New advances concerning critical sets based on Latin square autotopisms

Raúl M. Falcón*

Department of Applied Maths I.
Universidad de Sevilla.
rafalgan@us.es

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

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* **Joint work with Laura Johnson (University of St. Andrews) and Stephanie Perkins (University of South Wales).**

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1. Preliminaries.
2. Critical sets based on Latin square autotopisms.
3. Open questions.

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Ruth **Moufang** (1935)

A **quasigroup** is a pair $Q = (S, \cdot)$ formed by

- a finite set S
- a product $\cdot$

such that both equations

$$a \cdot x = b \text{ and } y \cdot a = b$$

have unique solutions $x, y \in S$, for all $a, b \in S$.

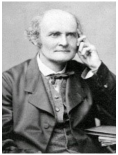
- The multiplication table of Q is a **Latin square**.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

$\in \mathrm{LS}_4$

- Associative quasigroup = **Group**.

Latin square.



	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	β	1	ϵ	γ	δ
β	β	1	α	δ	ϵ	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	ϵ	γ	β	1	α
ϵ	ϵ	γ	δ	α	β	1

Arthur **Cayley** (1854)

108 L. EULER RECHERCHES

directrices. Enfin le premier de ces trois espèces de quarrés, en changeant l'ordre des bandes horizontales fournit le quarré très remarquable suivant:

dans lequel non seulement les bandes verticales et horizontales renferment les différens lettres grecs et latins mais où même les diagonales et leurs parallèles complettées comme: 3¹ 1⁴ 4¹ 2¹ 5¹ satisfont aux conditions préscrites.

Cas de n=7.

§. 24. Le cas de n=7 nous fournit le quarré latin à simple marche de 49 cases suivant:

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

où la consideration des directrices croissantes en progression arithmetique (§. 24.)

SUR LES QUARRES MAGIQUES. 109

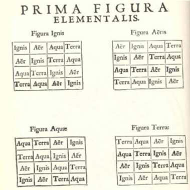
(§. 24.) nous fournit d'abord les formules directrices suivantes pour l'exposant 1:

dont la première croit de 2; la seconde de 3; la troisième de 4; la quatrième de 5 et la cinquième de 6. Mais il ne faut pas penser, que ce sont là toutes les directrices pour l'exposant 1; puisqu'en examinant le quarré plus soigneusement on trouve encore les 14 suivantes:

§. 25.

Leonhard **Euler** (1767)

Choe **Seok-Jeong** (1700)



PRIMA FIGURA ELEMENTALIS.

Figura Ignis

Ignis	Aër	Aqua	Terra
Aër	Ignis	Terra	Aqua
Aqua	Terra	Ignis	Aër
Terra	Aqua	Aër	Ignis

Figura Aëris

Aër	Ignis	Aqua	Terra
Ignis	Aër	Terra	Aqua
Aqua	Terra	Aër	Ignis
Terra	Aqua	Ignis	Aër

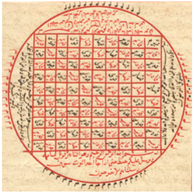
Figura Aquæ

Aqua	Terra	Aër	Ignis
Terra	Aqua	Ignis	Aër
Aër	Ignis	Aqua	Terra
Ignis	Aër	Terra	Aqua

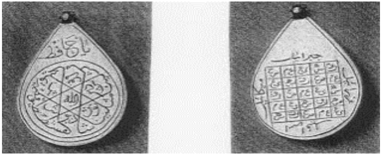
Figura Terræ

Terra	Aqua	Aër	Ignis
Aqua	Terra	Ignis	Aër
Aër	Ignis	Terra	Aqua
Ignis	Aër	Aqua	Terra

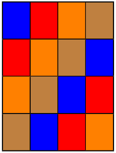
Ramón **Llull** (1283)



Ahmad **al-Buni** (1225)



Earrings founded in Damascus (1000)



$\equiv$

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Partial Latin square.

$L = (l_{ij}) \equiv$

1	2	3	·
·	3	·	·
3	·	1	4
·	1	·	3

Entry set: $\mathrm{Ent}(L) := \{(\text{row, column, symbol})\} = \{(i, j, l_{ij})\}$.

$$\mathrm{Ent}(L) = \{(1,1,1), (1,2,2), (1,3,3) \\ (2,2,3), \\ (3,1,3), (3,3,1), (3,4,4), \\ (4,2,1), (4,4,3)\}.$$

Critical set.

1	·	·	4
·	1	4	·
3	·	2	1
4	3	1	·

uniquely completable to

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

- **Critical set**: Uniquely completable PLS such that any proper subset of entries is not uniquely completable.

1	·	·	4
·	1	·	·
3	·	2	·
·	·	·	·

→

1	·	·	4
·	1	·	·
3	4	2	1
·	·	·	·

→

1	·	·	4
·	1	4	·
3	4	2	1
4	·	1	·

→

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

Latin square isotopism.

$S_n \equiv$ Symmetric group on $[n] := \{1, \ldots, n\}$.

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$S_n \times S_n \times S_n \equiv$ **Isotopism group**.

Latin square isotopism.

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$S_n \times S_n \times S_n \equiv$ **Isotopism group**.

- $\Theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$ and $L \in \mathrm{LS}_n$.

$$\mathrm{Ent}(L^{\Theta}) = \{(\alpha(i), \beta(j), \gamma(k) \mid (i, j, k) \in \mathrm{Ent}(L)\}.$$

Row-permutations (α), column-permutations (β), symbol-permutations (γ).

$S_n \equiv$ Symmetric group on $[n] := \{1, \ldots, n\}$.

$S_n \times S_n \times S_n \equiv$ **Isotopism group**.

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Row-permutations (α), column-permutations (β), symbol-permutations (γ).

$L \equiv$

·	2	4	·
·	1	·	·
3	5	2	4
·	·	·	·

$\{(1,2,2), (1,3,4), (2,2,1), (3,1,3)\} \subset \mathrm{Ent}(L)$.

$$\Downarrow \Theta = ((1\ 2)(3), (1\ 2\ 3\ 4), (1\ 2)(3\ 4\ 5)) \in S_3 \times S_4 \times S_5.$$

$L^\Theta \equiv$

·	·	2	·
·	·	1	5
5	4	3	1
·	·	·	·

$\{(2,3,1), (2,4,5), (1,3,2), (3,2,4)\} \subset \mathrm{Ent}(L)$.

Latin square autotopism.

- If $L^{\Theta} = L$, then Θ is called an autotopism of L.

- $\mathrm{Atop}(L)$={Autotopisms of L} $\equiv$ **Autotopism group**

$L \equiv$

3	·	2
·	3	1
2	1	·

$$\Theta = ((1\ 2)(3), (1\ 2)(3), (1\ 2)(3)) \in \mathrm{Atop}(L).$$

- If $L^{\Theta} = L$, then Θ is called an autotopism of L.

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$L \equiv$

3	·	2
·	3	1
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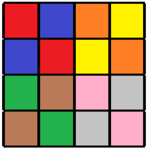
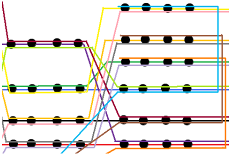
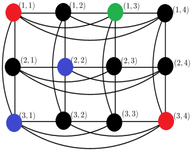
Applications:



Symmetries.



Incidence structures.



1		4	
	2		
2			1

Graph colouring games.

Latin square autotopism.

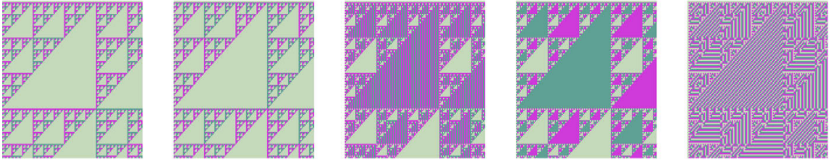
- If $L^{\Theta} = L$, then Θ is called an autotopism of L.
- $\mathrm{Atop}(L)$={Autotopisms of L} $\equiv$ **Autotopism group**

$L \equiv$

3	·	2
·	3	1
2	1	·

$$\Theta = ((1\ 2)(3), (1\ 2)(3), (1\ 2)(3)) \in \mathrm{Atop}(L).$$

Applications:



Pattern recognition (Cryptography).

- If $L^{\Theta} = L$, then Θ is called an autotopism of L.

- $\mathrm{Atop}(L)$={Autotopisms of L} $\equiv$ **Autotopism group**

$L \equiv$

3	·	2
·	3	1
2	1	·

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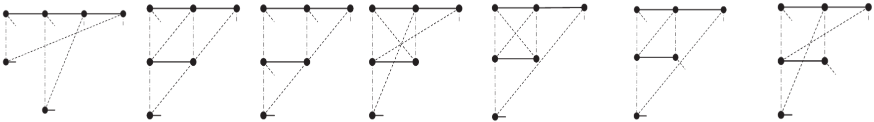
Applications:



$\mathcal{S}_{6,10}$ $\mathcal{S}_{6,11}$ $\mathcal{S}_{6,12}$ $\mathcal{S}_{6,13}$ $\mathcal{S}_{6,14}$ $\mathcal{S}_{6,15}$ $\mathcal{S}_{6,16}$

Enumeration and classification of discrete structures.

Origin of the concept of isotopism [Albert, 1942].

Abraham Adrian **Albert**

United States, 1905-1972

Two algebras $(A_1, \cdot)$ and $(A_2, \circ)$ are **isotopic** if there exist three nonsingular linear transformations

$$\alpha, \beta, \gamma : A_1 \to A_2$$

such that

$$\alpha(x) \circ \beta(y) = \gamma(x \cdot y), \text{ for all } x, y \in A_1.$$

- **Isotopism**: $\Theta = (\alpha, \beta, \gamma)$.
- If $\alpha = \beta = \gamma$, then this is an **isomorphism**.

Richard Hubert **Bruck**

United States, 1914-1991

Two quasigroups $(Q_1,\cdot)$ and $(Q_2,\circ)$ are **isotopic** if there exist three bijections

$$\alpha,\beta,\gamma: Q_1 \to Q_2$$

such that

$$\alpha(x)\circ\beta(y)=\gamma(x\cdot y),\text{ for all } x,y\in Q_1.$$

- **Isotopism**: $\Theta=(\alpha,\beta,\gamma)$.
- $\alpha=\beta=\gamma \Rightarrow$ **Isomorphic**.

In its related Latin square:

- α: Permutation of rows.
- β: Permutation of columns.
- γ: Permutation of symbols.

$L \equiv$

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

$\Theta=((1\ 2)(3\ 4),(2\ 3),\mathrm{Id})$

$\Rightarrow L^{\Theta} \equiv$

2	4	3	1
1	3	2	4
4	2	1	3
3	1	4	2

A historical perspective of the theory of isotopisms.

symmetry

MDPI

Article

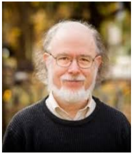
A Historical Perspective of the Theory of Isotopisms

Raúl M. Falcón [1,*], Óscar J. Falcón [2] and Juan Núñez [2]

Symmetry **2018**, *10*, 322; doi:10.3390/sym10080322 www.mdpi.com/journal/symmetry

LS isotopism $\equiv$ Graph isomorphism.

McKay, B.D., Meynert, A., Myrvold, W. *Small Latin Squares, Quasigroups and Loops*. J. Combin. Des. **15** (2007), 98–119.



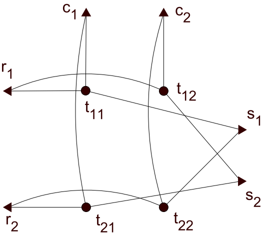
Brendan D. **McKay**

Alison M. **Meynert**

Wendy **Myrvold**

1	2
2	1

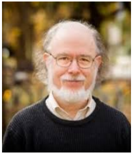
$\equiv$



Isotopic Latin squares $\equiv$ Isomorphic coloured graphs

LS isotopism $\equiv$ Graph isomorphism.

McKay, B.D., Meynert, A., Myrvold, W. *Small Latin Squares, Quasigroups and Loops*. J. Combin. Des. **15** (2007), 98–119.



Brendan D. **McKay**

Alison M. **Meynert**

Wendy **Myrvold**

Theorem 1. *Let L be a Latin square of order n and let* $(r, c, s) \in \mathrm{Is}(L)$ *be a non-trivial autotopism. Then one of the following is true.*

(i) *r, c, s have the same cycle structure with at least one and at most $\lfloor n/2 \rfloor$ fixed points.*

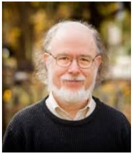
(ii) *One of r, c, s has at least one fixed point, and the other two have the same cycle structure without fixed points.*

(iii) *None of r, c, s has fixed points.*

Cycle structure: $\pi = (1234)(56)(78)(9) \rightarrow z_\pi = 4^1 2^2 1$.

LS isotopism $\equiv$ Graph isomorphism.

McKay, B.D., Meynert, A., Myrvold, W. *Small Latin Squares, Quasigroups and Loops*. J. Combin. Des. **15** (2007), 98–119.



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Cycle structure: $\pi = (1234)(56)(78)(9) \to z_\pi = 4^1 2^2 1$.

- Known for all LS of order $n \leq 17$ [Falcón, 2012; Stones, 2012].

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3. Open questions.

Θ-critical set.

$\Theta = ((12)(34), (12)(34), \mathrm{Id}) \in \mathrm{Atop}(L)$

1	·	·	4
·	1	4	·
3	·	2	1
4	3	1	·

Θ-uniquely completable to

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

Θ-critical set.

$\Theta = ((12)(34), (12)(34), \mathrm{Id}) \in \mathrm{Atop}(L)$

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4	·	·	·

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Θ-critical set.

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·	·	·	4
·	1	·	·
3	·	2	1
4	·	·	·

Θ-uniquely completable to

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

- **Critical set**: Uniquely completable PLS such that any proper subset of entries is not uniquely completable.

1	·	·	4
·	1	·	·
3	·	2	·
·	·	·	·

→

1	·	·	4
·	1	·	·
3	4	2	1
·	·	·	·

→

1	·	·	4
·	1	4	·
3	4	2	1
4	·	1	·

→

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

Θ-critical set.

$\Theta = ((12)(34), (12)(34), \mathrm{Id}) \in \mathrm{Atop}(L)$

·	·	·	4
·	1	·	·
3	·	2	1
4	·	·	·

Θ-uniquely completable to

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

- **Θ-critical set**: Uniquely Θ-completable PLS such that any proper subset of entries is not uniquely Θ-completable.

·	·	·	4
·	1	·	·
3	·	2	·
·	·	·	·

$\rightarrow$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

(**Θ-forced entries**).

Θ-orbit.

$$\Theta = (\alpha, \beta, \gamma) \in \mathrm{Atop}(L)$$

$$\mathrm{Orb}_\Theta((i,j,k)) := \{(\alpha^m(i), \beta^m(j), \gamma^m(k)) : \ m \geq 0\} \subseteq \mathrm{Ent}(L).$$

$$\Theta = ((12)(34),\, (12)(34),\, \mathrm{Id}) \in \mathrm{Atop}(L)$$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$\rightarrow$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$\mathrm{Orb}_\Theta((1,1,1)) = \mathrm{Orb}_\Theta((2,2,1))$ $\quad$ $\mathrm{Orb}_\Theta((1,2,2)) = \mathrm{Orb}_\Theta((2,1,2))$

$\mathrm{Orb}_\Theta((1,3,3)) = \mathrm{Orb}_\Theta((2,4,3))$ $\quad$ $\mathrm{Orb}_\Theta((1,4,4)) = \mathrm{Orb}_\Theta((2,3,4))$

$\mathrm{Orb}_\Theta((3,1,3)) = \mathrm{Orb}_\Theta((4,2,3))$ $\quad$ $\mathrm{Orb}_\Theta((3,2,4)) = \mathrm{Orb}_\Theta((4,1,4))$

$\mathrm{Orb}_\Theta((3,3,2)) = \mathrm{Orb}_\Theta((4,4,2))$ $\quad$ $\mathrm{Orb}_\Theta((3,4,1)) = \mathrm{Orb}_\Theta((4,3,1))$

Types of Θ-orbits.

- **Trivial**: Exactly one entry.
- **Principal**: Without entries sharing components.
- **Secondary**: It contains two distinct entries with one common component.
  - **Row-monotone**: All the entries are in the same row.
  - **Column-monotone**: All the entries are in the same column.
  - **Symbol-monotone**: All the entries have the same symbol.

$$\Theta = ((12)(3), (12)(3), (13)(2))$$

1	2	3
2	3	1
3	1	2

Types of Θ-orbits.

- Two row-monotone Θ-orbits are **parallel** if they share columns and symbols.
- Two column-monotone Θ-orbits are **parallel** if they share rows and symbols.

$$\Theta = ((1)(2)(345), (1)(2)(345), (1)(2)(345))$$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

Three main questions.

$$L \in \mathrm{LS}_n \qquad \text{and} \qquad \Theta \in \mathrm{Atop}(L)$$

Determine the exact values of

- $|\mathrm{CS}_\Theta(L)|$:= Number of Θ-critical sets of L.
- $\mathrm{scs}_\Theta(L)$:= Size of the smallest Θ-critical set of L.
- $\mathrm{lcs}_\Theta(L)$:= Size of the largest Θ-critical set of L.

Three main questions.

$$L \in \mathrm{LS}_n \qquad \text{and} \qquad \Theta \in \mathrm{Atop}(L)$$

Determine the exact values of

- $|\mathrm{CS}_\Theta(L)|$:= Number of Θ-critical sets of L.
- $\mathrm{scs}_\Theta(L)$:= Size of the smallest Θ-critical set of L.
- $\mathrm{lcs}_\Theta(L)$:= Size of the largest Θ-critical set of L.

Lemma

These values only depend on

- *The cycle structure of the autotopism Θ.*
- *The* **main class*** *of the Latin square L.*

* Two Latin squares belong to the same main class if they coincide up to isotopism and permutation among the three components of their entries.

The case $n \leq 5$.

$L_2 \equiv \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline\end{array}$ $L_3 \equiv \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline\end{array}$ $L_{4.1} \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline\end{array}$ $L_{4.2} \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 2 & 1 \\ \hline 4 & 3 & 1 & 2 \\ \hline\end{array}$

L	$\Theta \in \mathrm{Atop}(L)$	z_Θ	$\lvert\mathrm{CS}_\Theta(L)\rvert$	$\mathrm{scs}_\Theta(L)$	$\mathrm{lcs}_\Theta(L)$
L_2	$(\mathrm{Id}_2, \mathrm{Id}_2, \mathrm{Id}_2)$	$(1^2, 1^2, 1^2)$	4	1	1
	$((12), (12), \mathrm{Id}_2)$	$(2, 2, 1^2)$	4	1	1
L_3	$(\mathrm{Id}_3, \mathrm{Id}_3, \mathrm{Id}_3)$	$(1^3, 1^3, 1^3)$	27	2	3
	$((12), (12), (13))$	$(21, 21, 21)$	14	1	2
	$((123), (132), \mathrm{Id}_3))$	$(3, 3, 1^3)$	27	2	2
	$((123), (123), (132))$	$(3, 3, 3)$	9	1	1
$L_{4.1}$	$(\mathrm{Id}_4, \mathrm{Id}_4, \mathrm{Id}_4)$	$(1^4, 1^4, 1^4)$	576	5	7
	$((12)(34), (12)(34), \mathrm{Id}_4)$	$(2^2, 2^2, 1^4)$	192	4	4
	$((23), (14), (14))$	$(21^2, 21^2, 21^2)$	256	4	4
	$((12)(34), (13)(24), (14)(23))$	$(2^2, 2^2, 2^2)$	256	3	3
	$((243), (134), (134))$	$(31, 31, 31)$	90	2	2
	$((1234), (1234), (24))$	$(4, 4, 21^2)$	64	2	2
$L_{4.2}$	$(\mathrm{Id}_4, \mathrm{Id}_4, \mathrm{Id}_4)$	$(1^4, 1^4, 1^4)$	736	4	6
	$((12)(34), (12)(34), \mathrm{Id}_4)$	$(2^2, 2^2, 1^4)$	192	4	4
	$((13)(24), (14)(23), (34))$	$(2^2, 2^2, 21^2)$	224	3	3
	$((12), (12), (34))$	$(21^2, 21^2, 21^2)$	256	4	4
	$((1324), (1324), (12)(34))$	$(4, 4, 2^2)$	64	2	2
	$((1423), (1324), \mathrm{Id}_4)$	$(4, 4, 1^4)$	256	3	3

The case $n \leq 5$.

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L_{5.2} \equiv$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

L	$\Theta \in \mathrm{Atop}(L)$	z_Θ	$\lvert \mathrm{CS}_\Theta(L) \rvert$	$\mathrm{scs}_\Theta(L)$	$\mathrm{lcs}_\Theta(L)$
$L_{5.1}$	$(\mathrm{Id}_5, \mathrm{Id}_5, \mathrm{Id}_5)$	$(1^5, 1^5, 1^5)$	53250	6	10
	$((12)(35), (13)(45), (14)(23))$	$(2^2 1, 2^2 1, 2^2 1)$	3088	3	5
	$((2354), (1243), (1243))$	$(41, 41, 41)$	832	3	3
	$((12345), (15432), \mathrm{Id}_5)$	$(5, 5, 1^5)$	3125	4	4
	$((12345), (12345), (13524))$	$(5, 5, 5)$	250	2	2
$L_{5.2}$	$(\mathrm{Id}_5, \mathrm{Id}_5, \mathrm{Id}_5)$	$(1^5, 1^5, 1^5)$	48462	7	11
	$((13)(45), (25)(34), (13)(45))$	$(2^2 1, 2^2 1, 2^2 1)$	2896	3	5
	$((345), (345), (345))$	$(31^2, 31^2, 31^2)$	8424	5	6

The case $n = 2$.

$$L \in \mathrm{LS}_n \qquad \text{and} \qquad \Theta \in \mathrm{Atop}(L)$$

Proposition

$$0 < \mathrm{scs}_{\Theta}(L) \leq \mathrm{lcs}_{\Theta}(L) \leq |\mathrm{Orb}_{\Theta}(L)|.$$

$$\Theta = ((12), (12), \mathrm{Id})$$

$L_2 \equiv$

1	2
2	1

- $\mathrm{lcs}_{\Theta}(L_2) \leq 2$.
- Every entry is Θ-forced by any other given entry.

So, $\mathrm{scs}_{\Theta}(L_2) = \mathrm{lcs}_{\Theta}(L_2) = 1$ and $|\mathrm{CS}_{\Theta}(L_2)| = 4$.

The case $n = 3$.

Proposition

If there exist m secondary parallel Θ-orbits of the same type, then every Θ-critical set of L contains at least $m-1$ entries. Hence,

$$m - 1 \leq \mathrm{scs}_\Theta(L).$$

$$\Theta = ((123), (132), \mathrm{Id})$$

$L_3 \equiv$

1	2	3
2	3	1
3	1	2

- $2 \leq \mathrm{scs}_\Theta(L_3) \leq \mathrm{lcs}_\Theta(L_3) \leq 3$.
- Every entry is Θ-forced by any two entries of two distinct secondary Θ-orbits.

So, $\mathrm{scs}_\Theta(L_3) = \mathrm{lcs}_\Theta(L_3) = 2$ and $|\mathrm{CS}_\Theta(L_3)| = 27$.

$$L \in \mathrm{LS}_n \qquad \text{and} \qquad \Theta = (\alpha, \beta, \mathrm{Id}) \in \mathrm{Atop}(L)$$

Theorem

If $z_\alpha = z_\beta = n$, then

a) $\mathrm{scs}_\Theta(L) = \mathrm{lcs}_\Theta(L) = n - 1$.

b) $|CS_\Theta(L)| = n^n$.

$\Theta = ((12),(12),\mathrm{Id})$	$\Theta = ((123),(132),\mathrm{Id})$
1 2 / 2 1	1 2 3 / 2 3 1 / 3 1 2
$\mathrm{scs}_\Theta(L_2) = \mathrm{lcs}_\Theta(L_2) = 1$	$\mathrm{scs}_\Theta(L_3) = \mathrm{lcs}_\Theta(L_3) = 2$
$\lvert\mathrm{CS}_\Theta(L_2)\rvert = 4$	$\lvert\mathrm{CS}_\Theta(L_3)\rvert = 27$

$$\Theta = ((123), (123), (132))$$

$$L_3 \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}$$

Any of its entries gives rise to a Θ-critical set of order one. So, $\mathrm{scs}_\Theta(L_3) = \mathrm{lcs}_\Theta(L_3) = 1$ and $|\mathrm{CS}_\Theta(L_3)| = 9$.

$$\Theta = ((1234), (1234), (24))$$

$L_{4.1} \equiv$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Every Θ-critical set is a PLS of order two containing:

- One entry of a symbol-monotone Θ-orbit.
- One entry of a non-monotone Θ-orbit.

So, $\mathrm{scs}_\Theta(L_{4.1}) = \mathrm{lcs}_\Theta(L_{4.1}) = 2$ and $|\mathrm{CS}_\Theta(L_{4.1})| = 64$.

The case $n = 5$.

$$\Theta = ((12345), (12345), (13524))$$

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Every entry is Θ-forced by any two entries of a pair of different Θ-orbits.
So, $\mathrm{scs}_{\Theta}(L_{5.1}) = \mathrm{lcs}_{\Theta}(L_{5.1}) = 2$ and $|\mathrm{CS}_{\Theta}(L_{5.1})| = 250$.

Lemma

If there exists exactly one trivial orbit, then no Θ-critical set contains that entry.

$$\Theta = ((12),(12),(13))$$

$L_3 \equiv$

1	2	3
2	3	1
3	1	2

Every Θ-critical set of L_3 contains exactly either

- one entry of the principal Θ-orbit; or
- two entries of two distinct secondary Θ-orbits.

So, $\mathrm{scs}_\Theta(L_3) = 1 < \mathrm{lcs}_\Theta(L_3) = 2$ and $|\mathrm{CS}_\Theta(L_3)| = 14$.

1		

	2	3

	2	
3		

		3
3		

Trivial orbits.

$$\Theta = ((243), (134), (134))$$

$L_{4.1} \equiv$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Every Θ-critical set of $L_{4.1}$ contains exactly a pair of entries of two different non-trivial Θ-orbits. So, $\mathrm{scs}_\Theta(L_{4.1}) = \mathrm{lcs}_\Theta(L_{4.1}) = 2$ and $|\mathrm{CS}_\Theta(L_{4.1})| = 90$.

Trivial orbits: A more detailed example.

$$\Theta = ((2354), (1243), (1243))$$

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Every Θ-critical set of $L_{5.1}$ must contain at least one principal Θ-orbit.

1	2	3	4	5
3	4	2	5	1
4	1	5	3	2
2	5	4	1	3
5	3	1	2	4

$$\Theta = ((2354), (1243), (1243))$$

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

The knowledge of only one principal Θ-orbit is not enough.

3	1	4	2	5
2	5	3	1	4
1	4	2	5	3
5	3	1	4	2
4	2	5	3	1

Trivial orbits: A more detailed example.

$$\Theta = ((2354), (1243), (1243))$$

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

The knowledge of only a principal Θ-orbit and other distinct non-trivial Θ-orbit is similarly not enough

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
5	3	1	2	4
4	1	5	3	2

3	1	4	2	5
2	3	5	4	1
5	4	3	1	2
4	2	1	5	3
1	5	2	3	4

4	3	2	1	5
2	1	3	5	4
1	4	5	2	3
3	5	1	4	2
5	2	4	3	1

Trivial orbits: A more detailed example.

$$\Theta = ((2354), (1243), (1243))$$

$L_{5.1} \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

In fact, every Θ-critical sets of $L_{5.1}$ is a PLS of size three, whose entries are taken from

- three distinct principal Θ-orbits;
- two distinct principal Θ-orbits and the symbol-monotone Θ-orbit; or
- a principal Θ-orbit and two distinct secondary Θ-orbits.

So, $\mathrm{scs}_\Theta(L_{5.1}) = \mathrm{lcs}_\Theta(L_{5.1}) = 3$ and $|\mathrm{CS}_\Theta(L_{5.1})| = 832$.

CONTENTS

1. Preliminaries.
2. Critical sets based on Latin square autotopisms.
3. Open questions.

Open questions.

Question

What is the computational complexity of deciding whether a partial Latin square is Θ-completable?

Question

Does a Θ-critical set of size m exist for every $m \in \{scs_\Theta(L), \ldots, lcs_\Theta(L)\}$?

Question

Find lower and upper bounds for:

- $\mathrm{scs}_\Theta(n)$:= *Smallest size of a Θ-critical set for every LS of order n.*
- $\mathrm{lcs}_\Theta(n)$:= *Largest size of a Θ-critical set for every LS of order n.*

Question

Is it possible to find general formulas of $\mathrm{scs}_\Theta(L)$, $\mathrm{lcs}_\Theta(L)$ and $|\mathrm{CS}_\Theta(L)|$ for some specific cycle structures?

Open questions.

Question

The existence of non-monotone Θ-orbits is common for LS of higher orders.

$$\Theta = ((123456), (123)(456), (14)(25)(36)) \in S_6 \times S_6 \times S_6$$

1	2	3	4	5	6
6	4	5	3	1	2
2	3	1	5	6	4
4	5	6	1	2	3
3	1	2	6	4	5
5	6	4	2	3	1

- *How many types of non-monotone orbits exist?*
- *What is the role played by each one of them in the construction of Θ-critical sets?*

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Many thanks!

New advances concerning critical sets based on Latin square autotopisms

Raúl M. Falcón*

Department of Applied Maths I.
Universidad de Sevilla.
rafalgan@us.es

Kongunadu Arts and Science College. Tamil Nadu, India.
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