On Leibniz Superalgebras with Even Part Corresponding to $\mathfrak{sl}_2$

L. M. Camacho[1] · R. M. Navarro[2]

Abstract
In this paper we describe finite-dimensional complex Leibniz superalgebras whose even part is the simple Leibniz algebra corresponding to $\mathfrak{sl}_2$, i.e. its quotient algebra with respect to the Leibniz kernel I is isomorphic to $\mathfrak{sl}_2$. We classify these Leibniz superalgebras in several cases with arbitrary dimensions in which the odd part is essentially a Leibniz irreducible $(\mathfrak{sl}_2 \dotplus I)$-module or a finite direct sum of them.

Keywords Leibniz (super)algebra · Lie (super)algebra · Irreducible module · Simple Leibniz algebra

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1 Introduction

Since Loday's introduction of Leibniz algebras in 1993, many results of the theory of Lie algebras have been extended to Leibniz algebras. Nevertheless, a great deal of the results have been devoted to (co)homological problems [11, 16–18] or to the classification problems of nilpotent part and its subclasses [2, 3, 6, 20–24]. This is in contrast with the semisimple part of Leibniz algebras, which has been less studied. Hardly original consideration belongs to Dzhumadil'daev and Abdykassymova [4, 9] who suggested a notion of simple Leibniz algebra and studied its properties in characteristic p. However recently, in [19] it is been described the class of simple Leibniz algebras corresponding to $\mathfrak{sl}_2$, i.e.,

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whose quotient algebra with respect to the ideal I generated by squares (also called Leibniz kernel) is isomorphic to the simple Lie algebra $\mathfrak{sl}_2$.

On the other hand, in the last few years many authors have focused their study on the construction of Leibniz superalgebras from different approaches, see for instance [7, 8, 12]. Let us note that dealing with superalgebras is much more complicated than dealing with algebras, remark for instance that a decomposition similar to the Levi-Malcev decomposition in Lie algebras is not verified in the case of Leibniz superalgebras.

Thus, along the present work the study carried out for simple Leibniz algebras corresponding to $\mathfrak{sl}_2$ is extended to Leibniz superalgebras. Therefore, we describe finite-dimensional complex Leibniz superalgebras whose even part is the simple Leibniz algebra corresponding to $\mathfrak{sl}_2$. Recall that in order to obtain simple Lie superalgebras $G_0 \oplus G_1$ it makes perfect sense considering G_0 a semisimple Lie algebra and G_1 an irreducible G_0-module, see for instance Proposition 7.1 of [14]. This latter fact together with the description of Leibniz algebras corresponding to $\mathfrak{sl}_2$ developed in [19] lead us to consider Leibniz superalgebras $L_0 \oplus L_1$ with L_0 either $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \dotplus I$, where I is the Leibniz kernel. For the former we consider two cases; L_1 an indecomposable bimodule or a finite sum of Leibniz irreducible $\mathfrak{sl}_2$-modules, and for the latter we consider also two cases; L_1 with the structure of either a Leibniz irreducible $(\mathfrak{sl}_2 \dotplus I)$-module or a finite direct sum of them.

Throughout the present paper we will consider vector spaces and algebras over the field of complex numbers $\mathbb{C}$.

2 Basic Notions and Preliminaries Results

Note that the main difficulties dealing with Lie superalgebras, and so with Leibniz superalgebras, can be summarized in the following points (see for instance [5]):

1. A one-dimensional subspace of a Lie/Leibniz superalgebra L is not necessarily a Lie/Leibniz subsuperalgebra of L.
2. There is no analog to the Lie Theorem for solvable Lie superalgebras and neither for Leibniz superalgebras.
3. A decomposition similar to the Levi-Malcev decomposition in Lie algebras is not verified in the case of Lie superalgebras and therefore neither in the case of Leibniz superalgebras.

We suppose that the reader is familiarised with the basics of Leibniz algebras. However, let us recall briefly the concepts of representation, module and irreducible on Leibniz algebras.

Definition 2.1 [10] Let L be a Leibniz algebra, M a vector space over the field $\mathbb{K}$. Assume we have two $\mathbb{K}$-linear functions:

$$\rho,\ \lambda : L \longrightarrow \mathfrak{gl}(M)$$

Denote $\lambda(x)$ and $\rho(y)$ by λ_x and ρ_y for every $x,\ y \in L$. We say that M is a representation of L if the following properties are satisfied:

(1) $\rho_{[x,y]} = \rho_y \rho_x - \rho_x \rho_y$,
(2) $\lambda_{[x,y]} = \rho_y \lambda_x - \lambda_x \rho_y$,
(3) $\lambda_{[x,y]} = \rho_y \lambda_x + \lambda_x \lambda_y$ for every $x, y \in L$.

If M is a representation of L, then M becomes an L-module with the following $[\cdot,\cdot] : M \times L \longrightarrow M$ and $[\cdot,\cdot] : L \times M \longrightarrow M$ with the products: $[m, x] := \rho_x(m)$ and $[x, m] := \lambda_x(m)$ for every $x \in L,\ m \in M$. Conversely, for a given L-module M, we get the representation $\rho,\ \lambda : L \longrightarrow \mathfrak{gl}(M)$ with $\rho_x := [\cdot, x]$ and $\lambda_x := [x, \cdot]$ for all $x \in L$.

Definition 2.2 [10] Let $\mathbb{K}$ be a field, L a Leibniz algebra over $\mathbb{K}$, and V a vector space over $\mathbb{K}$. We say that the Leibniz representation $\rho,\ \lambda : L \longrightarrow \mathfrak{gl}(V)$ is irreducible (equivalently, V is an irreducible L-module), if ρ and λ are irreducible. In other words, if $U \subseteq V$ is an invariant subspace of ρ and λ ($\forall x \in L$, $\rho_x(U) \subseteq U$ and $\lambda_x(U) \subseteq U$), then $U = \{0\}$ or $U = V$.

Recall next some necessary basic definitions and notions regarding Leibniz superalgebras.

Definition 2.3 [1] A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}\ \mathit{for\ all}\ \alpha, \beta \in \mathbb{Z}_2$$

$$[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta}[[x, z], y] - \ \mathit{graded\ Leibniz\ identity}$$

for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$, $\alpha, \beta \in \mathbb{Z}_2$.

Note that if a Leibniz superalgebra L satisfies the identity $[x, y] = -(-1)^{\alpha\beta}[y, x]$ for all $x \in L_\alpha$, $y \in L_\beta$, then L becomes a Lie superalgebra. Therefore Leibniz superalgebras are a generalization of Lie superalgebras and many results and definitions of Lie superalgebras can be extended for Leibniz superalgebras. In the same way as for Lie superalgebras, isomorphisms are assumed to be consistent with the $\mathbb{Z}_2$-graduation.

If we denote by R_x the right multiplication operator, i.e., $R_x : L \to L$, then the graded Leibniz identity can be expressed in the following form: $R_{[x,y]} = R_y R_x - (-1)^{\alpha\beta} R_x R_y$ where $x \in L_\alpha$, $y \in L_\beta$.

We denote by $R(L)$ the set of all right multiplication operators. It is not difficult to prove that $R(L)$ with the multiplication defined by:

$$< R_a, R_b >:= R_a R_b - (-1)^{\alpha\beta} R_b R_a$$

for $R_a \in R(L)_\alpha$, $R_b \in R(L)_\beta$, becomes a Lie superalgebra.

For a Leibniz superalgebra $L = L_0 \oplus L_1$ it is defined the set $Ann(L)$, $Ann(L) = \{X \in L\ :\ [L, X] = 0\}$ which will be called the right annihilator of L.

It is easy to see that $Ann(L)$ is a two-sided ideal of L and $[X, X] \in Ann(L)$ for any $X \in L_0$, this notion is consistent with the right annihilator in Leibniz algebras. If we consider $I = ideal < [X, Y] + (-1)^{\alpha\beta}[Y, X]\ :\ X \in L_\alpha, Y \in L_\beta >$, then $I \subseteq Ann(L)$.

In order to provide an example of non-Lie Leibniz superalgebra, we can consider an associative superalgebra, $A = A_0 \oplus A_1$, and a linear mapping $D : A \to A$ satisfying the condition:

$$D(a(Db)) = DaDb = D((Da)b)$$

for all $a, b \in A$ and define a new multiplication over the underlying $\mathbb{Z}_2$-graded vector space, $<,>$, by:

$$< a, b >_D := a(Db) - (-1)^{\alpha\beta} D(b)a$$

for $a \in A_\alpha, b \in A_\beta$. Then A equipped with multiplication $<,>$ becomes a Leibniz superalgebra, which in general is not a Lie superalgebra.

Next we extend in a natural way the definition of simple Lie superalgebras to simple Leibniz superalgebras. Recall that a Lie superalgebra L is called simple if its ideals ($\mathbb{Z}_2$-graded ideals) are only $\{0\}$ and L, verifying $[L, L] \neq 0$, see for instance [13]. Likewise, we can introduce the concept of simple Leibniz superalgebras.

Definition 2.4 A Leibniz superalgebra L is called simple if its ideals ($\mathbb{Z}_2$-graded ideals) are only $\{0\}$, I, L and $[L, L] \neq I$.

Note that this definition agrees with that of simple Lie superalgebra whenever $I = 0$.

3 Leibniz Superalgebras with Even Part Corresponding to $\mathfrak{sl}_2$

Throughout the present paper we consider Leibniz superalgebras $L_0 \oplus L_1$ with L_0 either $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \dot{+} I$ where I is the Leibniz kernel. L_1, on the other hand, is essentially an indecomposable bimodule, or a finite sum of Leibniz irreducible $\mathfrak{sl}_2$-modules.

Regarding L_1 as a Leibniz irreducible module of $\mathfrak{sl}_2$, we can extend the study developed in [10]. Thus,

Proposition 3.1 *Let $L_0 \oplus L_1$ be a Leibniz superalgebra. Setting V_1 the underlying vector space of L_1, we have two linear functions ρ and λ:*

$\rho : L_0 \longrightarrow \mathfrak{gl}(V_1)$, *defined by* $\rho_x := [\cdot, x]$ *and*
$\lambda : L_0 \longrightarrow \mathfrak{gl}(V_1)$, *defined by* $\lambda_x := [x, \cdot]$
which constitutes a Leibniz representation of L_0 and therefore L_1 has the structure of L_0-module.

Proof From Definition 2.5 in [10], the conditions for ρ and λ are the following:

(1) $\rho_{[x,y]} = \rho_y \rho_x - \rho_x \rho_y$,
(2) $\lambda_{[x,y]} = \rho_y \lambda_x - \lambda_x \rho_y$,
(3) $\lambda_{[x,y]} = \rho_y \lambda_x + \lambda_x \lambda_y$ for every $x, y \in L_0$.

Firstly, note that the graded Leibniz identity is nothing but the Leibiz identity provided at least two of the three vectors considered belong to the even part of the Leibniz superalgebra. Thus, on account of $x, y \in L_0$, if we consider $z \in V_1$ we have $\rho_{[x,y]}(z) = [z, [x, y]]$ verifying Leibniz identity, i.e. $[z, [x, y]] = [[z, x], y] - [[z, y], x]$, but this last expression is exactly $(\rho_y \rho_x - \rho_x \rho_y)(z)$ and we obtain (1). Analogously (2) and (3) can be checked by means of Leibniz identity. $\square$

Remark 3.1 Note that this result is an extension of the equivalent one for Lie superalgebras. Any time we have a Lie superalgebra $G_0 \oplus G_1$, on one hand we have G_0 a Lie algebra and on the other hand, we have a representation of G_0 and therefore G_1 has the structure of G_0-module.

4 $L_0 = \mathfrak{sl}_2 \dot{+} I$, and L_1 as Leibniz Irreducible $(\mathfrak{sl}_2 \dot{+} I)$-Module

Consider $L_0 = \mathfrak{sl}_2 \dot{+} I$ where I is the Leibniz kernel, and L_1 has a structure of Leibniz irreducible $(\mathfrak{sl}_2 \dot{+} I)$-module.

Proposition 4.1 *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra such that L_0 is a simple Leibniz algebra with $dim(L_0) \geq 5$ with Lie part $\mathfrak{sl}_2$; and L_1 has the structure of Leibniz irreducible L_0-module. Then, there exists a basis $\{e, h, f, x_0, x_1, \dots x_m, y_0, y_1, \dots, y_n\}$ of L in which the multiplication table is as follows:*

$$
\begin{array}{ll}
[e,h] = 2e,\ [h,e] = -2e,\ [h,f] = 2f, & [f,h] = -2f,\ [e,f] = h,\ [f,e] = -h, \\
[x_k, h] = (m-2k)x_k, & 0 \leq k \leq m, \\
[x_k, f] = x_{k+1}, & 0 \leq k \leq m-1, \\
[x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leq k \leq m, \\
[y_j, h] = (n-2j)y_j, & [h, y_j] = -\delta(n-2j)y_j,\ 0 \leq j \leq n, \\
[y_j, f] = y_{j+1}, & [f, y_j] = -\delta y_{j+1},\ 0 \leq j \leq n-1, \\
[y_j, e] = -j(n+1-j)y_{j-1}, & [e, y_j] = \delta j(n+1-j)y_{j-1},\ 1 \leq j \leq n, \\
[y_i, y_j] = \alpha_{ij}e + \beta_{ij}h + \gamma_{ij}f + \sum_{k=0}^{m} \lambda_{ij}^k x_k,\ 0 \leq i, j \leq n, &
\end{array}
$$

with $\delta \in \{0, 1\}$ and the omitted products are zero.

Remark 4.1 Note that in particular $\{e, h, f, x_0, x_1, \dots x_m\}$ is a basis of L_0 and $\{y_0, y_1, \dots, y_n\}$ is a basis of L_1, being m and n positive integers.

Proof From [19] we can deduce the multiplication table for L_0, i.e. $\mathfrak{sl}_2 \dot{+} I$. Secondly, with respect to L_1 being an irreducible $(\mathfrak{sl}_2 \dot{+} I)$-module, from Theorem 3.4 of [10] we obtain that the representation $\lambda,\ \rho : \mathfrak{sl}_2 \dot{+} I \longrightarrow \mathfrak{gl}(V_1)$ verifies in particular that $\lambda\ |_I = \rho\ |_I = 0$ and then L_1 is in fact an irreducible $\mathfrak{sl}_2$-module. The condition $\lambda\ |_I = \rho\ |_I = 0$ leads, in terms of multiplication table, to $[x_i, y_j] = [y_j, x_i] = 0$ for all i, j. Finally, from [10] we can assert that given $V_1 = span\{y_0, y_1, \dots, y_n\}$ an $(n+1)$-dimensional complex vector space and $\lambda,\ \rho : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V_1)$ the irreducible representations of $\mathfrak{sl}_2$, then either $\rho + \lambda = 0$ or $\lambda = 0$. We can traduce this fact into the multiplication table as follows:

$$
\begin{array}{ll}
[y_j, h] = (n-2j)y_j, & [h, y_j] = -\delta(n-2j)y_j,\ 0 \leq j \leq n, \\
[y_j, f] = y_{j+1}, & [f, y_j] = -\delta y_{j+1},\ 0 \leq j \leq n-1, \\
[y_j, e] = -j(n+1-j)y_{j-1}, & [e, y_j] = \delta j(n+1-j)y_{j-1},\ 1 \leq j \leq n,
\end{array}
$$

with $\delta \in \{0, 1\}$. Note that $\delta = 1$ is exactly the case $\rho + \lambda = 0$ where multiplication between elements of $\mathfrak{sl}_2$ and V is skew-symmetric and V is in fact an irreducible Lie-module. The structure of a Lie irreducible $\mathfrak{sl}_2$-module is a well-known one and coincides with the above multiplication table provided $\delta = 1$. Recall that Lie algebras (resp. superalgebras) are particular cases of Leibniz algebras (resp. superalgebras). If $\delta = 0$, on the other hand, is the case $\lambda = 0$, and this is the only one non-Lie Leibniz irreducible module/representation. □

Theorem 4.1 *Let L be under the conditions of Proposition 4.1, then L is isomorphic to one of the two non-isomorphic Leibniz superalgebras, expressed in the basis* $\{e, h, f, x_0, x_1, \dots x_m, y_0, y_1, \dots, y_n\}$ *as follows:*

L^1 :

$$
\begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[x_k,h]=(m-2k)x_k,\ 0\le k\le m, & [y_j,h]=(n-2j)y_j,\ 0\le j\le n,\\
[x_k,f]=x_{k+1},\ 0\le k\le m-1, & [y_j,f]=y_{j+1},\ 0\le j\le n-1,\\
[x_k,e]=-k(m+1-k)x_{k-1},\ 1\le k\le m, & [y_j,e]=-j(n+1-j)y_{j-1},\ 1\le j\le n,
\end{array}
$$

L^2 :

$$
\begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[x_k,h]=(m-2k)x_k, & 0\le k\le m,\\
[x_k,f]=x_{k+1}, & 0\le k\le m-1,\\
[x_k,e]=-k(m+1-k)x_{k-1}, & 1\le k\le m,\\
[y_j,h]=(n-2j)y_j, & [h,y_j]=-(n-2j)y_j,\ 0\le j\le n,\\
[y_j,f]=y_{j+1}, & [f,y_j]=-y_{j+1},\ 0\le j\le n-1,\\
[y_j,e]=-j(n+1-j)y_{j-1}, & [e,y_j]=j(n+1-j)y_{j-1},\ 1\le j\le n,
\end{array}
$$

where the omitted products, in both superalgebras, are equal to zero.

Proof Let L be under the conditions of Proposition 4.1. Thus, assume

$$[y_i, y_j] = \alpha_{ij} e + \beta_{ij} h + \gamma_{ij} f + \sum_{k=0}^{m} \lambda_{ij}^k x_k,\ 0 \le i, j \le n.$$

By the graded Leibniz identity $[x_0, [y_i, y_j]] = [[x_0, y_i], y_j] + [[x_0, y_j], y_i]$ and thanks to the fact that the representation verifies the condition $\lambda\ |_I = \rho\ |_I = 0$ one derives $\beta_{ij} m x_0 + \gamma_{ij} x_1 = 0$, being then $\beta_{ij} = \gamma_{ij} = 0$ for all $i,\ j$. Next, the graded Leibniz identity $[x_1, [y_i, y_j]] = [[x_1, y_i], y_j] + [[x_1, y_j], y_i]$ leads to $-m\alpha_{ij} x_0 = 0$, and thus $\alpha_{ij} = 0$ for all $i,\ j$. Hence then, we can suppose without loss of generality that

$$[y_i, y_j] = \sum_{k=0}^{m} \lambda_{ij}^k x_k,\ 0 \le i, j \le n.$$

Using now the following graded Leibniz identity $[y_i, [y_j, h]] = [[y_i, y_j], h] - [[y_i, h], y_j]$ we have $[y_i, (n-2j)y_j] = [[y_i, y_j], h] - [(n-2i)y_i, y_j]$, that is $(n - 2j + n - 2i)[y_i, y_j] = \sum_{k=0}^{m} \lambda_{ij}^k (m-2k) x_k$ and then we obtain

$$(2n - 2j - 2i)(\sum_{k=0}^{m} \lambda_{ij}^k x_k) = \sum_{k=0}^{m} \lambda_{ij}^k (m-2k) x_k. \tag{4.1}$$

Next, we will distinguish several cases depending on m and n.

Case 1. m odd and n arbitrary.

In this case from the above equation we deduce $\lambda_{ij}^k = 0$ for all k on account of its coefficients on both sides of the equation, on the left-hand side the coeffients are always even numbers in contrast with the coefficients on the other side, odd ones. Consequently $[y_i, y_j] = 0$ for all $i,\ j$ obtaining then L^1 or L^2 depending on the value of δ.

Case 2. m even and n odd.

In this case, from Eq. 4.1 we get that instead of having all λ_{ij}^k zeroes, there is one potential λ_{ij}^k that remains, exactly the one with $k = \frac{m-2n+2i+2j}{2}$ if possible. Thus we have $[y_i, y_j] = \lambda_{ij}^k x_k$ with k as above. If $\delta = 0$, from the identity $[y_i, [h, y_j]] = [[y_i, h], y_j] - [[y_i, y_j], h]$ we have $0 = [(n-2i)y_i, y_j] - [\lambda_{ij}^k x_k, h] = (n-2i)\lambda_{ij}^k x_k - (m-2k)\lambda_{ij}^k x_k$. As $(n-2i) - (m-2k) \neq 0$ then $\lambda_{ij}^k = 0$ for all $i,\ j$, obtaining then L^1.

On the other hand, if $\delta = 1$ from the graded Leibniz identity $[h, [y_i, y_i]] = 2[[h, y_i], y_i]$ we get $0 = 2(2i-n)[y_i, y_i]$ and since $(2i-n) \neq 0$ then $[y_i, y_i] = 0$ with $0 \leq i \leq n$. Now the identity $[f, [y_i, y_i]] = 2[[f, y_i], y_i]$ leads to $0 = -2[y_{i+1}, y_i]$, thus $[y_{i+1}, y_i] = 0$ with $0 \leq i \leq n-1$. Repeating this last process but for $[y_{i+1}, y_i] = 0$, we obtain $[f, [y_{i+1}, y_i]] = [[f, y_{i+1}], y_i] + [[f, y_i], y_{i+1}]$, that is $0 = -[y_{i+2}, y_i] - [y_{i+1}, y_{i+1}]$ and since the last summand vanishes, remains $[y_{i+2}, y_i] = 0$ with $0 \leq i \leq n-2$. By repeating this process we finally have for all i, $[y_{i+p}, y_i] = 0$ with $0 \leq p \leq n-i$. Moreover, on account of the graded Leibniz identity $[h, [y_i, y_j]] = [[h, y_i], y_j] + [[h, y_j], y_i]$ we have $(2i-n)[y_i, y_j] = (n-2j)[y_j, y_i]$, which allows us to assert that $[y_i, y_{i+p}] = 0$ for all i and p as above. Therefore, $[y_i, y_j] = 0$ for all $i,\ j$, obtaining then L^2.

Case 3. m even and n even.

Analogously as the precent case we deduce from Eq. 4.1 that $[y_i, y_j] = \lambda_{ij} x_k$ with $k = \frac{m-2n+2i+2j}{2}$ if possible. If $\delta = 0$, using the identity $[y_i, [h, y_j]] = [[y_i, h], y_j] - [[y_i, y_j], h]$ we have $[(n-2i)-(m-2k)]\lambda_{ij} = 0$. Replacing k by its value it is obtained $(-n+2j)\lambda_{ij} = 0$, and therefore $\lambda_{ij} = 0$ provided $j \neq \frac{n}{2}$. Only remain $[y_i, y_{\frac{n}{2}}] = \lambda_{i\frac{n}{2}} x_k$ with $k = \frac{m-n}{2} + i$. By the graded Leibniz identity we have $[y_0, [y_{\frac{n}{2}}, f]] = [[y_0, y_{\frac{n}{2}}], f] - [[y_0, f], y_{\frac{n}{2}}] = [\lambda_{0\frac{n}{2}} x_k, f] - [y_1, y_{\frac{n}{2}}] = (\lambda_{0\frac{n}{2}} - \lambda_{1\frac{n}{2}}) x_{k+1}$ since $[y_0, [y_{\frac{n}{2}}, f]] = 0$ we have $\lambda_{0\frac{n}{2}} = \lambda_{1\frac{n}{2}}$. Repeating this process for i with $1 \leq i \leq n-1$, $[y_i, [y_{\frac{n}{2}}, f]]$, leads to

$$\lambda_{i\frac{n}{2}} = \lambda_{(i+1)\frac{n}{2}},\ \ 0 \leq i \leq n-1$$

for simplicity we will call it by λ and now from the equality

$$\lambda x_k = [y_0, y_{\frac{n}{2}}] = [y_0, [y_{\frac{n}{2}-1}, f]] = [[y_0, y_{\frac{n}{2}-1}], f] - [[y_0, f], y_{\frac{n}{2}-1}] = 0$$

we get $\lambda = 0$, obtaining then L^1.

On the other hand, if $\delta = 1$ by means of the following graded Leibniz identities we obtain the restrictions as follow:

$$\begin{aligned}
0 = [h, [y_i, y_i]] &= 2[[h, y_i], y_i] && \Longrightarrow [y_i, y_i] = 0,\ i \neq \tfrac{n}{2},\\
0 = [h, [y_i, y_{\frac{n}{2}}]] &= [[h, y_i], y_{\frac{n}{2}}] + [[h, y_{\frac{n}{2}}], y_i] && \Longrightarrow [y_i, y_{\frac{n}{2}}] = 0,\ i \neq \tfrac{n}{2},\\
0 = [f, [y_i, y_i]] &= 2[[f, y_i], y_i] && \Longrightarrow [y_{i+1}, y_i] = 0,\ 0 \leq i \leq n-1,\\
0 = [e, [y_i, y_i]] &= 2[[e, y_i], y_i] && \Longrightarrow [y_{i-1}, y_i] = 0,\ 1 \leq i \leq n,\\
0 = [f, [y_i, y_n]] &= [[f, y_i], y_n] + [[f, y_n], y_i] && \Longrightarrow [y_{i+1}, y_n] = 0,\ 0 \leq i \leq n-1,\\
0 = [e, [y_i, y_0]] &= [[e, y_i], y_0] + [[e, y_0], y_i] && \Longrightarrow [y_{i-1}, y_0] = 0,\ 1 \leq i \leq n,\\
[y_i, [f, y_{i+1}]] &= [[y_i, f], y_{i+1}] - [[y_i, y_{i+1}], f] && \Longrightarrow -[y_i, y_{i+2}] = [y_{i+1}, y_{i+1}],\\
& && \quad\ 0 \leq i \leq n-2,\\
0 = [f, [y_i, y_{i+1}]] &= [[f, y_i], y_{i+1}] + [[f, y_{i+1}], y_i] && \Longrightarrow -[y_{i+2}, y_i] = [y_{i+1}, y_{i+1}],\\
& && \quad\ 0 \leq i \leq n-2,
\end{aligned}$$

$$
\begin{aligned}
0=[f,[y_i,y_j]] &= [[f,y_i],y_j]+[[f,y_j],y_i] \implies -[y_{i+1},y_j]=[y_{j+1},y_i],\\
&\qquad 0\le i,j\le n-1,\\
0=[e,[y_i,y_j]] &= [[e,y_i],y_j]+[[e,y_j],y_i] \implies -a_i[y_{i-1},y_j]=a_j[y_{j-1},y_i],\\
&\qquad a_i=i(n+1-i)\neq 0\neq a_j,\\
&\qquad 1\le i,j\le n.
\end{aligned}
$$

It can be seen that the set of all of these restrictions combined leads to $[y_i, y_j] = 0$ except for those verifying $i + j = n$. More concretely, we can suppose

$$
\begin{aligned}
&[y_{\frac{n}{2}}, y_{\frac{n}{2}}] = \lambda x_{\frac{m}{2}},\\
&[y_i, y_{n-i}] = b_i\lambda x_{\frac{m}{2}},\ \text{with } b_i \neq 0,\ 1\le i\le n,\ i\neq \tfrac{n}{2}.
\end{aligned}
$$

Finally one derives $\lambda = 0$ from the equality

$$
0=[y_0,[y_n,f]]=[[y_0,y_n],f]-[[y_0,f],y_n]=[b_0\lambda x_{\frac{m}{2}},f]-[y_1,y_n]=b_0\lambda x_{\frac{m}{2}+1}
$$

obtaining then L^2.

This completes the proof on account of L^1 and L^2 are clearly non-isomorphic. □

5 $L_0 = \mathfrak{sl}_2$, and L_1 as an Indecomposable Leibniz $\mathfrak{sl}_2$-Bimodule

Next, we consider the case of $L_0 = \mathfrak{sl}_2$ and L_1 has the structure of an indecomposable Leibniz $\mathfrak{sl}_2$-bimodule.

Proposition 5.1 *Let $L = L_0 \oplus L_1 = \mathfrak{sl}_2 \oplus (V \oplus W)$ be a Leibniz superalgebra with V and W simple $\mathfrak{sl}_2$-modules and such that $L_1 = V \oplus W$ is an indecomposable Leibniz $\mathfrak{sl}_2$-bimodule. Then, L admits a basis $\{e, h, f, v_0, v_1, \dots v_n, w_0, w_1, \dots, w_{n-2}\}$ in which the multiplication table is one of the two non-isomorphic that follow:*

$$
(1):\ \begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[v_j,h]=(n-2j)v_j,\quad 0\le j\le n, & [h,v_j]=-(n-2j)v_j-2jw_{j-1},\\
[v_j,f]=v_{j+1},\quad 0\le j\le n-1, & [f,v_j]=-v_{j+1}+w_j,\\
[v_j,e]=-j(n+1-j)v_{j-1},\ 1\le j\le n, & [e,v_j]=j(n+1-j)v_{j-1}+j(j-1)w_{j-2},\\
[w_k,h]=(n-2-2k)w_k, & 0\le k\le n-2,\\
[w_k,f]=w_{k+1}, & 0\le k\le n-3,\\
[w_k,e]=-k(n-1-k)w_{k-1}, & 1\le k\le n-2,\\
[v_i,v_j]=\alpha_{ij}e+\beta_{ij}h+\gamma_{ij}f, & 0\le i,j\le n,\\
[v_i,w_j]=\alpha'_{ij}e+\beta'_{ij}h+\gamma'_{ij}f, & 0\le i\le n,\ 0\le j\le n-2,\\
[w_j,v_i]=\alpha''_{ij}e+\beta''_{ij}h+\gamma''_{ij}f, & 0\le i\le n,\ 0\le j\le n-2,\\
[w_i,w_j]=\alpha'''_{ij}e+\beta'''_{ij}h+\gamma'''_{ij}f, & 0\le i,j\le n-2,
\end{array}
$$

$$(2): \begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[v_j,h]=(n-2j)v_j, & 0\le j\le n,\\
[v_j,f]=v_{j+1}, & 0\le j\le n-1,\\
[v_j,e]=-j(n+1-j)v_{j-1}, & 1\le j\le n,\\
[w_k,h]=(n-2-2k)w_k,\ 0\le k\le n-2, & [h,w_k]=2(n-1-k)v_{k+1}-(n-2-2k)w_k,\\
[w_k,f]=w_{k+1},\ 0\le k\le n-2, & [f,w_k]=v_{k+2}-w_{k+1},\\
[w_k,e]=-k(n-1-k)w_{k-1},\ 0\le k\le n-2, & [e,w_k]=(n-1-k)((n-k)v_k+kw_{k-1}),\\
[v_i,v_j]=\alpha_{ij}e+\beta_{ij}h+\gamma_{ij}f, & 0\le i,j\le n,\\
[v_i,w_j]=\alpha'_{ij}e+\beta'_{ij}h+\gamma'_{ij}f, & 0\le i\le n,\ 0\le j\le n-2,\\
[w_j,v_i]=\alpha''_{ij}e+\beta''_{ij}h+\gamma''_{ij}f, & 0\le i\le n,\ 0\le j\le n-2,\\
[w_i,w_j]=\alpha'''_{ij}e+\beta'''_{ij}h+\gamma'''_{ij}f, & 0\le i,j\le n-2,
\end{array}$$

where the omitted products are equal to zero.

Proof The statement of the above proposition is a consequence of three facts. Firstly, from Proposition 3.1 any time we have a Leibniz superalgebra, the odd part has even-part module structure. Hence, from [15] we can deduce all the products $[\mathfrak{sl}_2, V]$, $[\mathfrak{sl}_2, W]$, $[V, \mathfrak{sl}_2]$ and $[W, \mathfrak{sl}_2]$ assuming that $w_j = 0$ if $j \notin \{0, 1, \dots n-2\}$. Finally since the underlying vector space of any Leibniz superalgebra is in fact a $\mathbb{Z}_2$-graded vector space we deduce the remaining products $[V, V]$, $[V, W]$, $[W, V]$ and $[W, W]$. $\square$

Theorem 5.1 *Let L be under the conditions of Proposition 5.1, then L is isomorphic to one of the two non-isomorhic Leisbniz superalgebras expressed in the basis* $\{e, h, f, v_0, v_1, \dots v_n, w_0, w_1, \dots, w_{n-2}\}$ *by the following multiplication tables respectively:*

$$M_1: \begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[v_j,h]=(n-2j)v_j,\quad 0\le j\le n, & [h,v_j]=-(n-2j)v_j-2jw_{j-1},\\
[v_j,f]=v_{j+1},\quad 0\le j\le n-1, & [f,v_j]=-v_{j+1}+w_j,\\
[v_j,e]=-j(n+1-j)v_{j-1},\ 1\le j\le n, & [e,v_j]=j(n+1-j)v_{j-1}+j(j-1)w_{j-2},\\
[w_k,h]=(n-2-2k)w_k, & 0\le k\le n-2,\\
[w_k,f]=w_{k+1}, & 0\le k\le n-3,\\
[w_k,e]=-k(n-1-k)w_{k-1}, & 1\le k\le n-2,
\end{array}$$

$$M_2: \begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e,\ [h,f]=2f, & [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[v_j,h]=(n-2j)v_j, & 0\le j\le n,\\
[v_j,f]=v_{j+1}, & 0\le j\le n-1,\\
[v_j,e]=-j(n+1-j)v_{j-1}, & 1\le j\le n,\\
[w_k,h]=(n-2-2k)w_k,\ 0\le k\le n-2, & [h,w_k]=2(n-1-k)v_{k+1}-(n-2-2k)w_k,\\
[w_k,f]=w_{k+1},\ 0\le k\le n-2, & [f,w_k]=v_{k+2}-w_{k+1},\\
[w_k,e]=-k(n-1-k)w_{k-1},\ 0\le k\le n-2, & [e,w_k]=(n-1-k)((n-k)v_k+kw_{k-1}),
\end{array}$$

where the omitted products are equal to zero.

Proof Next, we only present in details the case of M_1, which derives from the family (1) of Proposition 5.1. Analogously it could be obtained the Leibniz superalgebra M_2 from the family (2) of Proposition 5.1.

Thus, to prove the statement of the theorem we need to show that all the products involving only V and W vanish. Next we are goning to prove that $[W, W] = 0$, that is, $\alpha'''_{ij} = \beta'''_{ij} = \gamma'''_{ij} = 0$ for all $i,\ j$ such that $0 \le i, j \le n-2$.

By means of the graded Leibniz identity that follows: $[e, [w_i, w_j]] = [[e, w_i], w_j] + [[e, w_j], w_i]$, we obtain that $2\beta'''_{ij}e + \gamma'''_{ij}h$ which is the result of the left-hand side of the equation, equals zero, result of the other side of the equation. Therefore, $\beta'''_{ij} = \gamma'''_{ij} = 0$ for all $i,\ j$. Finally, the graded Leibniz identity $[h, [w_i, w_j]] = [[h, w_i], w_j] + [[h, w_j], w_i]$ leads to $-2\alpha'''_{ij}e = 0$ and then $\alpha'''_{ij} = 0$ for all $i,\ j$.

Application of the Leibniz superidentity constraint in the following cases we get the results given in the table.

Leibniz superidentity	Constraint	
$\{v_0, v_0, w_j\},\ 0 \le j \le n-2$	$\beta'_{0\,j} = \gamma'_{0\,j} = 0$	$[v_0, w_j] = \alpha'_{0\,j}e$
$\{v_0, v_0, v_0\}$	$\beta_{0\,0} = \gamma_{0\,0} = 0$	$[v_0, v_0] = \alpha_{0\,0}e$
$\{v_0, w_j, v_0\},\ 0 \le j \le n-2$	$\beta''_{0\,j} = \gamma''_{0\,j} = 0$	$[w_j, v_0] = \alpha''_{0\,j}e$
$\{v_0, v_0, v_j\},\ 1 \le j \le n$	$\beta_{0\,j} = \gamma_{0\,j} = 0,\ \alpha_{0\,0} = 0$	$[v_0, v_j] = \alpha_{0\,j}e,\ [v_0, v_0] = 0$
$\{v_0, v_j, v_0\},\ 1 \le j \le n$	$\beta_{j\,0} = \gamma_{j\,0} = 0$	$[v_j, v_0] = \alpha_{j\,0}e$
$\{h, v_0, w_j\},\ 0 \le j \le n-2$	$\alpha'_{0\,j} = 0$	$[v_0, w_j] = 0$
$\{h, w_j, v_0\},\ 0 \le j \le n-2$	$\alpha''_{0\,j} = 0$	$[w_j, v_0] = 0$
$\{e, v_0, w_j\},\ 2 \le j \le n$	$\alpha_{j\,0} = 0,\ 1 \le j \le n-1$	$[v_j, v_0] = 0,\ 1 \le j \le n-1$
$\{h, v_0, v_j\},\ 2 \le j \le n$	$\alpha_{0\,j} = 0,\ 1 \le j \le n-1$	$[v_0, v_j] = 0,\ 1 \le j \le n-1$
$\{v_0, v_n, h\}$	$\alpha_{0\,n} = 0$	$[v_0, v_n] = 0$
$\{v_n, v_0, h\}$	$\alpha_{n\,0} = 0$	$[v_n, v_0] = 0$

It can be summed up as $[v_0, v_j] = [v_j, v_0] = 0$ with $0 \le j \le n$ and $[v_0, w_k] = [w_k, v_0] = 0$ with $0 \le k \le n-2$.

Firstly, we are going to prove that $[v_i, v_j] = 0$ for $1 \le i, j \le n$, $[v_i, w_k] = 0$ for $1 \le i \le n$ and $0 \le k \le n-2$, using the induction method on i.

Fixed j and k ($0 \le j \le n$, $0 \le k \le n-2$), we suppose that $[v_{i-1}, v_j] = 0$ and $[v_{i-1}, w_k] = 0$. Consider the following Leibniz superidentities:

$$[v_{i-1}, [f, v_j]] - [[v_{i-1}, f], v_j] + [[v_{i-1}, v_j], f] = 0,$$

$$[v_{i-1}, [f, w_k]] - [[v_{i-1}, f], w_k] + [[v_{i-1}, w_k], f] = 0$$

we get $[v_i, v_j] = [v_i, w_k] = 0$.

Then, only rest to prove that $[w_k, v_i] = 0$ with $1 \le i \le n$ and $0 \le k \le n-2$. Analogously to before, we prove it using the induction method on i ($0 \le i \le n$). Recall that $[w_k, v_0] = 0$ with $0 \le k \le n-2$. Fix k ($0 \le k \le n-2$). We suppose that $[w_k, v_{i-1}] = 0$. Taking

$$[w_k, [f, v_{i-1}]] - [[w_k, f], v_{i-1}] + [[w_k, v_{i-1}], f] = 0$$

we get $[w_k, v_i] = 0$.

Thus, we obtain the Leibniz superalgebra of the statement of the theorem. □

6 $L_0 = \mathfrak{sl}_2$, and $L_1 = \bigoplus_{i=1}^{k} J_i$ with Each J_i a Leibniz Irreducible $\mathfrak{sl}_2$-Module

Next, we consider $L_0 = \mathfrak{sl}_2$ and L_1 with structure of a finite direct sum of Leibniz irreducible $\mathfrak{sl}_2$-modules.

Proposition 6.1 *Let $L = L_0 \oplus L_1 = \mathfrak{sl}_2 \oplus J$ be a Leibniz superalgebra with J a finite direct sum of Leibniz irreducible $\mathfrak{sl}_2$-modules, i.e. $J = \bigoplus_{i=1}^{k} J_i$ with J_i a Leibniz irreducible $\mathfrak{sl}_2$-module. Then, L admits a basis $\{e, h, f, v_0^i, v_1^i, \dots v_{n_i}^i;\ 1 \le i \le k\}$ with respect to which the multiplication table is as follows:*

$$
\begin{array}{ll}
[e,h] = 2e,\ [h,e] = -2e, & [h,f] = 2f,\ [f,h] = -2f,\ [e,f] = h,\ [f,e] = -h,\\
[v_j^i, h] = (n_i - 2j)v_j^i, & 0 \le j \le n_i,\ 1 \le i \le k,\\
[v_j^i, f] = v_{j+1}^i, & 0 \le j \le n_i - 1,\ 1 \le i \le k,\\
[v_j^i, e] = -j(n_i + 1 - j)v_{j-1}^i, & 1 \le j \le n_i,\ 1 \le i \le k,\\
[h, v_j^i] = -\delta_i(n_i - 2j)v_j^i, & 0 \le j \le n_i,\ 1 \le i \le k,\\
[f, v_j^i] = -\delta_i v_{j+1}^i, & 0 \le j \le n_i - 1,\ 1 \le i \le k,\\
[e, v_j^i] = \delta_i j(n_i + 1 - j)v_{j-1}^i, & 1 \le j \le n_i,\ 1 \le i \le k,\\
[v_m^i, v_j^l] = \alpha_{mj}^{il} e + \beta_{mj}^{il} h + \gamma_{mj}^{il} f, & 0 \le m \le n_i,\ 0 \le j \le n_l,\ 1 \le i, l \le k,
\end{array}
$$

where $\delta_i \in \{0, 1\}$, $1 \le i \le k$ and the omitted products are equal to zero.

Proof From [10] we deduce that given $V_i = span\{v_0^i, v_1^i, \dots, v_{n_i}^i\}$ a $(n_i + 1)$-dimensional complex vector space and $\lambda,\ \rho : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V_i)$ an irreducible representation of $\mathfrak{sl}_2$, then either $\rho + \lambda = 0$, or $\lambda = 0$. We can traduce this fact into the multiplication table as follows for each value of i:

$$
\begin{array}{ll}
[v_j^i, h] = (n_i - 2j)v_j^i, & [h, v_j^i] = -\delta_i(n_i - 2j)v_j^i,\ 0 \le j \le n_i,\\
[v_j^i, f] = v_{j+1}^i, & [f, v_j^i] = -\delta_i v_{j+1}^i,\ 0 \le j \le n_i - 1,\\
[v_j^i, e] = -j(n_i + 1 - j)v_{j-1}^i, & [e, v_j^i] = \delta_i j(n_i + 1 - j)v_{j-1}^i,\ 1 \le j \le n_i
\end{array}
$$

with $\delta_i \in \{0, 1\}$. Note that $\delta_i = 1$ is exactly the case $\rho + \lambda = 0$ where multiplication between elements of $\mathfrak{sl}_2$ and V_i is skew-symmetric and V_i is in fact an irreducible Lie $\mathfrak{sl}_2$-module. If $\delta_i = 0$, on the other hand, is the case $\lambda = 0$, and this case corresponds with the non-Lie Leibniz irreducible $\mathfrak{sl}_2$-module. Finally since the underlying vector space of any Leibniz superalgebra is in fact a $\mathbb{Z}_2$-graded vector space we deduce the expression of the remaining products. □

Theorem 6.1 *Let L be under the conditions of Proposition 6.1, then L is isomorphic to one of the following Leibniz superalgebras expressed with respect to the basis* $\{e, h, f, v_0^i, v_1^i, \dots v_{n_i}^i;\ 1 \le i \le k\}$ *by the following multiplication table:*

$$
\begin{array}{ll}
[e,h]=2e,\ [h,e]=-2e, & [h,f]=2f,\ [f,h]=-2f,\ [e,f]=h,\ [f,e]=-h,\\
[v_j^i,h]=(n_i-2j)v_j^i, & 0\le j\le n_i,\ 1\le i\le k,\\
[v_j^i,f]=v_{j+1}^i, & 0\le j\le n_i-1,\ 1\le i\le k,\\
[v_j^i,e]=-j(n_i+1-j)v_{j-1}^i, & 1\le j\le n_i,\ 1\le i\le k,\\
[h,v_j^i]=-\delta_i(n_i-2j)v_j^i, & 0\le j\le n_i,\ 1\le i\le k,\\
[f,v_j^i]=-\delta_i v_{j+1}^i, & 0\le j\le n_i-1,\ 1\le i\le k,\\
[e,v_j^i]=\delta_i j(n_i+1-j)v_{j-1}^i, & 1\le j\le n_i,\ 1\le i\le k,
\end{array}
$$

where $\delta_i \in \{0, 1\}$, $1 \le i \le k$ *and the omitted products are equal to zero.*

Remark 6.1 Note that the superalgebra from the statement of the theorem, with $\delta_i = 1$ for all i is the only one that is a Lie superalgebra, the remaining ones are non-Lie Leibniz superalgebras.

Proof Note that the proof is equivalent to the proof when $k = 2$, since in order to show that the products

$$[v_m^i, v_j^l] = \alpha_{mj}^{il} e + \beta_{mj}^{il} h + \gamma_{mj}^{il} f$$

vanish we only have to distinguish two cases $i = l$ (both vectors in the same subspace V_i, being V_i a $\mathfrak{sl}_2$-module) or $i \ne l$ (each vector in a different subspace, V_i and V_l respectively). Therefore without loss of generality we can suposse $k = 2$.

Therefore, let $\{e, h, f, v_0^1, v_1^1, \dots v_{n_1}^1, v_0^2, v_1^2, \dots v_{n_2}^2\}$ be a basis of L. In order to prove the theorem, we distinguish three cases: $a)$ $\delta_1 = \delta_2 = 1$; $b)$ $\delta_1 = 1,\ \delta_2 = 0$ (or $\delta_1 = 0,\ \delta_2 = 1$) and $c)$ $\delta_1 = \delta_2 = 0$. In all these cases, the used tools are based on Leibniz superidentity and the induction method. Next, we only present in details the first case and analogously (using the same techniques) it could be obtained the other two cases.

Case a: $\delta_1 = \delta_2 = 1$. Using Leibniz superidentity, we have the following constraints:

Leibniz superidentity	Constraint	
$\{v_0^1, h, v_j^1\},\ 0 \le j \le n_1,$	$\begin{cases} \alpha_{0j}^{11} = 0, & j \ne n_1 - 1, \\ \beta_{0j}^{11} = 0, & j \ne n_1, \\ \gamma_{0j}^{11} = 0, \end{cases}$	$\begin{cases} [v_0^1, v_j^1] = 0,\ 0 \le j \le n_1 - 2, \\ [v_0^1, v_{n_1-1}^1] = \alpha_{0n_1-1}^{11} e, \\ [v_0^1, v_{n_1}^1] = \beta_{0n_1}^{11} h, \end{cases}$
$\{v_0^1, v_0^1, v_{n_1}^1\},$	$\beta_{0n_1}^{11} = 0,$	$[v_0^1, v_{n_1}^1] = 0,$
$\{v_0^1, e, v_{n_1}^1\},$	$\alpha_{0n_1}^{11} = 0,$	$[v_0^1, v_{n_1-1}^1] = 0,$
$\{v_0^1, v_0^1, v_j^2\},\ 0 \le j \le n_2,$	$\beta_{0j}^{12} = \gamma_{0j}^{12} = 0,$	$[v_0^1, v_j^2] = \alpha_{0j}^{12} e,$
$\{v_0^1, v_i^1, v_j^1\},\ 0 \le i,$ $j \le n_1,\ i \ne 0,$	$\beta_{ij}^{11} = \gamma_{ij}^{11} = 0,$	$[v_i^1, v_j^1] = \alpha_{ij}^{11} e,$
$\{v_0^1, v_1^1, v_j^2\},\ 0 \le j \le n_2,$	$\beta_{1j}^{12} = \alpha_{0j}^{12},\ \gamma_{1j}^{12} = 0,$	

$\{v_0^1, v_2^1, v_j^2\},\ 0 \le j \le n_2,$	$\beta_{2j}^{12} = 0,\ \gamma_{2j}^{12} = 2(n_1 - 1)\alpha_{0j}^{12},$
$\{v_0^1, v_i^1, v_j^2\},\ 3 \le i \le n_1,\ 0 \le j \le n_2,$	$\beta_{ij}^{12} = \gamma_{ij}^{12} = \alpha_{0j}^{12} = 0,$
$\{v_0^1, v_i^2, v_j^1\},\ 1 \le i \le n_1,\ 0 \le j \le n_2,$	$\beta_{ij}^{21} = \gamma_{ij}^{21} = 0,$
$\{v_0^1, v_i^2, v_j^2\},\ 0 \le i, j \le n_2,$	$\beta_{ij}^{22} = \gamma_{ij}^{22} = 0.$

We summarize all these considerations in the following table:

$$\begin{aligned}
&[v_0^1, v_i^1] = [v_0^1, v_j^2] = 0,\ 0 \le i \le n_1,\ 0 \le j \le n_2,\\
&[v_i^1, v_j^{n_k}] = \alpha_{ij}^{1n_k} e,\ 1 \le i \le n_1,\ 0 \le j \le n_k,\ 1 \le k \le 2,\\
&[v_i^2, v_j^{n_k}] = \alpha_{ij}^{2n_k} e,\ 0 \le i \le n_2,\ 0 \le j \le n_k,\ 1 \le k \le 2.
\end{aligned}$$

For finish this proof, we should proceed by induction on i. We have that $[v_0^1, v_i^1] = [v_0^1, v_j^2] = 0$, $0 \le i \le n_1$ and $0 \le j \le n_2$. We fix j, $0 \le j \le n_l$ and $1 \le l \le 2$, and we supose that $[v_{i-1}^1, v_j^l] = 0$. Then by Leibniz superidentity on the following triple elements $\{v_{i-1}^1, f, v_j^l\}$ permits to get $[v_i^1, v_j^l] = 0$ for $1 \le i \le n_1$ and $0 \le j \le n_l$, $1 \le l \le 2$.

Finally, from Leibniz superidentity on $\{v_1^1, v_i^2, v_j^l\}$ with $0 \le i \le n_2$, $0 \le j \le n_l$ and $1 \le l \le 2$ we obtain $[v_i^2, v_j^l] = 0$. $\square$

7 $L_0 = \mathfrak{sl}_2 \dot{+} I$, and $L_1 = \bigoplus_{i=1}^{k} J_i$ with Each J_i a Leibniz Irreducible $(\mathfrak{sl}_2 \dot{+} I)$-Module

Now we consider $L_0 = \mathfrak{sl}_2 \dot{+} I$ where I is the Leibniz kernel and L_1 with structure of a finite direct sum of Leibniz irreducible $(\mathfrak{sl}_2 \dot{+} I)$-modules. It can be seen that following the spirit of Propositions 4.1 and 6.1 we get the next result.

Proposition 7.1 *Let $L = L_0 \oplus L_1 = (\mathfrak{sl}_2 \dot{+} I) \oplus J$ be a Leibniz superalgebra with J a finite direct sum of Leibniz irreducible L_0-modules, i.e. $J = \bigoplus_{i=1}^{k} J_i$ with J_i a Leibniz irreducible $(\mathfrak{sl}_2 \dot{+} I)$-module. Then, L admits a basis $\{e, h, f, x_0, x_1, \dots x_m, v_0^i, v_1^i, \dots v_{n_i}^i;$ $1 \le i \le k\}$ with respect to which the multiplication table is as follows:*

$$\begin{aligned}
&[e, h] = 2e,\ [h, e] = -2e,\ [h, f] = 2f,\ [f, h] = -2f,\ [e, f] = h,\ [f, e] = -h,\\
&[x_t, h] = (m - 2t)x_t, && 0 \le t \le m,\\
&[x_t, f] = x_{t+1}, && 0 \le t \le m - 1,\\
&[x_t, e] = -t(m + 1 - t)x_{t-1}, && 1 \le t \le m,\\
&[v_j^i, h] = (n_i - 2j)v_j^i, && 0 \le j \le n_i,\ 1 \le i \le k,\\
&[v_j^i, f] = v_{j+1}^i, && 0 \le j \le n_i - 1,\ 1 \le i \le k,
\end{aligned}$$

$$
\begin{aligned}
&[v^i_j, e] = -j(n_i + 1 - j)v^i_{j-1}, && 1 \le j \le n_i,\ 1 \le i \le k,\\
&[h, v^i_j] = -\delta_i(n_i - 2j)v^i_j, && 0 \le j \le n_i,\ 1 \le i \le k,\\
&[f, v^i_j] = -\delta_i v^i_{j+1}, && 0 \le j \le n_i - 1,\ 1 \le i \le k,\\
&[e, v^i_j] = \delta_i j(n_i + 1 - j)v^i_{j-1}, && 1 \le j \le n_i,\ 1 \le i \le k,\\
&[v^i_p, v^l_j] = \alpha^{il}_{pj} e + \beta^{il}_{pj} h + \gamma^{il}_{pj} f + \sum_{t=0}^{m} \lambda^{ilt}_{pj} x_t, && 0 \le p \le n_i,\ 0 \le j \le n_l,\ 1 \le i, l \le k,
\end{aligned}
$$

where $\delta_i \in \{0, 1\}$, $1 \le i \le k$ and the omitted products are equal to zero.

Theorem 7.1 *Let L be under the conditions of Proposition 7.1, with m an odd positive integer and all of the n_i with the same parity (simultaneously either odd positive integers or even ones), then L is isomorphic to one of the following Leibniz superalgebras expressed with respect to the basis $\{e, h, f, x_0, x_1, \dots x_m, v^i_0, v^i_1, \dots v^i_{n_i};\ 1 \le i \le k\}$ by the following multiplication table:*

$$
\begin{aligned}
&[e, h] = 2e,\ [h, e] = -2e,\ [h, f] = 2f,\ [f, h] = -2f,\ [e, f] = h,\ [f, e] = -h,\\
&[x_t, h] = (m - 2t)x_t, && 0 \le t \le m,\\
&[x_t, f] = x_{t+1}, && 0 \le t \le m - 1,\\
&[x_t, e] = -t(m + 1 - t)x_{t-1}, && 1 \le t \le m,\\
&[v^i_j, h] = (n_i - 2j)v^i_j, && 0 \le j \le n_i,\ 1 \le i \le k,\\
&[v^i_j, f] = v^i_{j+1}, && 0 \le j \le n_i - 1,\ 1 \le i \le k,\\
&[v^i_j, e] = -j(n_i + 1 - j)v^i_{j-1}, && 1 \le j \le n_i,\ 1 \le i \le k,\\
&[h, v^i_j] = -\delta_i(n_i - 2j)v^i_j, && 0 \le j \le n_i,\ 1 \le i \le k,\\
&[f, v^i_j] = -\delta_i v^i_{j+1}, && 0 \le j \le n_i - 1,\ 1 \le i \le k,\\
&[e, v^i_j] = \delta_i j(n_i + 1 - j)v^i_{j-1}, && 1 \le j \le n_i,\ 1 \le i \le k,
\end{aligned}
$$

where $\delta_i \in \{0, 1\}$, $1 \le i \le k$ and the omitted products are equal to zero.

Proof Notice that the proof is equivalent to the proof when $k = 2$, since while checking that the products

$$[v^i_p, v^l_j] = \alpha^{il}_{pj} e + \beta^{il}_{pj} h + \gamma^{il}_{pj} f + \sum_{t=0}^{m} \lambda^{ilt}_{pj} x_t$$

vanish we only have to distinguish two cases $i = l$ or $i \neq l$. Consequently, without loss of generality we can suppose $k = 2$. Moreover the case $i = l$ was already seen throughout the proof of Theorem 4.1, then only rest to study de case $i \neq l$, i.e., $i = 1$ and $l = 2$. Thus, suppose

$$[v^1_i, v^2_j] = \alpha^{12}_{ij} e + \beta^{12}_{ij} h + \gamma^{12}_{ij} f + \sum_{t=0}^{m} \lambda^{12t}_{ij} x_t,\ 0 \le i \le n_1,\ 0 \le j \le n_2.$$

By the graded Leibniz identity $[x_0, [v^1_i, v^2_j]] = [[x_0, v^1_i], v^2_j] + [[x_0, v^2_j], v^1_i]$ it is obtained $\beta^{12}_{ij} m x_0 + \gamma^{12}_{ij} x_1 = 0$, being then $\beta^{12}_{ij} = \gamma^{12}_{ij} = 0$ for all $i,\ j$. Next, the graded Leibniz identity $[x_1, [v_i^1, v_j^2]] = [[x_1, v_i^1], v_j^2] + [[x_1, v_j^2], v_i^1]$ leads to $-m\alpha_{ij}^{12}x_0 = 0$, and thus $\alpha_{ij}^{12} = 0$ for all $i,\ j$. Hence then, we can suppose without loss of generality that

$$[v_i^1, v_j^2] = \sum_{t=0}^{m} \lambda_{ij}^{12t} x_t,\ 0 \leq i \leq n_1,\ 0 \leq j \leq n_2.$$

Using now the following graded Leibniz identity $[v_i^1, [v_j^2, h]] = [[v_i^1, v_j^2], h] - [[v_i^1, h], v_j^2]$ we have $[v_i^1, (n_2 - 2j)v_j^2] = [[v_i^1, v_j^2], h] - [(n_1 - 2i)v_i^1, v_j^2]$, that is $(n_2 - 2j + n_1 - 2i)[v_i^1, v_j^2] = \sum_{t=0}^{m} \lambda_{ij}^{12t}(m - 2t)x_t$ and then we obtain

$$(n_2 - 2j + n_1 - 2i)(\sum_{t=0}^{m} \lambda_{ij}^{12t} x_t) = \sum_{t=0}^{m} \lambda_{ij}^{12t}(m - 2t)x_t. \tag{7.1}$$

Since m is odd and n_1 and n_2 are both at the same time either even or odd integers, then from the above equation we deduce $\lambda_{ij}^{12t} = 0$ for all t on account of its coefficients on both sides of the equation, on the left-hand side the coeffients are always even numbers in contrast with the coefficients on the other side, odd ones. Thus, $[v_i^1, v_j^2] = 0$ for all $i,\ j$. Analgously it can be shown that $[v_j^2, v_i^1] = 0$ for all $i,\ j$, obtaining then the (non-Lie) Leibniz superalgebras of the statement of the theorem. □

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