On Filiform and 2-Filiform Leibniz Algebras of Maximum Length

J. M. Cabezas*, **L. M. Camacho, and I. M. Rodríguez**

Abstract. Leibniz algebras appear as a generalization of Lie algebras. The classification of naturally graded p-filiform Lie algebras is known. Several authors have studied the naturally graded p-filiform Leibniz algebras for any p with $p \geq 0$.
Gómez, Jiménez-Merchán and Reyes have investigated families of nilpotent Lie algebras with other types of non-natural gradation, a gradation with a large number of subspaces. The algebras with maximum number of subspaces in the gradation will be called maximum length algebras.
In this work we deal with the classification of filiform and 2-filiform Leibniz algebras of maximum length.
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1. Introduction

In the course of her work on the cohomology on nilpotent Lie algebras [16], M. Verge classified naturally graded filiform Lie algebras.

In [11] the authors introduce the notion of the "length" of a Lie algebra, and study families of nilpotent Lie algebras with a gradation with a large number of subspaces. This condition facilitates the study of some cohomological properties for such algebras (see [4, 11, 14]). For such a length of the gradation, the main interest are those algebras whose length is as large as possible.

The natural gradation of nilpotent Leibniz algebras, the subspaces of gradation, and the existence of an appropriate homogeneous basis (needed to obtain the classification) are derived from the central descending sequence.

In a way, the gradations with n subspaces are the finite connected gradations with the greatest possible number of non-zero subspaces; they will be called maximum length gradations. The algebras with maximum length gradations will be called maximum length algebras.

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An algebra $\mathcal{L}$ over a field F is called Leibniz algebra if it verifies the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for any elements $x, y, z \in \mathcal{L}$ and where $[.,.]$ is a multiplication in $\mathcal{L}$.

When in $\mathcal{L}$ the identitiy $[x,x] = 0$ holds, then the Leibniz identity coincides with the Jacobi identity, thus we can say that Leibniz algebras are a generalization of Lie algebras.

For a given algebra we define the sequence

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad \text{with } k \geq 1$$

An n-dimensional Leibniz algebra $\mathcal{L}$ is called *zero-filiform* if $\dim(\mathcal{L}^i) = n+1-i$, $1 \leq i \leq n+1$.

An n-dimensional Leibniz algebra $\mathcal{L}$ is called *filiform* if $\dim(\mathcal{L}^i) = n - i$, with $2 \leq i \leq n$.

The natural gradation is defined as follows. For a Leibniz algebra $\mathcal{L}$ we consider $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1}$. Then we put $\mathrm{gr}(\mathcal{L}) \approx \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$. If $\mathcal{L} \approx \mathrm{gr}(\mathcal{L})$, we say that $\mathcal{L}$ is a *naturally graded Leibniz algebra.*

Let x be a nilpotent element of set $\mathcal{L} \setminus \mathcal{L}^2$. For the nilpotent operator of right multiplication R_x we define a decreasing sequence $C(x) = (n_1, \ldots, n_k)$ of the dimensions of Jordan blocks of the operator R_x. In the former set of sequences we consider lexicographic order.

The sequence $C(\mathcal{L}) = \max\{\ C(x)_{x \in \mathcal{L} \setminus \mathcal{L}^2}\}$ is called the *characteristic sequence* of the algebra $\mathcal{L}$.

A Leibniz algebra $\mathcal{L}$ is called *p-filiform* if $C(L) = (n-p, \underbrace{1, \ldots, 1}_{p})$, where $p \geq 0$. If $p = 2$, $\mathcal{L}$ is called *quasi-filiform.* If $p = 1$, $\mathcal{L}$ is *filiform* and if $p = 0$, *zero-filiform.*

The set $R(\mathcal{L}) = \{x \in \mathcal{L} : [y,x] = 0,\ \forall y \in \mathcal{L}\}$ is said to be the *right annihilator* of $\mathcal{L}$.

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded Leibniz algebra, that is, $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} V_i$, where $[V_i, V_j] \subseteq V_{i+j}$ for any $i, j \in \mathbb{Z}$ with a finite number of nonnull spaces V_i.

We will say that a nilpotent Leibniz algebra $\mathcal{L}$ admits the *connected gradation* $\mathcal{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$, if $V_{k_i} \neq 0$ for any i, $(k_1 \leqslant i \leqslant k_t)$.

The number $\mathrm{len}(\bigoplus \mathcal{L}) = k_t - k_1 + 1$ is called the *length of gradation.* A gradation is called of *maximum length,* if $\mathrm{len}(\bigoplus \mathcal{L}) = \dim \mathcal{L}$.

We define the length of $\mathcal{L}$ by $\mathrm{len}(\mathcal{L}) = \max\{\mathrm{len}(\bigoplus \mathcal{L}) = k_t - k_1 + 1$ such that
$\mathcal{L} = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_t}$ is a connected gradation$\}$.

A Leibniz algebra $\mathcal{L}$ is called of maximum length if $\mathrm{len}(\mathcal{L}) = \dim \mathcal{L}$.

Example Let ZF_n be the 0-filiform Leibniz algebra with dimension n [3]. This algebra ZF_n is an algebra of maximum length. In fact, taking $V_i = \langle e_i \rangle$, $(1 \leqslant i \leqslant n)$, we obtain $\mathcal{L} = V_1 \oplus V_2 \oplus \cdots V_n$, where $[V_i, V_j] \subseteq V_{i+j}$.

Note that ZF_n is the unique Leibniz algebra for which the naturally gradation coincides with the maximum gradation.

The cases of 0-filiform and 1-filiform have already been studied [3]. Quasifiliform Lie algebras have the characteristic sequence $(n-2,1,1)$. In Leibniz algebras there are two possibilities, $(n-2,1,1)$ (case 2-filiform) and $(n-2,2)$. The idea of length of an algebra is well known for Lie algebras and can be generalized to Leibniz algebras. In the case of Lie algebras some authors [8, 12] have considered on graded Lie algebras not only one natural gradation but a gradation with a large number of subspaces. Such a gradation facilitates the study of certain cohomological properties of these algebras.

The present work is devoted to the study of filiform and 2-filiform Leibniz algebras which admit a gradation by a maximum number of nonnull homogeneous spaces.

The following theorem is true.

Theorem 1.1. [12] *Let $\mathfrak{g}$ be a complex n-dimensional non split filiform Lie algebra of maximum length. Then, it is isomorphic to one of the following pairwise non isomorphic algebras*

$$L_n \left\{ [X_0, X_i] = X_{i+1}, \quad 1 \leqslant i \leqslant n-2. \right.$$

$$R_n \left\{ \begin{array}{ll} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2 \\ [X_1, X_j] = X_{2+j}, & 2 \leqslant j \leqslant n-3. \end{array} \right.$$

$K_n (n \geq 8)$:

$$\left\{ \begin{array}{ll} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2 \\ \left[X_i, X_{2\lfloor (n-2)/2 \rfloor -1-i}\right] = (-1)^{i-1} X_{2\lfloor (n-2)/2 \rfloor}, & 1 \leqslant i \leqslant \lfloor \frac{n-4}{2} \rfloor \\ \left[X_i, X_{2\lfloor (n-2)/2 \rfloor -i}\right] = (-1)^{i-1} (\lfloor (n-2)/2 \rfloor - i) X_{2\lfloor (n-2)/2 \rfloor +1}, & 1 \leqslant i \leqslant \lfloor \frac{n-4}{2} \rfloor \\ [X_i, X_{n-2-i}] = \frac{1}{2} (-1)^i (i-1)(n-3-i) \alpha X_{n-1}, & 2 \leqslant i \leqslant \frac{n-3}{2} \end{array} \right.$$

where $\alpha = 0$, if n is even and $\alpha = 1$ if n is odd.

$W_n (n \geq 7)$:

$$\left\{ \begin{array}{ll} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2 \\ [X_i, X_j] = \dfrac{6(i-1)!(j-1)!(j-i)}{(i+j)!} X_{i+j+1}, & 1 \leqslant i \leqslant \lfloor \frac{n-3}{2} \rfloor, \ i \leqslant j \leqslant n-2-i. \end{array} \right.$$

$Q_n (n \geq 7)$:

$$\left\{ \begin{array}{ll} [X_0, X_i] = X_{i+1}, & 1 \leqslant i \leqslant n-2 \\ [X_i, X_{n-2-i}] = (-1)^{i-1} X_{n-1}, & 1 \leqslant i \leqslant \frac{n-5}{2}. \end{array} \right.$$

where n is odd.

Theorem 1.2. [3] *An arbitrary n-dimensional naturally graded filiform com-*

plex non Lie Leibniz algebra is isomorphic to the following not isomorphic algebras

$$NGF1: \quad [e_1,e_1]=e_3, \quad [e_i,e_1]=e_{i+1}, \qquad 2\leqslant i\leqslant n-1.$$

$$NGF2: \quad [e_1,e_1]=e_3, \quad [e_i,e_1]=e_{i+1}, \qquad 3\leqslant i\leqslant n-1.$$

$$NGF3: \left\{\begin{array}{ll} [e_i,e_1]=-[e_1,e_i]=e_{i+1}, & 2\le i\le n-1 \\ [e_i,e_{n+1-i}]=-[e_{n+1-i},e_i]=\alpha(-1)^{i+1}e_n & 2\le i\le n-1. \end{array}\right.$$

(the rest of the products are zero).

From [6] we know that an arbitrary 2-filiform naturally graded non Lie Leibniz algebra is either split (i.e. $\mathcal{L}=ZF_{n-2}\oplus\mathbb{C}^2$ or $\mathcal{L}=NGF1_{n-1}\oplus\mathbb{C}$ or $\mathcal{L}=NGF2_{n-1}\oplus\mathbb{C}$ or $\mathcal{L}=NGF3_{n-1}\oplus\mathbb{C}$) or isomorphic to one of the following algebras ($n\ge 6$):

$$\mathrm{I}: \quad \left\{\begin{array}{ll} [e_i,e_1]=e_{i+1}, & 1\leqslant i\leqslant n-3 \\ [e_1,e_{n-1}]=e_n. & \end{array}\right.$$

$$\mathrm{II}: \quad \left\{\begin{array}{ll} [e_i,e_1]=e_{i+1}, & 1\leqslant i\leqslant n-3 \\ [e_1,e_{n-1}]=e_2+e_n & \\ [e_i,e_{n-1}]=e_{i+1} & 2\leqslant i\leqslant n-3. \end{array}\right.$$

2. Filiform Leibniz algebras of maximum length

Let $\mathcal{L}$ be an n-dimensional filiform Leibniz algebra of maximum length.

$$\mathcal{L}=V_{k_1}\oplus V_{k_2}\oplus\cdots\oplus V_{k_n}$$

where $[V_{k_i},V_{k_j}]\subseteq V_{k_i+k_j}$ and $V_{k_i}=\langle x_i\rangle$.

The results from [3] and [13] allow us to obtain the decomposition of all complex filiform Leibniz algebras into three disjoint classes.

Proposition 2.1. *Let $\mathcal{L}$ be an n-dimensional ($n>3$) complex filiform Leibniz algebra and let $\{e_1,e_2,\ldots,e_n\}$ be an adapted basis. Then, $\mathcal{L}$ is isomorphic to one* *of the following algebras:*

$$F_1 : \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1 \\ [e_1, e_2] = \alpha_4 e_4 + \alpha_5 e_5 + \cdots + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_2] = \alpha_4 e_{j+2} + \alpha_5 e_{j+3} + \cdots + \alpha_{n+2-j} e_n, & 3 \leq i \leq n \end{cases}$$

$$F_2 : \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1 \\ [e_1, e_2] = \beta_4 e_4 + \beta_5 e_5 + \cdots + \beta_n e_n, & \\ [e_2, e_2] = \gamma e_n, & \\ [e_j, e_2] = \beta_4 e_{j+2} + \beta_5 e_{j+3} + \cdots + \beta_{n+2-j} e_n, & 3 \leq i \leq n-1 \end{cases}$$

$$F_3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n-1 \\ [e_1, e_i] = -e_{i+1}, & 3 \leq i \leq n-1 \\ [e_1, e_1] = \theta_1 e_n, & \\ [e_1, e_2] = -e_3 + \theta_2 e_n, & \\ [e_2, e_2] = \theta_3 e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in lin\langle e_{i+j+1}, e_{i+j+2}, \ldots, e_n \rangle, & 2 \leq i \leq n-2, \\ & 2 \leq j \leq n-i \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1} e_n & 2 \leq i \leq n-1. \end{cases}$$

where omitted products are equal to zero, $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

Theorem 1.1 can be extended to the Leibniz algebras case, namely, the following theorem holds.

Theorem 2.2. *Let $\mathcal{L}$ be a n-dimensional non split filiform non Lie Leibniz algebra $\mathcal{L}$ of maximum length, then there exists a basis $\{y_1, y_2, \ldots, y_n\}$ of $\mathcal{L}$ such that its multiplication has the following form*

$$[y_i, y_1] = y_{i+1}, \ 2 \leqslant i \leqslant n-1$$

(the rest of the products are equal to zero).

Proof. Let $\mathcal{L}$ be an n-dimensional filiform Leibniz algebra of maximum length.

$$\mathcal{L} = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_n}$$

where $[V_{k_i}, V_{k_j}] \subseteq V_{k_i + k_j}$ and $V_{k_i} = \langle x_i \rangle$

Let $\mathcal{L}$ be isomorphic to one algebra of the family F_1.

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis where $x_i = \sum\limits_{k=1}^{n} \alpha_k^i e_k \Rightarrow \exists s \neq t : \alpha_1^s \neq 0 \neq \alpha_2^t$ $\Rightarrow \alpha_1^s = \alpha_2^t = 1$. Thus,

$$\begin{cases} x_s = e_1 + \sum\limits_{k=2}^{n} \alpha_k^s e_k \\ x_t = \alpha_1^t e_1 + e_2 + \sum\limits_{k=3}^{n} \alpha_k^t e_k \end{cases}$$

If we make the computation $[x_s, x_s]$, we obtain

$$[x_s, x_s] = (1 + \alpha_2^s)e_3 + \sum_{i\geq 4} A_i e_i$$
$$[x_t, x_t] = \alpha_1^t(1 + \alpha_1^t)e_3 + \sum_{i\geq 4} B_i e_i$$

It is possible to consider two cases:

1. $1 + \alpha_2^s \neq 0$.

Then, we denote

$$y_1 = x_s,\ y_2 = [x_s, x_s],\ \dots,\ y_{n-1} = [\dots, [x_s, x_s], \dots, x_s],\ y_n = x_t$$

with $y_i \neq 0,\ 1 \leq i \leq n-1$.

Using the table of multiplication from the family I, we obtain that $y_1 \in V_{k_s}$, $y_2 \in V_{2k_s}, \dots,\ y_{n-1} \in V_{(n-1)k_s}$.

Thus, $\mathcal{L} = V_{k_t} \oplus V_{k_s} \oplus V_{2k_s} \oplus \cdots V_{(n-1)k_s}$. According to the definition of the gradation, we have the following embedding:

$$[V_{pk_s}, V_{k_t}] \subseteq V_{pk_s+k_t} \subseteq V_{m_1k_s},$$
$$[V_{k_t}, V_{pk_s}] \subseteq V_{pk_s+k_t} \subseteq V_{m_2k_s},$$
$$[V_{k_t}, V_{k_t}] \subseteq V_{2k_s} \subseteq V_{m_3k_s}.$$

From $[V_{k_i}, V_{k_j}] \subseteq V_{k_i+k_j} \Rightarrow k_t = \delta k_s$ with $\delta \in \{-(n-2), -(n-3), \dots, -1, 0,\ 1, \dots, n-2\}$. We compute $[V_{k_t}, V_{k_t}]$, $[V_{pk_s}, V_{k_t}]$, $[V_{k_t}, V_{pk_s}]$.

1.1. If $\delta \in \{1, \dots, n-2\} \Rightarrow V_{k_t} = V_{\delta k_s} \Rightarrow \dim(V_{\delta k_s}) = 2$. That is contradiction, this case is not possible.

1.2. If $\delta \in \{-n+2, -n+3, \dots, 1, 0\}$, we have that

$$[V_{pk_s}, V_{k_t}] = [V_{k_t}, V_{pk_s}] = [\langle y_p\rangle, \langle y_n\rangle] \subset V_{(p+\delta)k_s} = \langle y_{p+\delta}\rangle$$

and $y_p \in \mathcal{L}^p \backslash \mathcal{L}^{p+1}$, $y_n \in \mathcal{L} \backslash \mathcal{L}^2$, $y_{p+\delta} \in \mathcal{L}^{p+\delta} \backslash \mathcal{L}^{p+\delta-1}$. Thus, $[y_p, y_n] \in \mathcal{L}^{p+1} \backslash \mathcal{L}^{p+2}$ as $p+1 > p+\delta$ we can conclude that $[V_{k_t}, V_{k_t}] = [V_{pk_s}, V_{k_t}] = [V_{k_t}, V_{pk_s}] = 0$. Then the Leibniz algebra of maximum length is split.

2. $\alpha_2^s = -1$.

Then,

$$x_s = e_1 - e_2 + \alpha_3^s e_3 + \cdots + \alpha_n^s e_n,$$
$$x_t = \alpha_1^t e_1 + e_2 + \alpha_3^t e_3 + \cdots + \alpha_n^t e_n.$$

$x_s, x_t \in \mathcal{L} \backslash \mathcal{L}^2$.

Computing the following multiplication, we have that

$$[x_s, x_s] = \sum_{i\geq 4} A_i e_i$$
$$[x_t, x_t] = \alpha_1^t(1 + \alpha_1^t)e_3 + \sum_{i\geq 4} B_i e_i$$
$$[x_s, x_t] = \sum_{i\geq 4} C_i e_i$$
$$[x_t, x_s] = (1 + \alpha_1^t)e_3 + \sum_{i\geq 4} D_i e_i$$

It is necessary to make out two cases:

2.1. $\alpha_1^t \neq 0$.
Note that $\alpha_1^t \neq -1$, if $\alpha_1^t = -1$ then x_t is not a generator of $\mathcal{L}$. Analogously to case 1 for x_t, a split Leibniz algebra is obtained.

2.2. $\alpha_1^t = 0$.
We have

$$x_s = e_1 - e_2 + \alpha_3^s e_3 + \cdots + \alpha_n^s e_n,$$
$$x_t = e_2 + \alpha_3^t e_3 + \cdots + \alpha_n^t e_n.$$

Putting $y_1 = x_s,\ y_2 = x_t,\ y_3 = [x_t, x_s],\ y_4 = [[x_t, x_s], x_s], \ldots,\ y_n = [\ldots[[x_t, x_s], x_s], \ldots, x_s]$ (multiplying by x_s $n-2$ times) and $y_i \neq 0,\ 1 \leq i \leq n,\ y_1, y_2 \in \mathcal{L}\backslash\mathcal{L}^2,\ y_i \in \mathcal{L}^{i-1}\backslash\mathcal{L}^i$ with $3 \leq i \leq n$. We have the following multiplication:

$$[y_i, y_1] = y_{i+1}, \qquad 2 \leq i \leq n-1$$

Thus, $y_1 \in V_{k_s},\ k_s \neq 0,\ y_2 \in V_{k_t},\ k_t \neq k_s\ \ldots,\ y_n \in V_{k_t+(n-2)k_s}$, so $\mathcal{L} = V_{k_t} \oplus V_{k_s} \oplus V_{k_t+k_s} \oplus V_{k_t+2k_s} \oplus \cdots \oplus V_{k_t+(n-2)k_s}$.
Following a similar reasoning, $[V_{k_i}, V_{k_j}] \subseteq V_{k_i+k_j} \Rightarrow k_t = \delta k_s$ with $\delta \in \{-(n-4), -(n-5), \ldots, -1, 0, 2, \ldots, n-3\}$.

2.2.1. $\delta \in \{2, \ldots, n-3\}$.

$$V_{k_t} = V_{\delta k_s} = [\ldots[[V_{k_s}, V_{k_s}], V_{k_s}], \ldots, V_{k_s}]\ (\delta \text{ times })$$

and that contradicts $V_{k_t} \in \mathcal{L}\backslash\mathcal{L}^2$.

2.2.2. $\delta \in \{4-n, \ldots, -1\}$.

$$[V_{k_t+pk_s}, V_{k_t}] = [V_{k_t+pk_s}, V_{\delta k_s}] \subset V_{k_t+(p+\delta k_s)}$$

and we have that $V_{k_t+pk_s} \in \mathcal{L}^{p+1}\backslash\mathcal{L}^{p+2},\ V_{\delta k_s} \in \mathcal{L}^1\backslash\mathcal{L}^2,\ V_{k_t+(p+\delta)k_s} \in \mathcal{L}^{p+\delta+2}$ but $p+2 \geq p+\delta+2$, that is not possible.

2.2.3. $\delta = 0,\ \ V_{k_t} = V_0 \Rightarrow V_{k_t+k_s} = \langle y_3 \rangle = V_{k_s} = \langle y_1 \rangle \Rightarrow \dim(V_{k_s}) = 2$. That is not possible.

Thus, we conclude that

$$[V_{k_s}, V_{k_s}] = [V_{k_t}, V_{k_t}] = [V_{k_t+pk_s}, V_{k_t}] = 0$$

obtaining the algebra of the theorem.

Let $\mathcal{L}$ be isomorphic to one algebra of the family F_2.

We consider a decomposition of the basis $\{x_1, x_2, \ldots, x_n\}$ into the basis $\{e_1, \ldots, e_n\}$. That is, $x_m = \sum_{i=1}^{n} \alpha_i^m e_i$. It is evident that there exists $s \in \mathbf{N}$ such that $\alpha_1^s \neq 0$. We can suppose that without loss of generality $\alpha_1^s = 1$. Then $x_s = e_1 + \alpha_2^s e_2 + \cdots + \alpha_n^s e_n$ and $x_s \in \mathcal{L}\backslash\mathcal{L}^2$.

Consider the product: $[x_s, x_s] = e_3 + \sum_{i\geq 4} e_i$. Furthermore, thinking as in case F_1, we obtain a split Leibniz algebra.

Let $\mathcal{L}$ be isomorphic to one algebra of the family F_3.

We choose generators from the homogeneous basis $x_s = e_1+a_2e_2+\cdots+a_ne_n$ and $x_t = b_1e_1 + e_2 + b_3e_3 + \cdots + b_ne_n$.

Consider the following products

$$[x_s, x_s] = (\theta_1 + a_2\theta_2 + a_2^2\theta_3)e_n,$$

$$[x_t, x_t] = (b_1^2\theta_1 + b_1\theta_2 + \theta_3)e_n,$$

$$[\ldots[x_t, \underbrace{x_s], \ldots, x_s]}_{i-times} = (1 - a_2b_1)e_{i+2} + (*)e_{i+3} + \cdots + (*)e_n,\ 1 \leq i \leq n-2,$$

where $1 - a_2b_1 \neq 0$.

We can choose $y_1 = x_s,\ y_2 = x_t,\ y_i = [\ldots[x_t, \underbrace{x_s], \ldots, x_s]}_{(i-2)-times},\ 3 \leq i \leq n$.

Therefore, $\langle y_1\rangle \subseteq V_{k_s},\ \langle y_2\rangle \subseteq V_{k_t},\ \langle y_i\rangle \subseteq V_{k_t+(i-2)k_s}$ $3 \leq i \leq n$ and $[y_i, y_1] = y_{i+1},\ 2 \leq i \leq n-1$.

If we compute $[y_1, y_1] = \frac{\theta_1+a_2\theta_2+a_2^2\theta_3}{1-a_2b_1} y_n = \theta y_n$.

1. $[y_1, y_1] \neq 0$. We have that $[y_1, y_1] \in V_{k_t+(n-2)k_s}$. Moreover, $[V_{k_s}, V_{k_s}] = V_{2k_s} = V_{k_t+(n-2)k_s} \Rightarrow 2k_s = k_t + (n-2)k_s \Rightarrow k_t = (4-n)k_s$. However, this is not possible because $V_{k_t+(n-3)k_s} = V_{k_s} \Rightarrow \dim(V_{k_s}) = 2$, that is a contradiction.

2. $[y_1, y_1] = 0$. We have that $[y_2, y_2] = \theta' y_n \in V_{k_t+(n-2)k_s}$.

2.1. $[y_2, y_2] \neq 0$. $y_2 \in V_{k_t} \Rightarrow [y_2, y_2] \in V_{2k_t}$ and $y_n \in V_{k_t+(n-2)k_s} \Rightarrow V_{k_t} = (n-2)k_s$. Thus, we have the homogeneous spaces V_{k_s}, $V_{(n-2)k_s}$, $V_{(n-1)k_s}$, V_{nk_s}, …, $V_{(2n-4)k_s}$. Since that is a conneted gradation, $n = 3$ and for $n = 3$ the family F_3 does not exist.

2.2. $[y_2, y_2] = 0$, since $[y_1, y_2] = -y_3 + \theta_2' y_n \Rightarrow [y_1, y_2] \in V_{k_t+k_s}$ but $-y_3 + \theta_2' y_n \in V_{k_t+k_s} + V_{k_t+(n-2)k_s} \Rightarrow \theta_2' = 0$ Using the Leibniz identity , the table of multiplication is:

$$\begin{array}{ll} [y_i, y_1] = y_{i+1} & 2 \leq i \leq n-1 \\ [y_1, y_i] = -y_{i+1} & 2 \leq i \leq n-1 \end{array}$$

and we obtain a Lie algebra. □

3. 2-filiform Leibniz algebras of maximum lenght

The notions 2-filiform and quasi-filiform for Lie algebras coincide. The classification of quasifiliform Lie algebras of maximum length is described in [12]. From now on we will only consider non Lie Leibniz algebras.

Let $\mathcal{L}$ be an n-dimensional 2-filiform non Lie Leibniz algebra of maximum length, then

$\mathcal{L} = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_t}$, where $[V_{k_i}, V_{k_j}] \subseteq V_{k_i+k_j}$ and $V_{k_i} = \langle X_i\rangle$.

Proposition 3.1. [3, 6] *Let $\mathcal{L}$ be a 2-filiform non Lie Leibniz algebra, then from the classification of 2-filiform naturally graded Leibniz algebras we have that $\mathcal{L}$ belongs to one of the following families*

$KF_1:$

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-3 \\ [e_i, e_{n-1}] = \alpha_{i,i+2}e_{i+2} + \alpha_{i,i+3}e_{i+3} + \ldots + \alpha_{i,n-2}e_{n-2}, & 1 \leqslant i \leqslant n-4 \\ [e_i, e_n] = \beta_{i,i+2}e_{i+2} + \beta_{i,i+3}e_{i+3} + \ldots + \beta_{i,n-2}e_{n-2}, & 1 \leqslant i \leqslant n-4 \\ [e_{n-1}, e_{n-1}] = \beta_3e_3 + \beta_4e_4 + \ldots + \beta_{n-2}e_{n-2}, & \\ [e_n, e_{n-1}] = \gamma_3e_3 + \gamma_4e_4 + \ldots + \gamma_{n-2}e_{n-2}, & \\ [e_n, e_n] = \theta_3e_3 + \theta_4e_4 + \ldots + \theta_{n-2}e_{n-2}, & \\ [e_{n-1}, e_n] = \lambda_3e_3 + \lambda_4e_4 + \ldots + \lambda_{n-2}e_{n-2} & \end{cases}$$

where $\mathrm{gr}(\mathcal{L}) = ZF_{n-2} \oplus \mathbb{C}^2,\ ZF_{n-2} = \langle e_1, e_2, \ldots, e_{n-2}\rangle,\ \mathbb{C}^2 = \langle e_{n-1}, e_n\rangle$

$KF_2:$

$$\begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-2 \\ [e_1, e_2] = \alpha_4e_4 + \alpha_5e_5 + \ldots + \alpha_{n-2}e_{n-2} + \theta e_{n-1}, & \\ [e_i, e_2] = \alpha_4e_{i+2} + \alpha_5e_{i+3} + \ldots + \alpha_{n+1-i}e_{n-1}, & 2 \leqslant i \leqslant n-3 \\ [e_1, e_n] = \beta_{1,4}e_4 + \beta_{1,5}e_5 + \ldots + \beta_{1,n-1}e_{n-1}, & \\ [e_i, e_n] = \beta_{i,i+2}e_{i+2} + \beta_{i,i+3}e_{i+3} + \ldots + \beta_{i,n-1}e_{n-1}, & 2 \leqslant i \leqslant n-3 \\ [e_n, e_n] = \theta_4e_4 + \theta_5e_5 + \ldots + \theta_{n-1}e_{n-1}, & \\ [e_n, e_2] = \lambda_4e_4 + \lambda_5e_5 + \ldots + \lambda_{n-1}e_{n-1}, & \\ [e_n, e_1] = \mu_4e_4 + \mu_5e_5 + \ldots + \mu_{n-1}e_{n-1} & \end{cases}$$

where $\mathrm{gr}(\mathcal{L}) = NGF1_{n-1} \oplus \mathbb{C},\ NGF1_{n-1} = \langle e_1, e_2, \ldots, e_{n-1}\rangle,\ \mathbb{C} = \langle e_n\rangle.$

$KF_3:$

$$\begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 3 \leqslant i \leqslant n-2 \\ [e_1, e_2] = \alpha_4e_4 + \alpha_5e_5 + \ldots + \alpha_{n-1}e_{n-1}, & \\ [e_2, e_2] = \gamma e_{n-1} & \\ [e_i, e_2] = \alpha_4e_{i+2} + \alpha_5e_{i+3} + \ldots + \alpha_{n+1-i}e_{n-1}, & 3 \leqslant i \leqslant n-3 \\ [e_1, e_n] = \beta_{1,4}e_4 + \beta_{1,5}e_5 + \ldots + \beta_{1,n-1}e_{n-1}, & \\ [e_i, e_n] = \beta_{i,i+2}e_{i+2} + \beta_{i,i+3}e_{i+3} + \ldots + \beta_{i,n-1}e_{n-1}, & 2 \leqslant i \leqslant n-3 \\ [e_n, e_n] = \theta_4e_4 + \theta_5e_5 + \ldots + \theta_{n-1}e_{n-1}, & \\ [e_n, e_2] = \lambda_4e_4 + \lambda_5e_5 + \ldots + \lambda_{n-1}e_{n-1}, & \\ [e_n, e_1] = \mu_4e_4 + \mu_5e_5 + \ldots + \mu_{n-1}e_{n-1} & \end{cases}$$

where $\mathrm{gr}(\mathcal{L}) = NGF2_{n-1} \oplus \mathbb{C},\ NGF2_{n-1} = \langle e_1, e_2, \ldots, e_{n-1}\rangle,\ \mathbb{C} = \langle e_n\rangle.$

$KF_4:$

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-3 \\ [e_1, e_{n-1}] = e_n + \alpha_3e_3 + \ldots + \alpha_{n-2}e_{n-2}, & \\ [e_{n-1}, e_{n-1}] = \beta_3e_3 + \beta_4e_4 + \ldots + \beta_{n-2}e_{n-2}, & \\ [e_i, e_{n-1}] = \beta_{i,i+2}e_{i+2} + \beta_{i,i+3}e_{i+3} + \ldots + \beta_{i,n-2}e_{n-2}, & 2 \leqslant i \leqslant n-4 \\ [e_n, e_{n-1}] = \gamma_4e_4 + \gamma_5e_5 + \ldots + \gamma_{n-2}e_{n-2} & \end{cases}$$

$KF_5:$

$$\begin{cases} [e_i,e_1]=e_{i+1}, & 1\leqslant i\leqslant n-3 \\ [e_1,e_{n-1}]=e_2+e_n+\alpha_3e_3+\alpha_4e_4+\ldots+\alpha_{n-2}e_{n-2}, & \\ [e_i,e_{n-1}]=e_{i+1}+\beta_{i,i+2}e_{i+2}+\beta_{i,i+3}e_{i+3}+\ldots+\beta_{i,n-2}e_{n-2}, & 2\leqslant i\leqslant n-4 \\ [e_{n-1},e_{n-1}]=\beta_3e_3+\beta_4e_4+\ldots+\beta_{n-2}e_{n-2}, & \\ [e_n,e_{n-1}]=\gamma_4e_4+\gamma_5e_5+\ldots+\gamma_{n-2}e_{n-2} & \end{cases}$$

$KF_6:$

$$\begin{cases} [e_i,e_1]=e_{i+1}, & 2\leqslant i\leqslant n-2 \\ [e_1,e_i]=-e_{i+1}, & 3\leqslant i\leqslant n-2 \\ [e_1,e_1]=\theta_1e_{n-1}, & \\ [e_1,e_2]=-e_3+\theta_2e_{n-1}, & \\ [e_2,e_2]=\theta_3e_{n-1} & \\ [e_i,e_j]=-[e_j,e_i]\in\{e_{i+j+1},e_{i+j+2},\ldots,e_{n-1}\}, & 2\leqslant i\leqslant n-4, \\ & 2\leqslant j\leqslant n-2-i \\ [e_1,e_n]=\beta_{1,4}e_4+\beta_{1,5}e_5+\ldots+\beta_{1,n-1}e_{n-1}, & \\ [e_i,e_n]=\beta_{i,i+2}e_{i+2}+\beta_{i,i+3}e_{i+3}+\ldots+\beta_{i,n-1}e_{n-1}, & 2\leqslant i\leqslant n-3 \\ [e_n,e_1]=\beta_{1,4}e_4+\beta_{1,5}e_5+\ldots+\beta_{1,n-1}e_{n-1}, & \\ [e_n,e_i]=\beta_{i,i+2}e_{i+2}+\beta_{i,i+3}e_{i+3}+\ldots+\beta_{i,n-1}e_{n-1}, & 2\leqslant i\leqslant n-3 \\ [e_n,e_n]=\theta_4e_4+\theta_5e_5+\ldots+\theta_{n-1}e_{n-1} & \end{cases}$$

where $gr(\mathcal{L})=NGF3_{n-1}\oplus\mathbb{C},\ NGF3_{n-1}=\langle e_1,e_2,\ldots,e_{n-1}\rangle,$ $\mathbb{C}=\langle e_n\rangle$, *i.e.,* $NGF3_{n-1}$ *is obtained from naturally graded filiform Lie algebras.*

Theorem 3.2. *Let $\mathcal{L}$ be a 2-filiform non-Lie Leibniz algebra of maximum length. Then $\mathcal{L}$ is isomorphic to the following algebra*

$$M:\begin{cases} [y_i,y_1] & = & y_{i+1}, \quad 1\leqslant i\leqslant n-3 \\ [y_1,y_{n-1}] & = & y_n \end{cases}$$

Proof. Let $\mathcal{L}$ be a 2-filiform Leibniz algebra. From Proposition 3.1 we have that $\mathcal{L}$ is isomorphic to $KF_1,\ KF_2,\ KF_3,\ KF_4,\ KF_5$, or KF_6.

First, we consider the case where $\mathcal{L}$ belongs to the family KF_1.

In this case $\mathcal{L}$ is a three generated algebra. Let $x_s,\ x_t,\ x_r$ be the generators of $\mathcal{L}$. Since $\mathcal{L}$ has also basis $\{e_1,\ e_2,\ldots,e_n\}$, then we have

$$\begin{aligned} x_s&=a_1e_1+a_2e_2+\ldots+a_{n-1}e_{n-1}+a_ne_n \\ x_t&=b_1e_1+b_2e_2+\ldots+b_{n-1}e_{n-1}+b_ne_n \\ x_r&=c_1e_1+c_2e_2+\ldots+c_{n-1}e_{n-1}+c_ne_n \end{aligned}$$

where $\begin{vmatrix} a_1 & a_{n-1} & a_n \\ b_1 & b_{n-1} & b_n \\ c_1 & c_{n-1} & c_n \end{vmatrix}\neq 0$. Without loss of generality we can assume that $a_1=b_{n-1}=c_n=1$.

Consider the products

$$
\begin{aligned}
[x_s, x_s] =& [e_1 + a_2e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n, e_1 + a_2e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n] = \\
&= e_2 + (*)e_3 + \ldots + (*)e_{n-2}, \\
[[x_s, x_s], x_s] &= [e_2 + (*)e_3 + \ldots + (*)e_{n-2}, e_1 + a_2e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n] = \\
&= e_3 + (*)e_4 + \ldots + (*)e_{n-2}, \\
&\vdots \\
\underbrace{[\ldots[x_s, x_s], \ldots, x_s]}_{n-2-times} &= e_{n-2}
\end{aligned}
$$

where the asterisks (*) denote the correspondent coefficient in the product. If we denote

$$y_1 = x_s,\ y_2 = [x_s, x_s], \ldots, y_{n-2} = [\ldots[x_s, x_s], \ldots, x_s], y_{n-1} = x_t,\ y_n = x_r$$

then we have that

$$V_{k_s} = \langle y_1 \rangle,\ V_{2k_s} = \langle y_2 \rangle, \ldots, V_{(n-2)k_s} = \langle y_{n-2} \rangle,\ V_{k_t} = \langle y_{n-1} \rangle,\ V_{k_r} = \langle y_n \rangle$$

1. $[V_{k_t}, V_{k_s}] = V_{\lambda k_s}$ or $[V_{k_r}, V_{k_s}] = V_{\mu k_s}$, with $1 \leqslant \lambda, \mu \leqslant n-2$.

We can suppose that $[V_{k_t}, V_{k_s}] \subseteq V_{\lambda k_s}$. However, it is evident that this is only possible in the case where $\lambda = 1$ and $k_t = 0$. Then, from decomposition of gradation and nilpotency of $\mathcal{L}$, we obtain that $[y_{n-1}, \mathcal{L}] = [\mathcal{L}, y_{n-1}] = 0$, i.e., $\mathcal{L}$ is split.

2. $[V_{k_t}, V_{k_s}] \neq V_{\lambda k_s}$ and $[V_{k_r}, V_{k_s}] \neq V_{\mu k_s}$, $1 \leqslant \lambda, \mu \leqslant n-2$.

Then, the condition $[V_{k_i}, V_{k_j}] \subseteq V_{k_i + k_j}$ implies that

$$
\begin{aligned}
&k_t = (n-1)k_s,\ k_r = nk_s \text{ and} \\
&\mathcal{L} = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1} \oplus V_n \text{ such that } V_i = \langle y_i \rangle,\ 1 \leqslant i \leqslant n.
\end{aligned}
$$

Thus, the product has the following form

$$
\begin{aligned}
&[y_i, y_1] = y_{i+1},\ 1 \leqslant i \leqslant n-3, \\
&[y_{n-1}, \mathcal{L}] = [\mathcal{L}, y_{n-1}] = [y_n, \mathcal{L}] = [\mathcal{L}, y_n] = 0
\end{aligned}
$$

That is, in this case $\mathcal{L}$ is split.

In case $\mathcal{L}$ belongs to KF_2 we have three generators $x_s,\ x_t,\ x_r$ again and we can express them with respect to the basis $\{e_1,\ e_2, \ldots, e_n\}$ of $\mathcal{L}$. Analogously,
$\begin{vmatrix} a_1 & a_2 & a_n \\ b_1 & b_2 & b_n \\ c_1 & c_2 & c_n \end{vmatrix} \neq 0.$

Now, without loss of generality we can assume that $a_1 = b_2 = c_n = 1$. So, the products are

$$[x_s, x_s] = [e_1 + a_2e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n, e_1 + a_2e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n] = \\ = (1 + a_2)e_3 + (*)e_4 + \ldots + (*)e_{n-1},$$

$$[[x_s, x_s], x_s] = [(1+a_2)e_3+(*)e_4+\ldots+(*)e_{n-1}, e_1+a_2e_2+\ldots+a_{n-1}e_{n-1}+a_ne_n] = \\ = (1 + a_2)e_4 + (*)e_5 + \ldots + (*)e_{n-1},$$

$$\vdots$$

$$\underbrace{[\ldots[x_s, x_s], \ldots, x_s]}_{n-2-times} = (1 + a_2)e_{n-1}$$

$$[x_s, x_t] = [e_1 + a_2e_2+\ldots+a_{n-1}e_{n-1} + a_ne_n, b_1e_1 + e_2+\ldots+b_{n-1}e_{n-1} + b_ne_n] = \\ = b_1(1 + a_2)e_3 + (*)e_4 + \ldots + (*)e_{n-1}.$$

1. $a_2 \neq -1$

We denote

$$y_1 = x_s,\ y_2 = [x_s, x_s], \ldots, y_{n-2} = [\ldots[x_s, x_s], \ldots, x_s], y_{n-1} = x_t,\ y_n = x_r$$

It is easy to see that $\{y_1,\ y_2, \ldots,\ y_n\}$ is a basis and we put

$$V_{k_s} = \langle y_1\rangle,\ V_{2k_s} = \langle y_2\rangle, \ldots, V_{(n-2)k_s} = \langle y_{n-2}\rangle, \\ V_{k_t} = \langle y_{n-1}\rangle,\ V_{k_r} = \langle y_n\rangle$$

Using similar arguments to the case where $\mathcal{L}$ is in KF_1, we obtain that $\mathcal{L}$ is split.

2. $a_2 = -1$.

Consider the products

$$[x_t, x_t] = [b_1e_1 + e_2 + \ldots + b_{n-1}e_{n-1} + b_ne_n, b_1e_1 + e_2 + \ldots + b_{n-1}e_{n-1} + b_ne_n] = \\ = b_1(1 + b_1)e_3 + (*)e_4 + \ldots + (*)e_{n-1}.$$

$$[[x_t, x_t], x_t] = [b_1(1+b_1)e_3+(*)e_4+\ldots+(*)e_{n-1}, b_1e_1+e_2+\ldots+b_{n-1}e_{n-1}+b_ne_n] \\ = b_1^2(1 + b_1)e_4 + (*)e_5 + \ldots + (*)e_{n-1}.$$

$$\vdots$$

$$\underbrace{[\ldots[x_t, x_t], \ldots, x_t]}_{n-2-times} = b_1^{n-3}(1 + b_1)e_{n-1}$$

Note that we can suppose that rank $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = 2$.

In the case $a_1 = b_2 = 1$ and $a_2 = -1 \Rightarrow b_1 \neq -1$.

2.1. $b_1 \neq 0$. Then, taking into account case 1 for x_t and x_s, we obtain a contradiction if $\mathcal{L}$ is non split.

2.2. $b_1 = 0$. Then, we write

$$x_s = e_1 - e_2 + \ldots + a_{n-1}e_{n-1} + a_ne_n \\ x_t = e_2 + \ldots + b_{n-1}e_{n-1} + b_ne_n \\ x_r = c_1e_1 + c_2e_2 + \ldots + c_{n-1}e_{n-1} + c_ne_n$$

Let us consider the products

$$
\begin{aligned}
&[x_t, x_s] = [e_2 + \ldots + b_{n-1}e_{n-1} + b_n e_n, e_1 - e_2 + \ldots + a_{n-1}e_{n-1} + a_n e_n] = \\
&= e_3 + (*)e_4 + \ldots + (*)e_{n-1}, \\
&[[x_t, x_s], x_s] = [e_3 + (*)e_4 + \ldots + (*)e_{n-1}, e_1 - e_2 + \ldots + a_{n-1}e_{n-1} + a_n e_n] \\
&= e_4 + (*)e_5 + \ldots + (*)e_{n-1}, \\
&\quad \vdots \\
&[\ldots[x_t, x_s], \ldots, x_s] = e_{n-1}
\end{aligned}
$$

In this case we denote $y_1 = x_s,\ y_2 = x_t,\ y_3 = [x_t, x_s], \ldots, y_{n-1} = [\ldots[x_t, x_s], \ldots, x_s], y_n = x_r$, and so we put

$$V_{k_s} = \langle y_1 \rangle,\ V_{k_t} = \langle y_2 \rangle,\ V_{k_t+k_s} = \langle y_3 \rangle, \ldots, V_{k_t+(n-3)k_s} = \langle y_{n-1} \rangle,\ V_{k_r} = \langle y_n \rangle$$

with $[y_i, y_1] = y_{i+1},\ 2 \leqslant i \leqslant n-2$.

2.2.1. $[V_{k_r}, V_{k_s}] \neq 0$ or $[V_{k_s}, V_{k_r}] \neq 0$.
Then $V_{k_r+k_s} = V_{k_t+\lambda k_s}, 0 \leqslant \lambda \leqslant n-3$ and so we have $k_r = (\lambda - 1)k_s + k_t$. That is a contradiction because x_t is a generator.

2.2.2. $[V_{k_r}, V_{k_s}] = [V_{k_s}, V_{k_r}] = 0$.
In this case we have $[y_1, y_n] = [y_n, y_1] = 0$ and using the multiplication in the family KF_2, we obtain $[\mathcal{L}, y_n] = [y_n, \mathcal{L}] = 0$, i.e., $\mathcal{L}$ is split.

Let $\mathcal{L}$ be isomorphic to one algebra of the family KF_3.

We have that $x_s,\ x_t,\ x_r$ are the generators of $\mathcal{L}$. Without loss of generality we suppose that $a_1 = b_2 = c_n = 1$.

Now we denote

$$y_1 = x_s,\ y_2 = [x_s, x_s], \ldots, y_{n-2} = [\ldots[x_s, x_s], \ldots, x_s], y_{n-1} = x_t,\ y_n = x_r$$

Evidently $y_1,\ y_2, \ldots, y_n$ form a basis and we have that

$$V_{k_s} = \langle y_1 \rangle,\ V_{2k_s} = \langle y_2 \rangle, \ldots, V_{(n-2)k_s} = \langle y_{n-2} \rangle,\ V_{k_t} = \langle y_{n-1} \rangle,\ V_{k_r} = \langle y_n \rangle$$

1. $[V_{k_r}, V_{k_s}] \neq 0$ or $[V_{k_s}, V_{k_r}] \neq 0$.

We can suppose that $[V_{k_r}, V_{k_s}] \neq 0$. Then $V_{k_r+k_s} = V_{\lambda k_s}, 1 \leqslant \lambda \leqslant n-2$. So $k_r = (\lambda - 1)k_s + k_t$ or $k_r + k_s = k_t$. But that is a contradiction because x_r and x_t are generators.

2. $[V_{k_r}, V_{k_s}] = [V_{k_s}, V_{k_r}] = 0$.

That is, $[y_1, y_n] = [y_n, y_1] = 0$, it is easy to see that $[\mathcal{L}, y_n] = [y_n, \mathcal{L}] = 0$, i.e. $\mathcal{L}$ is split.

Let $\mathcal{L}$ be isomorphic to one algebra of the family KF_4.

Note that $\mathcal{L}$ is a two generated algebra whose generators are x_s, x_t. Without loss of generality we can assume that $a_1 = b_{n-1} = 1$ with $a_1 b_{n-1} \neq 1$.

Let us reconsider the products $[x_s, x_s]$, $\dots$ $[\dots[x_s, x_s], \dots, x_s]$ and

$$[x_s, x_t] = b_1 e_2 + e_n + (*)e_3 + \dots + (*)e_{n-2}$$

By denoting $y_1 = x_s$, $y_2 = [x_s, x_s]$, $\dots$, $y_{n-2} = [\dots[x_s, x_s], \dots, x_s]$, $y_{n-1} = x_t$, $y_n = [x_s, x_t]$, we can write $V_{k_s} = \langle y_1 \rangle$, $V_{2k_s} = \langle y_2 \rangle$, $\dots$, $V_{(n-2)k_s} = \langle y_{n-2} \rangle$, $V_{k_t} = \langle y_{n-1} \rangle$, $V_{k_s+k_t} = \langle y_n \rangle$. Since $[V_{k_s+k_t}, V_{k_s}] \subseteq V_{2k_s+k_t}$, then we can consider the cases below:

1. $[V_{k_s+k_t}, V_{k_s}] = V_{\lambda k_s}$.
Then $[V_{k_s+k_t}, V_{k_s}] = V_{2k_s+k_t} = V_{\lambda k_s}$ where $1 \leqslant \lambda \leqslant n-2$ and we obtain that $k_t = (\lambda - 2)k_s$, but it is contradiction for $\lambda \neq 1$. Therefore we have the decomposition:

$$\mathcal{L} = V_{-1} \oplus V_0 \oplus V_1 \oplus \dots \oplus V_{n-2}$$

where $V_{-1} = \langle y_{n-1} \rangle$, $V_0 = \langle y_n \rangle$, $V_i = \langle y_i \rangle$, $1 \leqslant i \leqslant n-2$.

Checking all the bracket products with the restriction $b_1 a_{n-1} \neq 1$ and applying the restriction from the nilpotency, we obtain the algebra M.

2. $[V_{k_s+k_t}, V_{k_s}] \neq V_{\lambda k_s}$, $1 \leqslant \lambda \leqslant n-2$.
Then from condition $[V_{k_i}, V_{k_j}] \subseteq V_{k_i+k_j}$, we have that $k_t = (n-1)k_s$ so, $\mathcal{L} = V_1 \oplus V_2 \oplus \dots \oplus V_{n-1} \oplus V_n$ where $V_i = \langle y_i \rangle$, $1 \leqslant i \leqslant n$. It is not difficult to see that we obtain the algebra M.

If $\mathcal{L}$ belongs to the family KF_5, we use similar arguments to the above cases.

Consider the products:

$$[\dots[x_s, x_s], \dots, x_s] = (1 + a_{n-1})^{n-3} e_{n-2}$$
$$[\dots[x_t, x_t], \dots, x_t] = b_1(1 + b_1)^{n-3} e_{n-2}$$
$$[x_s, x_t] = (1 + b_1)e_2 + e_n + (*)e_3 + \dots + (*)e_{n-2},$$
$$[x_t, x_s] = b_1(1 + a_{n-1})e_2 + b_1 a_{n-1} e_n + (*)e_3 + \dots + (*)e_{n-2}.$$

1. $a_{n-1} \neq -1$
Then we put $y_1 = x_s$, $y_2 = [x_s, x_s]$, $\dots$, $y_{n-2} = [\dots[x_s, x_s], \dots, x_s]$, $y_{n-1} = x_t$, $y_n = [x_s, x_t]$

It is evident that y_1, y_2, $\dots$, y_n are linearly independent. Therefore we can take it as a basis of $\mathcal{L}$ and we obtain

$$V_{k_s} = \langle y_1 \rangle, \ V_{2k_s} = \langle y_2 \rangle, \dots, V_{(n-2)k_s} = \langle y_{n-2} \rangle, V_{k_t} = \langle y_{n-1} \rangle, \ V_{k_s+k_t} = \langle y_n \rangle$$

Checking all the products we get again the algebra M.

2. $a_{n-1} = -1$
Since $1 - a_{n-1}b_1 \neq 0$, then $b_1 \neq -1$

2.1. $b_1 \neq 0$.
Making the change $x'_s = \frac{x_t}{b_1},\ x'_t = -x_s$ we have case 1.

2.2. $b_1 = 0$ In a similar way we can denote

$$y_1 = x_t,\ y_2 = x_s,\ y_3 = [x_s, x_t], \dots,$$
$$y_{n-1} = [\dots[x_s, x_t], \dots, x_t],\ y_n = [x_s, x_s]$$

In this case we do not obtain any new algebra with maximum length.

Let $\mathcal{L}$ be isomorphic to one algebra of the family KF_6.

Then $x_s,\ x_t,\ x_r$ are generators of $\mathcal{L}$. In a similar way we study the different possibilities for the gradation, and we obtain a Lie algebra. □

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J. M. Cabezas
Dpto. de Matemática Aplicada
Universidad del País Vasco
C/ Nieves Cano 12
01006 Vitoria, Álava, Spain
jm.cabezas@ehu.esw

L. M. Camacho
Dpto. Matemática Aplicada I
Universidad de Sevilla
Avda. Reina Mercedes s/n
41012 Sevilla, Spain
lcamacho@us.es

I. Rodríguez
Dpto. de Matemáticas
Universidad de Huelva
Ctra. de Palos de la Frontera s/n
21071 Rábida. Huelva, Spain
rodgar@uhu.es