leibniz algebras. Since the method of the reconstruction of solvable Lie algebras from their nilpotent radicals (see [25]) was extended to the Leibniz algebras [10], the main problem of the description of finite-dimensional Leibniz algebras consists of the study of nilpotent Leibniz algebras. Numerous works are devoted to the description of solvable Lie and Leibniz algebras with a given nilpotent radical (see [1, 5, 6, 9, 19, 26] and reference therein).

It is known that any Leibniz algebra law can be considered as a point of an affine algebraic variety defined by the polynomial equations coming from the Leibniz identity for a given basis. This way provides a description of the difficulties in classification problems referring to the classes of nilpotent and solvable Leibniz algebras. The orbits under the base change action of the general linear group correspond to the isomorphism classes of Leibniz algebras therefore, the classification problems (up to isomorphism) can be reduced to the classification of these orbits. An affine algebraic variety is a union of a finite number of

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irreducible components and the Zariski open orbits provide interesting classes of Leibniz algebras to be classified. The Leibniz algebras of this class are called rigid.

In the study of nilpotent Lie algebras a very useful tool is characteristic sequence, which a priori gives the multiplication on one basis element. Recently, in the paper [2] it was considered a finite-dimensional solvable Lie algebra $\mathfrak{r}_c$ whose nilpotent radical $\mathfrak{n}_c$ has the simplest structure with a given characteristic sequence $c = (n_1, n_2, \dots, n_k, 1)$. Using Hochschild – Serre factorization theorem the authors established that for the algebra $\mathfrak{r}_c$ low order cohomology groups with coefficient in itself are trivial.

In this paper we consider the family of nilpotent Leibniz algebras such that its corresponding Lie algebra is $\mathfrak{n}_c$. Further, solvable Leibniz algebras with such nilpotent radicals and $(k+1)$-dimensional complementary subspaces to the nilpotent radicals are described. Namely, we prove that such solvable Leibniz algebra is unique and centerless. For this Leibniz algebra the triviality of the first and the second cohomology groups with coefficient in itself is established as well.

2. PRELIMINARIES

Throughout the paper, all vector spaces and algebras considered are finite-dimensional over the field of complex numbers $\mathbb{C}$. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero.

In this section we give necessary definitions and results on solvable Leibniz algebras and its construction with a given nilpotent radical.

Definition 1. An algebra $(L, [\cdot, \cdot])$ is called a Leibniz algebra if it satisfies the property

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \text{ for all } x, y \in L,$$

which is called Leibniz identity.

The Leibniz identity is a generalization of the Jacobi identity since under the condition of anti-symmetricity of the product "$[\cdot\,,\,\cdot]$" this identity changes to the Jacobi identity. In fact, Leibniz algebras is characterized by the property that any right multiplication operator is a derivation.

For a Leibniz algebra L, a subspace generated by squares of its elements $\mathcal{I} = \text{span}\{[x, x] : x \in L\}$ is a two-sided ideal, and the quotient $\mathcal{G}_L = L/\mathcal{I}$ is a Lie algebra called corresponding Lie algebra (sometimes also called by liezation) of L.

For a given Leibniz algebra L we can define the following two-sided ideals

$$\text{Ann}_r(L) = \{x \in L \mid [y, x] = 0, \text{ for all } y \in L\},$$

$$\text{Center}(L) = \{x \in L \mid [x, y] = [y, x] = 0, \text{ for all } y \in L\}$$

called the *right annihilator* and the *center* of L, respectively.

Applying the Leibniz identity we obtain that for any two elements x, y of an algebra the elements $[x, x], [x, y] + [y, x]$ in $\text{Ann}_r(L)$.

The notion of a derivation for Leibniz algebras is defined in a usual way and the set of all derivations of L (denoted by $\mathfrak{Der} L$) forms a Lie algebra with respect to the commutator. Moreover, the operator of right multiplication on an element $x \in L$ (further denoted by $\mathcal{R}_x$) is a derivation, which is called *inner derivation*.

Definition 2. A Leibniz algebra L is called *complete* if $\text{Center}(L) = 0$ and all derivations of L are inner.

For a Leibniz algebra L we define the *lower central* and the *derived series* as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1, \qquad L^{[1]} = L,\ L^{[s+1]} = [L^{[s]}, L^{[s]}],\ s \geq 1,$$

respectively.

Definition 3. A Leibniz algebra L is called *nilpotent* (respectively, *solvable*), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

The maximal nilpotent ideal of a Leibniz algebra is said to be the *nilpotent radical* of the algebra.

Further we shall need the following result from [3]. It is an extension of the similar result for Lie algebras.

Theorem 4. *Let L be a finite-dimensional solvable Leibniz algebra over a field of characteristic zero. Then L is solvable if and only if L^2 is nilpotent algebra.*

An analogue of Mubarakzjanov's methods has been applied for solvable Leibniz algebras which shows the importance of the consideration of nilpotent Leibniz algebras and its nil-independent derivations [10].

Definition 5. Let $d_1, d_2, \dots, d_n$ be derivations of a Leibniz algebra L. The derivations $d_1, d_2, \dots, d_n$ are said to be nil-independent if $\alpha_1 d_1 + \alpha_2 d_2 + \ldots + \alpha_n d_n$ is not nilpotent for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$, which are not all zero.

In the paper of [21] it is proved the following theorem.

Theorem 6. *Let $R = N \oplus Q$ be a solvable Lie algebra such that $\dim Q = \dim N/N^2 = k$. Then R admits a basis $\{e_1, e_2, \dots, e_n, x_1, x_2, \dots, x_k\}$ such that the table of multiplication in R has the following form:*

$$\begin{cases} [e_i, e_j] = \sum\limits_{t=k+1}^{n} \gamma_{i,j}^t e_t, & 1 \le i, j \le n, \\ [e_i, x_i] = e_i, & 1 \le i \le k, \\ [e_i, x_j] = \alpha_{i,j} e_i, & k+1 \le i \le n, 1 \le j \le k, \end{cases}$$

where $\alpha_{i,j}$ is the number of entries of a generator basis element e_j involved in forming non generator basis element e_i.

For a nilpotent Leibniz algebra L and $x \in L \setminus L^2$ we consider the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ with respect to the lexicographical order of the dimensions Jordan's blocks of the operator $\mathcal{R}_x$.

Definition 7. The sequence $C(L) = \max\limits_{x \in L \setminus L^2} C(x)$ is called the *characteristic sequence* of the Leibniz algebra L.

In the paper [2] it is considered the cohomological properties of a solvable Lie algebra whose nilpotent radical has a given characteristic sequence $(n_1, n_2, \dots, n_k, 1)$ and complementary subspace to nilpotent radical has dimension equal to $k + 1$.

For characteristic sequence $(n_1, n_2, \dots, n_k, 1)$ we consider the model nilpotent Lie algebra $\mathfrak{n}_c$ given by its non-zero products:

$$\begin{array}{ll} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n_1, \\ [e_{n_1+\ldots+n_j+i}, e_1] = -[e_1, e_{n_1+\ldots+n_j+i}] = e_{n_1+\ldots+n_j+1+i}, & 2 \le i \le n_{j+1},\ 1 \le j \le k-1. \end{array}$$

Due to Theorem 6 a solvable Lie algebra with nilpotent radical $\mathfrak{n}_c$ and $(k+1)$-dimensional complementary subspace to $\mathfrak{n}_c$ is unique. For our convenience we present its table of multiplication in the following way:

$$\mathfrak{r}_c : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n_1, \\ [e_{n_1+\ldots+n_j+i}, e_1] = -[e_1, e_{n_1+\ldots+n_j+i}] = e_{n_1+\ldots+n_j+1+i}, & 2 \le i \le n_{j+1}, \\ [e_1, x_1] = -[x_1, e_1] = e_1, & \\ [e_i, x_1] = -[x_1, e_i] = (i-2)e_i, & 3 \le i \le n_1 + 1, \\ [e_{n_1+\ldots+n_j+i}, x_1] = -[x_1, e_{n_1+\ldots+n_j+i}] = (i-2)e_{n_1+\ldots+n_j+i} & 2 \le i \le n_{j+1}, \\ [e_i, x_2] = -[x_2, e_i] = e_i, & 2 \le i \le n_1 + 1, \\ [e_{n_1+\ldots+n_j+i}, x_{j+2}] = -[x_{j+2}, e_{n_1+\ldots+n_j+i}] = e_{n_1+\ldots+n_j+i}, & 2 \le i \le n_{j+1}. \end{cases}$$

where $1 \le j \le k-1$.

Here we present the main result of the paper [2].

Theorem 8. *For any characteristic sequence $(n_1, \dots, n_k, 1)$, the model nilpotent Lie algebra $\mathfrak{n}_c$ arises as the nilpotent radical of a solvable Lie algebra $\mathfrak{r}_c$ such that*

$$H^a(\mathfrak{r}_c, \mathfrak{r}_c) = 0, \quad 0 \le a \le 3.$$

2.1. **Cohomology of Leibniz algebras.**

We call a vector space M a *module over a Leibniz algebra* L if there are two bilinear maps:

$$[-,-]\colon L\times M\to M \qquad \text{and} \qquad [-,-]\colon M\times L\to M$$

satisfying the following three axioms

$$\begin{aligned}
[m,[x,y]] &= [[m,x],y]-[[m,y],x],\\
[x,[m,y]] &= [[x,m],y]-[[x,y],m],\\
[x,[y,m]] &= [[x,y],m]-[[x,m],y],
\end{aligned}$$

for any $m\in M$, $x,y\in L$.

For a Leibniz algebra L and module M over L we consider the spaces

$$\mathrm{CL}^0(L,M)=M,\quad \mathrm{CL}^n(L,M)=\mathrm{Hom}(L^{\otimes n},M),\ n>0.$$

Let $d^n:\mathrm{CL}^n(L,M)\to \mathrm{CL}^{n+1}(L,M)$ be an $\mathbb{C}$-homomorphism defined by

$$\begin{aligned}
(d^n\varphi)(x_1,\dots,x_{n+1}) := [x_1,\varphi(x_2,\dots,x_{n+1})] &+ \sum_{i=2}^{n+1}(-1)^i[\varphi(x_1,\dots,\widehat{x}_i,\dots,x_{n+1}),x_i]\\
&+\sum_{1\le i<j\le n+1}(-1)^{j+1}\varphi(x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\widehat{x}_j,\dots,x_{n+1}),
\end{aligned}$$

where $\varphi\in\mathrm{CL}^n(L,M)$ and $x_i\in L$. The property $d^{n+1}\circ d^n=0$ leads that the derivative operator $d=\sum\limits_{i\ge 0}d^i$ satisfies the property $d\circ d=0$. Therefore, the n-th cohomology group is well defined by

$$\mathrm{HL}^n(L,M):=\mathrm{ZL}^n(L,M)/\mathrm{BL}^n(L,M),$$

where the elements $\mathrm{ZL}^n(L,M) := \mathrm{Ker}\ d^{n+1}$ and $\mathrm{BL}^n(L,M) := \mathrm{Im}\ d^n$ are called *n-cocycles* and *n-coboundaries*, respectively.

In the case of $n=2$ we give explicit expressions for elements $\mathrm{ZL}^2(L,L)$ and $\mathrm{BL}^2(L,L)$. Namely, elements $\psi\in\mathrm{BL}^2(L,L)$ and $\varphi\in\mathrm{ZL}^2(L,L)$ are defined by:

(1) $$\psi(x,y)=[d(x),y]+[x,d(y)]-d([x,y]) \text{ for some linear map } d\in\mathrm{Hom}(L,L),$$

(2) $$[x,\varphi(y,z)]-[\varphi(x,y),z]+[\varphi(x,z),y]+\varphi(x,[y,z])-\varphi([x,y],z)+\varphi([x,z],y)=0.$$

In terms of cohomology groups the notion of completeness of a Leibniz algebra L means that it is centerless and $\mathrm{HL}^1(L,L)=0$.

Definition 9. A Leibniz algebra L is called cohomologically rigid if $\mathrm{HL}^2(L,L)=0$.

Remark 10. For a centerless Lie algebra G it is known that $\mathrm{H}^2(G,G)=\mathrm{HL}^2(G,G)$ (see Corollary 2 of [14]).

3. Main Part

Let us consider the following family of nilpotent Leibniz algebras $L(\alpha_i, \beta_j)$ with $1 \le i \le k+1, 1 \le j \le k$ with a given table of multiplications:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n_1, \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n_1, \\ [e_{n_1+\ldots+n_j+i}, e_1] = e_{n_1+\ldots+n_j+1+i}, & 2 \le i \le n_{j+1},\ 1 \le j \le k-1, \\ [e_1, e_{n_1+\ldots+n_j+i}] = -e_{n_1+\ldots+n_j+1+i}, & 3 \le i \le n_{j+1},\ 1 \le j \le k-1, \\ [e_1, e_1] = \alpha_1 h, & \\ [e_2, e_2] = \alpha_2 h, & \\ [e_{n_1+\ldots+n_i+2}, e_{n_1+\ldots+n_i+2}] = \alpha_{i+2} h, & 1 \le i \le k-1. \\ [e_1, e_2] = -e_3 + \beta_1 h, & \\ [e_1, e_{n_1+\ldots+n_i+2}] = -e_{n_1+\ldots+n_i+3} + \beta_{i+1} h, & 1 \le i \le k-1, \end{cases}$$

where $n_1 \ge n_2 \ge \ldots n_k \ge 1$ and at least one of the parameters α_i, β_j is non-zero.

One can assume that $\alpha_1 \neq 0$. Indeed, if $\alpha_1 = 0$, then taking the following change of the basis

$$e_1' = A_1 e_1 + A_2 e_2 + \sum_{i=1}^{k-1} B_i e_{n_1+\ldots+n_i+2}, \quad e_2' = e_2, \quad e_{i+1}' = [e_i', e_1'], \quad 2 \le i \le n_1, \quad h' = h,$$

$$e'_{n_1+\ldots+n_j+2} = e_{n_1+\ldots+n_j+2}, \quad e'_{n_1+\ldots+n_j+1+i} = [e'_{n_1+\ldots+n_j+i}, e_1'], \quad 2 \le i \le n_{j+1},\ 1 \le j \le k-1,$$

we have

$$[e_1', e_1'] = (A_2^2 \alpha_2 + \sum_{i=1}^{k-1} B_i^2 \alpha_{i+2} + A_1 A_2 \beta_1 + A_1 \sum_{i=1}^{k-1} B_i^2 \alpha_{i+2} \beta_{i+1}) h'.$$

Taking into account that at least one of the parameters α_i, β_j is non-zero, we always can chose values A_1, A_2, B_i such that

$$A_2^2 \alpha_2 + \sum_{i=1}^{k-1} B_i^2 \alpha_{i+2} + A_1 A_2 \beta_1 + A_1 \sum_{i=1}^{k-1} B_i^2 \alpha_{i+2} \beta_{i+1} \neq 0.$$

Therefore, we can conclude that parameter α_1 is non-zero. Now, scaling the basis element h we can assume that $\alpha_1 = 1$, i.e., $[e_1, e_1] = h$.

Thus, we consider the family of nilpotent Leibniz algebras $L(\alpha_i, \beta_i)$ with $1 \le i \le k$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n_1, \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n_1, \\ [e_{n_1+\ldots+n_j+i}, e_1] = e_{n_1+\ldots+n_j+1+i}, & 2 \le i \le n_{j+1},\ 1 \le j \le k-1, \\ [e_1, e_{n_1+\ldots+n_j+i}] = -e_{n_1+\ldots+n_j+1+i}, & 3 \le i \le n_{j+1},\ 1 \le j \le k-1, \\ [e_1, e_1] = h, & \\ [e_2, e_2] = \alpha_1 h, & \\ [e_{n_1+\ldots+n_i+2}, e_{n_1+\ldots+n_i+2}] = \alpha_{i+1} h, & 1 \le i \le k-1. \\ [e_1, e_2] = -e_3 + \beta_1 h, & \\ [e_1, e_{n_1+\ldots+n_i+2}] = -e_{n_1+\ldots+n_i+3} + \beta_{i+1} h, & 1 \le i \le k-1, \end{cases}$$

where $n_1 \ge n_2 \ge \ldots n_k \ge 1$.

3.1. **Particular case.**

In order to avoid routine calculations which involve many indexes we limit ourselves to the family $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ with the following table of multiplications:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n_1, \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n_1, \\ [f_i, e_1] = f_{i+1}, & 1 \le i \le n_2 - 1, \\ [e_1, f_i] = -f_{i+1}, & 2 \le i \le n_2 - 1, \\ [e_1, e_1] = h, & [e_2, e_2] = \alpha_2 h, \qquad [f_1, f_1] = \alpha_3 h, \\ [e_1, e_2] = -e_3 + \beta_1 h, & [e_1, f_1] = -f_2 + \beta_2 h. \end{cases}$$

Proposition 11. *Any derivation of the algebra $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ has the following matrix form:*

$$\mathbb{D} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \textit{where}$$

$$A = \sum_{j=1}^{n_1+1} \lambda_j e_{1,j} + \sum_{i=2}^{n_1+1} ((i-2)\lambda_1 + \gamma_2) e_{i,i} + \sum_{i=2}^{n_1} \sum_{j=i+1}^{n_1+1} \gamma_{j-i+2} e_{i,j}, \quad C = \sum_{i=1}^{n_2} \sum_{j=i+1}^{n_1+1} \theta_{j-i+1} e_{i,j},$$

$$B = \sum_{j=1}^{n_2} \mu_j e_{1,j} + \sum_{i=1}^{2} c_i e_{i,n_2+1} + (\lambda_2 \alpha_1) e_{3,n_2+1} + \sum_{i=2}^{n_2} \sum_{j=i-1}^{n_2} \delta_{j-i+2} e_{i,j},$$

$$D = \sum_{i=1}^{n_2} ((i-1)\lambda_1 + \nu_1) e_{i,i} + c_3 e_{1,n_2+1} + \sum_{i=2}^{n_2} \sum_{j=i+1}^{n_2} \nu_{j-i+1} e_{i,j} + (\mu_1 \alpha_2) e_{2,n_2+1} + m e_{n_2+1,n_2+1}$$

$m = (2\lambda_1 + \lambda_2\beta_1 + \mu_1\beta_2)$, $A \in M_{n_1+1,n_1+1}$, $B \in M_{n_1+1,n_2+1}$, $C \in M_{n_2+1,n_1+1}$, $D \in M_{n_2+1,n_2+1}$ *and matrix units $e_{i,j}$ and with the restrictions:*

$$\begin{cases} \alpha_1\theta_2 + \alpha_2\delta_1 = 0, \\ -2\lambda_2\alpha_1 + \lambda_1\beta_1 + \lambda_2\beta_1^2 + \mu_1\beta_1\beta_2 - \gamma_2\beta_1 - \delta_1\beta_2 = 0, \\ -2\mu_1\alpha_2 + \lambda_1\beta_2 + \lambda_2\beta_1\beta_2 + \mu_1\beta_2^2 - \theta_2\beta_1 - \nu_1\beta_2 = 0, \\ \alpha_1(2\lambda_1 + \lambda_2\beta_1 + \mu_1\beta_2 - 2\gamma_2) = 0, \\ \alpha_2(2\lambda_1 + \lambda_2\beta_1 + \mu_1\beta_2 - 2\nu_1) = 0. \end{cases}$$

Moreover, if $n_1 > n_2$, then $\theta_j = 0$ with $2 \le j \le n_1 - n_2 + 1$.

Proof. The proof is carried out by straightforward checking the derivation property and using the table of multiplications of the algebras $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$. □

Lemma 12. *Let d be a derivation of the algebra $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Then we have that coefficient $d(h)|_h$ is $\epsilon_1 + \epsilon_2$, where $\epsilon_k \in \{\nu_1, \lambda_1, \gamma_2\}$.*

Proof. Let us consider the following cases:

(1) $\alpha_2 \neq 0$. In this case, by applying the derivation conditions we have $2\lambda_1 + \lambda_2\beta_1 + \mu_1\beta_2 - 2\nu_1 = 0$ then $d(h) = 2\nu_1 h$.

(2) $\alpha_2 = 0$ and $\alpha_1 \neq 0$. Similar to the above case we have $d(h) = 2\gamma_2 h$.

(3) $\alpha_2 = 0$ and $\alpha_1 = 0$. We consider the following:

   (a) $\beta_2 \neq 0$. Making the following change of basis: $e_1' = e_1,\ e_i' = \beta_2 e_i - \beta_1 f_{i-1},\ 2 \le i \le n_2 + 1$, $e_i = \beta_2 e_i, n_2 + 2 \le i \le n_1 + 1$ and $f_i' = f_i,\ 1 \le i \le n_2$, we can suppose $\beta_1' = 0$ and by restrictions we have that $\mu_1\beta_2 = \nu_1 - \lambda_1$. Hence, $d(h) = (\lambda_1 + \nu_1)h$.

   (b) $\beta_2 = \beta_1 = 0$. Then $d(h) = 2\lambda_1 h$.

   (c) $\beta_2 = 0,\ \beta_1 \neq 0$. By restrictions we have that $\lambda_2\beta_1 = \gamma_2 - \lambda_1$. Therefore, $d(h) = (\lambda_1 + \gamma_2)h$.

□

Lemma 13. *The number of nil-independent derivations of the algebra $L(\alpha_1,\alpha_2,\beta_1,\beta_2)$ is equal to 4.*

Proof. We are going to prove that the matrix $\mathbb{D}$ is a nilpotent matrix if and only if $\lambda_1=\gamma_2=\nu_1=\delta_1\theta_2=0$. By Lemma 12, we have that $d(h)=(a_1\lambda_1+a_2\gamma_2+\alpha_3\nu_1)h.$

According Proposition 11 we have

$$\mathbb{D}=\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix} A_1+A_2 & B\\ C & D_1+D_2\end{pmatrix}, \text{ where}$$

$$A_1=diag\{\lambda_1,\gamma_2,\lambda_1+\gamma_2,\dots,(n_1-1)\lambda_1+\gamma_2\},$$
$$D_1=diag\{\nu_1,\lambda_1+\nu_1,2\lambda_1+\nu_1,\dots,(n_2-1)\lambda_1+\nu_1,a_1\lambda_1+a_2\gamma_2+\alpha_3\nu_1\}$$

are diagonal matrices, A_2,D_2,C are strictly upper triangular matrices and the matrix B is upper triangular matrix with non-zero diagonal under the main diagonal such that $A\in M_{n_1+1,n_1+1},\ B\in M_{n_1+1,n_2+1},$ $C\in M_{n_2+1,n_1+1}$ and $D\in M_{n_2+1,n_2+1}$.

Note that matrices $A_1A_2,\ A_2^2,\ D_1D_2,\ D_1^2$ are nilpotent, the matrices $C(A_1+A_2)$ and $(D_1+D_2)C$ have the same type pattern as C (that is if any entry of C is 0, the entry of $C(A_1+A_2)$ and the entry of $(D_1+D_2)C$ at the same position is zero, as well). Likewise, the matrix $(A_1+A_2)B$ and $B(D_1+D_2)$ has the same pattern as B.

It is easy to see that $BC=K_1+K_2$ with diagonal matrix $K_1=diag\{\underbrace{0,\delta_1\theta_2,\delta_1\theta_2,\dots,\delta_1\theta_2}_{n_2},0,\dots,0\}$ and strictly upper triangular matrix K_2. Similarly, $CB=Z_1+Z_2$ with diagonal matrix $Z_1=diag\{\delta_1\theta_2,\delta_1\theta_2,\dots,\delta_1\theta_2,0\}$ and strictly upper triangular matrix Z_2.

According to the above arguments we have the following formula:

$$\mathbb{D}^2=\begin{pmatrix} \widetilde{A}_1+\widetilde{A}_2 & \widetilde{B}\\ \widetilde{C} & \widetilde{D}_1+\widetilde{D}_2\end{pmatrix},$$

where $\widetilde{A}_2,\ \widetilde{D}_2-$ nilpotent matrices and the matrices $\widetilde{B}$ and $\widetilde{C}$ are the same type as B and C, respectively and $\widetilde{A}_1$ and $\widetilde{D}_1$ are the following diagonal matrices:

$$\widetilde{A}_1=diag\{\lambda_1^2,\gamma_2^2+\delta_1\theta_2,(\lambda_1+\gamma_2)^2+\delta_1\theta_2,\dots,((n_2-2)\lambda_1+\gamma_2)^2+\delta_1\theta_2,$$
$$((n_2-1)\lambda_1+\gamma_2)^2,\dots,((n_1-1)\lambda_1+\gamma_2)^2\},$$
$$\widetilde{D}_1=diag\{\nu_1^2+\delta_1\theta_2,(\lambda_1+\nu_1)^2+\delta_1\theta_2,(2\lambda_1+\nu_1)^2+\delta_1\theta_2,\dots,$$
$$\dots,((n_2-1)\lambda_1+\nu_1)^2+\delta_1\theta_2,(a_1\lambda_1+a_2\gamma_2+\alpha_3\nu_1)^2\}.$$

To continue iteration we conclude that in the main diagonal of the matrix $\mathbb{D}^k$ will be equal to zero if and only if $\lambda_1=\gamma_2=\nu_1=\delta_1\theta_2=0$. Thus, the nilpotency of the matrix $\mathbb{D}$ implies $\lambda_1=\gamma_2=\nu_1=\delta_1\theta_2=0$.

Let us assume now that $\lambda_1=\gamma_2=\nu_1=\delta_1\theta_2=0$. Then we obtain that matrices $\widetilde{A}_1+\widetilde{A}_2,\ \widetilde{D}_1+\widetilde{D}_2,\widetilde{C}$ are strictly upper triangular and the matrix $\widetilde{B}$ is upper triangular. Therefore, the matrix $\mathbb{D}^2$ is nilpotent and hence, $\mathbb{D}$ is nilpotent. $\square$

Let R be a solvable Leibniz algebra whose nilpotent radical is the algebras from $L(\alpha_1,\alpha_2,\beta_1,\beta_2)$. We denote by Q the complementary subspace to a nilpotent radical of R. Due to work [10] we have that dimension of Q is bounded by number of nil-independent derivations of $L(\alpha_1,\alpha_2,\beta_1,\beta_2)$.

Let us introduce denotations

$$d_1\in\mathfrak{Der}L(\alpha_1,\alpha_2,\beta_1,\beta_2) \text{ with } \lambda_1\neq0,\ \gamma_2=\nu_1=\delta_1\theta_2=0,$$

$$d_2\in\mathfrak{Der}L(\alpha_1,\alpha_2,\beta_1,\beta_2) \text{ with } \gamma_2\neq0,\ \lambda_1=\nu_1=\delta_1\theta_2=0,$$

$$d_3\in\mathfrak{Der}L(\alpha_1,\alpha_2,\beta_1,\beta_2) \text{ with } \nu_1\neq0,\ \lambda_1=\gamma_2=\delta_1\theta_2=0,$$

$$d_4\in\mathfrak{Der}L(\alpha_1,\alpha_2,\beta_1,\beta_2) \text{ with } \delta_1\theta_2\neq0,\ \lambda_1=\gamma_2=\nu_1=0.$$

Proposition 14. $\dim Q\le 3.$

Proof. Due to Lemma 13 we have that the number of nil-independent derivations of $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is equal to 4 and they are depends on parameters $\lambda_1, \gamma_2, \nu_1, \theta_2, \delta_1$. Let us assume that $\dim Q = 4$, that is, $Q = \{x_1, x_2, x_3, x_4\}$. Then

$$\mathcal{R}_{x_i|L(\alpha_1,\alpha_2,\beta_1,\beta_2)} = d_i, \ i = 1, \dots, 4.$$

By scaling of the basis elements $x_i, \ 1 \le i \le 4$ one can assume that $\lambda_1 = 1$ in d_1, $\gamma_2 = 1$ in d_2, $\nu_1 = 1$ in d_3, respectively.

Let us assume that $\theta_2 \neq 0$ (recall that this case is impossible when $n_1 > n_2$). Thanks to Theorem 4 we have $R^2 \subseteq L(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Applying this embedding in the following equalities:

$$(*)f_2 = [f_1, [x_2, x_4]] = [[f_1, x_2], x_4] - [[f_1, x_4], x_2] = -e_2 + L(\alpha_i, \beta_j)^2$$

we get a contradiction with the assumption that $\theta_2 \neq 0$. Thus, we obtain $\dim Q \le 3$. □

The following theorem describes solvable Leibniz algebras with nilpotent radical $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and maximal possible dimension of Q.

Theorem 15. *Solvable Leibniz algebra with nilpotent radical $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and three-dimensional complementary subspace is isomorphic to the algebra:*

$$R: \begin{cases} [e_1, e_1] = h, & [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n_1, \\ [h, x_1] = 2h, & [f_i, e_1] = -[e_1, f_i] = f_{i+1}, & 1 \le i \le n_2 - 1, \\ [e_1, x_1] = -[x_1, e_1] = e_1, & [e_i, x_1] = -[x_1, e_i] = (i-2)e_i, & 3 \le i \le n_1 + 1, \\ [f_i, x_1] = -[x_1, f_i] = (i-1)f_i, & 2 \le i \le n_2, & \\ [e_i, x_2] = -[x_2, e_i] = e_i, & 2 \le i \le n_1 + 1, & \\ [f_i, x_3] = -[x_3, f_i] = f_i, & 1 \le i \le n_2. & \end{cases}$$

Proof. Let $R = L(\alpha_1, \alpha_2, \beta_1, \beta_2) \oplus Q$ with $\{e_1, e_2, \dots, e_{n_1+1}, f_1 \dots, f_{n_2}, x_1, x_2, x_3\}$ such that

$$\mathcal{R}_{x_i|L(\alpha_1,\alpha_2,\beta_1,\beta_2)} = d_i, \ i = 1, 2, 3.$$

Due to Proposition 11 we have the products $[L(\alpha_1, \alpha_2, \beta_1, \beta_2), x_j], 1 \le j \le 3$.

From the table of multiplications of the algebra $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$ we derive that

$$e_i \notin \mathrm{Ann}_r(R) \ \text{ with } \ 1 \le i \le n_1, \quad f_j \notin \mathrm{Ann}_r(R) \ \text{ with } 1 \le j \le n_2 - 1.$$

Taking into account that for any $x, y \in R$ we have $[x, y] + [y, x] \in Ann_r(R)$ we conclude that

$$[x_1, e_i] + [e_i, x_1] = (*)e_{n_1+1} + (*)f_{n_2} + (*)h, \quad 1 \le i \le n_1,$$
$$[x_1, f_i] + [f_i, x_1] = (*)e_{n_1+1} + (*)f_{n_2} + (*)h, \quad 1 \le i \le n_2 - 1.$$

Consider

$$\begin{aligned} [x_1, e_{n_1+1}] &= [x_1, [e_{n_1}, e_1]] = [[x_1, e_{n_1}], e_1] - [[x_1, e_1], e_{n_1}] = \\ &= [-[e_{n_1}, x_1], e_1] - [-[e_1, x_1], e_{n_1}] = -(n_1 - 1)e_{n_1+1}, \\ [x_1, f_{n_2}] &= [x_1, [f_{n_2-1}, e_1]] = [[x_1, f_{n_2-1}], e_1] - [[x_1, e_1], f_{n_2-1}] = \\ &= [-[f_{n_2-1}, x_1], e_1] - [-[e_1, x_1], f_{n_2-1}] = \\ &- \sum_{k=n_2+2}^{n_1+1} \theta_{k-n_2+1,1} e_k - (n_2 - 1)f_{n_2}. \end{aligned}$$

This imply that $e_{n_1+1}, f_{n_2} \notin \mathrm{Ann}_r(R)$

We claim that $span\langle e_i, f_j \mid 1 \le i \le n_1 + 1, \ 1 \le j \le n_2 \rangle \cap \mathrm{Ann}_r(R) = \{0\}$. Indeed, let

$$z = a_1 e_1 + a_2 e_2 + \ldots + a_{n_1+1} e_{n_1+1} + b_1 f_1 + \ldots + b_{n_2} f_{n_2} + c_1 x_1 + c_2 x_2 + c_3 x_3 \in \mathrm{Ann}_r(R).$$

Then considering the products

$$0 = [x_1, z] = [x_2, z] = [x_3, z] = [e_1, z] = [e_2, z] = [f_1, z]$$

we derive $z = 0$.

Thus, we obtain $\mathrm{Ann}_r(R) = \langle h \rangle$ and

$$\begin{array}{lll} [x_j, e_i] = -[e_i, x_j] + (*)h, & 1 \leq i \leq n_1 + 1, & 1 \leq j \leq 3, \\ [x_j, f_i] = -[f_i, x_j] + (*)h, & 1 \leq i \leq n_2, & 1 \leq j \leq 3, \\ [x_i, x_j] = -[x_j, x_i] + (*)h, & 1 \leq i, j \leq 3. & \end{array}$$

It is easy to see that the quotient algebra $R/\mathrm{Ann}_r(R)$ is a particular case of the Lie algebra $\mathfrak{r}_c$. Namely, the quotient Lie algebra has nilpotent radical $\mathfrak{n}_c$ with characteristic sequence $(n_1, n_2, 1)$ and its table of multiplication has the following form:

$$\begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n_1, \\ [f_i, e_1] = -[e_1, f_i] = f_{i+1}, & 1 \leq i \leq n_2 - 1, \\ [e_1, x_1] = -[x_1, e_1] = e_1, & \\ [e_i, x_1] = -[x_1, e_i] = (i-2)e_i, & 3 \leq i \leq n_1 + 1, \\ [f_i, x_1] = -[x_1, f_i] = (i-1)f_i, & 2 \leq i \leq n_2, \\ [e_i, x_2] = -[x_2, e_i] = e_i, & 2 \leq i \leq n_1 + 1, \\ [f_i, x_3] = -[x_3, f_i] = f_i, & 1 \leq i \leq n_2. \end{cases}$$

If we now rise up to the initial algebra R, then we get the following table of multiplications (we omit the bracket of the family $L(\alpha_1, \alpha_2, \beta_1, \beta_2)$):

$$\begin{array}{ll} [e_1, x_j] = \delta_{1j} e_1 + a_j h, & [x_j, e_1] = -\delta_{1j} e_1 + \widetilde{a}_j h, \ 1 \leq j \leq 3 \\ [e_2, x_j] = \delta_{2j} e_2 + b_j h, & [x_j, e_2] = -\delta_{2j} e_2 + \widetilde{b}_j h, \ 1 \leq j \leq 3 \\ [e_3, x_j] = (1 - \delta_{3,j}) e_3 + c_j h, & 1 \leq j \leq 3 \\ [e_i, x_1] = (i-2)e_i, & 4 \leq i \leq n_1 + 1, \\ [e_i, x_2] = e_i, & 4 \leq i \leq n_1 + 1, \\ [f_1, x_j] = \lambda f_1 + d_j h, & [x_j, f_1] = -\lambda f_1 + \widetilde{d}_1 h, \\ [f_2, x_j] = (1 - \delta_{2j}) f_2 + g_j h, & \\ [f_i, x_1] = (i-1) f_i, & 3 \leq i \leq n_2, \\ [f_i, x_3] = f_i, & 3 \leq i \leq n_2, \\ [h, x_j] = m_j h, & 1 \leq j \leq 3 \\ [x_i, x_j] = \varphi_{i,j} h, & 1 \leq i, j \leq 3. \end{array}$$

with δ_{ij} the Kronecker symbol, $\lambda = 0$ if $j = 1, 2$ and $\lambda = 1$ if $j = 3$.

The Leibniz identity on the following triples imposes further constraints on the above family.

Leibniz identity		Constraint	Leibniz identity		Constraint
$\{e_1, e_1, x_1\}$,	$\Rightarrow$	$m_1 = 2$,	$\{e_1, e_2, x_1\}$,	$\Rightarrow$	$\beta_1 = 0$,
$\{e_1, e_1, x_i\}, \ 2 \leq i \leq 3$	$\Rightarrow$	$m_i = 0$,	$\{e_1, f_1, x_1\}$,	$\Rightarrow$	$\beta_2 = 0$,
$\{e_2, e_2, x_1\}$,	$\Rightarrow$	$\alpha_1 = 0$,	$\{e_2, e_1, x_i\}, \ 1 \leq i \leq 3$	$\Rightarrow$	$c_i = 0$,
$\{f_1, f_1, x_1\}$,	$\Rightarrow$	$\alpha_2 = 0$,	$\{f_1, e_1, x_i\}, \ 1 \leq i \leq 3$	$\Rightarrow$	$g_i = 0$.

At that time, the following change of basis

$$e_1' = e_1 - a_1 h, \quad e_2' = e_2 - \frac{b_1}{2} h, \quad f_1' = f_1 - \frac{d_1}{2} h, \quad x_i' = x_i - \frac{\varphi_{i,1}}{2} h, \ 1 \leq i \leq 3$$

allows to assume that $a_1 = b_1 = d_1 = \varphi_{i,1} = 0$ for $1 \leq i \leq 3$.

We again apply the Leibniz identity and we have the following:

Leibniz identity		Constraint
$\{x_i, e_1, x_1\},\ 1 \leq i \leq 3$	$\Rightarrow$	$\widetilde{a}_i = 0,\ 1 \leq i \leq 3,$
$\{x_i, e_2, x_1\},\ 1 \leq i \leq 3$	$\Rightarrow$	$\widetilde{b}_i = 0,\ 1 \leq i \leq 3,$
$\{f_1, x_i, x_1\},\ 2 \leq i \leq 3$	$\Rightarrow$	$d_i = 0,\ 2 \leq i \leq 3,$
$\{x_i, f_1, x_1\},\ 2 \leq i \leq 3$	$\Rightarrow$	$\widetilde{d}_i = 0,\ 1 \leq i \leq 3,$
$\{e_1, x_i, x_1\},\ 2 \leq i \leq 3$	$\Rightarrow$	$a_i = 0,\ 2 \leq i \leq 3,$
$\{e_2, x_i, x_1\},\ 2 \leq i \leq 3$	$\Rightarrow$	$b_i = 0,\ 2 \leq i \leq 3,$
$\{x_i, x_j, x_1\},\ 1 \leq i \leq 3,\ 2 \leq j \leq 3$	$\Rightarrow$	$\varphi_{i,j} = 0,\ 1 \leq i \leq 3,\ 2 \leq j \leq 3.$

Finally, if we consider the equalities $[x_j, e_i] = [x_j, [e_{i-1}, e_1]]$ with $3 \leq i \leq n_1 + 1$ and $[x_j, f_i] = [x_j, [f_{i-1}, e_1]]$ with $2 \leq i \leq n_2,\ 1 \leq j \leq 3$ we obtain the algebra of the theorem statement.

□

The next result establish the completeness of the algebra R.

Theorem 16. *The solvable Leibniz algebra R is complete.*

Proof. Centerless of the algebra R is immediately follows from the table of multiplications in Theorem 15. Note that $\langle h\rangle$ forms an ideal of R.

The quotient algebra $R/\langle h\rangle$ is the algebra $\mathfrak{r}_c$, which is complete due to Theorem 8. Applying this result in the following equalities by modulo of an ideal $\langle h\rangle$:

$$\begin{array}{rcl}
d(e_1) & = & d([e_1, x_1]) = [d(e_1), x_1] + [e_1, d(x_1)] \not\equiv 0 \Rightarrow d(e_1) \equiv \mathcal{R}_{\alpha e_1}(e_1) = \alpha h,\ \alpha \in \mathbb{C},\\
0 & = & d([e_2, x_1]) = [d(e_2), x_1] + [e_2, d(x_1)] \equiv [d(e_2), x_1] \Rightarrow d(e_2) \equiv 0,\\
d(e_{i+1}) & = & d([e_i, e_1]) = [d(e_i), e_1] + [e_i, d(e_1)] \equiv 0 \Rightarrow d(e_{i+1}) \equiv 0,\ 2 \leq i \leq n_1,\\
0 & = & d([f_1, x_1]) = [d(f_1), x_1] + [f_1, d(x_1)] \equiv [d(f_1), x_1] \Rightarrow d(f_1) \equiv 0,\\
d(f_{i+1}) & = & d([f_i, e_1]) = [d(f_i), e_1] + [f_i, d(e_1)] \equiv 0 \Rightarrow d(f_{i+1}) \equiv 0,\ 1 \leq i \leq n_2 - 1,\\
0 & = & d([x_i, x_1]) = [d(x_i), x_1] + [x_i, d(x_1)] \equiv [d(x_i), x_1] \Rightarrow d(x_i) \equiv 0, 1 \leq i \leq 3.
\end{array}$$

and in the chain of equalities

$$\begin{array}{rl}
2[d(e_1), e_1] + 2[e_1, d(e_1)] & = 2d([e_1, e_1]) = 2d(h) = d([h, x_1]) = [d(h), x_1] + [h, d(x_1)]\\
& \Rightarrow d(h) = \mathcal{R}_{\beta x_1}(h) = 2\beta h,\ \beta \in \mathbb{C}
\end{array}$$

we conclude that any derivation of R is inner.

□

Now we prove the triviality of the second group of cohomology for the algebra R with coefficient itself (that is $\mathrm{HL}^2(R, R) = 0$). Since $J := \langle h\rangle$ is an ideal of R and quotient algebra R/J is the Lie algebra $\mathfrak{r}_c$, we get a decomposition $R = \mathfrak{r}_c \oplus J$ as the direct sum of the vector spaces (here we identify the space of the quotient space $\mathfrak{r}_c$ and its preimage under the natural homomorphism). Hence, for any $x, y \in R$ and $\varphi(x, y) \in \mathrm{ZL}^2(R, R)$ one has

$$[x, y] = [x, y]_{\mathfrak{r}_c} + [x, y]_J, \quad \varphi(x, y) = \varphi(x, y)_{\mathfrak{r}_c} + \varphi(x, y)_J,$$

with $[x, y]_{\mathfrak{r}_c} \in \mathfrak{r}_c,\ [x, y]_J \in J$ and $\varphi(x, y)_{\mathfrak{r}_c} \in \mathfrak{r}_c,\ \varphi(x, y)_J \in J$.

For an arbitrary elements $x, y, z \in \mathfrak{r}_c$ and $\varphi \in \mathrm{ZL}^2(R, R)$ using (1) we consider the chain of equalities:

$$\begin{array}{rl}
0 & = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y) =\\
& = [x, \varphi(y, z)_{\mathfrak{r}_c}]_{\mathfrak{r}_c} - [\varphi(x, y)_{\mathfrak{r}_c}, z]_{\mathfrak{r}_c} + [\varphi(x, z)_{\mathfrak{r}_c}, y]_{\mathfrak{r}_c} + \varphi(x, [y, z]_{\mathfrak{r}_c})_{\mathfrak{r}_c} - \varphi([x, y]_{\mathfrak{r}_c}, z)_{\mathfrak{r}_c} +\\
& + \varphi([x, z]_{\mathfrak{r}_c}, y)_{\mathfrak{r}_c} + [x, \varphi(y, z)_{\mathfrak{r}_c}]_J + [x, \varphi(y, z)_J]_J - [\varphi(x, y)_{\mathfrak{r}_c}, z]_J - [\varphi(x, y)_J, z]_J +\\
& + [\varphi(x, z)_{\mathfrak{r}_c}, y]_J + [\varphi(x, z)_J, y]_J + \varphi(x, [y, z]_{\mathfrak{r}_c})_J + \varphi(x, [y, z]_J)_{\mathfrak{r}_c} + \varphi(x, [y, z]_J)_J -\\
& - \varphi([x, y]_{\mathfrak{r}_c}, z)_J - \varphi([x, y]_J, z)_{\mathfrak{r}_c} - \varphi([x, y]_J, z)_J + \varphi([x, z]_{\mathfrak{r}_c}, y)_J + \varphi([x, z]_J, y)_{\mathfrak{r}_c} +\\
& + \varphi([x, z]_J, y)_J.
\end{array}$$

From this we obtain

$$(3)\quad \begin{cases} [x,\varphi(y,z)_{\mathfrak{r}_c}]_{\mathfrak{r}_c} - [\varphi(x,y)_{\mathfrak{r}_c},z]_{\mathfrak{r}_c} + [\varphi(x,z)_{\mathfrak{r}_c},y]_{\mathfrak{r}_c} + \\ \varphi(x,[y,z]_{\mathfrak{r}_c})_{\mathfrak{r}_c} - \varphi([x,y]_{\mathfrak{r}_c},z)_{\mathfrak{r}_c} + \varphi([x,z]_{\mathfrak{r}_c},y)_{\mathfrak{r}_c} + \\ \varphi(x,[y,z]_J)_{\mathfrak{r}_c} - \varphi([x,y]_J,z)_{\mathfrak{r}_c} + \varphi([x,z]_J,y)_{\mathfrak{r}_c} = 0, \end{cases}$$

$$(4)\quad \begin{cases} [\varphi(x,z)_{\mathfrak{r}_c},y]_J + [x,\varphi(y,z)_J]_J + [x,\varphi(y,z)_{\mathfrak{r}_c}]_J + [\varphi(x,z)_J,y]_J + \\ \varphi(x,[y,z]_{\mathfrak{r}_c})_J - [\varphi(x,y)_{\mathfrak{r}_c},z]_J - [\varphi(x,y)_J,z]_J + \varphi(x,[y,z]_J)_J - \\ \varphi([x,y]_{\mathfrak{r}_c},z)_J - \varphi([x,y]_J,z)_J + \varphi([x,z]_{\mathfrak{r}_c},y)_J + \varphi([x,z]_J,y)_J = 0. \end{cases}$$

Note that the first six terms of the equality (3) define a Leibniz 2-cocycle for the quotient Lie algebra $\mathfrak{r}_c$. Therefore, Leibniz 2-cocycles of the Lie algebra $\mathfrak{r}_c$ with its trivial extensions on domains $J\otimes R$, $R\otimes J$, $J\otimes J$ are included into $\mathrm{ZL}^2(R,R)$ (the same is true for 2-coboundaries of the algebra $\mathfrak{r}_c$). Moreover, the last three terms in (3) appear only for the triples $\{e_1,e_1,a\},\ \{e_1,a,e_1\},\ \{a,e_1,e_1\}$ with $a\in\mathfrak{r}_c$.

Proposition 17. *The following* 2*-cochains together with a basis of* $\mathrm{ZL}^2(\mathfrak{r}_c,\mathfrak{r}_c)$*:*

$$\varphi_1(e_1,e_1)=h,\quad \varphi_2(x_1,e_1)=\varphi_2(e_1,x_1)=h,\quad \varphi_3(x_1,x_1)=h,\quad \varphi_4(x_3,x_1)=h,$$

$$\{\varphi_5(f_1,x_1)=-2h,\quad \varphi_5(f_1,x_3)=-\varphi_{11}(x_3,f_1)=h,$$

$$\begin{cases} \varphi_6(x_2,e_2)=-\varphi_6(e_2,x_2)=h, \\ \varphi_6(e_2,x_1)=h, \end{cases} \qquad \begin{cases} \varphi_7(f_1,x_3)=-\varphi_7(x_3,f_1)=h, \\ \varphi_7(f_1,x_1)=(-2)h, \end{cases}$$

$$\begin{cases} \varphi_8(e_1,e_1)=\frac{1}{2}x_3, \\ \varphi_8(h,x_1)=x_3, \\ \varphi_8(h,f_i)=-\varphi_8(f_i,h)=\frac{1}{2}f_i, \\ 1\le i\le n_2, \end{cases} \qquad \begin{cases} \varphi_9(e_1,e_1)=\frac{1}{2}x_2, \\ \varphi_9(h,x_1)=x_2, \\ \varphi_9(h,e_i)=-\varphi_9(e_i,h)=\frac{1}{2}e_i, \\ 2\le i\le n_1+1, \end{cases}$$

$$\begin{cases} \varphi_{10}(h,h)=-h, \\ \varphi_{10}(h,x_1)=x_1, \\ \varphi_{10}(e_1,e_1)=\frac{1}{2}x_1, \\ \varphi_{10}(h,e_1)=-\varphi_{10}(e_1,h)=\frac{1}{2}e_1, \\ \varphi_{10}(h,e_i)=-\varphi_{10}(e_i,h)=\frac{i-2}{2}e_i, \\ \varphi_{10}(h,f_j)=-\varphi_{10}(f_j,h)=\frac{i-1}{2}f_j, \\ 3\le i\le n_1+1,\ 2\le j\le n_2, \end{cases} \qquad \begin{cases} \varphi_{11}(e_1,e_1)=e_1, \\ \varphi_{11}(e_1,h)=-\varphi_{11}(h,e_1)=h, \\ \varphi_{11}(h,x_1)=\varphi_{11}(x_1,h)=e_1, \\ \varphi_{11}(h,e_i)=-\varphi_{11}(e_i,h)=e_{i+1}, \\ \varphi_{11}(h,f_j)=-\varphi_{11}(f_j,h)=f_{j+1}, \\ 2\le i\le n_1,\ 1\le j\le n_2-1, \end{cases}$$

$$\begin{cases} \varphi^{12}_j(e_1,e_1)=e_j, \\ \varphi^{12}_j(e_1,h)=-\varphi^{12}_j(h,e_1)=e_{j+1}, \\ \varphi^{12}_j(x_1,h)=(j-2)e_j, \\ \varphi^{12}_j(h,x_1)=-(j-4)e_j, \\ \varphi^{12}_j(x_2,h)=-\varphi^{12}_j(h,x_2)=e_j, \\ 2\le j\le n_1+1, \end{cases} \qquad \begin{cases} \varphi^{13}_j(e_1,e_1)=f_j, \\ \varphi^{13}_j(e_1,h)=-\varphi^{13}_j(h,e_1)=f_{j+1}, \\ \varphi^{13}_j(x_1,h)=(j-1)f_j, \\ \varphi^{13}_j(h,x_1)=(3-j)f_j, \\ \varphi^{13}_j(x_3,h)=-\varphi^{13}_j(h,x_3)=f_j, \\ 1\le j\le n_2, \end{cases}$$

$$\begin{cases} \varphi^{14}_j(e_1,e_j)=-\varphi^{14}_j(e_j,e_1)=h, \\ \varphi^{14}_j(x_1,e_{j+1})=(j-1)h, \\ \varphi^{14}_j(e_{j+1},x_1)=(3-j)h, \\ \varphi^{14}_j(x_2,e_{j+1})=-\varphi^{14}_j(e_{j+1},x_2)=h, \\ 3\le j\le n_1, \end{cases} \qquad \begin{cases} \varphi^{15}_j(e_1,f_j)=-\varphi^{15}_j(f_j,e_1)=h, \\ \varphi^{15}_j(x_1,f_{j+1})=\varphi^{15}_j(f_{j+1},x_1)=jh, \\ \varphi^{15}_j(x_3,f_{j+1})=-\varphi^{15}_j(f_{j+1},x_3)=h, \\ 1\le j\le n_2-1, \end{cases}$$

form a basis of spaces $\mathrm{ZL}^2(R,R)$ *and* $\mathrm{BL}^2(R,R)$.

Proof. The proof of this proposition is carried out by straightforward calculations of (1) and (2) by using result of Theorem 8. In fact, due to Remark 10 and centerlessness of the Lie algebra $\mathfrak{r}_c$ we conclude that

$\mathrm{H}^2(\mathfrak{r}_c, \mathfrak{r}_c) = \mathrm{HL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$, that is, $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c) = \mathrm{BL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$. Taking into account that $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$ is isomorphically embedded into $\mathrm{ZL}^2(R, R)$ (respectively, $\mathrm{BL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$ is isomorphically embedded into $\mathrm{BL}^2(R, R)$) we need to find a basis of complementary subspaces to $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$ (respectively, to $\mathrm{BL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$).

Further, we consider the equalities $(d^2\varphi)(x, y, z) = 0$ for the following cases:

$$\begin{array}{llll} x, y, z \in J, & x \in \mathfrak{r}_c, y, z \in J, & x, z \in J, y \in \mathfrak{r}_c, & x, y \in J, z \in \mathfrak{r}_c, \\ x, y \in \mathfrak{r}_c, z \in J, & x, z \in \mathfrak{r}_c, y \in J & x \in J, y, z \in \mathfrak{r}_c, & \end{array}$$

from where we get the relations similar to the equations (3) and (4). In addition, calculations of (3) for the triples $\{e_1, e_1, a\},\ \{e_1, a, e_1\},\ \{a, e_1, e_1\}$ with $a \in \mathfrak{r}_c$ and (4) for $x, y, z \in \mathfrak{r}_c$ give us some additional relations for complementary subspace to $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$.

Finally, combining all restrictions on 2-cocycles and identifying the basis of complementary subspace to $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$ in $\mathrm{ZL}^2(R, R)$ we get the required basis of $\mathrm{ZL}^2(R, R)$.

Applying the same arguments for 2-coboundaries we complete the proof of theorem. □

Remark 18. In the above proposition we simplified the calculations using the results for the quotient Lie algebra $\mathfrak{r}_c$. In fact, we exclude calculation of equalities (4) for the triples $x, y, z \in \mathfrak{r}_c$ except $\{e_1, e_1, a\},\ \{e_1, a, e_1\},\ \{a, e_1, e_1\}$ with $a \in \mathfrak{r}_c$. Thus, instead of $(\dim \mathfrak{r}_c)^3$ triples we calculated just $3 \dim \mathfrak{r}_c$ triples in (4).

As a consequence of Proposition 17 we get the following main result.

Theorem 19. *The solvable Leibniz algebra R is a cohomologically rigid algebra.*

4. General case

In this section we present results similar to obtained in particular case for solvable Leibniz algebras with nilpotent radical $L(\alpha_i, \beta_i),\ 1 \leq i \leq k$ and $(k+1)$-dimensional complementary subspace.

Taking into account that the general case is analogous to a special case we omit routine calculations using indexes n_i and induction in the proofs of results below, we just give short sketch their proofs.

The sketch consists of the following steps:

(1) Firstly, we compute the space $\mathfrak{Der}(L(\alpha_i, \beta_i))$ with $1 \leq i \leq k$. Further, we indicate $(k+1)$-pieces nil-independent derivations, which are depends on only non-zero parameters in the diagonal of the general matrix form of derivations.

(2) Secondly, we construct the solvable Leibniz algebra $\mathbf{R} = L(\alpha_i, \beta_i) \oplus Q$ with $Q = \langle x_1, \dots, x_{k+1} \rangle$ such that $\mathcal{R}_{x_s|_{L(\alpha_i, \beta_i)}} = d_s$, where $d_s,\ 1 \leq s \leq k+1$ are the nil-independent derivations indicated in the first step. Next, applying the Leibniz identity, the appropriate basis transformations and the mathematical induction we obtain the statement of Theorem 20.

(3) In order to prove the completeness of the solvable Leibniz algebra $\mathbf{R}$ (the first assertion of Theorem 21) we just need to verify the table of multiplications of $\mathbf{R}$ obtained in the second step and using the fact that any derivation of the quotient Lie algebra $\mathfrak{r}_c = \mathbf{R}/\langle h \rangle$ is inner together with arguments applied in the proof of the particular case (see Theorem 16) allow us to prove the completeness of the algebra $\mathbf{R}$.

(4) Finally, in the study of the second cohomology group of the algebra $\mathbf{R}$ we also use the triviality of the second group of cohomologies for the quotient algebra $\mathfrak{r}_c$, that is, we use the equality $\mathrm{Z}^2(\mathfrak{r}_c, \mathfrak{r}_c) = \mathrm{B}^2(\mathfrak{r}_c, \mathfrak{r}_c)$. By arguments applied in before Proposition 17 and due to Remark 10 we conclude

$$\mathrm{Z}^2(\mathfrak{r}_c, \mathfrak{r}_c) = \mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c) \subseteq \mathrm{ZL}^2(\mathbf{R}, \mathbf{R}), \quad \mathrm{B}^2(\mathfrak{r}_c, \mathfrak{r}_c) = \mathrm{BL}^2(\mathfrak{r}_c, \mathfrak{r}_c) \subseteq \mathrm{BL}^2(\mathbf{R}, \mathbf{R})$$

we only need to compute the dimensions of complementary subspaces to $\mathrm{ZL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$ (respectively, to $\mathrm{BL}^2(\mathfrak{r}_c, \mathfrak{r}_c)$) in $\mathrm{ZL}^2(\mathbf{R}, \mathbf{R})$ (respectively, in $\mathrm{BL}^2(\mathbf{R}, \mathbf{R})$). Thus, the proof of triviality of the second cohomology group for the algebra $\mathbf{R}$ with coefficient itself is completed by computations of dimensions of the mentioned complementary subspaces.

Theorem 20. *Solvable Leibniz algebra with nilpotent radical* $L(\alpha_i, \beta_i),\ 1 \le i \le k$ *and* $(k+1)$*-dimensional complementary subspace is isomorphic to the algebra:*

$$\mathbf{R}: \begin{cases} [e_1, e_1] = h, \quad [h, x_1] = 2h, & \\ [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n_1, \\ [e_{n_1+\ldots+n_j+i}, e_1] = -[e_1, e_{n_1+\ldots+n_j+i}] = e_{n_1+\ldots+n_j+1+i}, & 2 \le i \le n_{j+1}, \\ [e_1, x_1] = -[x_1, e_1] = e_1, & \\ [e_i, x_1] = -[x_1, e_i] = (i-2)e_i, & 3 \le i \le n_1+1, \\ [e_{n_1+\ldots+n_j+i}, x_1] = -[x_1, e_{n_1+\ldots+n_j+i}] = (i-2)e_{n_1+\ldots+n_j+i} & 2 \le i \le n_{j+1}, \\ [e_i, x_2] = -[x_2, e_i] = e_i, & 2 \le i \le n_1+1, \\ [e_{n_1+\ldots+n_j+i}, x_{j+2}] = -[x_{j+2}, e_{n_1+\ldots+n_j+i}] = e_{n_1+\ldots+n_j+i}, & 2 \le i \le n_{j+1}. \end{cases}$$

where $1 \le j \le k-1$.

Theorem 21. *The solvable Leibniz algebra* $\mathbf{R}$ *is complete and its second group of cohomologies in coefficient itself is trivial.*

From the results of the paper [4] we obtain rigidity of the algebra $\mathbf{R}$.

Corollary 22. *The solvable Leibniz algebra* $\mathbf{R}$ *is rigid.*

Remark 23. Note that the structure of the rigid algebra $\mathbf{R}$ depends on the given decreasing sequence $(n_1, n_2, \ldots, n_k)$. Set $p(n)$ the number of such sequences, that is, $p(x)$ is the number of integer solutions of the equation $n_1 + n_2 + \ldots + n_k = n$ with $n_1 \ge n_2 \ge \ldots \ge n_k \ge 0$. The asymptotic value of $p(n)$, given in [17] by the expression $p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$, (where $a(n) \approx b(n)$ means that $\lim\limits_{n\to\infty} \frac{a(n)}{b(n)} = 1$) get the existence of at least $p(n)$ irreducible components of the variety of Leibniz algebras of dimension $n+k+3$.

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